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Nonuniqueness of Best Complex Rational Approximations to Real Functions on Real Intervals

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It seems not to be well known that best *complex* rational approximations, in the uniform norm, to a real function on a real finite interval need *not* be unique. Because of this, the purpose of this note is to highlight this nonuniqueness with three different examples.

1. INTRODUCTION

It is well known that for any real continuous function $f(x)$ on $[-1, +1]$ and for any pair (m, n) of nonnegative integers, the best uniform approximation to $f(x)$ on $[-1, +1]$ by a real rational function in $\pi^r_{m,n}$ (defined below) is *unique* (cf. Meinardus [2, p. 161], Rice [4, p. 77]). On the other hand, Walsh [5, p. 356] has given an example of a continuous complex-valued function, namely $f(z) := z + z^{-1}$, on a certain closed crescent-shaped region in the complex plane, whose best uniform complex rational approximation of order 1 is *not* unique.

While Walsh's example is of great importance, what is interesting is that this nonuniqueness of best complex rational approximation can hold *even* in the situation of most common interest, namely, in approximating *real* functions on finite *real* intervals. Moreover, this nonuniqueness can in fact

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be exhibited for best uniform complex rational approximation of order n, with n an arbitrary positive integer, which extends Walsh's example in another direction. The purpose of this note, then, is to give three elementary examples for which best uniform complex rational approximation to a real continuous function on $[-1, +1]$ is *not* unique. These examples point out a somewhat curious fact, namely, that complex rational functions may give *closer* uniform approximations to a real function on $[-1, +1]$ than the best real rational function of the same type.

For the remainder of this section, we give necessary background and notation. To begin, for an arbitrary nonnegative integer ν, $\pi_\nu{}^r$ denotes the collection of all polynomials (in the variable z or x) of degree at most ν having *real* coefficients, while $\pi_\nu{}^c$ is the analogous collection of polynomials with *complex* coefficients. For each pair (m, n) of nonnegative integers, $\pi^r_{m,n}$ denotes the collection of all rational functions which can be written in the form p_m/q_n with $p_m \in \pi_m{}^r$ and $q_n \in \pi_n{}^r$, while $\pi^c_{m,n}$ is analogously the collection when $p_m \in \pi_m{}^c$ and $q_n \in \pi_n{}^c$. Obviously, $\pi^r_{m,n} \subset \pi^c_{m,n}$.

If $\mathbb{C}_r[-1, +1]$ denotes the collection of all *real* continuous functions on $[-1, +1]$ and if $\| g \| := \sup_{x\in[-1,+1]} | g(x)|$ for any real or complex-valued function g defined on $[-1, +1]$, we further set

$$E^r_{m,n}(f) := \inf_{g\in\pi^r_{m,n}} \|f - g\|; \qquad E^c_{m,n}(f) := \inf_{g\in\pi^c_{m,n}} \|f - g\| \tag{1.1}$$

for any $f \in \mathbb{C}_r[-1, +1]$. As previously remarked, for any $f \in \mathbb{C}_r[-1, +1]$ and for any nonnegative integers (m, n), there is a *unique* $R_{m,n} \in \pi^r_{m,n}$ such that

$$E^r_{m,n}(f) = \|f - R_{m,n}\|. \tag{1.2}$$

Moreover, this best approxiation in $\pi^r_{m,n}$ is characterized by the following property: If $R_{m,n} = P/Q$ with $P \in \pi_m$ and $Q \in \pi_n$ having no common factors, $f - R_{m,n}$ has an alternation set (cf. [2, p. 161]) of length l, consisting of l distinct points on $[-1, +1]$, with

$$l \geqslant 2 + \max\{m + \deg Q; n + \deg P\}, \tag{1.3}$$

(where we adopt the convention that if $P \equiv 0$, we take $\deg P = -\infty$, $\deg Q = 0$, so that $l \geqslant 2 + m$ in this case). We also remark that (cf. Walsh [5, p. 351]) for any $f \in \mathbb{C}_r[-1, +1]$ and for any nonnegative integers (m, n), there always exists an $\hat{R}_{m,n} \in \pi^c_{m,n}$ for which

$$E^c_{m,n}(f) = \|f - \hat{R}_{m,n}\|, \tag{1.4}$$

and it is convenient to distinguish the collection of such best uniform rational approximants $\hat{R}_{m,n}$ by

$$B^c_{m,n}(f) := \{R^c_{m,n} \in \pi^c_{m,n} : \|f - R^c_{m,n}\| = E^c_{m,n}(f)\}, \tag{1.5}$$

where the analogous definition for $B^r_{m,n}(f)$ is used. If $|A|$ denotes the *cardinality* of the set A, i.e., the number of its elements, then, as previously remarked, $B^r_{m,n}(f) = \{R_{m,n}(f)\}$ and $|B^r_{m,n}(f)| = 1$, while the statement

$$|B^c_{m,n}(f)| > 1 \tag{1.6}$$

implies simply that there is no unique best uniform rational approximation in $\pi^c_{m,n}$ to f on $[-1, +1]$.

2. Main Results and Examples

In this section, we state some propositions which form the basis for our three examples on nonuniqueness. These examples, along with some discussion and some open problems, are also given in this section, but the proofs for the propositions and examples are given in Section 3. For convenience, we shall deal only with the classes $\pi_{m,n}$ (and $\pi^c_{m,n}$) for which $m = n$.

Proposition 1. *Given any $f \in \mathbb{C}_r[-1, +1]$ and given any nonnegative integer n,*

$$E^r_{2n,2n}(f) \leqslant \inf_{g \in \pi^c_{n,n}} \|f - \operatorname{Re} g\| \leqslant E^c_{n,n}(f) \leqslant E^r_{n,n}(f). \tag{2.1}$$

Furthermore, if $E^r_{2n,2n}(f) = E^r_{n,n}(f)$, then $B^c_{n,n}(f) = B^r_{n,n}(f) = \{R_{n,n}\}$.

Corollary 1. *Given $f \in \mathbb{C}_r[-1, +1]$, let $R_{n,n} = P/Q \in B^r_{n,n}(f)$, and set $d := \max(\deg P; \deg Q)$, where P and Q have no common factors. If $f - R_{n,n}$ has an alternation set of length at least $2 + 2n + d$, then $B^c_{n,n}(f) = B^r_{n,n}(f) = \{R_{n,n}\}$.*

Proposition 2. *Given $f \in \mathbb{C}_r[-1, +1]$, let $R_{n,n} = P/Q \in B^r_{n,n}(f)$ and let $d := \max(\deg P; \deg Q)$ where P and Q have no common factors. If every alternation set for $f - R_{n,n}$ has length at most $1 + 2n - d$, then*

$$E^c_{n,n}(f) < E^r_{n,n}(f) \tag{2.2}$$

and

$$|B^c_{n,n}(f)| > 1. \tag{2.3}$$

Example 1. *Let $f \in \mathbb{C}_r[-1, +1]$* be even, and monotone and nonconstant on $[0, 1]$. Then

$$E^c_{1,1}(f) < E^r_{1,1}(f) \tag{2.4}$$

and

$$|B^c_{1,1}(f)| > 1. \tag{2.5}$$

We remark that Example 1 includes as a special case the function $f_\alpha(x) := |x^\alpha|$, $\alpha > 0$, on $[-1, +1]$. Moreover, it is easy to see that $E^r_{1,1}(f_\alpha) = \frac{1}{2}$ for any $\alpha > 0$. On the other hand, for $\alpha = 2$ and for the specific function

$$r_{1,1}(x) := \left(\frac{x + (2^{1/2} - 1)\,i}{x + i}\right) \in \pi^c_{1,1}, \tag{2.6}$$

a short calculation shows that $\| x^2 - r_{1,1} \| = 2^{1/2} - 1 \doteqdot 0.4142$, so that by definition

$$E^c_{1,1}(x^2) < E^r_{1,1}(x^2) = \tfrac{1}{2}. \tag{2.7}$$

Also, we have kindly been informed by Professor Colin Bennett (cf. [1]) that $E^c_{1,1}(x^2) \leqslant (4/27)^{1/2} \doteqdot 0.3849$.

EXAMPLE 2. *For* $m > 1$, *let* $T_m(x)$ be the Chebyshev polynomial (of the first kind) of degree m, and set $k: = [(m = 1)/2]$. Then, for each n with $0 \leqslant n \leqslant k$, $|B^c_{n,n}(T_m)| = 1$ and

$$B^c_{n,n}(T_m) = \{g(x) \equiv 0\}, \qquad 0 \leqslant n \leqslant k, \tag{2.8}$$

while for $k + 1 \leqslant n \leqslant m - 1$,

$$E^c_{n,n}(T_m) < E^r_{n,n}(T_m), \tag{2.9}$$

and

$$|B^c_{n,n}(T_m)| > 1. \tag{2.10}$$

EXAMPLE 3. There exists an entire function $f \in \mathbb{C}_r[-1, +1]$ and an infinite sequence of distinct positive integers $\{n_i\}_{i=1}^\infty$ such that

$$E^c_{n_i,n_i}(f) < E^r_{n_i,n_i}(f), \qquad \forall i \geqslant 1, \tag{2.11}$$

and

$$|B^c_{n_i,n_i}(f)| > 1, \qquad \forall i \geqslant 1. \tag{2.12}$$

We now list some open questions suggested by the above examples.

(1) What is $\max_{f \in \mathbb{C}_r[-1,+1]} |B^c_{n,n}(f)|$? Can this quantity be in fact infinite?

(2) Does there exist an $f \in \mathbb{C}_r[-1, +1]$ for which $E^c_{n,n}(f) = E^r_{n,n}(f)$, but for which $|B^c_{n,n}(f)| > 1$?

(3) Find $\gamma(n) := \inf\{E^c_{n,n}(f)/E^r_{n,n}(f) : f \in \mathbb{C}_r[-1, +1] \text{ with } f \notin \pi^r_{n,n}(f)\}$.

(4) Does there exist an entire function $f \in \mathbb{C}_r[-1, +1]$ such that (2.11) to (2.12) are valid for *all* $n \geqslant 1$?

(5) Is $|B^c_{n,n}(|x|)| > 1$ for all $n \geqslant 1$?

3. Proofs

We now give the justifications for the statements made in the previous section.

The proof of Proposition 1 follows immediately from the fact that if $t \in \pi^c_{n,n}$, then $\operatorname{Re} r^c(x)$ and $\operatorname{Im} r^c(x)$ belong to $\pi^r_{2n,2n}$. Corollary 1 is then a trivial consequence of Proposition 1 and the alternation property (1.3).

Proof of Proposition 2. Let l be the length of any longest alternation set for $f - R_{n,n}$ on $[-1, +1]$, and call H the set of extreme points of $f - R_{n,n}$ on $[-1, +1]$:

$$H := \{x \in [-1, +1]: |(f - R_{n,n})(x)| = E^r_{n,n}(f)\}.$$

Obviously, $|H| \geqslant l$. From the knowledge of the set H, $l - 1$ distinct points x_i with $-1 < x_1 < x_2 < \cdots < x_{l-1} < 1$ can be determined such that if $s(x) := \prod_{i=1}^{l-1} (x - x_i)$, then $[(f - R_{n,n})\, s](x)$ is of one sign on H. Without loss of generality, we may assume that $(f - R_{n,n})\, s$ is positive on H, and hence, by continuity, there is an open set $\mathcal{O}$ in $[-1, +1]$ with $H \subset \mathcal{O}$ for which

$$[(f - R_{n,n})\, s](x) > 0 \qquad \text{for all } x \in \mathcal{O}. \tag{3.1}$$

Next, set

$$p(x) := \prod_{k=1}^{l+d-n-1} (x - x_i) \qquad \text{and} \qquad q(x) := s(x)/p(x), \tag{3.2}$$

and as $l \geqslant 2 + n + d$ from (1.3), then $l + d - n - 1 \geqslant 1 + 2d \geqslant 1$, and as $d \leqslant n$ by definition, then $l + d - n - 1 \leqslant l - 1$, showing that the polynomials p and q are well defined. Next, with $R_{n,n} = P/Q$ and $\delta > 0$, set

$$\begin{aligned} r_\delta(x) &:= \frac{P(x)}{Q(x)} + \delta \frac{p(x)}{Q(x)(q(x) + i)} \\ &= \frac{P(x)(q(x) + i) + \delta p(x)}{Q(x)(q(x) + i)}. \end{aligned} \tag{3.3}$$

As $l \leqslant 1 + 2n - d$ by hypothesis and as $\deg q = n - d$, we see that $r_\delta(x) \in \pi^c_{n,n}$ for any $\delta > 0$.

Now, with $e_{n,n}(x) := f(x) - R_{n,n}(x) = f(x) - (P(x)/Q(x))$, then for any $x \in [-1, +1]$,

$$\begin{aligned} |f(x) - r_\delta(x)|^2 &= \left| e_{n,n}(x) - \delta \frac{p(x)}{Q(x)(q(x) + i)} \right|^2 \\ &= e^2_{n,n}(x) - \frac{2\delta e_{n,n}(x)\, p(x)}{Q(x)} \operatorname{Re}\left(\frac{1}{q(x) + i}\right) + \frac{\delta^2 p^2(x)}{Q^2(x)(q^2(x) + 1)} \\ &= e^2_{n,n}(x) + \frac{\delta^2 p^2(x)}{Q^2(x)(q^2(x) + 1)} - \frac{2\delta e_{n,n}(x)\, s(x)}{Q(x)(q^2(x) + 1)}. \end{aligned} \tag{3.4}$$

Since we may assume that $Q(x) > 0$ on $[-1, +1]$, we then have from (3.1) that $2\delta e_{n,n}(x)\, s(x)/\{Q(x)[q^2(x) + 1]\}$ is positive at each point of $\mathcal{O}$. Consequently, for $\delta > 0$ sufficiently small, it follows from (3.4) that

$$\|f - r_\delta\| < E^r_{n,n}(f),$$

whence $E^c_{n,n}(f) < E^r_{n,n}(f)$, giving (2.2) of Proposition 2.

To establish nonuniqueness, consider any $r \in B^c_{n,n}(f)$. Clearly, the conjugate $\bar{r}$ (on $[-1, +1]$) is also an element of $B^c_{n,n}(f)$. But $\bar{r} \not\equiv r$, since the contrary assumption would imply that $E^c_{n,n}(f) = E^r_{n,n}(f)$, which would contradict (2.2). Thus, $|B^c_{n,n}(f)| > 1$. ▮

We remark that Proposition 2 could also be deduced from Meinardus and Schwedt [3] (cf. Meinardus [2, p. 136]), but for the sake of completeness, a short proof of Proposition 2 was included here.

Proof of Example 1. Since f is even on $[-1, +1]$, it follows that $R_{1,1}(x) \in B^r_{1,1}(f)$ necessarily reduces to a constant, and since f is monotone and nonconstant on $[0, 1]$, it is easy to see that $f - R_{1,1}$ has an alternation set on $[-1, +1]$ of longest length $l = 3$. Thus, the hypotheses of Proposition 2 are fulfilled with $n = 1$ and $d = 0$. ▮

Proof of Example 2. Since $T_m(x) = T_m(x) - 0$ has an alternation set on $[-1, +1]$ of longest length $l = m + 1$, then for $R_{n,n}(x) \equiv 0$ for $0 \leqslant n \leqslant k$ where $k := [(m-1)/2]$, $T_m - R_{n,n}$ satisfies the hypotheses of Corollary 1 with $d = 0$, and thus $|B^c_{n,n}(T_m)| = 1$ for all $0 \leqslant n \leqslant k$. On the other hand, since $T_m(x) - 0$ satisfies the hypotheses of Proposition 2 with $d = 0$ and $k + 1 \leqslant n \leqslant m - 1$, then (2.9) and (2.10) follow directly from (2.2) and (2.3) of Corollary 2. ▮

Proof of Example 3. Let $f(x) := \sum_{\nu=1}^{\infty} \epsilon_\nu T_{2m_\nu - 1}(x)$, where $m_\nu = 3^\nu$ for $\nu \geqslant 1$, and where the nonnegative real numbers ϵ_ν are defined recursively by means of

$$\epsilon_1 = 1, \qquad \epsilon_\nu = \frac{\min(\delta_\nu\,;\,\epsilon_{\nu-1})}{(2m_\nu - 1)!}, \qquad \forall \nu \geqslant 2, \tag{3.5}$$

where

$$\delta_1 = 1, \qquad \delta_\nu = E^r_{4m_{\nu-2}-1}(f_{\nu-1}) - E^c_{4m_{\nu-2}-1}(f_{\nu-1}), \qquad \forall \nu \geqslant 2; \tag{3.6}$$

and where $f_{\nu-1}(x) := \sum_{k=1}^{\nu-1} \epsilon_k T_{2m_k - 1}(x)$. (Note that we are, for convenience, now writing E_j for $E_{j,j}$.) It is clear from (3.5) and (3.6) that f is entire, and that $f \in \mathbb{C}_r[-1, +1]$. We shall show that

$$E^c_{4m_\nu - 1}(f) < E^r_{4m_\nu - 1}(f), \qquad \forall \nu \geqslant 0. \tag{3.7}$$

For this purpose, we first prove that $\delta_\nu > 0$ for all $\nu \geqslant 1$. By definition, this is true for $\nu = 1$, and we inductively assume that $\delta_k > 0$ for all $1 \leqslant k \leqslant \nu - 1$. From (3.5), then $\epsilon_k > 0$ for $1 \leqslant k \leqslant \nu - 1$. Next, we show that $f_{\nu-2}$ is the unique best uniform rational function in $\pi^r_{4m_{\nu-2}-1,4m_{\nu-2}-1}$ to $f_{\nu-1}$ on $[-1, +1]$. Indeed,

$$f_{\nu-1}(x) - f_{\nu-2}(x) = \epsilon_{\nu-1} T_{2m_{\nu-1}-1}(x)$$

has an alternation set on $[-1, +1]$ of longest length $l = 2m_{\nu-1} = 2 \cdot 3^{\nu-1}$, and we note that (cf. (1.3))

$$l = 2m_{\nu-1} = 2 + \max\{4m_{\nu-2} - 1 + \deg f_{\nu-2}\,;\, 4m_{\nu-2} - 1 + 0\}.$$

Furthermore,

$$l = 2m_{\nu-1} = 1 + 2(4m_{\nu-2} - 1) - \deg f_{\nu-2}\,,$$

so that from (2.2) of Proposition 2 (with $n = 4m_{\nu-2} - 1$ and $d = \deg f_{\nu-2}$) and (3.6),

$$0 < E^r_{4m_{\nu-2}-1}(f_{\nu-1}) - E^c_{4m_{\nu-2}-1}(f_{\nu-1}) := \delta_\nu\,.$$

This completes the induction, and also shows that $\epsilon_\nu > 0$ for all $\nu \geqslant 1$.

To prove (3.7), we first note by the triangle inequality that

$$E^c_{4m_\nu-1}(f) \leqslant E^c_{4m_\nu-1}(f_{\nu+1}) + \Big\| \sum_{k=\nu+2}^{\infty} \epsilon_k T_{2m_k-1} \Big\|, \tag{3.8}$$

and that

$$E^r_{4m_\nu-1}(f) \geqslant E^r_{4m_\nu-1}(f_{\nu+1}) - \Big\| \sum_{k=\nu+2}^{\infty} \epsilon_k T_{2m_k-1} \Big\|. \tag{3.9}$$

Hence, (3.7) will be satisfied if

$$\Big\| \sum_{k=\nu+2}^{\infty} \epsilon_k T_{2m_k-1} \Big\| < \tfrac{1}{2}[E^r_{4m_\nu-1}(f_{\nu+1}) - E^c_{4m_\nu-1}(f_{\nu+1})] = \tfrac{1}{2}\delta_{\nu+2}\,. \tag{3.10}$$

But from (3.5), we have that

$$\Big\| \sum_{k=\nu+2}^{\infty} \epsilon_k T_{2m_k-1} \Big\| \leqslant \sum_{k=\nu+2}^{\infty} \epsilon_k \leqslant \frac{\delta_{\nu+2}}{(2m_{\nu+2} - 1)!} + \epsilon_{\nu+2} \sum_{k=\nu+3}^{\infty} \frac{1}{(2m_k - 1)!}$$

$$\leqslant \frac{\delta_{\nu+2}}{(2m_{\nu+2} - 1)!} \left\{ 1 + \sum_{k=\nu+3}^{\infty} \frac{1}{(2m_k - 1)!} \right\}$$

$$< \frac{\delta_{\nu+2}}{2}$$

since $m_\nu = 3^\nu$, and the proof is complete. ∎

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Note added in proof. 1) It has been shown by A. Ruttan [6] that there is a real continuous function $[-1, +1]$ for which $| B^c_{1,1}(f)| = \infty$, which settles open problem (1). 2) A. A. Gončar has recently informed us that his paper "The Rate of Approximation by Rational Fractions and the Properties of Functions" (1968) mentions the possibility of nonuniqueness in a footnote. Gončar's student K. N. Lungu studied the problem in more detail. He showed [*Matematicheskie Zametki*, Vol. 10, No. 1, pp. 11–15, July (1971)] that if for a continuous function $f(x)$ on $[-1, 1]$ we set $M := \max_{[-1,1]} f(x)$, $m := \min_{[-1,1]} f(x)$ and if there exist points $-1 \leqslant x_0 < x_1 < \cdots < x_n \leqslant 1$, $n > 1$, such that the set $\{x: f(x) = M\}$ consists of x_k with even subscripts and the set $\{x: f(x) = m\}$ consists of x_k with odd subscripts, then

$$E^c_{n-1,n-1}(f) < E^r_{n-1,n-1}(f).$$

This result is but a special case of our Proposition 2 because under the above hypotheses, $R_{n-1,n-1}(x) \equiv (M + m)/2$ and $f - R_{n-1,n-1}$ has largest alternation set of length $n + 1$, which is not greater than $1 + 2(n - 1) - 0 = 2n - 1$ for $n > 1$. For the Chebyshev polynomials $T_m(x)$, Lungu's result gives $E^c_{n,n}(T_m) < E^r_{n,n}(T_m)$ only for $n = m - 1$, while our result gives *all* the integers n for which this inequality holds, namely $[\![(m - 1)/2]\!] \leqq n \leqq m - 1$.

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