J. Math. Anal. Appl. 515 (2022) 126367

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Regular Articles

On the solution of a Riesz equilibrium problem and integral identities for special functions

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ARTICLE INFO

Article history:
Received 1 August 2021
Available online 25 May 2022
Submitted by M.J. Schlosser

Keywords:
Potential theory
Equilibrium measure
Riesz kernel
Integral identities / special functions
Hypergeometric function
Elliptic integral

ABSTRACT

The aim of this note is to provide a full space quadratic external field extension of a classical result of Marcel Riesz for the equilibrium measure on a ball with respect to Riesz s-kernels. We address the case $s = d - 3$ for arbitrary dimension d, in particular the logarithmic kernel in dimension 3. The equilibrium measure for this full space external field problem turns out to be a radial arcsine distribution supported on a ball with a special radius. As a corollary, we obtain new integral identities involving special functions such as elliptic integrals and more generally hypergeometric functions. It seems that these identities are not found in the existing tables for series and integrals, and are not recognized by advanced mathematical software. Among other ingredients, our proofs involve the Euler – Lagrange variational characterization, the Funk – Hecke formula, the Weyl regularity lemma, the maximum principle, and special properties of hypergeometric functions.

1. Introduction and main results

The goal of this note is to provide a full space quadratic external field extension (Theorem 1.4 below) of a classical result of Marcel Riesz (Theorem 1.1 below) for the equilibrium measure on a ball in arbitrary dimensions with respect to Riesz s-kernels, including the logarithmic kernel. The equilibrium measure turns out to be a radial arcsine distribution. As corollaries, we obtain new integral identities involving special functions such as elliptic integrals and more generally hypergeometric functions; see, for example, Corollar-

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https://doi.org/10.1016/j.jmaa.2022.126367

ies 1.3, 1.5, and 1.6 below. It seems that these identities are not found in the existing tables for series and integrals, and are not recognized by advanced mathematical software.

Before we present our results and identities, we recall some basic notions from potential theory. Throughout this note, we denote by d the Euclidean dimension, which is always a positive integer, and by $s \in (-2, +\infty)$ the Riesz parameter. For $x \in \mathbb{R}^d$, $x \neq 0$, the Riesz s-kernel is defined by

$$K_s(x) := \begin{cases} \operatorname{sign}(s)\,|x|^{-s} & \text{if } -2 < s < 0 \text{ or } s > 0 \\ -\log|x| & \text{if } s = 0 \end{cases}, \tag{1.1}$$

where $|x| := \sqrt{x_1^2 + \cdots + x_d^2}$ is the Euclidean norm. It is the Coulomb or Newton kernel if $s = d-2$. Let $\mathcal{M}_1$ be the set of probability measures on $\mathbb{R}^d$ and let $V : \mathbb{R}^d \mapsto (-\infty, +\infty]$ be a lower semicontinuous function, which will play the role of an external field. In this note we only deal with either an external field constant on a centered ball and infinite outside the ball, or with a quadratic external field of the form $V(\cdot) = \gamma\,|\cdot|^2$, $\gamma > 0$. The energy of $\mu \in \mathcal{M}_1$ with external field V is defined by

$$\mathrm{I}(\mu) := \iint_{\mathbb{R}^d \times \mathbb{R}^d} (K_s(x-y) + V(x) + V(y))\mu(\mathrm{d}x)\mu(\mathrm{d}y) \in (-\infty, +\infty]. \tag{1.2}$$

For $s \in (-2, d)$, with our choices of V, the integrand in the double integral in (1.2) is bounded below, I is strictly convex[1] on $\mathcal{M}_1$ and lower semicontinuous with compact level sets.[2] It has a unique global minimizer called the "equilibrium measure" $\mu_{\mathrm{eq}} \in \mathcal{M}_1$; in other words,

$$\mathrm{I}(\mu_{\mathrm{eq}}) = \min_{\mu \in \mathcal{M}_1} \mathrm{I}(\mu) \quad \text{and} \quad \mathrm{I}(\mu) > \mathrm{I}(\mu_{\mathrm{eq}}) \text{ for all } \mu \neq \mu_{\mathrm{eq}},\ \mu \in \mathcal{M}_1. \tag{1.3}$$

Moreover, μ_{eq} is compactly supported with finite energy $\mathrm{I}(\mu_{\mathrm{eq}}) < +\infty$. We refer to [21] and [5] for more details. If $s < 0$, then K_s is not singular and we could have $\mathrm{I}(\mu) < \infty$ for a $\mu \in \mathcal{M}_1$ having Dirac masses; in particular μ_{eq} could conceivably have Dirac masses. In contrast, if $s \geq 0$ then K_s is singular and $\mathrm{I}(\mu) = +\infty$ whenever μ has Dirac masses; consequently μ_{eq} cannot have such masses.

We first recall a classical result of M. Riesz for the equilibrium measure with constant external field in a closed ball and infinite outside the ball. For $R > 0$, let

$$B_R := \{x \in \mathbb{R}^d : |x| \leq R\} \quad \text{and} \quad S_R := \{x \in \mathbb{R}^d : |x| = R\}$$

denote the ball and sphere of radius R centered at the origin. In particular $S_1 = \mathbb{S}^{d-1}$ is the unit sphere, with surface area $|\mathbb{S}^{d-1}| = 2\pi^{d/2}/\Gamma(d/2)$. For a subset S of $\mathbb{R}^d$, we denote, when it makes sense, by σ_S the uniform probability measure on S (normalized trace of Lebesgue measure).

Theorem 1.1 *(Riesz [29]). Suppose that $d \in \{2, 3, 4, \ldots\}$ and* $V = \begin{cases} 0 & \text{on } B_R \\ +\infty & \text{outside } B_R \end{cases}$, $R > 0$.

- *If $-2 < s \leq d-2$, then $\mu_{\mathrm{eq}} = \sigma_{S_R}$,*
- *If $d-2 < s < d$, then μ_{eq} is the probability measure*

$$\mu_{\mathrm{eq}}(\mathrm{d}x) = \frac{\Gamma(1+\frac{s}{2})}{R^s \pi^{\frac{d}{2}} \Gamma(1+\frac{s-d}{2})} \frac{\mathbf{1}_{|x| \leq R}}{(R^2 - |x|^2)^{\frac{d-s}{2}}} \mathrm{d}x = \frac{2\Gamma(1+\frac{s}{2})}{R^s \Gamma(1+\frac{s-d}{2}) \Gamma(\frac{d}{2})} \frac{r^{d-1} \mathbf{1}_{r \leq R}}{(R^2 - r^2)^{\frac{d-s}{2}}} \mathrm{d}r \mathrm{d}\sigma_{S_1}, \tag{1.4}$$

[1] In other words, K_s is conditionally strictly positive in the sense of Bochner, see for instance [5, Section 4.4].

[2] We follow the probability theory standard and equip the convex set $\mathcal{M}_1$ with the topology of weak convergence with respect to continuous and bounded test functions, in other words the weak-$*$ convergence.

where $\mathrm{d}x$ *and* $\mathrm{d}r$ *denote the Lebesgue measures on* $\mathbb{R}^d$ *and on* $[0,+\infty)$ *respectively.*
Moreover, the equilibrium potential $U^{\mu_{\mathrm{eq}}}$ *satisfies, for* $x \in B_R$*,*

$$U^{\mu_{\mathrm{eq}}}(x) := (K_s * \mu_{\mathrm{eq}})(x) = \int_{\mathbb{R}^d} K_s(x-y)\mu_{\mathrm{eq}}(\mathrm{d}y) = \mathrm{I}(\mu_{\mathrm{eq}}) = \frac{\Gamma(1+\frac{s}{2})\Gamma(\frac{d-s}{2})}{R^s\Gamma(\frac{d}{2})}. \tag{1.5}$$

The case $d-2 < s < d$ in Theorem 1.1 is a direct consequence of the following formula.

Lemma 1.2 *(Riesz formula [29]). If* $d \in \{2,3,4,\ldots\}$*,* $0 \le d-2 < s < d$*, and* $R > 0$*, then for* $x \in B_R$*,*

$$\int_{\mathbb{R}^d} \frac{|x-y|^{-s}}{(R^2-|y|^2)^{\frac{d-s}{2}}} \mathbf{1}_{|y|\le R} \mathrm{d}y = \frac{\pi^{\frac{d}{2}+1}}{\Gamma(\frac{d}{2})\sin(\frac{\pi}{2}(d-s))}. \tag{1.6}$$

The proof of Theorem 1.1 and Lemma 1.2 can be found, together with some geometric aspects, in the works of M. Riesz [28, p. 438–439] and [29, § 16, Eq. (1)], where it is mentioned that the cases $d = 1,2,3$ were already considered by Pólya and Szegő in [27]. It can also be found in the book [21, § II.3.13, p. 163–164, and Appendix, p. 399 − 400], and is stated in [5, Eq. (4.6.13)]. The proof sketched by Riesz, with a bit more detail by Landkof, involves first a geometric inversion transforming the integral on the ball into an integral on its complement, and second a trigonometric substitution which has a geometric interpretation, both steps being inspired by the analytic-geometric techniques used classically for elliptic integrals since the eighteenth century. For the reader's convenience, a detailed proof of Lemma 1.2 is given in Appendix C.

Our first result, Corollary 1.3, is a simple consequence of Theorem 1.1. It relates an equilibrium measure of potential theory with an integral identity for special functions (here a ${}_2F_1$ hypergeometric function). Before stating it, let us recall the *Newton binomial series*

$$\frac{1}{(1-z)^\alpha} = \sum_{n=0}^{\infty} (\alpha)_n \frac{z^n}{n!}, \quad \alpha, z \in \mathbb{C},\ |z| < 1, \tag{1.7}$$

where $(\alpha)_n := \alpha(\alpha+1)\cdots(\alpha+n-1)$ is the *Pochhammer symbol* for the rising factorial, with the convention $(\alpha)_0 := 1$ if $\alpha \neq 0$. If $\Re(\alpha) > 0$, then $(\alpha)_n = \Gamma(\alpha+n)/\Gamma(\alpha)$. More generally, the *hypergeometric function* with parameters $(a_1,\ldots,a_p) \in \mathbb{C}^p$ and $(b_1,\ldots,b_q) \in \mathbb{C}^q$, at $z \in \mathbb{C}$, $|z| < 1$, is given (when it makes sense) by the series

$${}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z) := \sum_{n=0}^{\infty} \frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\frac{z^n}{n!}. \tag{1.8}$$

Special choices of the parameters $a_1,\ldots,a_p$ and $b_1,\ldots,b_q$ allow us to recover many special functions, for instance ${}_2F_1(\alpha,\beta;\beta;z) = (1-z)^{-\alpha}$, ${}_2F_1(1,1;2;-z) = \frac{\log(1+z)}{z}$, and ${}_2F_1(\frac{1}{2},\frac{1}{2};\frac{3}{2};z^2) = \frac{\arcsin(z)}{z}$. Actually one of the main historical motivations for the introduction and study of hypergeometric functions is the unification of as many as possible special functions via series expansions. For instance the complete *elliptic integral* of first and second kind, K and E respectively, satisfy for $z \in [0,1]$,

$$K(z) := \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{\sqrt{1-z\sin^2(\theta)}} = \int_0^1 \frac{\mathrm{d}t}{\sqrt{1-zt^2}\sqrt{1-t^2}} = \frac{\pi}{2}\,{}_2F_1\Big(\frac{1}{2},\frac{1}{2};1;z\Big) \tag{1.9}$$

and

$$E(z) := \int_0^{\frac{\pi}{2}} \sqrt{1 - z\sin^2(\theta)}\,\mathrm{d}\theta = \int_0^1 \frac{\sqrt{1-zt^2}}{\sqrt{1-t^2}}\,\mathrm{d}t = \frac{\pi}{2}\,{}_2F_1\Big(-\frac{1}{2},\frac{1}{2};1;z\Big). \tag{1.10}$$

They can be extended to the complex plane, with a branch cut discontinuity running from 1 to ∞. For more basic facts about these functions, we refer for instance to the classical books [15,7]. Here we only remark that $K, E \geq 0$ on the interval $[0,1]$, $K(0) = E(0) = \frac{\pi}{2}$, $K(1) = \infty$, and $E(1) = 1$.

The following identity is an easy consequence of Theorem 1.1 (see Section 2.1 and Remark 2.4).

Corollary 1.3 *(Special function identity[3]). For $d \in \{2,3,4,\ldots\}$, $d-2 < s < d$, $\lambda \in [0,1]$,*

$$\int_0^1 {}_2F_1\left(\frac{s}{4},\frac{s+2}{4};\frac{d}{2};\frac{4r^2\lambda^2}{(\lambda^2+r^2)^2}\right)\frac{r^{d-1}}{(\lambda^2+r^2)^{\frac{s}{2}}(1-r^2)^{\frac{d-s}{2}}}\,\mathrm{d}r = \frac{\pi}{2\sin(\frac{\pi}{2}(d-s))}. \tag{1.11}$$

Here are some special cases worth noting for the hypergeometric function in (1.11):

- for $(d,s) = (2,1)$ we get ${}_2F_1\left(\frac{s}{4},\frac{s+2}{4};\frac{d}{2};z\right) = {}_2F_1\left(\frac{1}{4},\frac{3}{4};1;z\right) = \frac{2K\left(\frac{2\sqrt{z}}{\sqrt{z}+1}\right)}{\pi\sqrt{\sqrt{z}+1}}$.
- for $(d,s) = (3,2)$ we get ${}_2F_1\left(\frac{s}{4},\frac{s+2}{4};\frac{d}{2};z\right) = {}_2F_1\left(\frac{1}{2},1;\frac{3}{2};z\right) = \frac{\tanh^{-1}(\sqrt{z})}{\sqrt{z}}$.
- for $(d,s) = (5,4)$ we get ${}_2F_1\left(\frac{s}{4},\frac{s+2}{4};\frac{d}{2};z\right) = {}_2F_1\left(1,\frac{3}{2};\frac{5}{2};z\right) = \frac{3(\sqrt{z}\tanh^{-1}(\sqrt{z})-z)}{z^2}$.

Our main potential theoretic result is the following external field version of Theorem 1.1.

Theorem 1.4 *(Main result). Suppose that $d \in \{2,3,4,\ldots\}$ and $s = d-3$, namely*

$$(d,s) \in \{(2,-1),(3,0),(4,1),\ldots\}.$$

Let

$$R := \left(\frac{c_{d,d-3}\sqrt{\pi}}{4\gamma}\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d+2}{2}\right)}\right)^{\frac{1}{d-1}}, \quad \text{where} \quad c_{d,s} := \begin{cases} |s|(d-2-s) & \text{if } s \neq 0 \\ d-2 & \text{if } s = 0 \end{cases}. \tag{1.12}$$

If $V = \gamma\,|\cdot|^2$, $\gamma > 0$, then the equilibrium measure μ_{eq} for the minimum energy problem on $\mathbb{R}^d$ (1.2)–(1.3) with kernel K_s and external field V is the "radial arcsine distribution"

$$\mu_{\mathrm{eq}}(\mathrm{d}x) = \frac{\Gamma\left(\frac{d+1}{2}\right)}{\pi^{\frac{d+1}{2}}R^{d-1}}\frac{\mathbf{1}_{|x|\leq R}}{\sqrt{R^2-|x|^2}}\mathrm{d}x = \frac{2\Gamma\left(\frac{d+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{d}{2}\right)R^{d-1}}\frac{r^{d-1}\mathbf{1}_{r\leq R}}{\sqrt{R^2-r^2}}\,\mathrm{d}r\,\mathrm{d}\sigma_{S_1}, \tag{1.13}$$

where $\mathrm{d}x$ and $\mathrm{d}r$ are the Lebesgue measures on $\mathbb{R}^d$ and on $[0,\infty)$ respectively. Moreover, this μ_{eq} is also the equilibrium measure in Theorem 1.1 with $s = d-1$ and R as in (1.12).

Theorem 1.4 is proved in Section 2.2.

Several extensions of Theorem 1.4 for more general (d,s) or V are considered in [10].

[3] It is worth noting that $4r^2\lambda^2 \leq (\lambda^2+r^2)^2$ with equality if and only if $\lambda = r$, so that the radius of convergence of the ${}_2F_1$ in (1.11), which is equal to 1, is reached in the interior of the interval of integration over r when $\lambda \in [0,1)$.

Table 1
Some special values of the critical radius R in Theorem 1.4, with $\gamma = 1$.

$d=2, s=-1$	$d=3, s=0$	$d=4, s=1$	$d=\infty, s=\infty-3$
$\frac{\pi}{8} \approx 0.392699$	$\frac{1}{\sqrt{3}} \approx 0.57735$	$\frac{1}{2}\sqrt[3]{\frac{3\pi}{4}} \approx 0.665335$	1

Table 1 gives values of the radius R in (1.12) for $\gamma = 1$ and various values of d. For $\gamma = 1$, integer $d \geq 2$, the function $d \mapsto R$ achieves its minimum ≈ 0.392699 at $d = 2$ ($s = -1$) and its maximum ≈ 1.04747 at $d = 16$ ($s = 13$), and these values are the unique extreme points. Regarding high dimensional behavior or asymptotic analysis, we have $\lim_{d=s+3\to\infty} R = 1$.

As we shall verify, Theorem 1.4 yields the following integral formulas.

Corollary 1.5 *(Integral formula). Let $d \in \{2, 3, 4, \ldots\}$ and $\lambda \in [0, 1]$. Then*

$$\int_0^1 \mathcal{S}_{d-3}\Big(\frac{4\lambda r}{(\lambda+r)^2}\Big)\frac{(\lambda+r)^{3-d}r^{d-1}\,\mathrm{d}r}{\sqrt{1-r^2}} = \frac{\pi^{\frac{3}{2}}\Gamma(\frac{d-1}{2})}{2^{d+1}\Gamma(\frac{d}{2})}\Big(\Big(\frac{3}{d}-1\Big)\lambda^2+1\Big), \tag{1.14}$$

where

$$\mathcal{S}_s(z) := \int_0^{\frac{\pi}{2}} \frac{\sin^{s+1}(2\alpha)\mathrm{d}\alpha}{2^{s+1}(1-z\sin^2(\alpha))^{\frac{s}{2}}} = \int_0^1 \frac{t^{s+1}(1-t^2)^{\frac{s}{2}}\mathrm{d}t}{(1-zt^2)^{\frac{s}{2}}} = \frac{\Gamma(\frac{s+2}{2})^2}{2\Gamma(s+2)}\,{}_2F_1\Big(\frac{s+2}{2},\frac{s}{2};s+2;z\Big). \tag{1.15}$$

Equivalently, for $d \in \{2, 3, 4, \ldots\}$ and $\lambda \in [0, 1]$,

$$\int_0^1 {}_2F_1\Big(\frac{d-1}{2},\frac{d-3}{2};d-1;\frac{4\lambda r}{(\lambda+r)^2}\Big)\frac{(\lambda+r)^{3-d}r^{d-1}}{\sqrt{1-r^2}}\,\mathrm{d}r = \frac{\pi}{4}\Big(\Big(\frac{3}{d}-1\Big)\lambda^2+1\Big). \tag{1.16}$$

Corollary 1.5 is proved in Section 2.3.

The formula (1.16) comes from the Euler–Lagrange characterization related to Theorem 1.4. Numerical experiments suggest that (1.14) and (1.16) remain valid whenever the parameter $d > 1$ is real.

It is tempting to regard $\mathcal{S}_s$ as a *special function* in its own right. When $(d, s) = (2, -1)$, it becomes the complete elliptic integral of the second kind, namely $\mathcal{S}_{-1} = E$, and (1.14) becomes (1.19) below.

Corollary 1.6 *(More integral formulas[4]). For $\lambda \in [0, 1]$,*

$$\int_0^1 \big((\lambda+r)^2\log(\lambda+r)-(\lambda-r)^2\log|\lambda-r|\big)\frac{r\,\mathrm{d}r}{\sqrt{1-r^2}} = \pi\Big(\frac{\lambda^3}{3}+(1-\log 2)\lambda\Big), \tag{1.17}$$

$$\int_0^1 \big((\lambda+r)\log(\lambda+r)-(\lambda-r)\log|\lambda-r|\big)\frac{r\,\mathrm{d}r}{\sqrt{1-r^2}} = \frac{\pi}{2}\Big(\lambda^2+\frac{1}{2}-\log 2\Big), \tag{1.18}$$

$$\int_0^1 E\left(\frac{4\lambda r}{(\lambda+r)^2}\right)\frac{(\lambda+r)r\,\mathrm{d}r}{\sqrt{1-r^2}} = \frac{\pi^2}{8}\Big(\frac{\lambda^2}{2}+1\Big), \tag{1.19}$$

[4] Note that $\frac{4r\lambda}{(\lambda+r)^2} = \frac{4x}{(1+x)^2}$ with $x := \frac{\lambda}{r}$. The map $x \mapsto \frac{4x}{(1+x)^2}$ is the *Landen transform* of elliptic integrals.

$$\int_0^1 K\left(\frac{4\lambda r}{(r+\lambda)^2}\right)\frac{(\lambda-r)r\,\mathrm{d}r}{\sqrt{1-r^2}}=\frac{\pi^2}{8}\Big(\frac{3\lambda^2}{2}-1\Big), \tag{1.20}$$

where E and K are the special functions defined in (1.10) *and* (1.9)*.*

Corollary 1.6 is proved in Section 2.4 by applying further transformations to the Euler–Lagrange conditions of Theorem 1.4 in the special cases $(d,s)\in\{(2,-1),(3,0)\}$.

To the best of our knowledge, the formulas provided by Corollaries 1.3, 1.5, and 1.6 are not found in the existing catalogs of identities and tables for series and integrals such as [12], [26], and [6], and are not recognized by advanced software such as Maplesoft Maple and Wolfram Mathematica. However it is worth noting that these softwares do recognize the first parts of (1.17) and (1.18) in terms of ${}_3F_2$:

$$\begin{aligned}&\int_0^1(\lambda+r)^2\log(\lambda+r)\frac{r\,\mathrm{d}r}{\sqrt{1-r^2}}\\&\qquad=\frac{\pi\,{}_3F_2\left(\frac12,1,1;2,3;\frac{1}{\lambda^2}\right)}{16\lambda}-\frac{2\,{}_3F_2\left(1,1,\frac32;\frac52,\frac72;\frac{1}{\lambda^2}\right)}{45\lambda^2}+\frac{\pi\lambda}{4}+\frac{3\lambda(2\lambda+\pi)+4}{6}\log(\lambda)+1\end{aligned}\tag{1.21}$$

and

$$\begin{aligned}&\int_0^1(\lambda+r)\log(\lambda+r)\frac{r\,\mathrm{d}r}{\sqrt{1-r^2}}\\&\qquad=\frac{32\lambda\,{}_3F_2\left(\frac12,1,1;\frac32,\frac52;\frac{1}{\lambda^2}\right)-3\pi\,{}_3F_2\left(1,1,\frac32;2,3;\frac{1}{\lambda^2}\right)+24\lambda^2((4\lambda+\pi)\log(\lambda)+\pi)}{96\lambda^2}.\end{aligned}\tag{1.22}$$

2. Proofs

2.1. Proof of Corollary 1.3

Let us consider the settings of Theorem 1.1 in the case $d-2<s<d$. By scaling we can assume without loss of generality that $R=1$. Then, using the Funk–Hecke formula (A.11), we get, for $x\in\mathbb{R}^d$, $x\neq0$, denoting $\lambda:=|x|$ and $C:=\frac{\Gamma(1+\frac{s}{2})}{\pi^{\frac{d}{2}}\Gamma(1+\frac{s-d}{2})}$,

$$\begin{aligned}U^{\mu_{\mathrm{eq}}}(x)&=C\int_{|y|\le1}\frac{|x-y|^{-s}}{(1-|y|^2)^{\frac{d-s}{2}}}\mathrm{d}y\\&=C\int_0^1\Big(\int_{S_1}\frac{(\lambda^2+r^2-2\lambda r\frac{x}{|x|}\cdot u)^{-\frac{s}{2}}}{(1-r^2)^{\frac{d-s}{2}}}r^{d-1}\mathrm{d}u\Big)\,\mathrm{d}r\\&=C\tau_{d-1}|S_1|\int_0^1\Big(\int_{-1}^1\frac{(1-t^2)^{\frac{d-3}{2}}}{(\lambda^2+r^2-2\lambda rt)^{\frac{s}{2}}}\mathrm{d}t\Big)\frac{r^{d-1}}{(1-r^2)^{\frac{d-s}{2}}}\,\mathrm{d}r.\end{aligned}$$

Now, since $\frac{2\lambda r}{(\lambda^2+r^2)}\in[0,1/2]$, using the Newton binomial series (1.7),

$$\int_{-1}^1\frac{(1-t^2)^{\frac{d-3}{2}}}{(\lambda^2+r^2-2\lambda rt)^{\frac{s}{2}}}\mathrm{d}t=\frac{1}{(\lambda^2+r^2)^{\frac{s}{2}}}\int_{-1}^1\frac{(1-t^2)^{\frac{d-3}{2}}}{(1-\frac{2\lambda r}{\lambda^2+r^2}t)^{\frac{s}{2}}}\mathrm{d}t$$

$$= \frac{1}{(\lambda^2+r^2)^{\frac{s}{2}}} \sum_{n=0}^{\infty} \frac{(\frac{s}{2})_{2n}}{(2n)!} \left(\frac{4r^2\lambda^2}{(\lambda^2+r^2)^2}\right)^n \int_{-1}^{1} (1-t^2)^{\frac{d-3}{2}} t^{2n} \mathrm{d}t$$

$$= \frac{1}{(\lambda^2+r^2)^{\frac{s}{2}}} \sum_{n=0}^{\infty} \frac{(\frac{s}{2})_{2n}}{(2n)!} \left(\frac{4r^2\lambda^2}{(\lambda^2+r^2)^2}\right)^n \frac{\Gamma(\frac{d-1}{2})\Gamma(n+\frac{1}{2})}{\Gamma(\frac{d}{2}+n)}$$

$$= \frac{\sqrt{\pi}\Gamma(\frac{d-1}{2})}{\Gamma(\frac{d}{2})} \frac{{}_2F_1\left(\frac{s}{4}, \frac{s+2}{4}; \frac{d}{2}; \frac{4r^2\lambda^2}{(r^2+\lambda^2)^2}\right)}{(\lambda^2+r^2)^{\frac{s}{2}}},$$

where we have use the identities (related to the Legendre duplication formula (A.2))

$$\left(\frac{s}{2}\right)_{2n} = 2^{2n}\left(\frac{s}{4}\right)_n\left(\frac{s}{4}+\frac{1}{2}\right)_n \quad \text{and} \quad 2^{2n}\left(\frac{1}{2}\right)_n = \frac{(2n)!}{n!}.$$

Note that we could alternatively proceed as in the proof of Lemma B.2 and use the Euler integral formula (A.10). Next, using the fact that $\tau_{d-1} = \frac{\Gamma(\frac{d}{2})}{\sqrt{\pi}\Gamma(\frac{d-1}{2})}$, we get, for $x \in \mathbb{R}^d$, $x \neq 0$,

$$U^{\mu_{\mathrm{eq}}}(x) = C|S_1| \int_0^1 {}_2F_1\left(\frac{s}{4}, \frac{s+2}{4}; \frac{d}{2}; \frac{4r^2\lambda^2}{(\lambda^2+r^2)^2}\right) \frac{r^{d-1}}{(\lambda^2+r^2)^{\frac{s}{2}}(1-r^2)^{\frac{d-s}{2}}} \mathrm{d}r.$$

(This last integral may be compared with extensions of the Beta integral in [13, Th. 5.1] and [31]).

Now, the Euler–Lagrange conditions (A.14) and the continuity of $U^{\mu_{\mathrm{eq}}}$ give that this quantity is constant on B_1. Finally the value of the constant can be obtained from (1.6). □

2.2. Proof of Theorem 1.4

We split the proof into several subsections.

2.2.1. Computation of critical radius and candidate equilibrium measure

From the uniqueness property we know that the equilibrium measure μ_{eq} is radially symmetric. Let us make a succession of assumptions to extract a candidate for μ_{eq}, and we will then check that it indeed satisfies the Euler–Lagrange conditions (A.14). We start by observing from (A.14) that, for $x \in S_* := \mathrm{supp}(\mu_{\mathrm{eq}})$,

$$\int K_s(x-y)\mu_{\mathrm{eq}}(\mathrm{d}y) + \gamma|x|^2 = c. \tag{2.1}$$

Applying the Laplacian operator to (2.1) and assuming it can be taken inside the integral, we get from (2.1) and (A.15) that for all x in the interior of S_*,

$$-c_{d,s}\int K_{s+2}(x-y)\mu_{\mathrm{eq}}(\mathrm{d}y) + 2\gamma d = 0. \tag{2.2}$$

In our case $s = d-3$, so $c_{d,s} = c_{d,d-3}$ is equal to $|d-3|$ if $d \neq 3$ while it is equal to 1 if $d = 3$.

Next suppose that $S_* = B_R$ for some $R > 0$. Let ν_R be the equilibrium measure for the minimum energy problem on B_R with kernel $K_{s+2} = |\cdot|^{-(s+2)}$ and $V = 0$. Observing that ν_R is the dilation by a factor of R of ν_1, we see from Theorem 1.1 that ν_R is the “radial arcsine distribution”; in other words, the measure

$$\nu_R(\mathrm{d}x) = \frac{C_{d,R}}{\sqrt{R^2-|x|^2}}\mathbf{1}_{|x|\le R}\,\mathrm{d}x, \quad \text{where} \quad C_{d,R} = \frac{2\Gamma\left(\frac{d+1}{2}\right)}{|S_1|\sqrt{\pi}\Gamma\left(\frac{d}{2}\right)R^{d-1}} = \frac{\Gamma\left(\frac{d+1}{2}\right)}{R^{d-1}\pi^{\frac{d+1}{2}}}. \tag{2.3}$$

In particular, the support of ν_R is all of B_R. Next, by definition of ν_R, the associated Euler–Lagrange conditions state that, for some constant W_R,

$$\int K_{s+2}(x-y)\nu_R(\mathrm{d}x) = W_R, \quad \text{for } y \in B_R. \tag{2.4}$$

As $d = s+3$ we obtain (using $y = 0 \in B_R$)

$$W_R = \frac{W_1}{R^{s+2}} = \frac{W_1}{R^{d-1}} \quad \text{and} \quad W_1 = C_{d,1}|S_1| \int_0^1 \frac{r^{-(s+2)}r^{d-1}}{\sqrt{1-r^2}}\,\mathrm{d}r = \sqrt{\pi}\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}. \tag{2.5}$$

To derive the value of R we first integrate (2.2) with respect to $\nu_R(\mathrm{d}x)$ and swap the integrals, assuming that this is legal, giving

$$-c_{d,s}\int\Big(\int K_{s+2}(x-y)\nu_R(\mathrm{d}x)\Big)\mu_{\mathrm{eq}}(\mathrm{d}y) + 2\gamma d = 0. \tag{2.6}$$

Then, using (2.4) and (2.5) in (2.6), we get

$$\frac{c_{d,s}\sqrt{\pi}\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)} R^{-s-2} = 2\gamma d.$$

Finally, from the formula $z\Gamma(z) = \Gamma(z+1)$ with $z = d/2$, we derive the desired formula for R, namely

$$R = \left(\frac{c_{d,d-3}\sqrt{\pi}\Gamma(\frac{d+1}{2})}{4\gamma\Gamma(\frac{d+2}{2})}\right)^{\frac{1}{d-1}} = \left(\frac{c_{s+3,s}\sqrt{\pi}\Gamma(\frac{s+4}{2})}{4\gamma\Gamma(\frac{s+5}{2})}\right)^{\frac{1}{s+2}}. \tag{2.7}$$

See also Remark 2.4 for an alternative way to compute this critical value of R.

2.2.2. Euler–Lagrange characterization

The probability measure ν_R in (2.3) with R as in (2.7) satisfies the Frostman conditions (A.14) with kernel K_s and $V = \gamma\,|\cdot|^2$ thanks to Lemma 2.1 below.

Lemma 2.1 *(Potentials). Let R be as in (2.7) and define $\Phi := K_s * \nu_R + \gamma\,|\cdot|^2$. Then,*

- *Φ is continuous on $\mathbb{R}^d$;*
- *$\Phi = \Phi(0)$ on B_R;*
- *$\Phi \geq \Phi(0)$ outside B_R.*

Proof. As $K_s * \nu_R$ is radially symmetric, so is Φ. Thus we define for any $\lambda \geq 0$,

$$\varphi(\lambda) := \Phi\left(\lambda R\widehat{x}\right) \qquad \text{for any } \widehat{x} \in \mathbb{R}^d \text{ with } |\widehat{x}| = 1. \tag{2.8}$$

Using from Lemma A.1 that $K_s * \nu_R \in L^1_{\mathrm{loc}}(\mathbb{R}^d, \mathrm{d}x)$ and we can swap the Laplacian and the Riesz potential, we get

$$\Delta\Phi = -c_{s+3,s}K_{s+2} * \nu_R + 2d\gamma. \tag{2.9}$$

Moreover, ν_R and the radius R have been chosen in the preceding subsection precisely in such a way that on $\mathrm{int}(B_R) := \{x \in \mathbb{R}^d : |x| < R\}$,

$$\Delta\Phi = -c_{s+3,s}K_{s+2} * \nu_R + 2d\gamma = 0. \tag{2.10}$$

Continuity of Φ. At this step, let us remark that when $d < 6$, Lemma A.1 *(iv)* gives that $K_s * \nu_R$ is continuous on $\mathbb{R}^d$ since $\frac{\mathrm{d}\nu_R}{\mathrm{d}x} \in L^p(\mathbb{R}^d, \mathrm{d}x)$, $2 > p > d/(d-s) = d/3$ (recall that $s = d-3$).

Actually $K_s * \nu_R$ is continuous on $\mathbb{R}^d$ for arbitrary dimension d. Indeed, this can be checked directly, in the case $(d,s) = (2,-1)$, since K_{-1} is not singular. This can also be checked directly in the case $(d,s) = (3,0)$ from the formula provided by Lemma B.1. Finally, in the case $s = d-3 > 0$, using Lemma B.1 and Lemma B.2 and the change of variable $r = \sin(\theta)$ to remove the singularity at the edge $r = 1$, we get, for $x \in \mathbb{R}^d$, with $\lambda = |x|/R$, for some constant $C_s > 0$,

$$(K_s * \nu_R)(x) = C_s \int_0^{\frac{\pi}{2}} {}_2F_1\Big(\frac{s}{2}+1, \frac{s}{2}; s+2; \frac{4\lambda\sin(\theta)}{(\lambda+\sin(\theta))^2}\Big)\frac{\sin(\theta)^{s+2}}{(\lambda+\sin(\theta))^s}\mathrm{d}\theta. \tag{2.11}$$

The continuity of $K_s * \nu_R$ follows then from the uniform continuity of the hypergeometric function. Indeed, by (A.4) the series that defines ${}_2F_1(a,b;c;z)$ converges absolutely for $z \in [0,1]$ (and remarkably for $z = 1$) when $c-a-b > 0$ as in our case $c-a-b = 1$. The hypergeometric function ${}_2F_1\left(\frac{s}{2}+1, \frac{s}{2}; s+2; z\right)$ is uniformly continuous on $[0,1]$ since it is clearly analytic on $[0,1)$ and it is also continuous at $z = 1$; the latter assertion follows from Abel's Limit Theorem [1, Sec. 2.5] and the fact that ${}_2F_1\left(\frac{s}{2}+1, \frac{s}{2}; s+2; 1\right)$ is finite. Furthermore $\lambda = 0$ is not a problem as soon as we establish the fact that Φ is harmonic (in fact constant) in the unit disk (see below!).

Constantness on B_R. It follows from Lemma 2.2.

Behavior outside B_R. Since φ defined in (2.8) is continuous on $[0,+\infty)$ and differentiable on $(1,+\infty)$, the Frostman condition outside B_R is realized if we show that $\varphi'(\lambda) \geq 0$ for $\lambda > 1$.

Let us first consider the case $(d,s) = (2,-1)$. By Lemma B.3, φ is convex on $[1,+\infty)$ since

$$\varphi''(\lambda) = R\frac{\sqrt{1-\frac{1}{\lambda^2}}}{2\lambda} - R\frac{\arcsin\left(\frac{1}{\lambda}\right)}{2} + 2\gamma R^2 \geq 0 - \frac{\pi}{8\gamma}\frac{\frac{\pi}{2}}{2} + 2\frac{\pi^2}{64\gamma} = 0, \quad \lambda > 1. \tag{2.12}$$

Moreover since (B.8) gives $\lim_{\lambda\to 1^+}\varphi'(\lambda) = -\frac{\pi}{16\gamma}\frac{\pi}{2} + 2\frac{\pi^2}{64\gamma} = 0$, it follows that $\varphi'(\lambda) \geq 0$ when $\lambda > 1$.

Finally, consider the case $(d,s) = (3,0)$. By Lemma B.4, if $\lambda > 1$,

$$\varphi'(\lambda) = \frac{2\sqrt{\lambda^2-1}^3}{3\gamma\lambda^2} \geq 0. \tag{2.13}$$

Let us now consider the case $s = d-3 > 0$. We can rewrite (2.11) as

$$\varphi(\lambda) = C_s \int_0^1 h(\lambda, r)\,\frac{r^{d-1}}{\sqrt{1-r^2}}\,\mathrm{d}r, \tag{2.14}$$

where

$$h(\lambda, r) := {}_2F_1\left(\frac{s}{2}+1, \frac{s}{2}; s+2; \frac{4\lambda r}{(\lambda+r)^2}\right)(\lambda+r)^{-s} \tag{2.15}$$

$$= {}_2F_1\left(\frac{s}{4}, \frac{s+2}{4}; \frac{s+3}{2}; \frac{4\lambda^2 r^2}{(\lambda^2+r^2)^2}\right)\left(\lambda^2+r^2\right)^{-\frac{s}{2}} \tag{2.16}$$

where the last equality comes from the quadratic transformation (A.6). Differentiating (2.15) with

$$z = \frac{4\lambda r}{(\lambda + r)^2}, \qquad \frac{\partial z}{\partial \lambda} = \frac{4r(r-\lambda)}{(\lambda+r)^3}$$

and using the derivative formula (A.9) for ${}_2F_1$ we get

$$\frac{\partial h}{\partial \lambda}(\lambda, r) = sr(r-\lambda)(\lambda+r)^{-s-3}\ {}_2F_1\left(\frac{s}{2}+2, \frac{s}{2}+1; s+3; \frac{4\lambda r}{(\lambda+r)^2}\right) - s(\lambda+r)^{-s-1}\ {}_2F_1\left(\frac{s}{2}+1, \frac{s}{2}; s+2; \frac{4\lambda r}{(\lambda+r)^2}\right). \quad (2.17)$$

The only potential difficulties are when the argument of the hypergeometric functions is $z = 1$. As before $z \in [0,1]$ and $z = 1 \iff \lambda = r$. The parameters in the first hypergeometric function in (2.17) satisfy $c - a - b = 0$, so by the property (A.5) of ${}_2F_1$, we get

$$\lim_{\lambda \to r} (r-\lambda)\ {}_2F_1\left(\frac{s}{2}+2, \frac{s}{2}+1; s+3; \frac{4\lambda r}{(\lambda+r)^2}\right) = 0.$$

As before, the second hypergeometric function in (2.17) has parameters which satisfy $c - a - b = 1 > 0$. Thus $\frac{\partial h}{\partial \lambda}(\lambda, r)$ is uniformly continuous for $r \in [0,1]$ and $\lambda \geq 0$, so by the Leibniz integral rule

$$\varphi'(\lambda) = C_s \int_0^{\frac{\pi}{2}} \frac{\partial h}{\partial \lambda}(\lambda, \sin(\theta))\ \sin(\theta)^{d-1}\, \mathrm{d}\theta$$

and $\varphi'(\lambda)$ is continuous for $\lambda \geq 0$. In particular

$$\lim_{\lambda \to 1^+} \varphi'(\lambda) = \varphi'(1).$$

Let us show now that $\varphi'(\lambda) \geq 0$ for $\lambda \geq 0$.

Since $s + 2 = d - 1 > d - 2$, the function $K_{s+2} * \nu_R$ is subharmonic outside the support B_R of ν_R, see for instance [21, Th. I.1.4, p. 66]. Since it is continuous everywhere in $\mathbb{R}^d$ even at ∞, it follows by the maximum principle applied on the complement of B_R, that for $|x| \geq R$,

$$I(\nu_R) \geq U^{\nu_R}(x) = \int_{\mathbb{R}^d} \frac{1}{|x-y|^{s+2}}\ \nu_R(\mathrm{d}y) \quad (2.18)$$

(equality holds for $|x| = R$ by Theorem 1.1). It follows by using (1.5) and (2.9) that $\Delta\Phi(x) \geq 0$ for $|x| > R$. Next, using the radial form of the Laplacian,

$$\frac{1}{\lambda^{d-1}} \left(\lambda^{d-1}\varphi'(\lambda)\right)' \geq 0 \quad \text{for} \quad \lambda > 1.$$

Thus $\int_\rho^\lambda [\tau^{d-1}\varphi'(\tau)]'\, \mathrm{d}\tau \geq 0$ for $\lambda \geq \rho > 1$, and so

$$\lambda^{d-1}\varphi'(\lambda) \geq \rho^{d-1}\varphi'(\rho).$$

Finally, letting $\rho \to 1^+$ we get, as $\varphi'(1) = 0$,

$$\varphi'(\lambda) \geq 0 \quad \text{for} \quad \lambda \geq 1.$$

We also know that φ is constant for $0 \le \lambda \le 1$, so $\varphi'(\lambda) \ge 0$ for $\lambda \ge 0$. □

Lemma 2.2 *(Laplacian inversion or Liouville lemma). Let $\Phi : \mathrm{int}(B_R) \to \mathbb{R}$, $d \ge 2$, $R > 0$. If*

- *(local integrability) $\Phi \in L^1_{\mathrm{loc}}(\mathrm{d}x)$;*
- *(weak harmonicity) $\Delta\Phi = c$ for a constant c, in the sense of Schwartz distributions;*
- *(radial symmetry) Φ is equal to a constant on the sphere S_r for all $r < R$;*

then Φ is $\mathcal{C}^\infty$ and is given by $\Phi = \frac{c}{2d}|\cdot|^2 + \Phi(0)$. In particular Φ is constant when $c = 0$.

Related statements can be found in [30, Sec. 0.3] for $d = 2$, and in [21, Th. 3.3, Ch. III, p. 183].

Remark 2.3 *(Extension).* Lemma 2.2 extends to the case where $\Delta\Phi$ is a $\mathcal{C}^\infty$ radial function on B_R, say $\Delta\Phi = A(|\cdot|)$. Indeed the same proof gives $\Phi = \Phi(0) + B(|\cdot|)$, where B solves $rB''(r) + (d-1)B'(r) = A(r)$, $0 < r < R$, with $B(0) = B'(0) = 0$, which gives $B(r) = \int_0^r u^{1-d}\Big(\int_0^u v^{d-1}A(v)\mathrm{d}v\Big)\mathrm{d}u$. Thus

$$B(r) = \int_0^r (k(r) - k(v))v^{d-1}A(v)\mathrm{d}v \quad \text{with} \quad k(v) := \begin{cases} \frac{v^{2-d}}{2-d} & \text{if } d \ne 2 \\ \log(v) & \text{if } d = 2 \end{cases}.$$

If A is a polynomial of degree m, then B is a polynomial of degree $m+2$, while if A is a hypergeometric series, then B is also a hypergeometric series. For an arbitrary integer $m \ge 1$, repeating this procedure gives a symmetric polynomial in d variables Φ such that $\Delta^m\Phi = c$. See for instance [14] and references therein for a link with Jacobi and Zernike orthogonal polynomials and hypergeometric functions.

Proof of Lemma 2.2. By a version of the Weyl lemma expressing the Hörmander hypoellipticity of the Laplacian operator, see for instance Stroock's expository note [34], we get that Φ is $\mathcal{C}^\infty(\mathrm{int}(B_R))$. Next, by radial symmetry $\Phi(x) = \psi(r)$ where $r = |x|$. Using $\Delta = \partial_r^2 + \frac{d-1}{r}\partial_r + \frac{1}{r^2}\Delta_{S_1}$ we get that

$$c = \Delta\Phi(x) = \psi''(r) + \frac{d-1}{r}\psi'(r) = \frac{(r^{d-1}\psi'(r))'}{r^{d-1}},$$

and thus $r^{d-1}\psi'(r) = \frac{c}{d}r^d$ (note that we use here the fact that $d > 1$ to get that $r^{d-1}\psi'(r) = 0$ when $r \to 0$). Hence $\psi(r) = \frac{c}{2d}r^2 + \psi(0)$. Note that $\Delta\Phi = c = \Delta(\frac{c}{2d}|\cdot|^2)$ gives $\Delta(\Phi - \frac{c}{2d}|\cdot|^2) = 0$. □

2.2.3. Completion of proof

To complete the proof of Theorem 1.4 note that we can reinterpret (2.2) as Frostman conditions (A.14): $\mu_{\mathrm{eq}} = \nu_R$ is seen as an equilibrium measure for kernel $\widetilde{K} = K_{s+2}$ with external field $\widetilde{V}$ equal to 0 on B_R and to $+\infty$ outside, connecting with Theorem 1.1. □

Remark 2.4 *(Alternative motivation of the Frostman condition on B_R).* Following [14, Lemma 2.3] or [8, Sec. 4], Riesz's formula (1.6) with $d \ge 2$, $d-2 < s < d$, $R > 0$ gives, using (A.15) and (A.17), for $x \in \mathrm{int}(B_R)$,

$$\Delta \int_{|y| \le R} \frac{\mathrm{d}y}{|x-y|^{s-2}(R^2 - |y|^2)^{\frac{d-s}{2}}} = \int_{|y| \le R} \frac{c_{d,s-2}\mathrm{d}y}{|x-y|^s(R^2 - |y|^2)^{\frac{d-s}{2}}} = \frac{c_{d,s-2}\pi^{\frac{d}{2}+1}}{\Gamma(\frac{d}{2})\sin((d-s)\frac{\pi}{2})}. \tag{2.19}$$

Now, inverting the Laplacian as in Lemma 2.2 and using Lemma 2.1 for continuity at the boundary, we get

$$\int_{|y| \le R} \frac{\mathrm{d}y}{|x-y|^{s-2}(R^2 - |y|^2)^{\frac{d-s}{2}}} = \frac{\pi^{\frac{d}{2}+1}}{\Gamma(\frac{d}{2})\sin((d-s)\frac{\pi}{2})}\Big(\frac{c_{d,s-2}|x|^2}{2d} + \frac{d-s}{2}R^2\Big). \tag{2.20}$$

Replacing $s-2$ by s gives, for $d \geq 2$ and $d-4 < s < d-2$, $R > 0$, $x \in B_R$,

$$\int\limits_{|y|\leq R} \frac{\mathrm{d}y}{|x-y|^s (R^2-|y|^2)^{\frac{d-s}{2}-1}} = \frac{\pi^{\frac{d}{2}+1}}{\Gamma(\frac{d}{2})\sin((d-s)\frac{\pi}{2})}\Big(\frac{c_{d,s}}{2d}|x|^2 - \frac{d-s-2}{2}R^2\Big). \tag{2.21}$$

The left-hand side can be normalized using the fact that for $0 < \beta < 1$,

$$Z_\beta := \int\limits_{|y|\leq R} \frac{\mathrm{d}y}{(R^2-|y|^2)^\beta} = |S_1|R^{d-2\beta}\int\limits_0^1 \frac{r^{d-1}\mathrm{d}r}{(1-r^2)^\beta} = \frac{R^{d-2\beta}\pi^{\frac{d}{2}}\Gamma(1-\beta)}{\Gamma(1-\beta+\frac{d}{2})}. \tag{2.22}$$

Indeed, with $d-s=3$ and $\beta = \frac{d-s}{2}-1 = \frac{1}{2}$, we get $Z_{\frac{1}{2}} = \frac{R^{s+2}\pi^{\frac{s+4}{2}}}{\Gamma(\frac{s+4}{2})}$, and (2.21) gives

$$\frac{\Gamma(\frac{s+4}{2})}{R^{s+2}\pi^{\frac{s+4}{2}}}\int\limits_{|y|\leq R} \frac{\mathrm{d}y}{|x-y|^s\sqrt{R^2-|y|^2}} + \frac{\Gamma(\frac{s+4}{2})\sqrt{\pi}c_{s+3,s}}{4R^{s+2}\Gamma(\frac{s+5}{2})}|x|^2 = \frac{\Gamma(\frac{s+4}{2})\sqrt{\pi}}{2R^s\Gamma(\frac{s+3}{2})}. \tag{2.23}$$

When R is equal to the critical value (2.7), the prefactor of $|x|^2$ in (2.23) is equal to γ and (2.23) becomes the Frostman condition on B_R for Theorem 1.4. Note also that taking $(d,s) = (3,0)$ in (2.23) is allowed but produces a trivial kernel inside the integral in the left-hand side. It is also possible to take $d = 3$ and $s \to 0$ while keeping $s \neq 0$, and use, for $x \neq 0$,

$$\lim_{\substack{s\to 0\\ s\neq 0}} |s|^{-1}c_{3,s} = 1 \quad \text{and} \quad \lim_{\substack{s\to 0\\ s\neq 0}}\Big(\frac{1}{s|x|^s} - \frac{1}{s}\Big) = \lim_{s\to 0}\frac{|x|^{-s}-1}{s-0} = -\log|x| \tag{2.24}$$

to recover the logarithmic kernel in this case. In another direction, note also that repeating the process that we used to get (2.21) to reach higher powers provides a family of generalizations of (2.21) involving Jacobi polynomials in the right-hand side, and even more generally hypergeometric ${}_2F_1$ functions, see for instance [14,8,17], producing potential extensions of Theorem 1.4.

Remark 2.5 *(Hypergeometric formulas outside B_R).* As pointed out by the referee we can give hypergeometric formulas for the integrals (1.11) and (1.14) when $\lambda = |x|/R \geq 1$. More precisely, for $d \in \{2,3,4,\ldots\}$, $d-2 < s < d$, $\lambda \geq 1$,

$$\int\limits_{\mathbb{R}^d} \frac{|x-y|^{-s}}{(R^2-|y|^2)^{\frac{d-s}{2}}}\mathbf{1}_{|y|\leq R}\mathrm{d}y = \frac{1}{\lambda^s}\frac{\pi^{\frac{d}{2}}\Gamma(\frac{s-d+2}{2})}{\Gamma(\frac{s+2}{2})}{}_2F_1\Big(\frac{s}{2},\frac{s-d+2}{2};\frac{s+2}{2};\frac{1}{\lambda^2}\Big) \tag{2.25}$$

and, using the function $\mathcal{S}_{d-3}$ defined in (1.15),

$$\int\limits_0^1 \mathcal{S}_{d-3}\Big(\frac{4\lambda r}{(\lambda+r)^2}\Big)\frac{(\lambda+r)^{3-d}r^{d-1}\mathrm{d}r}{\sqrt{1-r^2}} = \frac{1}{2\lambda^{d-3}}\frac{\Gamma(\frac{d-1}{2})^2\Gamma(\frac{d}{2})\Gamma(\frac{3}{2})}{\Gamma(d-1)\Gamma(\frac{d+1}{2})}{}_2F_1\Big(\frac{d-3}{2},-\frac{1}{2};\frac{d+1}{2};\frac{1}{\lambda^2}\Big). \tag{2.26}$$

Indeed, to get (2.25), we start from the integral in the left hand side of (1.11) by using, for $0 \leq r < \lambda$,

$${}_2F_1\Big(\frac{s}{4},\frac{s+2}{4};\frac{d}{2};\frac{4\lambda^2r^2}{(\lambda^2+r^2)^2}\Big) = \frac{(\lambda^2+r^2)^{\frac{s}{2}}}{\lambda^s}{}_2F_1\Big(\frac{s}{2},\frac{s-d+2}{2};\frac{d}{2};\frac{r^2}{\lambda^2}\Big), \tag{2.27}$$

which comes from the quadratic transformation (A.7) with $z = r^2/\lambda^2 < 1$. With this replacement the left-hand side of (1.11) takes the form of the following Euler beta integral formula (see [18] and [3, eq. (2.2.2)])

$$\int_0^\infty u^{\alpha-1}(1-u)^{\gamma-\alpha-1}{}_2F_1(a_1,a_2;b_1;tu)\mathrm{d}u = \frac{\Gamma(\alpha)\Gamma(\gamma-\alpha)}{\Gamma(\gamma)}{}_3F_2(a_1,a_2;\alpha;b_1;\gamma;t), \tag{2.28}$$

which leads to (2.25) after cancellation of upper and lower ${}_3F_2$ parameters. Similarly, to get (2.26), we start from the integral on the left-hand side of (1.14) by using, for $0 \le r < \lambda$, the relation[5]

$${}_2F_1\Big(\frac{s}{2}, \frac{s+2}{2}; s+2; \frac{4\lambda r}{(\lambda+r)^2}\Big) = \frac{(\lambda+r)^s}{\lambda^s}{}_2F_1\Big(\frac{s}{2}, -\frac{1}{2}; \frac{s+3}{2}; \frac{r^2}{\lambda^2}\Big), \tag{2.29}$$

which comes from the quadratic transformation (A.8) with $z = r/\lambda < 1$. This leads to (2.26) via (2.28).

2.3. Proof of Corollary 1.5

The function φ defined in (2.8) is continuous on $[0, 1]$ and differentiable on $(0, 1)$. From the proof of Theorem 1.4, the Frostman condition states that φ is constant and equal to $\varphi(0)$ on $\lambda \in [0, 1]$, namely $\varphi'(\lambda) = 0$ for $\lambda \in (0, 1)$. Now the formula (1.14) in Corollary 1.5 comes from the combination of equation $\varphi(\lambda) = \varphi(0)$ together with the formulas for φ provided by Lemmas B.1 and B.2 of Appendix B. Note that (1.14) is trivial when $d = 3$. The formula (1.16) is obtained from (1.14) by using (1.15) and the Legendre duplication formula (A.2).

2.4. Proof of Corollary 1.6

Let us keep the notation used in Appendix B. First of all, the formulas (1.17)–(1.18) in Corollary 1.6 come from the Frostman condition $\varphi(\lambda) = \varphi(0)$ and its reformulation $\varphi'(\lambda) = 0$, and the formula for φ provided by Lemma B.1.

The formula (1.19) in Corollary 1.6 is obtained by further reformulating φ when $(d, s) = (2, -1)$ in terms of special functions using Lemma B.3 below. The formula for φ' provided by this lemma gives

$$\int_0^1 \left[(\lambda+r)E\left(\frac{4\lambda r}{(r+\lambda)^2}\right) + (\lambda-r)K\left(\frac{4\lambda r}{(r+\lambda)^2}\right)\right]\frac{r\,\mathrm{d}r}{\sqrt{1-r^2}} = \frac{\pi^2}{4}\lambda^2. \tag{2.30}$$

Next, following [19,20,2], the Landen transform for E and K gives, for $z \in [-1, 1]$,

$$K\Big(\frac{4z}{(1+z)^2}\Big) = (1+z)K(z^2) \quad \text{and} \quad E\Big(\frac{4z}{(1+z)^2}\Big) = \frac{2}{1+z}E(z^2) - (1-z)K(z^2). \tag{2.31}$$

Now, the formula (1.20) of Corollary 1.5 comes by combining (1.19) and (2.30) with $z = \frac{r}{\lambda}$.

[5] The relation (2.29) can also be derived using Erdélyi [12, 3.3(13)] which is Whipple's relation between the Legendre functions of the first and second kinds, rewritten in terms of ${}_2F_1$. Both the first-kind P_ν^μ and the second-kind Q_ν^μ have ${}_2F_1$ representations; see [11, 14.3.6 and 14.3.7].

Acknowledgments

The authors are grateful to Doug Hardin for the suggestion of using the spherical Laplacian, to Bent Fuglede and Wolfgang Wendland for helpful comments on the convolution of distributions, to Franck Wielonsky for historical references, and to Ekatrina Karatsuba for the references to tables and related literature. The manuscript further benefited from enriching comments by an anonymous reviewer (see Remark 2.5), as well as from the analytic argument based on Gegenbauer polynomials in Remark C.5 which was contributed by Christophe Vignat.

Appendix A. Useful tools

Let us recall the Euler reflection formula for the Gamma function, valid for $z \notin \{-1, -2, \ldots\}$,

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}, \tag{A.1}$$

and the Legendre duplication formula, valid for $2z \notin \{-0, -1, -2, -3, \ldots\}$,

$$\sqrt{\pi}\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma\Big(z + \frac{1}{2}\Big). \tag{A.2}$$

A.1. Hypergeometric identities

- The hypergeometric function ${}_2F_1$ can be written as (see [11, (15.2(i))])

$${}_2F_1\left(a, b; c; z\right) := \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)} \frac{z^k}{k!}. \tag{A.3}$$

- If $\Re(c - a - b) > 0$ then (A.3) converges absolutely for $|z| \leq 1$ and (see [11, (15.4.20)])

$${}_2F_1\left(a, b; c; 1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{A.4}$$

- If $c = a + b$ then ([11, (15.4.21)]):

$$\lim_{z \to 1^-} \frac{{}_2F_1\left(a, b; a+b; z\right)}{-\log(1-z)} = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}. \tag{A.5}$$

- Quadratic transformation (see [11, (15.8.13)] or [12, 2.11(4)]): if $|\mathrm{phase}(1-z)| < \pi$ then

$${}_2F_1\left(\frac{a}{2}, \frac{1}{2} + \frac{a}{2}; \frac{1}{2} + b; \frac{z^2}{(2-z)^2}\right) = \left(1 - \frac{z}{2}\right)^a {}_2F_1\left(a, b; 2b; z\right). \tag{A.6}$$

- Quadratic transformation ([12, 2.11(34)]): if $0 \leq z \leq 1$ then

$${}_2F_1\Big(\frac{a}{2}, \frac{a+1}{2}; a-b+1; \frac{4z}{(1+z)^2}\Big) = (1+z)^a\, {}_2F_1(a, b; a-b+1; z) \tag{A.7}$$

  (there is a typo in [12, 2.11(34)]: $a - b - 1$ has been corrected here to $a - b + 1$).
- Quadratic transformation (see [12, 2.11(5)]): if $0 \leq z \leq 1$ then

$${}_2F_1\Big(a, b; 2b; \frac{4z}{(1+z)^2}\Big) = (1+z)^{2a}\, {}_2F_1\Big(a, a + \frac{1}{2} - b; b + \frac{1}{2}; z^2\Big). \tag{A.8}$$

- Derivative formula (see [11, (15.5.1)]):

$$\frac{\mathrm{d}}{\mathrm{d}z}{}_2F_1\left(a,b;c;z\right)=\left(\frac{ab}{c}\right){}_2F_1\left(a+1,b+1;c+1;z\right). \tag{A.9}$$

- Euler integral formula (see [4, p. 4-5] and [11, (15.6.1)]):

$${}_2F_1\left(a,b;c;z\right)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1\frac{u^{b-1}(1-u)^{c-b-1}}{(1-zu)^a}\mathrm{d}u, \tag{A.10}$$

  provided that $\Re(b)>0$, $\Re(c)>0$, and $|\mathrm{phase}(1-z)|<\pi$.

A.2. Funk–Hecke formula

Let $d\geq 2$ and σ_{S_1} denote the uniform probability measure on the unit centered sphere $S_1=\{x\in\mathbb{R}^d : |x|=1\}$. Then, for $z\in\mathbb{R}^d$ with $|z|=1$,

$$\int_{S_1}f(z\cdot x)\sigma_{S_1}(\mathrm{d}x)=\tau_{d-1}\int_0^\pi f(\cos(\theta))\sin^{d-2}(\theta)\mathrm{d}\theta=\tau_{d-1}\int_{-1}^1 f(t)(1-t^2)^{\frac{d-3}{2}}\mathrm{d}t \tag{A.11}$$

where

$$\tau_{d-1}:=\left(\int_0^\pi\sin^{d-2}(\theta)\mathrm{d}\theta\right)^{-1}=\left(\int_{-1}^1(1-t^2)^{\frac{d-3}{2}}\mathrm{d}t\right)^{-1}=\frac{\Gamma(\frac{d}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{d-1}{2})}. \tag{A.12}$$

The Funk–Hecke formula (A.11) is a useful tool to reduce multivariate integrals into univariate integrals. It gives the projection on any diameter of the uniform law on the sphere. If X is a random vector in $\mathbb{R}^d$ uniformly distributed on S_1 then for $z\in S_1$, the law of $z\cdot X$ has density $\tau_{d-1}(1-t^2)^{\frac{d-3}{2}}\mathbf{1}_{t\in[-1,1]}$. This is an arcsine law when $d=2$, a uniform law when $d=3$, a semicircle law when $d=4$, and more generally, for an arbitrary $d\geq 2$, the image by the map $u\mapsto\sqrt{u}$ of the beta law $\mathrm{Beta}(\frac{1}{2},\frac{d-1}{2})$. We refer to [25, p. 18] or [5, Eq. (5.1.9), p. 197] for a proof.

A.3. Euler–Lagrange characterization of equilibrium measure (Frostman conditions)

For $\mu\in\mathcal{M}_1$ such that $K_s(x)\mathbf{1}_{|x|>1}\mathbf{1}_{s\leq 0}\in L^1(\mu)$, we define the s-Riesz potential at point $x\in\mathbb{R}^d$ by

$$U^\mu(x):=(K_s*\mu)(x)=\int K_s(x-y)\mu(\mathrm{d}y)\in(-\infty,+\infty]. \tag{A.13}$$

The Euler–Lagrange characterization of the equilibrium measure μ_{eq}, also known as *Frostman conditions* in potential theory, states that a necessary and sufficient condition for such an element μ of $\mathcal{M}_1$ to be an equilibrium measure is that for some finite constant c we have (see, for example, [21])

$$U^\mu+V\begin{cases}\leq c & \text{on the support of }\mu\\ \geq c & \text{quasi-everywhere on }\mathbb{R}^d\end{cases}; \tag{A.14}$$

by “quasi-everywhere” we mean except on a set for which every probability measure supported on it has infinite energy. This condition holds everywhere when V is continuous. It is customary to say that c is the *modified Robin constant* and we have $c=\int U^{\mu_{\mathrm{eq}}}\mathrm{d}\mu_{\mathrm{eq}}+\int V\mathrm{d}\mu_{\mathrm{eq}}=\mathrm{I}(\mu_{\mathrm{eq}})-\int V\mathrm{d}\mu_{\mathrm{eq}}$.

A.4. Integrability and regularity of Riesz potentials

The following Lemma summarizes key regularity properties of the Riesz kernel, some of which are classical. We give a proof for the reader's convenience. On this topic, we also refer to the works of Mizuta such as [22–24].

Lemma A.1 *(Integrability and regularity of Riesz potentials).*

(i) $K_s \in L^1_{\mathrm{loc}}(\mathbb{R}^d, \mathrm{d}x)$ *if and only if* $s = 0$ *or* $s \neq 0$ *and* $s < d$.

(ii) *If* $s < d - 2$ *then, in the sense of distributions, and in the sense of functions on* $\{x \in \mathbb{R}^d : x \neq 0\}$,

$$\Delta K_s = -c_{d,s} K_{s+2} \quad \textit{where} \quad c_{d,s} := \begin{cases} |s|(d-2-s) & \textit{if } s \neq 0 \\ d-2 & \textit{if } s = 0 \end{cases}. \tag{A.15}$$

(iii) *Suppose that* $s = 0$ *or* $s \neq 0$ *and* $s < d$. *Let* μ *be a compactly supported probability measure on* $\mathbb{R}^d$. *Then the following function is well defined and belongs to* $L^1_{\mathrm{loc}}(\mathbb{R}^d, \mathrm{d}x)$*:*

$$x \in \mathbb{R}^d \mapsto (K_s * \mu)(x) := \int K_s(x-y)\mu(\mathrm{d}y). \tag{A.16}$$

Moreover, in the sense of distributions,

$$\Delta(K_s * \mu) = (\Delta K_s) * \mu = -c_{d,s} K_{s+2} * \mu. \tag{A.17}$$

(iv) *Suppose that* $s = 0$ *or* $s \neq 0$ *and* $s < d$. *If* μ *is a compactly supported probability measure on* $\mathbb{R}^d$ *such that* $\mu(\mathrm{d}x) = f(x)\mathrm{d}x$, $f \in L^p(\mathbb{R}^d, \mathrm{d}x)$, *and* $p > d/(d-s)$, *then* $K_s * \mu$ *is continuous on* $\mathbb{R}^d$.

Note that $K_{s+2} \in L^1_{\mathrm{loc}}(\mathbb{R}^d, \mathrm{d}x)$ implies $s + 2 < d$; in other words $s < d - 2$. Furthermore, the condition $s < d - 2$ is sharp for (A.15), indeed; in the sense of distributions, we have $\Delta K_{d-2} = -c_d \delta_0$ (Coulomb kernel). This suggests defining $K_d := \delta_0$ to make the formula (A.15) valid for the critical case $s = d - 2$, provided that we also set $c_{d,d-2} := c_d$.

We remark that (A.15) is a special case of (A.17) which corresponds to taking $\mu = \delta_0$ and that (A.17) goes beyond [33, Eq. (7), p. 118] and [16, Eq. (85), p. 136]. Note also that the distribution $\Delta\mu$ equals the convolution $(\Delta\delta_0) * \mu$, see [32, end of Ch. VI, Sec. 3; notably eq. (VI, 3; 34–35)]. From this point of view, it follows that (A.17) is a consequence of the associative law for convolution of three distributions, two of which have compact support, see [32, Ch. VI, Sec. 3, Th. VII] and [21, Lemma 0.6]. We give however a direct short proof of (A.17) below.

Proof of Lemma A.1. *Proof of (i).* It suffices to check local integrability in the neighborhood of the origin. We have

$$\int_{|x|\leq 1} |K_s(x)|\mathrm{d}x = \begin{cases} 2\pi \displaystyle\int_0^1 r^{-s+d-1}\mathrm{d}r < \infty & \text{if } s \neq 0 \text{ and } d > s \\ 2\pi \displaystyle\int_0^1 \log(r) r^{d-1}\mathrm{d}r < \infty & \text{if } s = 0 \end{cases}.$$

Proof of (ii). On $\mathbb{R}^d \setminus \{0\}$, the function K_s is $\mathcal{C}^\infty$ and a computation reveals that

$$\Delta K_s = -c_{d,s} K_{s+2}.$$

It follows that this equality also holds in the sense of distributions for test functions supported away from the origin. For general test functions, we proceed by integration by parts outside a centered ball of small radius. Namely, let φ be a compactly supported $\mathcal{C}^\infty$ test function, and let $\varepsilon > 0$. By the Green integration by parts formula for the open set $\{x \in \mathbb{R}^d : |x| > \varepsilon\}$, denoting $n(x) = -x|x|^{-1}$ the inner unit normal vector to the sphere $\{x \in \mathbb{R}^d : |x| = \varepsilon\}$ at the point x,

$$\int\limits_{|x|\geq\varepsilon} \Delta\varphi(x)K_s(x)\mathrm{d}x - \int\limits_{|x|\geq\varepsilon} \varphi(x)\Delta K_s(x)\mathrm{d}x = \int\limits_{|x|=\varepsilon} K_s(x)\nabla\varphi(x)\cdot n(x)\mathrm{d}\sigma_\varepsilon(x) - \int\limits_{|x|=\varepsilon} \varphi(x)\nabla K_s(x)\cdot n(x)\mathrm{d}\sigma_\varepsilon(x).$$

If $s \neq 0$ and $d > s+1$ then

$$\begin{aligned}\int\limits_{|x|=\varepsilon} K_s(x)\nabla\varphi(x)\cdot n(x)\mathrm{d}\sigma_\varepsilon(x) &= \varepsilon^{-s}\int\limits_{|x|=\varepsilon} \nabla\varphi(x)\cdot n(x)\mathrm{d}\sigma_\varepsilon(x)\\ &= \varepsilon^{-s}O(\varepsilon^{d-1}) = O(\varepsilon^{-s+d-1}) = o_{\varepsilon\to0^+}(1),\end{aligned}$$

while, using $\nabla K_s(x) = -sx|x|^{-(s+2)} = -sxK_{s+2}(x)$ and $x\cdot n_x = -|x|$, if $d > s+2$,

$$\begin{aligned}\int\limits_{|x|=\varepsilon} \varphi(x)\nabla K_s(x)\cdot n(x)\mathrm{d}\sigma_\varepsilon(x) &= s\varepsilon^{1-(s+2)}\int\limits_{|x|=\varepsilon} \varphi(x)\mathrm{d}\sigma_\varepsilon(x)\\ &= s\varepsilon^{1-(s+2)}O(\varepsilon^{d-1}) = O(\varepsilon^{d-(s+2)}) = o_{\varepsilon\to0^+}(1).\end{aligned}$$

Finally a careful analysis reveals that the conditions on d are the same in the case $s = 0$.

Proof of (iii). If $s < 0$, then $K_s \leq 0$, and hence $K_s * \mu$ is well defined and takes its values in $[-\infty, 0]$. Similarly, if $s > 0$, then $K_s \geq 0$, and hence $K_s * \mu$ is well defined and takes its values in $[0, +\infty]$. If $s = 0$ then $K_0\mathbf{1}_{|\cdot|\leq1} \geq 0$ while $\sup_{\mathbb{R}^d} K_0\mathbf{1}_{|\cdot|\geq1}/\log(1+|\cdot|) < \infty$, hence $K_0 * \mu$ is well defined and takes its values in $(-\infty, +\infty]$. Next, by the Fubini–Tonelli theorem, for $R > 0$, using (i) and the compactness of support of μ (note that this can be weakened into integrability of $\log(1+|\cdot|)\mathbf{1}_{s=0}$),

$$\iint |K_s(x-y)|\mathbf{1}_{|x|\leq R}\mu(\mathrm{d}y)\mathrm{d}x = \int\Big(\int |K_s(x)|\mathbf{1}_{|x+y|\leq R}\mathrm{d}x\Big)\mu(\mathrm{d}y) < \infty.$$

It follows that $K_s * \mu$ belongs to $L^1_{\mathrm{loc}}(\mathbb{R}^d, \mathrm{d}x)$.

For the differentiability, let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be a $\mathcal{C}^\infty$ and compactly supported test function. By the Fubini–Tonelli theorem, the Green integration by parts formula, and (ii), we have

$$\begin{aligned}\int(K_s*\mu)(x)\Delta\varphi(x)\mathrm{d}x &= \int\Big(\int K_s(x-y)\mu(\mathrm{d}y)\Big)\Delta\varphi(x)\mathrm{d}x\\ &= \int\Big(\int K_s(x-y)\Delta\varphi(x)\mathrm{d}x\Big)\mu(\mathrm{d}y)\\ &= -c_{d,s}\int\Big(\int K_{s+2}(x-y)\varphi(x)\mathrm{d}x\Big)\mu(\mathrm{d}y)\\ &= -c_{d,s}\int\varphi(x)\Big(\int K_{s+2}(x-y)\mu(\mathrm{d}y)\Big)\mathrm{d}x\\ &= -c_{d,s}\int\varphi(x)(K_{s+2}*\mu)(x)\mathrm{d}x.\end{aligned}$$

Proof of (iv). For the continuity, we follow closely the cutoff argument used in [9, Lem. 4.3], see also [22, Th. 1], [23], and [24, Sec. 5.3]. Namely, let us consider first the case $s > 0$. For $n \geq 1$ and $x \in \mathbb{R}^d$, let us define

$$R_n(x) := \int f(y) K_s(x-y) \mathbf{1}_{|K_s(x-y)| \leq n} \mathrm{d}y$$

and

$$T_n(x) := (K_s * \mu)(x) - R_n(x) = \int f(y) K_s(x-y) \mathbf{1}_{|K_s(x-y)| \geq n} \mathrm{d}y.$$

By the dominated convergence theorem, R_n is continuous on $\mathbb{R}^d$. Let us show now that $\lim_{n\to\infty} T_n = 0$ uniformly on compact subsets, which will prove the continuity of $K_s * \mu$. Let $q := p/(p-1)$ be the Hölder conjugate exponent of p. Now, by the Hölder inequality, using the fact that $K_s = |\cdot|^{-s}$, $s > 0$,

$$0 \leq T_n(x) = \int f(y) \frac{\mathbf{1}_{|x-y| \leq n^{-1/s}}}{|x-y|^s} \mathrm{d}y \leq \|f\|_{p,B(x,1)} \, \varepsilon_n^{1/q}$$

where $B(x,r) := \{x \in \mathbb{R}^d : |x| \leq r\}$ is the closed centered ball of radius r, where $\|\cdot\|_{p,C}$ denotes the L^p norm with respect to the trace of the Lebesgue measure on C, where

$$\varepsilon_n := |\mathbb{S}^{d-1}| \int_0^{n^{-1/s}} \frac{\mathrm{d}r}{r^{qs-d+1}},$$

and where $|\mathbb{S}^{d-1}|$ is the surface area of the unit sphere $\{x \in \mathbb{R}^d : |x| = 1\}$. The condition $p > d/(d-s)$, which is equivalent to $qs - d + 1 < 1$, ensures that ε_n is finite for all n and that $\lim_{n\to\infty} \varepsilon_n = 0$. Hence, if $C \subset \mathbb{R}^d$ is a compact set, then, denoting $C_1 := \{x \in \mathbb{R}^d : \mathrm{dist}(x, C) \leq 1\}$, we have

$$\sup_{x \in C} |T_n(x)| \leq \|f\|_{p,K_1} \, \varepsilon_n^{1/q} \underset{n\to\infty}{\longrightarrow} 0,$$

which completes the proof of the continuity of $K_s * \mu$. The case $s < 0$ is entirely similar up to a sign. It remains to examine the case $s = 0$. Let us write $K_0 = K_0^+ - K_0^-$ with $K_0^\pm \geq 0$, namely $K_0^+ = -\log|\cdot| \, \mathbf{1}_{|\cdot| \leq 1}$ and $K_0^- = \log|\cdot| \, \mathbf{1}_{|\cdot|>1}$. To establish the continuity of $K_0^+ * \mu$ we write

$$0 \leq T_n^+(x) := \int f(y) \log \frac{1}{|x-y|} \mathbf{1}_{|x-y| \leq 1} \mathbf{1}_{|x-y| \leq \mathrm{e}^{-n}} \mathrm{d}y \leq \|f\|_{p,B(x,1)} \, (\varepsilon_n^+)^{1/q}$$

where

$$\varepsilon_n^+ = -|\mathbb{S}^{d-1}| \int_0^{\mathrm{e}^{-n}} r^{d-1} \log(r) \mathrm{d}r \underset{n\to\infty}{\longrightarrow} 0.$$

On the other hand, the continuity of $K_0^- * \mu$ follows from that of K_0^-. Hence $K_0 * \mu$ is continuous. □

Appendix B. Key formulas for potential plus external field

Let d, s, $\mu_{\mathrm{eq}} = \nu_R$, R, and σ_{S_1} be as in Theorem 1.4. For $x \in \mathbb{R}^d$, the quantity $\Phi(x) := (K_s * \mu_{\mathrm{eq}})(x) + \gamma |x|^2$ depends only on $\lambda := |x|/R$ and we can define

$$\varphi(\lambda) := \Phi(x) = U^{\mu_{\mathrm{eq}}}(x) + \gamma |x|^2 = U^{\mu_{\mathrm{eq}}}(x) + \gamma R^2 \lambda^2. \tag{B.1}$$

The following lemmas provide key formulas for the potential plus external field φ.

Lemma B.1 *(Integral formula for potential). For $\lambda \geq 0$, denoting $c_d := \dfrac{2\,\mathrm{sign}(d-3)\Gamma(\frac{d+1}{2})}{\pi\Gamma(\frac{d-1}{2})}$,*

$$\varphi(\lambda) = \begin{cases} \dfrac{1}{R^{d-3}}\left(c_d \displaystyle\int_0^1 \Big(\int_0^\pi \frac{\sin^{d-2}(\theta)}{(\lambda^2 - 2r\lambda\cos(\theta) + r^2)^{\frac{d-3}{2}}}\mathrm{d}\theta\Big)\frac{r^{d-1}\,\mathrm{d}r}{\sqrt{1-r^2}} + \gamma R^{d-1}\lambda^2\right) & \text{if } d \neq 3 \\ -\displaystyle\int_0^1 \frac{(\lambda+r)^2\log(\lambda+r) - (\lambda-r)^2\log|\lambda-r|}{\pi\lambda}\frac{r\,\mathrm{d}r}{\sqrt{1-r^2}} - \log R + \frac{1}{2} + \gamma R^2\lambda^2 & \text{if } d = 3 \end{cases}.$$

Note that γR^{d-1} does not depend on γ.

Proof. By the Funk–Hecke formula (A.11), for $x \in \mathbb{R}^d$, $s \neq 0$, with $C_d := \mathrm{sign}(s)\frac{\Gamma(\frac{d+1}{2})}{\pi^{\frac{d+1}{2}}}$,

$$\begin{aligned} U^{\mu_{\mathrm{eq}}}(x) &= C_d \int_{|y|\leq 1} \frac{\mathrm{d}y}{|x-Ry|^s\sqrt{1-|y|^2}} \\ &= C_d|S_1| \int_0^1 \int_{S_1} \frac{\sigma_{S_1}(\mathrm{d}y) r^{d-1}\,\mathrm{d}r}{(|x|^2 - 2rR\langle x,y\rangle + r^2R^2)^{\frac{s}{2}}\sqrt{1-r^2}} \\ &= \frac{C_d|S_1|\tau_{d-1}}{R^s}\int_0^1 \Big(\int_{-1}^1 \frac{(1-t^2)^{\frac{s}{2}}}{(\lambda^2 - 2r\lambda t + r^2)^{\frac{s}{2}}}\mathrm{d}t\Big)\frac{r^{d-1}}{\sqrt{1-r^2}}\,\mathrm{d}r \\ &= \frac{2\mathrm{sign}(s)\Gamma(\frac{d+1}{2})}{\pi\Gamma(\frac{d-1}{2})R^{d-3}}\int_0^1\Big(\int_0^\pi \frac{\sin^{s+1}(\theta)}{(\lambda^2 - 2r\lambda\cos(\theta)+r^2)^{\frac{s}{2}}}\mathrm{d}\theta\Big)\frac{r^{d-1}}{\sqrt{1-r^2}}\,\mathrm{d}r, \end{aligned} \tag{B.2}$$

while if $s = 0$ ($d = 3$),

$$\begin{aligned} U^{\mu_{\mathrm{eq}}}(x) &= -\frac{1}{\pi^2}\int_{|y|\leq 1} \frac{\log|x-Ry|}{\sqrt{1-|y|^2}}\mathrm{d}y \\ &= -\frac{2}{\pi}\int_0^1 \Big(\int_{\mathbb{S}^2} \log\big(\lambda^2R^2 - 2rR\langle x,y\rangle + r^2R^2\big)\sigma_{S_1}(\mathrm{d}y)\Big)\frac{r^2}{\sqrt{1-r^2}}\,\mathrm{d}r \\ &= -\frac{1}{\pi}\int_0^1 \Big(4\log R + \int_{-1}^1 \log(\lambda^2 - 2r\lambda t + r^2)\mathrm{d}t\Big)\frac{r^2}{\sqrt{1-r^2}}\,\mathrm{d}r. \end{aligned}$$

Finally we observe that

$$\int_{-1}^1 \log(\lambda^2 - 2r\lambda t + r^2)\mathrm{d}t = \frac{(\lambda+r)^2\log(\lambda+r) - (\lambda-r)^2\log|\lambda-r|}{r\lambda} - 2. \quad \square$$

Lemma B.2 *(Landen transform and a special function). For $d \in \{2,3,\dots\}$, $\lambda \geq 0$, and $r \in [0,1]$,*

$$\int_0^\pi \frac{\sin^{d-2}(\theta)}{(\lambda^2 - 2r\lambda\cos(\theta) + r^2)^{\frac{d-3}{2}}}\mathrm{d}\theta = \frac{2^{d-1}}{(\lambda+r)^{d-3}}\mathcal{S}_{d-3}\left(\frac{4\lambda r}{(\lambda+r)^2}\right),$$

where for $z \in [0,1]$,

$$\mathcal{S}_s(z) := \int_0^{\frac{\pi}{2}} \frac{\sin^{s+1}(\alpha)\cos^{s+1}(\alpha)}{(1 - z\sin^2(\alpha))^{\frac{s}{2}}}\mathrm{d}\alpha = \int_0^1 \frac{t^{s+1}(1-t^2)^{\frac{s}{2}}}{(1-zt^2)^{\frac{s}{2}}}\mathrm{d}t = \frac{\Gamma(\frac{s}{2}+1)^2}{2\Gamma(s+2)}\,{}_2F_1\Big(\frac{s}{2}+1, \frac{s}{2}; s+2; z\Big).$$

Proof. We set $\rho_1 := \frac{2\lambda r}{\lambda^2+r^2}$, $\rho_2 := \frac{2\rho_1}{1+\rho_1} = \frac{4\lambda r}{(\lambda+r)^2}$, which gives $(\lambda^2+r^2)(1+\rho_1) = (\lambda+r)^2$. Using the change of variable $\theta = 2\alpha$, and $\cos(\theta) = 1 - 2\sin^2(\alpha)$, $\sin(\theta) = 2\sin(\alpha)\cos(\alpha)$, we get

$$\begin{aligned}
\int_0^\pi \frac{\sin^{s+1}(\theta)}{(\lambda^2 - 2r\lambda\cos(\theta) + r^2)^{\frac{s}{2}}}\mathrm{d}\theta &= \frac{1}{(\lambda^2+r^2)^{\frac{s}{2}}}\int_0^\pi \frac{\sin^{s+1}(\theta)}{(1-\rho_1\cos(\theta))^{\frac{s}{2}}}\mathrm{d}\theta \\
&= \frac{2^{s+2}}{(\lambda^2+r^2)^{\frac{s}{2}}}\int_0^{\frac{\pi}{2}} \frac{\sin^{s+1}(\alpha)\cos^{s+1}(\alpha)}{(1-\rho_1(1-2\sin^2(\alpha)))^{\frac{s}{2}}}\mathrm{d}\alpha \\
&= \frac{2^{s+2}}{(\lambda^2+r^2)^{\frac{s}{2}}(1-\rho_1)^{\frac{s}{2}}}\int_0^{\frac{\pi}{2}} \frac{\sin^{s+1}(\alpha)\cos^{s+1}(\alpha)}{(1+\frac{2\rho_1}{1-\rho_1}\sin^2(\alpha))^{\frac{s}{2}}}\mathrm{d}\alpha \\
&= \frac{2^{s+2}}{(\lambda+r)^s}\frac{(1+\rho_1)^{\frac{s}{2}}}{(1-\rho_1)^{\frac{s}{2}}}\mathcal{S}_s\Big(-\frac{2\rho_1}{1-\rho_1}\Big).
\end{aligned}$$

But for $z \in [0,1]$,

$$\mathcal{S}_s(-z) = \int_0^{\frac{\pi}{2}} \frac{\sin^{s+1}(\alpha)\cos^{s+1}(\alpha)}{(1+z\sin^2(\alpha))^{\frac{s}{2}}}\mathrm{d}\alpha = \frac{1}{(1+z)^{\frac{s}{2}}}\int_0^{\frac{\pi}{2}} \frac{\sin^{s+1}(\alpha)\cos^{s+1}(\alpha)}{(1-\frac{z}{1+z}\cos^2(\alpha))^{\frac{s}{2}}}\mathrm{d}\alpha = \frac{1}{(1+z)^{\frac{s}{2}}}\mathcal{S}_s\Big(\frac{z}{1+z}\Big).$$

In particular, with $z = \frac{2\rho_1}{1-\rho_1}$, we get $1+z = \frac{1+\rho_1}{1-\rho_1}$ and $\frac{z}{1+z} = \frac{2\rho_2}{1+\rho_1} = \rho_2$; therefore

$$\mathcal{S}_s\Big(-\frac{2\rho_1}{1-\rho_1}\Big) = \frac{(1-\rho_1)^{\frac{s}{2}}}{(1+\rho_1)^{\frac{s}{2}}}\mathcal{S}_s(\rho_2),$$

hence the desired integral formula in terms of $\mathcal{S}_{d-3}$. Finally the hypergeometric formula for $\mathcal{S}_s$ follows from $\mathcal{S}_s(z) = \frac{1}{2}\int_0^1 u^{\frac{s}{2}}(1-u)^{\frac{s}{2}}(1-zu)^{-\frac{s}{2}}\mathrm{d}u$ and the Euler integral formula (A.10). Note that we could alternatively proceed as in the proof of Corollary 1.3 via the Newton binomial series (1.7). □

Lemma B.3 *($d = 2, s = -1$). If $(d,s) = (2,-1)$, then*

$$\varphi(\lambda) = -\frac{1}{4\gamma}\int_0^1 (\lambda+r)E\left(\frac{4\lambda r}{(\lambda+r)^2}\right)\frac{r}{\sqrt{1-r^2}}\,\mathrm{d}r + \frac{\pi^2}{64\gamma}\lambda^2, \quad \lambda \geq 0,$$

$$\varphi'(\lambda) = -\frac{1}{8\gamma}\int_0^1 \left[\Big(1+\frac{r}{\lambda}\Big)E\left(\frac{4\lambda r}{(r+\lambda)^2}\right) + \Big(1-\frac{r}{\lambda}\Big)K\left(\frac{4\lambda r}{(r+\lambda)^2}\right)\right]\frac{r}{\sqrt{1-r^2}}\,\mathrm{d}r + \frac{\pi^2}{32\gamma}\lambda, \quad \lambda > 0,$$

$$\varphi(\lambda) = -\frac{\pi}{8\gamma}\lambda\,{}_2F_1\Big(-\frac{1}{2},-\frac{1}{2};\frac{3}{2};\frac{1}{\lambda^2}\Big) + \frac{\pi^2}{64\gamma}\lambda^2, \quad \lambda \geq 1$$

$$\varphi'(\lambda) = -\frac{\pi}{16\gamma}\left(\sqrt{1-\frac{1}{\lambda^2}} - \frac{\arcsin\left(\frac{1}{\lambda}\right)}{\frac{1}{\lambda}}\right) + \frac{\pi^2}{32\gamma}\lambda, \quad \lambda > 1.$$

Proof. By combining Lemmas B.1 and B.2 with $d = 2$, we get, for $\lambda \geq 0$,

$$\varphi(\lambda) = -\frac{R}{2\pi}\int_0^1 (\lambda + r)E\left(\frac{4\lambda r}{(\lambda+r)^2}\right)\frac{r}{\sqrt{1-r^2}}\,\mathrm{d}r + \gamma R^2\lambda^2. \tag{B.3}$$

Next, by using the well-known ordinary differential equations (for $0 < z < 1$)

$$K'(z) = \frac{E(z) - (1-z)K(z)}{2(1-z)z} \quad \text{and} \quad E'(z) = \frac{E(z) - K(z)}{2z} \tag{B.4}$$

we get, after some algebra,

$$\varphi'(\lambda) = -\frac{R}{\pi}\int_0^1 \left[\Big(1+\frac{r}{\lambda}\Big)E\left(\frac{4\lambda r}{(r+\lambda)^2}\right) + \Big(1-\frac{r}{\lambda}\Big)K\left(\frac{4\lambda r}{(r+\lambda)^2}\right)\right]\frac{r}{\sqrt{1-r^2}}\,\mathrm{d}r + 2\gamma R^2\lambda. \tag{B.5}$$

By combining (1.10) with the quadratic transformation (A.8) we get, for $z \in [0, 1)$,

$$(1+z)E\Big(\frac{4z}{(1+z)^2}\Big) = (1+z)\frac{\pi}{2}\,{}_2F_1\Big(-\frac{1}{2},\frac{1}{2};1;\frac{4z}{(1+z)^2}\Big) = \frac{\pi}{2}\,{}_2F_1\Big(-\frac{1}{2},-\frac{1}{2};1;z^2\Big). \tag{B.6}$$

Thus (B.3) gives, with $z = \frac{r}{\lambda} \in [0, 1)$, $r \in [0, 1]$, and $\lambda > 1$,

$$\begin{aligned}
\frac{2}{\pi}\int_0^1 \Big(1+\frac{r}{\lambda}\Big)E\Big(\frac{4\lambda r}{(\lambda+r)^2}\Big)\frac{r}{\sqrt{1-r^2}}\mathrm{d}r &= \frac{2}{\pi}\int_0^1 (1+z)E\Big(\frac{4z}{(1+z)^2}\Big)\frac{r}{\sqrt{1-r^2}}\mathrm{d}r \\
&= \int_0^1 {}_2F_1\Big(-\frac{1}{2},-\frac{1}{2};1;z^2\Big)\frac{r}{\sqrt{1-r^2}}\mathrm{d}r \\
&= \sum_{n=0}^{\infty}\frac{(-\frac{1}{2})_n^2}{n!^2}\Big(\frac{1}{\lambda}\Big)^{2n}\int_0^1 \frac{r^{2n+1}}{\sqrt{1-r^2}}\mathrm{d}r \\
&= \sum_{n=0}^{\infty}\frac{(-\frac{1}{2})_n^2}{(\frac{3}{2})_n n!}\Big(\frac{1}{\lambda}\Big)^{2n} = {}_2F_1\Big(-\frac{1}{2},-\frac{1}{2};\frac{3}{2};\Big(\frac{1}{\lambda}\Big)^2\Big),
\end{aligned}$$

and therefore, using (B.3), we get, when $\lambda > 1$,

$$\varphi(\lambda) = -R\lambda\,{}_2F_1\Big(-\frac{1}{2},-\frac{1}{2};\frac{3}{2};\frac{1}{\lambda^2}\Big) + \gamma R^2\lambda^2, \tag{B.7}$$

hence

$$\begin{aligned}
\varphi'(\lambda) &= -R\,{}_2F_1\Big(-\frac{1}{2},-\frac{1}{2};\frac{3}{2};\frac{1}{\lambda^2}\Big) + \frac{1}{3}\frac{R}{\lambda^2}\,{}_2F_1\Big(\frac{1}{2},\frac{1}{2};\frac{5}{2};\frac{R^2}{\lambda^2}\Big) + 2\gamma R^2\lambda \\
&= -\frac{R}{2}\left(\sqrt{1-\frac{1}{\lambda^2}} - \frac{\arcsin\left(\frac{1}{\lambda}\right)}{\frac{1}{\lambda}}\right) + 2\gamma R^2\lambda.
\end{aligned} \tag{B.8}$$

Finally, it remains to recall that $R = \frac{\pi}{8\gamma}$. □

Lemma B.4 *($d = 3$, $s = 0$). If $(d, s) = (3, 0)$ then, for $\lambda \geq 0$,*

$$\varphi(\lambda) = \frac{1 + \log(3\gamma)}{2} - \frac{1}{2\pi\lambda} \int_0^1 \frac{\big((\lambda + r)^2 \log((\lambda + r)^2) - (\lambda - r)^2 \log\big((\lambda - r)^2\big)\big) r}{\sqrt{1 - r^2}} \mathrm{d}r + \frac{\lambda^2}{3}$$

Moreover, if $\lambda \geq 1$,

$$\varphi(\lambda) = -\frac{1}{2} + \frac{\log(3\gamma)}{2} + \frac{\lambda^2 + 2}{3\lambda}\sqrt{\lambda^2 - 1} - \log \frac{\lambda + \sqrt{\lambda^2 - 1}}{2}.$$

Proof. From Lemma B.1 and with the formulas ($b \in (-a, a)$)

$$\int_{-1}^1 \log(a - bt)\mathrm{d}t = \frac{(a + b)\log(a + b) - (a - b)\log(a - b)}{b} - 2 \quad \text{and} \quad \int_0^1 \frac{r^2}{\sqrt{1 - r^2}} \mathrm{d}r = \frac{\pi}{4}$$

with $a = \lambda^2 + r^2$ and $b = 2r\lambda$, we obtain, for $\lambda \geq 0$,

$$\varphi(\lambda) = \frac{1}{2} - \log(R) - \int_0^1 \frac{(\lambda + r)^2 \log((\lambda + r)^2) - (\lambda - r)^2 \log\big((\lambda - r)^2\big)}{2\pi\lambda} \frac{r}{\sqrt{1 - r^2}} \mathrm{d}r + \gamma R^2 \lambda^2.$$

Recall that $R = \frac{1}{\gamma\sqrt{3}}$. It follows that if $\lambda > 1$,

$$\begin{aligned}\varphi(\lambda) - \frac{\lambda^2}{3} = \frac{1}{2} + \frac{\log(3\gamma)}{2} &- \frac{\lambda}{\pi} \int_0^1 \frac{\log(\lambda + r) - \log(\lambda - r)}{\sqrt{1 - r^2}} r\mathrm{d}r \\ &- \frac{2}{\pi} \int_0^1 \frac{\log(\lambda + r) + \log(\lambda - r)}{\sqrt{1 - r^2}} r^2\mathrm{d}r \\ &- \frac{1}{\pi\lambda} \int_0^1 \frac{\log(\lambda + r) - \log(\lambda - r)}{\sqrt{1 - r^2}} r^3\mathrm{d}r.\end{aligned}$$

These three last integrals can be explicitly computed and we obtain the desired formula. □

Appendix C. Proof of Riesz formula

C.1. Cross-ratio

Recall that in projective geometry, the *cross-ratio* (*birapport* in French) of four distinct points z_1, z_2, z_3, z_4 on the Riemann sphere $\mathbb{C} \cup \{\infty\}$ is defined by

$$[z_1, z_2; z_3, z_4] = \frac{z_3 - z_1}{z_3 - z_2} \Big/ \frac{z_4 - z_1}{z_4 - z_2} = \frac{(z_3 - z_1)(z_4 - z_2)}{(z_3 - z_2)(z_4 - z_1)},$$

where each length is removed from the formula if it involves the point at infinity. The following lemma is a classical and important result of projective geometry.

Lemma C.1 *(Cross-ratio invariance). The cross-ratio is invariant under the Möbius transform*

$$z \mapsto \frac{az+b}{cz+d}, \quad ad-bc \neq 0,$$

and thus its modulus is invariant under the "conjugated Möbius transform" $z \mapsto \dfrac{a\bar{z}+b}{c\bar{z}+d}$, $ad-bc \neq 0$.

C.2. Inversions

In $\mathbb{R}^d$, $d \geq 1$, the inversion with center x_0 and radius $R > 0$ is the transform that maps $x \neq x_0$ to $T(x)$ on the half line started from x_0 and passing through x, in such a way that

$$|x - x_0| \ |T(x) - x_0| = R^2.$$

The circle centered at x_0 and of radius R is pointwise invariant under the transformation in the sense that all its elements are fixed points of the transformation. The transformation maps the interior of this circle to its exterior, and vice versa. In projective geometry, this transformation is extended to the d-dimensional sphere by mapping x_0 to the point at infinity ∞, and vice versa. We have

$$T(x) - x_0 = \frac{R^2}{|x-x_0|^2}(x - x_0),$$

which exchanges x_0 and ∞. In dimension $d = 2$, using complex numbers, $T(z) - z_0 = R^2/(\overline{z - z_0})$, which is a special case of the conjugated Möbius transform $z \mapsto \frac{\alpha\bar{z}+\beta}{\gamma\bar{z}+\delta}$. It is worth mentioning that inversions are geometric transformations at the basis of the Kelvin transform of functions $\mathbb{R}^d \to \mathbb{R}$.

Lemma C.2 *(Classical properties of inversions). Let T be the inversion of $\mathbb{R}^d$, $d \geq 1$, with center $x_0 \in \mathbb{R}^d$ and radius $R > 0$. Then we have the following properties.*

1. *For all x, $|x - T(x)| = \dfrac{|R^2 - |x-x_0|^2|}{|x-x_0|}$.*
2. *For all x, y, $|T(x) - T(y)| = R^2 \dfrac{|x-y|}{|x-x_0|\,|y-x_0|}$.*
3. *As differential forms $\dfrac{\mathrm{d}T(x)}{|T(x)-x_0|^d} = \dfrac{\mathrm{d}x}{|x-x_0|^d}$.*
4. *The modulus of the cross-ratio of distinct coplanar points x_1, x_2, x_3, x_4 is invariant under T.*

Proof. We can assume without loss of generality that $x_0 = 0$.

1. Since $0, x, T(x)$ are aligned with 0 at the edge we have

$$|x - T(x)| = ||x| - |T(x)|| = \left| |x| - \frac{R^2}{|x|} \right| = \frac{||x|^2 - R^2|}{|x|}.$$

2. We have

$$\begin{aligned} |T(x) - T(y)|^2 &= |T(x)|^2 + |T(y)|^2 - 2\langle T(x), T(y)\rangle \\ &= \frac{R^4}{|x|^2} + \frac{R^4}{|y|^2} - 2\frac{R^4}{|x|^2|y|^2}\langle x, y\rangle = \frac{R^4}{|x|^2|y|^2}|x-y|^2. \end{aligned}$$

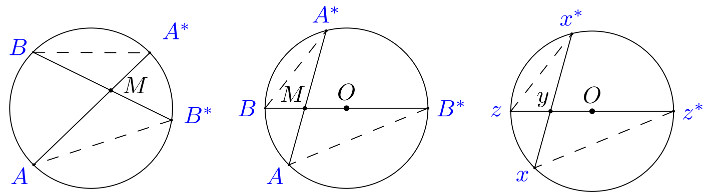

Fig. 1. Intersecting chords of a circle, AA^* and BB^* in the first two pictures, xx^* and zz^* for the third. On the two last pictures, the chords BB^* and zz^* are diameters of the circle. On the right, $x, y \in \mathbb{R}^d$, $d \geq 2$, $|x| = r$, $|y| < r$, x^* is aligned with x and y, y separates x and x^*.

3. We have $\mathrm{Jac}(T)(x) = \dfrac{R^2}{|x|^2}(I_d + u \otimes v)$, $u = \dfrac{x}{|x|}$, $v = -2\dfrac{x}{|x|}$, which gives then

$$|\det \mathrm{Jac}(T)(x)| = \left(\frac{R^2}{|x|^2}\right)^d = \left(\frac{|T(x)|}{|x|}\right)^d,$$

   via the "matrix determinant lemma" $\det(A + u \otimes v) = (1 + u \cdot A^{-1}v)\det(A)$, the determinant analogue of the Sherman–Morrison formula $(A + u \otimes v)^{-1} = A^{-1} - \frac{A^{-1}u \otimes vA^{-1}}{1+v \cdot A^{-1}u}$.
4. Follows from the fact that T restricted to the plane is a conjugated Möbius transform. □

C.3. Intersecting chords

The *intersecting chords theorem* in Euclidean (planar) geometry states that if AA^* and BB^* are two chords of a circle, intersecting at the point M, see Fig. 1, then

$$\mathrm{AM} \times \mathrm{MA}^* = \mathrm{BM} \times \mathrm{MB}^*.$$

Indeed, the triangles A^*MB and AMB^* are similar, identical up to rotation and scaling, more precisely they have two equal angles: $\widehat{A^*MB} = \widehat{AMB^*}$ (opposite angles) and $\widehat{MA^*B} = \widehat{MB^*A}$ (subtend the same arc).

Suppose now that the circle has center O, radius r, that BB^* is a diameter, and that M belongs to the segment OB (instead OB^*). Then $\mathrm{BM} = r - \mathrm{OM}$ while $\mathrm{MB}^* = \mathrm{OM} + r$ and thus

$$\mathrm{BM} \times \mathrm{MB}^* = (r - \mathrm{OM})(\mathrm{OM} + r) = r^2 - \mathrm{OM}^2.$$

In Euclidean geometry, this quantity is known as the Laguerre power of the point M with respect to the circle. We deduce immediately the following lemma.

Lemma C.3 *(Intersecting chords). For every chord AA^* of a circle with center O and radius r, intersecting an arbitrary diameter at point M, see Fig. 1, we have*

$$\mathrm{AM} \times \mathrm{MA}^* = r^2 - \mathrm{OM}^2.$$

C.4. Riesz geometric argument

The argument is essentially two-dimensional and involves projective geometry. Fix $r > 0$ and $x, y \in \mathbb{R}^d$, $d \geq 2$, with $|y| < r$. Let us define the map $S : x \mapsto S(x) = x^*$ where $x^* \in \mathbb{R}^d$ is the point aligned with x, y such that y separates x and x^* and

$$|x - y|\,|y - x^*| = r^2 - |y|^2.$$

The map S is the composition of an inversion centered at y of radius $\sqrt{r^2-|y|^2}$ and the central symmetry centered at y (recall that y separates x and x^*). Moreover, by Lemma C.3, see also Fig. 1, we have $|x|=r$ if and only if $|x^*|=r$, namely the centered sphere of radius r is globally invariant under S. The points y and ∞ are mapped to each other by S.

Let T be the inversion centered at the origin and with radius r. By Lemma C.2, the modulus of the cross-ratio of the coplanar points $x, T(y), y, T(x)$ satisfies

$$|[x,T(y);y,T(x)]| = \frac{|x-y|\,|T(x)-T(y)|}{|x-T(x)|\,|y-T(y)|} = \frac{|x-y|^2r^2}{|r^2-|x|^2|\,|r^2-|y|^2|}.$$

Note that since x, y, x^* are aligned, the points $x, y, x^*, T(x), T(y)$ are coplanar.

Lemma C.4 *(Commutation). S and T commute.*

This is related to the fact that S leaves globally invariant the fixed points (circle) of T.

Proof. Using complex coordinates $T(z)=r^2/\overline{z}$ while $T(z)-z_0=-(r^2-|z_0|^2)/(\overline{z-z_0})$, where z_0 stands for y. Now we have

$$T(S(z)) = \frac{r^2}{\overline{z_0}-\frac{r^2-|z_0|^2}{z-z_0}} = \frac{r^2(z-z_0)}{\overline{z_0}z-r^2} \quad \text{and} \quad S(T(z)) = z_0 - \frac{r^2-|z_0|^2}{\frac{r^2}{\overline{z}}-z_0} = \frac{r^2(z_0-z)}{r^2-\overline{z_0}z}. \qquad \square$$

Since S is the composition of an inversion and a central symmetry, it is a special case of a conjugate Möbius transform, and then, by Lemma C.1, $|[x,T(y);y,T(x)]| = |[S(x),S(T(y));S(y),S(T(x))]|$. Since S and T commute (Lemma C.4), we have, using Lemma C.2 for the final step,

$$\begin{aligned}|[x,T(y);y,T(x)]| &= |[S(x),T(S(y));S(y),T(S(x))]| = |[x^*,T(\infty);\infty,T(x^*)]| = |[x^*,0;\infty,T(x^*)]|\\ &= \frac{|T(x^*)|}{|T(x^*)-x^*|} = \frac{|T(x^*)||x^*|}{|r^2-|x^*|^2|} = \frac{r^2}{|r^2-|x^*|^2|}.\end{aligned}$$

It follows that in the case $|x|<r$ (in other words $|x^*|>r$) we get (recall that $|y|<r$)

$$\frac{|x-y|^2}{(r^2-|x|^2)(r^2-|y|^2)} = \frac{1}{|x^*|^2-r^2} \quad \text{hence} \quad \frac{1}{(r^2-|x|^2)^{\frac{\alpha}{2}}|x-y|^{-\alpha}} = \frac{(r^2-|y|^2)^{\frac{\alpha}{2}}}{(|x^*|^2-r^2)^{\frac{\alpha}{2}}}.$$

Finally, using this formula, we get, for all $y\in\mathbb{R}^d$, $|y|\leq r$, and all $\alpha\geq 0$, $d\geq 2$,

$$I(y) := \int\limits_{|x|\leq r} \frac{\mathrm{d}x}{(r^2-|x|^2)^{\frac{\alpha}{2}}|x-y|^{d-\alpha}} = (r^2-|y|^2)^{\frac{\alpha}{2}} \int\limits_{|x^*|\geq r} \frac{\mathrm{d}x^*}{(|x^*|^2-r^2)^{\frac{\alpha}{2}}|x^*-y|^d},$$

where the differential identity $\frac{\mathrm{d}x}{|x-y|^d} = \frac{\mathrm{d}x^*}{|x^*-y|^d}$ comes from Lemma C.2 applied to S which is not an inversion but which is the composition of an inversion with an isometry (central symmetry).

Using spherical coordinates with $\rho=|x_*|$ and the Funk−Hecke formula (A.11) we get

$$I(y) = (r^2-|y|^2)^{\frac{\alpha}{2}} \int\limits_{|x^*|\geq r} \frac{\mathrm{d}x^*}{(|x^*|^2-r^2)^{\frac{\alpha}{2}}(|x^*|^2-2x^*\cdot y+|y|^2)^{\frac{d}{2}}}$$

$$= |S_1| \frac{\Gamma(\frac{d}{2})}{\sqrt{\pi}\Gamma(\frac{d-1}{2})}(r^2 - |y|^2)^{\frac{\alpha}{2}} \int_r^\infty \int_0^\pi \frac{\rho^{d-1}\sin^{d-2}(\theta)\mathrm{d}\rho\mathrm{d}\theta}{(\rho^2 - r^2)^{\frac{\alpha}{2}}(\rho^2 - 2\rho|y|\cos(\theta) + |y|^2)^{\frac{d}{2}}}$$

$$= |S_1| \frac{\Gamma(\frac{d}{2})}{\sqrt{\pi}\Gamma(\frac{d-1}{2})}(r_1^2 - 1)^{\frac{\alpha}{2}} \int_{r_1}^\infty \frac{\rho_1^{d-1}}{(\rho_1^2 - r_1^2)^{\frac{\alpha}{2}}} \Big(\int_0^\pi \frac{\sin^{d-2}(\theta)\mathrm{d}\theta}{(\rho_1^2 - 2\rho_1\cos(\theta) + 1)^{\frac{d}{2}}} \Big) \mathrm{d}\rho_1 \tag{C.1}$$

where $r := r_1|y|$ and $\rho := \rho_1|y|$. Note that $r_1 \geq 1$ and $\rho_1 \geq 1$.

C.5. Trigonometric change of variable

Let us show that for $d > 1$ and $\rho_1 > 1$,

$$i_d := \int_0^\pi \frac{\sin^{d-2}(\theta)}{(\rho_1^2 - 2\rho_1\cos(\theta) + 1)^{\frac{d}{2}}}\mathrm{d}\theta = \frac{\rho_1^{2-d}}{\rho_1^2 - 1} \int_0^\pi \sin^{d-2}(\alpha)\mathrm{d}\alpha = \frac{\rho_1^{2-d}}{\rho_1^2 - 1}\sqrt{\pi}\frac{\Gamma\left(\frac{d-1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}. \tag{C.2}$$

We first give a historical geometric argument. We then give in Remark C.5 an analytic argument using properties of the Gegenbauer polynomials. The second equality in (C.2) follows from the fact that the middle integral becomes an Euler beta integral after the change of variable $u = \sin(\alpha)$. To prove the first equality in (C.2), we follow [21, p. 400], and we use the change of variable

$$\frac{\sin(\theta)}{\sqrt{\rho_1^2 - 2\rho_1\cos(\theta) + 1}} = \frac{\sin(\alpha)}{\rho_1},$$

see Fig. 2 for a geometric interpretation.[6] Following this figure, we have the identity

$$\rho_1^2 - 2\rho_1\cos(\theta) + 1 = A(\alpha)^2 \quad \text{where} \quad A(\alpha) = \sqrt{\rho_1^2 - \sin^2(\alpha)} + \cos(\alpha),$$

hence $2\rho_1\sin(\theta)\mathrm{d}\theta = 2A(\alpha)A'(\alpha)\mathrm{d}\alpha$ and by using the formula for the change of variable this gives

$$\mathrm{d}\theta = \frac{A'(\alpha)}{\sin(\alpha)}\mathrm{d}\alpha = \frac{-\sin(\alpha) - \frac{\sin(\alpha)\cos(\alpha)}{\sqrt{\rho_1^2 - \sin^2(\alpha)}}}{\sin(\alpha)}\mathrm{d}\alpha = -\left(\frac{\sqrt{\rho_1^2 - \sin^2(\alpha)} + \cos(\alpha)}{\sqrt{\rho_1^2 - \sin^2(\alpha)}}\right)\mathrm{d}\alpha.$$

Therefore, we obtain, noting that $\theta = 0 \iff \alpha = \pi$ and $\theta = \pi \iff \alpha = 0$ (see Fig. 2),

$$i_d = \int_0^\pi \Big(\frac{\sin(\alpha)}{\rho_1}\Big)^{d-2} \frac{1}{\left(\cos(\alpha) + \sqrt{\rho_1^2 - \sin^2(\alpha)}\right)^2} \frac{\sqrt{\rho_1^2 - \sin^2(\alpha)} + \cos(\alpha)}{\sqrt{\rho_1^2 - \sin^2(\alpha)}}\mathrm{d}\alpha$$

$$= \int_0^\pi \Big(\frac{\sin(\alpha)}{\rho_1}\Big)^{d-2} \frac{1}{\cos(\alpha) + \sqrt{\rho_1^2 - \sin^2(\alpha)}} \frac{1}{\sqrt{\rho_1^2 - \sin^2(\alpha)}}\mathrm{d}\alpha$$

$$= \int_0^\pi \Big(\frac{\sin(\alpha)}{\rho_1}\Big)^{d-2} \frac{\cos(\alpha) - \sqrt{\rho_1^2 - \sin^2(\alpha)}}{\cos^2(\alpha) - (\rho_1^2 - \sin^2(\alpha))} \frac{1}{\sqrt{\rho_1^2 - \sin^2(\alpha)}}\mathrm{d}\alpha$$

[6] It is mentioned in [21, p. 400] that this change of variable was suggested S.I. Greenberg. Nevertheless such geometric reasoning goes back at least to the works on elliptic integrals of the 19-th century, see [15].

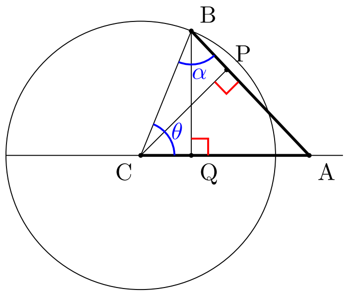

Fig. 2. Geometric interpretation of the θ to α change of variables for i_d. The angles and distances are $ACB = \theta$, $CBA = \alpha$, $CB = 1$ and $CA = \rho_1$. The right-angled triangle ABQ has hypotenuse AB, thus $AB^2 = BQ^2 + AQ^2 = \sin^2(\theta) + (AC - QC)^2 = \sin^2(\theta) + (\rho_1 - \cos(\theta))^2 = \rho_1^2 - 2\rho_1\cos(\theta) + 1$. The sine rule then gives $\sin(\alpha)/\rho_1 = \sin(\theta)/\sqrt{\rho_1^2 - 2\rho_1\cos(\theta) + 1}$. On the other hand, we also have $\sqrt{\rho_1^2 - 2\rho_1\cos(\theta) + 1} = AB = AP + PB = \sqrt{\rho_1^2 - \sin^2(\alpha)} + \cos(\alpha)$.

$$= \frac{1}{\rho_1^{d-2}(1-\rho_1^2)} \int_0^\pi (\sin(\alpha))^{d-2} \left(\frac{\cos(\alpha)}{\sqrt{\rho_1^2 - \sin^2(\alpha)}} - 1 \right) \mathrm{d}\alpha$$

$$= \frac{1}{\rho_1^{d-2}(\rho_1^2 - 1)} \int_0^\pi (\sin(\alpha))^{d-2} \mathrm{d}\alpha,$$

where the last equality follows from the antisymmetry of cos around $\pi/2$. This proves (C.2).

Remark C.5 *(Proof of* (C.2) *using Gegenbauer polynomials).* Let $\rho = \frac{1}{\rho_1} < 1$. Using the generating function for Gegenbauer polynomials $(1 - 2\rho\cos\theta + \rho^2)^{-\frac{d}{2}} = \sum_{n=0}^\infty C_n^{(\frac{d}{2})}(\cos\theta)\,\rho^n$ gives

$$\int_0^\pi \frac{\sin^{d-2}\theta}{(1 - 2\rho\cos\theta + \rho^2)^{\frac{d}{2}}}\,\mathrm{d}\theta = \sum_{n=0}^\infty \rho^n \int_0^\pi \sin^{d-2}\theta\, C_n^{(\frac{d}{2})}(\cos\theta)\,\mathrm{d}\theta. \tag{C.3}$$

The integral on the right-hand side vanishes for odd degree n since the Gegenbauer polynomials are odd functions of $\cos\theta$. For even degree $n = 2k$, the integral can be computed as

$$\int_0^\pi \sin^{d-2}\theta\, C_{2k}^{(\frac{d}{2})}(\cos\theta)\,\mathrm{d}\theta = \int_0^\pi \sin^{d-2}\theta\,\mathrm{d}\theta = \sqrt{\pi}\frac{\Gamma\left(\frac{d-1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}. \tag{C.4}$$

To establish (C.4), use the recurrence relation [11, (18.9.7)]

$$C_{2k}^{(\frac{d}{2})}(x) = C_{2k-2}^{(\frac{d}{2})}(x) + \frac{2k + \frac{d}{2} - 1}{\frac{d}{2} - 1} C_{2k}^{(\frac{d}{2}-1)}(x)$$

and integrate against $\sin^{d-2}\theta$ to produce

$$\int_0^\pi \sin^{d-2}\theta\, C_{2k}^{(\frac{d}{2})}(\cos\theta)\,\mathrm{d}\theta = \int_0^\pi \sin^{d-2}\theta\, C_{2k-2}^{(\frac{d}{2})}(\cos\theta)\,\mathrm{d}\theta + \frac{2k + \frac{d}{2} - 1}{\frac{d}{2} - 1} \int_0^\pi \sin^{d-2}\theta\, C_{2k}^{(\frac{d}{2}-1)}(\cos\theta).$$

The second integral on the right-hand side vanishes by orthogonality, so

$$\int_0^\pi \sin^{d-2}\theta\, C_{2k}^{(\frac{d}{2})}(\cos\theta)\,\mathrm{d}\theta = \int_0^\pi \sin^{d-2}\theta\, C_{2k-2}^{(\frac{d}{2})}(\cos\theta)\,\mathrm{d}\theta,$$

with repeated application giving the first equality in (C.4). Using (C.4) in (C.3) then gives

$$\int_0^\pi \frac{\sin^{d-2}\theta}{(1-2\rho\cos\theta+\rho^2)^{\frac{d}{2}}}\,\mathrm{d}\theta = \sqrt{\pi}\frac{\Gamma\left(\frac{d-1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}\sum_{k=0}^{\infty}\rho^{2k} = \sqrt{\pi}\frac{\Gamma\left(\frac{d-1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}\frac{1}{1-\rho^2}.$$

The substitution $\rho = \frac{1}{\rho_1}$ then gives (C.2).

C.6. Conclusion

By combining (C.1) and (C.2), using the successive changes of variables $t=\rho_1^2-r_1^2$, $t_1=t/(r_1^2-1)$, and $u=1/(1+t_1)$, and the Euler reflection formula (A.1), we get

$$\begin{aligned}
I(y) &= |S_1|(r_1^2-1)^{\frac{\alpha}{2}}\int_{r_1}^{\infty}\frac{\rho_1\mathrm{d}\rho_1}{(\rho_1^2-r_1^2)^{\frac{\alpha}{2}}(\rho_1^2-1)} \\
&= |S_1|\frac{(r_1^2-1)^{\frac{\alpha}{2}}}{2}\int_0^{\infty}\frac{\mathrm{d}t}{t^{\frac{\alpha}{2}}(t+r_1^2-1)} \\
&= \frac{|S_1|}{2}\int_0^{\infty}\frac{\mathrm{d}t_1}{t_1^{\frac{\alpha}{2}}(t_1+1)} \\
&= \frac{|S_1|}{2}\int_0^{1}\frac{u^{\frac{\alpha}{2}-1}\mathrm{d}u}{(1-u)^{\frac{\alpha}{2}}} = \frac{|S_1|\Gamma(\frac{\alpha}{2})\Gamma(1-\frac{\alpha}{2})}{2} = \frac{\pi^{\frac{d}{2}+1}}{\Gamma(\frac{d}{2})\sin(\frac{\pi}{2}\alpha)}.
\end{aligned}$$

This completes the proof of (1.6) and thus of Lemma 1.2.

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