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SOME OPEN PROBLEMS CONCERNING POLYNOMIALS AND RATIONAL FUNCTIONS

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We gather below a list of some open problems and conjectures concerning polynomials and rational functions, along with related references and comments.

1. If $\pi_{n,n}$ denotes the set of all real rational functions of degree n, set

$$\lambda_{n,n} := \inf_{r_n \in \pi_{n,n}} \| e^{-x} - r_n(x) \|_{L_\infty[0,+\infty)}.$$

Based on numerical results (cf. Cody, Meinardus, and Varga, J. Approximation Theory 2 (1969), 50-65) it is conjectured that

$$\lim_{n\to\infty} \lambda_{n,n}^{1/n} = \frac{1}{9}.$$

For related upper and lower bounds for $\lambda_{n,n}^{1/n}$, see Q.I. Rahman and G. Schmeisser, these Proceedings, and the references given there.

2. Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be an entire function of finite order $\rho > 0$, with $a_k > 0$ for all $k \geq 0$, and with f being of perfectly regular growth. If $s_n(z;f) := \sum_{k=0}^{n} a_k z^k$, it is known (cf. Meinardus and Varga, J. Approximation Theory 3 (1970), 300-309) that

$$\lim_{n\to\infty} \left\| \frac{1}{f(x)} - \frac{1}{s_n(x;f)} \right\|_{L_\infty[0,+\infty)}^{1/n} = \frac{1}{2^{1/\rho}}.$$

A particular function for which the above is valid is $f(z) = e^z$. If we denote generically

$$\lambda_{0,n}(f) := \inf_{p_n \in \pi_n} \left\| \frac{1}{f} - \frac{1}{p} \right\|_{L_\infty[0,+\infty)},$$

where π_n is the collection of real polynomials of degree $\leq n$, then Schönhage (J. Approximation Theory 7 (1973), 395-398) showed that

$$\lim_{n\to\infty} [\lambda_{0,n}(e^z)]^{1/n} = \frac{1}{3} .$$

It is conjectured that for any entire function f of finite positive order ρ, having positive Maclaurin coefficients and possessing perfectly regular growth that

$$\lim_{n\to\infty} [\lambda_{0,n}(f)]^{1/n} = \frac{1}{3^{1/\rho}} .$$

For related results, see A.R. Reddy and O. Shisha, J. Approximation Theory 12 (1974), 425-434.

3. Given any τ with $0 \leq \tau \leq 2$, and given any $K > 0$, define

$$S(\tau;K) := \{z=x+iy: |y| < Kx^{1-(\tau/2)}, x > 0\},$$

and for any θ with $0 \leq \theta < 2\pi$, define

$$S(\tau;K;\theta) := \{z: ze^{-i\theta} \varepsilon S(\tau;K)\}.$$

Given any entire function $f(z) = \sum_{k=0}^{\infty} a_k z^k$ with order $\rho>\tau$, it is conjectured that there is no $S(\tau;K;\theta)$ which is devoid of all zeros of all partial sums $s_n(z;f) := \sum_{k=0}^{n} a_k z^k$, $n \geq 1$.

For the case when $\tau = 0$, $S(0;K;\theta)$ is just a proper sector with vertex at $z = 0$, and the result of F. Carlson (Arkiv for Mathematik, Astronomi O. Fysik, Bd. 35A, No. 14 (1948), 1-18) precisely gives the truth of the conjecture for $\tau = 0$. Thus what is conjectured is a generalization of Carlson's result. The above conjecture is related to the width conjecture of E.B. Saff and R.S. Varga (cf. Pacific J. Math. 62 (1976), 523-549). Note also that for $f(z) = e^z$, for which $\rho = 1$, it is known (cf. Saff and Varga, SIAM J. Math. Anal. 7 (1976), 344-357) that no partial sum $s_n(z) = \sum_{k=0}^{n} z^k/k!$ for any $n \geq 1$ has zeros in the parabolic region

$$P_1 := \{z=x+iy\colon\ y^2 \le 4(x+1),\ x > -1\}.$$

Because the boundary points of this parabola satisfy $|y| \sim 2x^{1/2}$, as $x \to +\infty$, we see that the above conjecture cannot in general be valid with $\tau = \rho$.

4. Let

$$\tilde{\lambda}_{n-1,n} := \inf_{p_{n-1}\in\pi_{n-1}} \{ \| e^{-x} - p_{n-1}(x) / \prod_{i=1}^{n} (1+b_i x) \|_{L_\infty[0,+\infty)} :$$

$$b_i > 0 \text{ for } i=1,2,\ldots,n\},$$

where π_{n-1} denotes the collection of all real polynomials of degree $\le n-1$. It is known (cf. Saff, Schönhage, and Varga, Numer. Math. 25 (1976), 307-322) that these restricted rational approximations possess geometric convergence; specifically

$$\overline{\lim_{n\to\infty}} (\tilde{\lambda}_{n-1,n})^{1/n} \le \frac{1}{2}.$$

An open problem is to determine the exact geometric convergence rate of these restricted approximations. Next, it is known (cf. Kaufman and Taylor, these Proceedings) that there is a rational function

$$\tilde{r}_{n-1,n}(x) := \tilde{p}_{n-1}(x) / \prod_{i=1}^{n} (1+\tilde{b}_i(n)x)$$

with $b_i(n) > 0$, $1 \le i \le n$, such that

$$\tilde{\lambda}_{n-1,n} = \| e^{-x} - \tilde{r}_{n-1,n} \|_{L_\infty[0,+\infty)}.$$

<u>We and others</u> (E. Kaufman, T.C.-Y. Lau, A. Schönhage and G. Taylor) <u>conjecture that</u>

(i) $\tilde{b}_1(n) = \tilde{b}_2(n) = \ldots = \tilde{b}_n(n)$, for all $n \ge 1$,

(ii) $\lim_{n\to\infty} \tilde{b}_n(n) \cdot n = 1.$

Numerical results of Kaufman and Taylor (these Proceedings) and T.C.-Y. Lau (preprint) confirm (i) above for small values of n.

5. In the statement of the open problems of this section the following notation is used: $C_r[-1,1]$ denotes the collection of all real continuous functions on $[-1,1]$; π_ν^r is the collection of all polynomials of degree at most ν having real coefficients; π_ν^c is the analogous collection of polynomials with complex coefficients; $\pi_{n,n}^r$ is the collection of rational functions of the form p/q with $p,q \in \pi_n^r$; $\pi_{n,n}^c$ is the analogous collection with $p,q \in \pi_n^c$. For any real or complex-valued function g defined on $[-1,1]$ we also set

$$||g|| := \sup_{x\in[-1,1]} |g(x)|.$$

Finally for any $f \in C_r[-1,1]$ we put

$$E_{n,n}^r(f) := \inf_{R\in\pi_{n,n}^r} ||f-R||; \quad E_{n,n}^c(f) := \inf_{R\in\pi_{n,n}^c} ||f-R||;$$

and denote the corresponding collections of best approximates by

$$B_{n,n}^r(f) = \{R \in \pi_{n,n}^r : ||f-R|| = E_{n,n}^r(f)\},$$

$$B_{n,n}^c(f) = \{R \in \pi_{n,n}^c : ||f-R|| = E_{n,n}^c(f)\}.$$

It is well-known that $|B_{n,n}^r(f)|$ (the cardinality of $B_{n,n}^r(f)$) is one, i.e. uniqueness holds. In contrast, it has been shown by Saff and Varga (Bull. Amer. Math Soc., 83 (1977) No. 3) that there are classes of functions $f \in C_r[-1,1]$ for which $E_{n,n}^c(f) < E_{n,n}^r(f)$ and $|B_{n,n}^c(f)| \geq 2$. See also Bennett, Rudnick, and Vaaler (these Proceedings), and Arden Ruttan (these Proceedings).

We now list some open questions concerning real versus complex rational approximation to real functions on the real interval $[-1,1]$.

(i) Does there exist an $f \in C_r[-1,1]$ for which $E_{n,n}^c(f) = E_{n,n}^r(f)$, but for which $|B_{n,n}^c(f)| \geq 2$?

(ii) Find $\gamma(n) := \inf\{E^c_{n,n}(f)/E^r_{n,n}(f) : f \in C_r[-1,1]$ with $f \notin \pi^r_{n,n}(f)\}$.

(iii) Does there exist an entire function $f \in C_r[-1,1]$ such that $E^c_{n,n}(f) < E^r_{n,n}(f)$ for every $n \geq 1$? It has been shown by Saff and Varga (Bull. Amer. Math. Soc., 83 (1977) No. 3) that there exists an entire function $f \in C_r[-1,1]$ such that $E^c_{n,n}(f) < E^r_{n,n}(f)$ for infinitely many n.

(iv) Is $|B^c_{n,n}(|x|)| \geq 2$ for all $n \geq 1$?

6. A well-known conclusion of the Kakeya-Eneström Theorem is that if $p_n(z) = \sum_{i=0}^{n} a_i z^i$ with $a_i > 0$ for all $0 \leq i \leq n$, then the zeros $\hat{z}$ of $p_n(z)$ satisfy

$$|\hat{z}| \leq \max_{1 \leq i \leq n} \left(\frac{a_{i-1}}{a_i}\right).$$

In this form, the above result is sharp for every $n \geq 1$. When applied, say, to the Padé numerators $P_{n,\nu}(z)$ of type (n,ν) to e^z, the Kakeya-Eneström inequality yields

$$|\hat{z}| \leq n(\nu+1) \quad \text{for all } n \geq 1, \text{ all } \nu \geq 0.$$

On the other hand, it is known (Saff and Varga, these Proceedings) that the n zeros $\hat{z}$ of $P_{n,\nu}(z)$ satisfy

$$|\hat{z}| < n + \nu + \frac{4}{3},$$

which is a better estimate for these zeros than the Kakeya-Eneström bounds, except for small values of n and ν. This suggests the following open problem which would generalize the Kakeya-Eneström Theorem. Given $p_n(z) := \sum_{i=0}^{n} a_i z^i$ with $a_i > 0$ for all $0 \leq i \leq n$, assume in addition that all zeros $\hat{z}$ of $p_n(z)$ lie in the symmetric sector

$$S(\psi) := \{z : |\arg z| \geq \psi > 0, \text{ where } -\pi \leq \arg z \leq \pi\}.$$

Then, derive a bound $M(\psi; a_i)$ such that $|\hat{z}| \leq M(\psi; a_i)$ for all zeros $\hat{z}$ of p_n and such that

$$M(\psi;a_i) \leq \max_{1\leq i\leq n} \left(\frac{a_{i-1}}{a_i}\right) .$$

7. Let K be a compact set in the complex plane without isolated points, and let $||\cdot||_K$ denote the uniform norm on K. For each $n \geq 1$ denote by $T_n(K;z) = z^n + \cdots$ the Chebyshev polynomial of degree n for K, i.e.,

$$||T_n(K;z)||_K = \inf\{||P_n||_K : P_n \text{ monic of degree } n\}.$$

It is well-known that for $K = [-1,1]$ there holds for any polynomial p of degree $\leq n$

$$||p'||_{[-1,1]} \leq ||p||_{[-1,1]} \cdot \frac{||T_n'||_{[-1,1]}}{||T_n||_{[-1,1]}} = n^2||p||_{[-1,1]},$$

where $T_n(x) := T_n([-1,1];x)$ is the classical Chebyshev polynomial of degree n. This suggests the following open question: For what types of sets K is it true that for any polynomial p of degree $\leq n$

$$||p'||_K \leq ||p||_K \cdot \frac{||T_n'(K;z)||_K}{||T_n(K;z)||_K} \quad ?$$

For example, does the above inequality hold when K is a finite union of disjoint compact intervals?

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