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Inequalities for Polynomials with a Prescribed Zero

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§1. Introduction

In [11, prob. 8.2] Halász posed the following problem concerning polynomials having a prescribed zero at $z=1$: For each $n \geqq 1$, find

$$\lambda_n := \sup \left\{ \frac{|f(0)|}{\max_{|z|=1} |f(z)|} : f \in \pi_n,\ f(1)=0 \right\}, \tag{1.1}$$

where π_n denotes the collection of all complex polynomials of degree at most n. As stated by Halász, the solution to (1.1) has application to certain inequalities of Turán for lacunary polynomials. Bounds for λ_n were obtained by Halász [4], by Rahman and Stenger [14], and by Rahman and Schmeisser [12, 13], the latter authors establishing that

$$1 - \frac{\pi^2}{8n} \leqq \lambda_n \leqq 1 - \frac{1.03369}{n} + O(1/n^2), \qquad n \geqq 1. \tag{1.2}$$

Blatt [1] in a different connection raised the following problem: For each $n \geqq 1$, find

$$\mu_n := \min \{ \|p\|_{|z|=1} : p(1)=0,\ p(z)=z^n+\ldots \in \pi_n \}, \tag{1.3}$$

where $\|p\|_{|z|=1} := \sup \{ |p(z)| : |z|=1 \}$. Because $p \in \pi_n$ and its reciprocal polynomial (cf. (2.9)) p^* satisfy $\|p\|_{|z|=1} = \|p^*\|_{|z|=1}$, the problems of Halász and Blatt are equivalent in the sense that

$$\lambda_n = 1/\mu_n, \qquad n \geqq 1. \tag{1.4}$$

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The purpose of the present paper is to study the following more general problem: For each pair of nonnegative integers s, m, find

$$\min\{\|(z-1)^s q_m(z)\|_{|z|=1}: q_m(z)=z^m+\ldots\in\pi_m\}. \tag{1.5}$$

As a consequence of our results, we solve exactly the problems of Halász and Blatt by proving that

$$\lambda_n=\left[\cos\frac{\pi}{2(n+1)}\right]^{n+1}, \quad n\geqq 1. \tag{1.6}$$

In addition, we determine growth estimates for polynomials having prescribed order zeros at $z=1$, and these estimates are shown, in a limiting sense, to be best possible.

§ 2. Notation and Equivalence Theorems

As stated earlier, π_n shall denote the collection of all complex polynomials of degree at most n. We let $\pi_n^{\mathbf{C}}$ be the subset of π_n consisting of complex polynomials of degree at most n all of whose zeros lie on the circle $\mathbf{C}$: $|z|=1$. Polynomials in $\pi_n^{\mathbf{C}}$ shall be referred to as $\mathbf{C}$-*polynomials*. For each pair s, m of nonnegative integers, we further put

$$\pi_{s,m}:=\{(z-1)^s q_m(z): q_m\in\pi_m\}\subset\pi_{s+m}. \tag{2.1}$$

Finally, for any continuous function g defined on a compact set B of the plane, we set

$$\|g\|_B:=\max\{|g(z)|: z\in B\}. \tag{2.2}$$

We now state three different but related problems for constrained polynomials.

Problem I. For any nonnegative integers s, m, determine

$$e_{s,m}:=\min\{\|p\|_{\mathbf{C}}: p(z)=(z-1)^s q_m(z),\ q_m(z)=z^m+\ldots\in\pi_m\}, \tag{2.3}$$

where $\mathbf{C}$: $|z|=1$.

The analogous problem for $\mathbf{C}$-polynomials is stated as

Problem II. For any nonnegative integers s, m, determine

$$E_{s,m}:=\min\{\|P\|_{\mathbf{C}}: P(z)=(z-1)^s Q_m(z),\ Q_m(z)=z^m+\ldots\in\pi_m^{\mathbf{C}}\}. \tag{2.4}$$

The third extremal problem concerns constrained polynomials on the real interval $[-1,1]$.

Problem III. For any nonnegative real numbers α, β, and any nonnegative integer m, determine

$$\varepsilon(\alpha,\beta,m):=\min\{\|T\|_{[-1,1]}: T(x)=(1-x)^\alpha(1+x)^\beta\tau_m(x),\ \tau_m(x)=x^m+\ldots\in\pi_m\}. \tag{2.5}$$

The solution to Prob. I is known to be unique (cf. Walsh [18], p. 363) and will be denoted by $p_{s,m}(z)$, i.e.,

$$e_{s,m} = \|p_{s,m}\|_{\mathsf{C}}. \tag{2.6}$$

Prob. III also has a unique solution, which is characterized by an equioscillation property (cf. [9]), and we denote it by $T_m^{(\alpha,\beta)}(x)$. Thus,

$$\varepsilon(\alpha, \beta, m) = \|T_m^{(\alpha,\beta)}\|_{[-1,1]}. \tag{2.7}$$

When $s=0$, Prob. II has infinitely many solutions of the form $z^m - e^{i\theta}$, $0 \leqq \theta < 2\pi$; however, if $s \geqq 1$ we shall show below that Prob. II has a unique solution and it will be denoted by $P_{s,m}(z)$. Thus

$$E_{s,m} = \|P_{s,m}\|_{\mathsf{C}}, \quad s \geqq 1, \ m \geqq 0. \tag{2.8}$$

To establish the relationship between the solutions of Problems I and II, we require two lemmas. The first is a slight generalization of an exercise in [10], which is stated without proof.

Lemma 2.1 (Pólya-Szegö [10], v. 1, p. 108). *Let $p(z)$ be a polynomial of degree n all of whose zeros lie in $|z| \leqq 1$ and let $p^*(z)$ denote the reciprocal polynomial of $p(z)$, defined by*

$$p^*(z) := z^n \overline{p(1/\bar{z})}. \tag{2.9}$$

Then, the polynomial $z^k p(z) + e^{i\theta} p^*(z)$, for k any nonnegative integer and θ real, has all its zeros on C: $|z|=1$.

The second result which we need is the well-known Erdös-Lax theorem.

Lemma 2.2 (Lax [7]). *Let $P(z)$ be a polynomial of degree n all of whose zeros lie on or exterior to C: $|z|=1$. Then*

$$\|P'\|_{\mathsf{C}} \leqq \frac{n}{2} \|P\|_{\mathsf{C}}. \tag{2.10}$$

Furthermore, if P has all its zeros on C, then equality holds in (2.10).

We now relate the solutions of Prob. I with those of Prob. II by

Theorem 2.3. *For each pair s, m, of nonnegative integers, let $p_{s,m}(z) = (z-1)^s(z^m + \ldots) \in \pi_{s,m}$ be the unique solution to Prob.* I, *and let $P_{s+1,m}(z) = (z-1)^{s+1}(z^m + \ldots)$ be any polynomial in π^{C}_{s+m+1} for which $\|P_{s+1,m}\| = E_{s+1,m}$. Then,*

$$p_{s,m}(z) \equiv P'_{s+1,m}(z)/(s+m+1), \tag{2.11}$$

$$e_{s,m} = E_{s+1,m}/2. \tag{2.12}$$

Consequently, $P_{s+1,m}(z)$ is unique.

Proof. Since $p_{s,m}(z)$ is unique, its coefficients must be real, and so its nonreal roots occur in complex conjugate pairs. Furthermore, $p_{s,m}(z)$ has a zero of precise multiplicity s at $z=1$ and its m remaining zeros lie interior to C, as we now show.

For $m \geqq 1$, write $p_{s,m}(z)=(z-1)^s \prod_{k=1}^{m}(z-\alpha_k)$, and assume that, for some k, we have $|\alpha_k| \geqq 1$. If, for $\delta > 0$, we set

$$r(z;\delta):=p_{s,m}(z)\left(\frac{z-(1-\delta)\alpha_k}{z-\alpha_k}\right),$$

then it is easy to see that for δ sufficiently small $\|r(\cdot\,;\delta)\|_{\mathsf{C}} < e_{s,m} = \|p_{s,m}\|_{\mathsf{C}}$, which is a contradiction. Hence, $|\alpha_k| < 1$ for $1 \leqq k \leqq m$.

Next, we define the polynomial $Q(z)$ by

$$Q(z):=z\,p_{s,m}(z)+(-1)^{s+1}\,p^*_{s,m}(z), \tag{2.13}$$

where $p^*_{s,m}(z)$ is the reciprocal polynomial of $p_{s,m}(z)$ (cf. (2.9)). It is quickly verified that $Q(z)$ is a monic polynomial of exact degree $s+m+1$ with a zero of multiplicity $s+1$ at $z=1$. Also, since all the zeros of $p_{s,m}(z)$ lie in $|z| \leqq 1$, it follows from Lemma 2.1 that $Q(z) \in \pi^{\mathsf{C}}_{s+m+1}$.

Now as $Q'(z)/(s+m+1) \in \pi_{s,m}$ is monic, it is a competitor of $p_{s,m}(z)$. Hence, on applying Lemma 2.2, we obtain

$$e_{s,m} = \|p_{s,m}\|_{\mathsf{C}} \leqq \left\|\frac{Q'}{s+m+1}\right\|_{\mathsf{C}} = \tfrac{1}{2}\|Q\|_{\mathsf{C}} \leqq \|p_{s,m}\|_{\mathsf{C}}, \tag{2.14}$$

where the last inequality follows by applying the triangle inequality to (2.13). But, since $p_{s,m}(z)$ is unique, we have

$$p_{s,m}(z) \equiv Q'(z)/(s+m+1). \tag{2.15}$$

Next we show that $Q(z) \equiv P_{s+1,m}(z)$. From the definition of $P_{s+1,m}(z)$ as an extremal for Prob. II, we have

$$E_{s+1,m} = \|P_{s+1,m}\|_{\mathsf{C}} \leqq \|Q\|_{\mathsf{C}}. \tag{2.16}$$

On the other hand, as $P'_{s+1,m}(z)/(s+m+1)$ is an admissible polynomial for Prob. I, we have from (2.15) that

$$e_{s,m} = \|p_{s,m}\|_{\mathsf{C}} = \left\|\frac{Q'}{s+m+1}\right\|_{\mathsf{C}} \leqq \left\|\frac{P'_{s+1,m}}{s+m+1}\right\|_{\mathsf{C}}. \tag{2.17}$$

But since all the zeros of $Q(z)$ and $P_{s+1,m}(z)$ lie on C, it follows from (2.17) and the second part of Lemma 2.2 that

$$\|Q\|_{\mathsf{C}} \leqq \|P_{s+1,m}\|_{\mathsf{C}}. \tag{2.18}$$

Consequently, from (2.16), we have that equality holds in (2.18) and, a fortiori, in (2.17). Thus, from the uniqueness of solutions to Prob. I, we have

$$Q(z)=(s+m+1)\int_1^z p_{s,m}(t)\,dt = P_{s+1,m}(z), \tag{2.19}$$

which proves that $P_{s+1,m}(z)$ is unique. Finally, Eq. (2.12) follows from (2.19) and (2.14). ∎

From the previous discussion, for each pair of integers $s \geqq 0$, $m \geqq 0$, the polynomial $p_{s,m}(z)$ has a zero of precise order s at $z=1$ and its m remaining zeros lie interior to C. By Theorem 2.3 then, the C-polynomial $P_{s+1,m}(z)$ must have real coefficients with a zero of order exactly $s+1$ at $z=1$, and its nonreal zeros must be simple, occurring in complex conjugate pairs on C. Under the mapping $x=(z+z^{-1})/2$, a typical factor of $P_{s+1,m}(z)$ of the form $(z-e^{i\phi})(z-e^{-i\phi})$ becomes $2z(x-\cos\phi)$. This fact immediately enables us to relate the solutions of Prob. II with those of Prob. III.

Theorem 2.4. *For each pair of integers $s \geqq 1$, $m \geqq 0$, we have*

$$P_{s,m}(z)=\begin{cases}(-1)^{s/2}(2z)^{(s+m)/2}\, T^{(s/2,0)}_{m/2}((z+z^{-1})/2), & m \text{ even};\\ (-1)^{s/2}(2z)^{(s+m)/2}\, T^{(s/2,1/2)}_{(m-1)/2}((z+z^{-1})/2), & m \text{ odd};\end{cases} \tag{2.20}$$

$$E_{s,m}=\begin{cases}2^{(s+m)/2}\,\varepsilon(s/2,0,m/2), & m \text{ even};\\ 2^{(s+m)/2}\,\varepsilon(s/2,1/2,(m-1)/2), & m \text{ odd},\end{cases} \tag{2.21}$$

where $P_{s,m}(z)$ denotes the solution to Prob. II *and $T^{(\alpha,\beta)}_m(x)$ denotes the solution of Prob.* III.

Now, in order to solve the problem of Halász and Blatt, we need only determine the value $e_{1,m}=\mu_{m+1}$ for each integer $m \geqq 0$. However, in the light of equations (2.12) and (2.21), this is equivalent to solving for $\varepsilon(1,0,\nu)$ and $\varepsilon(1,1/2,\nu)$, for $\nu=0,1,2,\ldots$. In so doing, we shall establish

Theorem 2.5. *For each nonnegative integer m,*

$$e_{1,m}=\left[\cos\frac{\pi}{2(m+2)}\right]^{-(m+2)}. \tag{2.22}$$

Consequently, recalling (1.3) *and* (1.4), *the solution to the Halász problem is*

$$\lambda_n=\frac{1}{e_{1,n-1}}=\left[\cos\frac{\pi}{2(n+1)}\right]^{n+1}, \quad n \geqq 1. \tag{2.23}$$

Proof. For the case $m=2\nu$, we determine $\varepsilon(1,0,\nu)$ by exhibiting the real function $T^{(1,0)}_\nu(x)$ in terms of the classical Chebyshev polynomial of the first kind, $\mathsf{T}_{\nu+1}(x) := \cos(\nu+1)\theta$, with $x=\cos\theta$ and $\nu \geqq 0$. It is well-known (cf. [9], p. 32) that $\mathsf{T}_{\nu+1}(x)$ has an alternation set of precisely $\nu+2$ points in the interval $[-1,1]$, and its largest zero occurs at $x_\nu := \cos[\pi/(2(\nu+1))]$. Thus, with $t_\nu(x) := (x_\nu - 1 + (x_\nu+1)x)/2$, the polynomial $\mathsf{T}_{\nu+1}(t_\nu(x))$ vanishes at $x=1$ and has exactly $\nu+1$ alternants in $[-1,1]$. After normalizing the leading coefficient, we have

$$T^{(1,0)}_\nu(x)=-2(1+x_\nu)^{-(\nu+1)}\,\mathsf{T}_{\nu+1}(t_\nu(x)), \tag{2.24}$$

$$\varepsilon(1,0,\nu)=2(1+x_\nu)^{-(\nu+1)}=2^{-\nu}\left[\cos\frac{\pi}{4(\nu+1)}\right]^{-2(\nu+1)}. \tag{2.25}$$

For the case when $m=2\nu-1$, we obtain $\varepsilon(1,1/2,\nu-1)$ by considering modified Chebyshev polynomials of the second kind, $\mathsf{U}_{2\nu}(x) := \sin[(2\nu+1)\theta]/\sin\theta$, with $x=\cos\theta$ and $v\geqq 1$. $\mathsf{U}_{2v}(x)$ is an *even* polynomial which, when weighted by the factor $\sqrt{1-x^2}$, has an alternation set of exactly $2v+1$ points in $[-1,1]$. Further, $\mathsf{U}_{2v}(x)$ has its smallest positive zero at $y_v := \cos[v\pi/(2v+1)]$. Thus, for x in the positive interval $[y_v^2, 1]$, we put $\tilde{\mathsf{U}}_v(x) := \mathsf{U}_{2v}(\sqrt{x})$. The real polynomial $\tilde{\mathsf{U}}_v(x)\in\pi_v$ vanishes at $x=y_v^2$ and the real function $\sqrt{1-x}\ \tilde{\mathsf{U}}_v(x)$ has exactly v alternants in $[y_v^2, 1]$. Hence, with the transformation $r_v(x) := (y_v^2+1+(y_v^2-1)x)/2$, and after the normalization of the leading coefficient, we obtain

$$T_{v-1}^{(1,1/2)}(x) = (-1)^{v-1}2^{-v+1/2}(1-y_v^2)^{-v-1/2}\sqrt{1-r_v(x)}\ \tilde{\mathsf{U}}_v(r_v(x)), \tag{2.26}$$

$$\varepsilon(1,1/2,v-1) = 2^{-v+1/2}(1-y_v^2)^{-v-1/2} = 2^{-v+1/2}\left[\cos\frac{\pi}{2(2v+1)}\right]^{-(2v+1)}. \tag{2.27}$$

Finally, from Eqs. (2.12) and (2.21), we have

$$\begin{cases} e_{1,2v} = E_{2,2v}/2 = 2^v\varepsilon(1,0,v), & v\geqq 0, \\ e_{1,2v-1} = E_{2,2v-1}/2 = 2^{v-1/2}\varepsilon(1,1/2,v-1), & v\geqq 1, \end{cases} \tag{2.28}$$

which, utilizing (2.25) and (2.27), gives the desired Eq. (2.22). ∎

Employing Theorems (2.3), (2.4), and (2.5), we can explicitly determine the extremal polynomials $p_{1,m}(z)$ of Prob. I. These are given in Table 2.1, for $m=0,1,2,3$.

Table 2.1. $p_{1,m}(z)$, $m=0,1,2,3$.

$p_{1,0}(z)=(z-1)$
$p_{1,1}(z)=(z-1)(z+1/3)$
$p_{1,2}(z)=(z-1)\{z^2-(8-6\sqrt{2})z+(3-2\sqrt{2})\}$
$p_{1,3}(z)=(z-1)\{5z^3+(8\sqrt{5}-15)z^2+(2\sqrt{5}-3)z+(5-2\sqrt{5})\}/5$

We remark from (2.23) that $\lambda_n \sim 1-\pi^2/8n$ as $n\to\infty$, which shows that the lower estimate of (1.2) found by Rahman and Schmeisser [12] was in fact best possible, asymptotically.

As an easy application of Theorem 2.5 we have

Corollary 2.6. *For each pair of integers* $s\geqq 1$, $n\geqq 0$, *the solution to the extremal problem*

$$\gamma_{s,n} := \min\{\|(z^s-1)q_n(z)\|_{\mathsf{C}} : q_n(z)=z^n+\ldots\in\pi_n\} \tag{2.29}$$

is given by $\gamma_{s,n}=e_{1,m}=\{\cos(\pi/2(m+2))\}^{-(m+2)}$, *where* $m=[n/s]$ *is the integer part of* n/s.

The proof of Corollary 2.6 follows from the fact that $(z^s-1)q_n(z)=z^r p_{1,m}(z^s)$ is the unique extremal polynomial for (2.29) when $n=ms+r$.

Concerning the value of $e_{s,m}$ in the case when $s>1$, we have not been able to determine an explicit representation for $e_{s,m}$. However, we do give upper and lower estimates in

Theorem 2.7. *For each pair of integers* $s>1$, $m\geqq 0$, *we have*

$$L_{s,m}\leqq e_{s,m}\leqq\sqrt{s+m+1}\,L_{s,m}, \tag{2.30}$$

where $L_{s,m}:=\left[\binom{m+2s}{s}\Big/\binom{m+s}{s}\right]^{1/2}$.

Proof. For s fixed, let $\{\phi_m(z)\}_{m=0}^{\infty}$ denote the sequence of the monic polynomials of respective degree m which are orthogonal on C: $|z|=1$ with respect to the weight function $|z-1|^{2s}$. Erdös and Turán [2] showed that

$$\frac{1}{2\pi}\int_{\mathsf{C}}|z-1|^{2s}|\phi_m(z)|^2\,|dz|=\binom{m+2s}{s}\binom{m+s}{s}^{-1}=L_{s,m}^2.$$

We obtain the left-hand inequality of (2.30), since (cf. [17], p. 289)

$$L_{s,m}^2\leqq\frac{1}{2\pi}\int_{\mathsf{C}}|p_{s,m}(z)|^2\,|dz|\leqq e_{s,m}^2. \tag{2.31}$$

The right-hand inequality of (2.30) follows from (cf. [17], p. 290)

$$e_{s,m}^2\leqq\|(z-1)^s\,\phi_m(z)\|_{\mathsf{C}}^2\leqq L_{s,m}^2\sum_{k=0}^{s+m}\|z^k\|_{\mathsf{C}}^2=(s+m+1)\,L_{s,m}^2. \quad \blacksquare \tag{2.32}$$

In our next two results, using the equivalence theorems of this section, we discuss the location of the zeros of the solutions to Prob. II and Prob. III. We postpone the discussion about the zeros of the solutions to Prob. I until §3. We first examine the location of the zeros of $T_m^{(\alpha,\beta)}(x)$, the solution of Prob. III. Because of the similarity of this result with those of [6] and [16] we state without proof

Lemma 2.8. *Let* $\alpha,\beta\geqq 0$, *and let* m *be any nonnegative integer. Then* $T_m^{(\alpha,\beta)}(x)$ *has an alternation set of exactly* $m+1$ *points in the interval* $[a,b]$, *where* a *and* b *are given by*

$$\begin{aligned}a&=a(\alpha,\beta,m):=\mu\nu-\sqrt{(1-\mu^2)(1-\nu^2)},\\ b&=b(\alpha,\beta,m):=\mu\nu+\sqrt{(1-\mu^2)(1-\nu^2)},\end{aligned} \tag{2.33}$$

with $\mu:=(\beta+\alpha)/(\alpha+\beta+m)$ *and* $\nu:=(\beta-\alpha)/(\alpha+\beta+m)$. *Furthermore, as* $T_m^{(\alpha,\beta)}(x)$ *has exactly* m *zeros interior to* $[-1,1]$, *they must lie in* (a,b).

As an easy consequence of Lemma 2.8 and Eq. (2.20) of Theorem 2.4, we have, for the solutions $P_{s,m}(z)$ of Prob. II,

Corollary 2.9. *For each pair of integers* $s\geqq 1$, $m\geqq 0$, *the zeros of* $P_{s,m}(z)$ *different from* $z=1$ *all lie on the arc*

$$\{z\colon |z|=1,\ \operatorname{Re} z\leqq 1-2(s/(s+m))^2\}. \tag{2.34}$$

In proving Corollary 2.9, one only needs to observe that

$$b(s/2,1/2,(m-1)/2)<b(s/2,0,m/2)=1-2(s/(s+m))^2. \tag{2.35}$$

§ 3. Growth Estimates

In this section, we determine growth estimates for polynomials from $\pi_{s,m}$ (cf. (2.1)). These upper estimates are obtained with respect to the unit circle, and are shown, in a limiting sense, to be best possible. For analogous results with respect to real intervals, we refer the reader to [5, 6, 8, 15], and [16].

Before stating our estimates we introduce some needed notation. For $0<\theta<1$ fixed, we let A_θ denote the arc of the unit circle given by

$$A_\theta := \{z: |z|=1,\ \operatorname{Re} z \leqq 1-2\theta^2\}, \tag{3.1}$$

and let $\mathbb{C}^*$ denote the extended complex plane. Then the function

$$w=\phi(z)=\phi(z;\theta):=\frac{\sqrt{1+\theta}\,\Psi(\zeta(z))-i\sqrt{1-\theta}}{\sqrt{1-\theta}\,\Psi(\zeta(z))-i\sqrt{1+\theta}}, \tag{3.2}$$

where $\zeta(z)=\zeta(z;\theta):=i\theta(1-\theta^2)^{-1/2}(z+1)/(z-1)$ and $\Psi(\zeta):=\zeta+\sqrt{\zeta^2-1}$ (with the branch chosen so that $|\Psi(\zeta)|>1$), maps $\mathbb{C}^*\setminus A_\theta$ in the z-plane conformally onto $|w|>1$ in the w-plane so that the points at infinity correspond to each other. Next, we put

$$G(z;\theta):=\begin{cases} |\phi(z)|\left|\dfrac{\phi(z)-\phi(1)}{\overline{\phi(1)}\,\phi(z)-1}\right|^\theta, & z\in\mathbb{C}^*\setminus A_\theta,\\ 1, & z\in A_\theta,\end{cases} \tag{3.3}$$

where $\phi(z)=\phi(z;\theta)$ is defined in Eq. (3.2). We further extend the definition of $G(z;\theta)$ continuously to the case $\theta=1$ by setting

$$G(z;1):=|z-1|/2. \tag{3.4}$$

With the above definitions, we now prove

Theorem 3.1. *If* $p(z)\in\pi_{s,m}$ (cf. (2.1)), *where* $s\geqq 1$, $m\geqq 0$, *then*

$$|p(z)|\leqq \|p\|_{A_{s/(s-m)}}[G(z;s/(s+m))]^{s+m}, \quad \textit{for all } z\in\mathbb{C}^*. \tag{3.5}$$

Consequently,

$$|p(z)|\leqq \|p\|_{\mathrm{C}}[G(z;s/(s+m))]^{s+m}, \quad \textit{for all } z\in\mathbb{C}^*. \tag{3.6}$$

Proof. As the inequality (3.5) follows trivially from (3.4) when $m=0$, we assume $m>0$. For each θ, $0<\theta<1$, we combine two lemmas of Walsh ([18], p. 250) to obtain the estimate

$$|p(z)|\leqq \|p\|_{A_\theta}|\phi(z)|^{s+m}\left|\frac{\phi(z)-\phi(1)}{\overline{\phi(1)}\,\phi(z)-1}\right|^s,$$

where $z\in\mathbb{C}^*\setminus A_\theta$ and $\phi(z)=\phi(z;\theta)$ (cf. [5, 6]). Selecting $\theta=s/(s+m)$ in the last inequality, and recalling the definition in (3.3), we deduce the desired inequality (3.5). ∎

We remark that although inequality (3.6) follows from (3.5) by the apparently crude observation that $A_{s/(s+m)} \subset \mathsf{C}$, we shall show in Theorem 3.3 that, in fact, (3.6) is *sharp* in a certain limiting sense. Moreover, even in the case $s=1$, the estimate of (3.6) appears to improve upon known bounds derived by Giroux and Rahman [3].

We next mention some properties of the function $G(z;\theta)$. Clearly $G(1;\theta)=0$, and $G(z;\theta)\to\infty$ as $|z|\to\infty$. It is also straightforward to verify that the level curve $G(z;\theta)=1$ is a simple closed curve consisting, in part, of the arc A_θ and containing in its interior the complementary arc, $\mathsf{C}\setminus A_\theta$, as well as the open unit disk. We denote this interior by $\Lambda(\theta)$, i.e.,

$$\Lambda(\theta) := \{z\colon G(z;\theta)<1\}. \tag{3.7}$$

Figure 3.1 illustrates the particular level curve $G(z;\frac{1}{2})=1$.

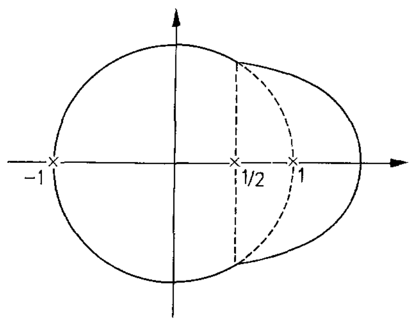

Fig. 3.1. $G(z;\frac{1}{2})=1$ is indicated by a solid line

With the above remarks, Theorem 3.1 immediately implies

Corollary 3.2. *If $p(z)\in\pi_{s,m}$, where $s\geqq 1$, $m\geqq 0$, and $p(z)$ is not identically zero, then*

$$|p(z)| < \|p\|_{\mathsf{C}}, \quad \textit{for all } z\in\Lambda(s/(s+m)). \tag{3.8}$$

Furthermore, if $\xi\in\mathsf{C}$ and $|p(\xi)|=\|p\|_{\mathsf{C}}$, then ξ lies on the arc $A_{s/(s+m)}$ (cf. (3.1)).

We next show that the estimates of (3.6) (and hence (3.5)) are best possible in the limit as s and m become large. For this purpose, we employ the solutions $P_{s,m}(z)$ to Prob. II (cf. (2.4)) and recall (cf. (2.8)) that $E_{s,m}=\|P_{s,m}\|_{\mathsf{C}}$.

Theorem 3.3. *Let $0<\theta\leqq 1$ be fixed and let $\{(s_i,m_i)\colon s_i\geqq 1,\ m_i\geqq 0\}_{i=1}^{\infty}$ be any infinite sequence of ordered pairs of integers for which*

$$\lim_{i\to\infty}(s_i+m_i)=\infty \quad \textit{and} \quad \lim_{i\to\infty} s_i/(s_i+m_i)=\theta. \tag{3.9}$$

Then,

$$\lim_{i\to\infty}(E_{s_i,m_i})^{1/(s_i+m_i)}=\sqrt{(1+\theta)^{1+\theta}(1-\theta)^{1-\theta}}=:\Delta(\theta), \tag{3.10}$$

and

$$\lim_{i\to\infty} |P_{s_i,m_i}(z)|^{1/(s_i+m_i)} = \Delta(\theta)\cdot G(z;\theta), \quad \textit{for all } z\in\mathbb{C}^*\setminus A_\theta, \tag{3.11}$$

the convergence in (3.11) *being uniform on any compact set omitting* A_θ.

Proof. Equation (3.10) follows, recalling (2.12) and (2.30), from an application of Stirling's formula. The equality of (3.11) follows, after taking logarithms, by imitating the normal families argument given in [6] or [16]. This argument is facilitated by the fact (cf. Cor. 2.9) that, except for $z=1$, the zeros of $P_{s_i,m_i}(z)$ all lie on the arc $A_{s_i/(s_i+m_i)}$, so that $\ln|P_{s_i,m_i}(z)|$ is harmonic in the complement of this arc, except at $z=1$ and $z=\infty$. ∎

In §2, we discussed the behavior of zeros of the solutions to Prob. II and Prob. III. Our next result contains information regarding the behavior of the zeros of the solutions $p_{s,m}(z)$ to Prob. I (cf. (2.3)), for large s and m.

Theorem 3.4. *Let* θ *and* $\{(s_i, m_i)\colon s_i\geqq 1, m_i\geqq 0\}_{i=1}^{\infty}$ *be as in Theorem* 3.3. *Then,*

$$\lim_{i\to\infty} |p_{s_i,m_i}(z)|^{1/(s_i+m_i)} = \Delta(\theta)\cdot G(z;\theta), \quad \textit{for all } z\in\mathbb{C}^*\setminus A_\theta, \tag{3.12}$$

uniformly on any compact set omitting A_θ. *Moreover, the limit points of the zeros of the* $p_{s_i,m_i}(z)$, *different from* $z=1$, *all lie on the arc* A_θ.

Remark. From the proof of Theorem 2.3, the non-unity zeros of each $p_{s_i,m_i}(z)$ lie *in* $|z|<1$, but we shall prove that the limit points of these zeros are, in fact, *all* on the arc A_θ of $|z|=1$.

Proof. For an appropriate choice of branches, Eq. (3.11) implies that

$$\lim_{i\to\infty} (P_{s_i+1,m_i}(z))^{1/(s_i+m_i+1)} = \Delta(\theta)\, e^{U+iV}, \tag{3.13}$$

uniformly on each closed disk in the complement of $A_\theta\cup\{1\}$, where $U=U(z;\theta)$: $=\ln G(z;\theta)$ and $V=V(z;\theta)$ denotes a harmonic conjugate of $U(z;\theta)$. Taking the logarithmic derivative in (3.13), and recalling the relationship (2.11), we have

$$\lim_{i\to\infty} \frac{p_{s_i,m_i}(z)}{P_{s_i+1,m_i}(z)} = U_x(z;\theta) - i\,U_y(z;\theta) =: H(z;\theta), \tag{3.14}$$

locally uniformly in the complement of $A_\theta\cup\{1\}$. It can be verified that the product $(z-1)\,H(z;\theta)$ is analytic and nonzero for $z\notin A_\theta\cup\{\infty\}$, even at $z=1$. Hence, $(z-1)\,p_{s_i,m_i}(z)/P_{s_i+1,m_i}(z)$ tends to $(z-1)\,H(z;\theta)$, locally uniformly in $\mathbb{C}\setminus A_\theta$. Furthermore, a slight modification of the proof of (3.11) shows that $|P_{s_i+1,m_i}(z)/(z-1)|^{1/(s_i+m_i)}\to\Delta(\theta)\,G(z;\theta)$, the convergence being locally uniform in $\mathbb{C}^*\setminus A_\theta$. Thus, (3.12) follows upon writing

$$\lim_{i\to\infty} |p_{s_i,m_i}(z)|^{1/(s_i+m_i)} = \lim_{i\to\infty} \left|\frac{P_{s_i+1,m_i}(z)}{z-1}\cdot\frac{(z-1)\,p_{s_i,m_i}(z)}{P_{s_i+1,m_i}(z)}\right|^{1/(s_i+m_i)}. \quad ∎ \tag{3.15}$$

We conclude with an application of Theorems 3.1 and 3.3.

Corollary 3.5. *Let* θ *and* $\{(s_i, m_i)\colon s_i \geqq 1,\ m_i \geqq 0\}_{i=1}^{\infty}$ *be as in Theorem* 3.3. *If* $\{p_i(z)\}_{i=1}^{\infty}$ *is any sequence of polynomials such that* $p_i(z) \in \pi_{s_i, m_i}$ *for each* i (cf. (2.1)) *and*

$$\limsup_{i\to\infty} \|p_i\|_{\mathsf{C}}^{1/(s_i+m_i)} \leqq 1, \tag{3.16}$$

then

$$\lim_{i\to\infty} p_i(z) = 0, \qquad \text{for all } z \in \Lambda(\theta), \tag{3.17}$$

the convergence in (3.17) *being uniform on each compact subset of* $\Lambda(\theta)$. *Moreover,* $\Lambda(\theta)$ *is the largest possible open set of convergence to zero for the class of such polynomial sequences* $\{p_i(z)\}$.

Proof. Let K be any compact subset of $\Lambda(\theta)$. For each $i \geqq 1$, we apply inequality (3.6) to obtain

$$\|p_i\|_K \leqq \|p_i\|_{\mathsf{C}} \, \|G(\cdot\,; s_i/(s_i+m_i))\|_K^{s_i+m_i}.$$

Thus, from the hypotheses (3.9) and (3.16), we have

$$\limsup_{i\to\infty} [\|p_i\|_K]^{1/(s_i+m_i)} \leqq 1 \cdot \limsup_{i\to\infty} \|G(\cdot\,; s_i/(s_i+m_i))\|_K = \|G(\cdot\,; \theta)\|_K, \tag{3.18}$$

where the last equality follows from the continuity of the function G. Since $\|G(\cdot\,; \theta)\|_K < 1$, it follows that $\|p_i\|_K \to 0$ as $i \to \infty$.

Finally, if we select $\tilde{p}_i(z) := P_{s_i, m_i}(z)/E_{s_i, m_i}$, $i \geqq 1$, then this sequence satisfies the hypotheses of Corollary 3.5 and, from (3.11), we have

$$\lim_{i\to\infty} |\tilde{p}_i(z)|^{1/(s_i+m_i)} = G(z; \theta), \qquad \text{for all } z \in \mathbb{C}^* \setminus A_\theta.$$

Hence, $\{\tilde{p}_i(z)\}_1^{\infty}$ converges to zero in $\Lambda(\theta)$, but *diverges* for z in the complement of the closure $\overline{\Lambda(\theta)}$. ∎

We remark that, from Corollary 3.2, the hypothesis (3.16) is equivalent to the assumption

$$\limsup_{i\to\infty} \|p_i\|_{\mathsf{A}_i}^{1/(s_i+m_i)} \leqq 1, \tag{3.19}$$

where $\mathsf{A}_i := A_{s_i/(s_i+m_i)}$, $i \geqq 1$.

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