Riesz energy problems with external fields and related theory

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Abstract

In this paper, we investigate Riesz energy problems on unbounded conductors in $\mathbb{R}^d$ in the presence of general external fields Q, not necessarily satisfying the growth condition $Q(x) \to \infty$ as $|x| \to \infty$ assumed in several previous studies. We provide sufficient conditions on Q for the existence of an equilibrium measure and the compactness of its support. Particular attention is paid to the case of the hyperplanar conductor $\mathbb{R}^d$, embedded in $\mathbb{R}^{d+1}$, when the external field is created by the potential of a signed measure ν outside of $\mathbb{R}^d$. Simple cases where ν is a discrete measure are analyzed in detail. New theoretic results for Riesz potentials, in particular an extension of a classical theorem by de La Vallée-Poussin, are established. These results are of independent interest.

1 Introduction

This paper is devoted to the study of weighted Riesz s-equilibrium measures in $\mathbb{R}^d$, $d \geq 2$. We will be dealing with their associated Riesz s-potentials, of the form

$$U^\sigma(x) := \int \frac{d\sigma(t)}{|x-t|^s}, \qquad 0 < s < d, \tag{1.1}$$

for measures σ supported in $\mathbb{R}^d$, where $|x|$ denotes the euclidean norm of x in $\mathbb{R}^d$. Potentials as in (1.1) will also serve to define our external fields, this time with measures σ supported in the ambient space $\mathbb{R}^{d+1}$.

Note that

$$U_0^\sigma(x) := -\int \log|x-t|\, d\sigma(t)\,,$$

is the limit case of the Riesz potentials as $s \to 0^+$ (see e.g. [22]); hereafter, we refer to this case as the log case or, simply, as $s = 0$. In the present paper we assume $0 < s < d$, and in most cases we restrict ourselves to $d - 2 \leq s < d$.

In the case of logarithmic potentials it is common to study energy problems on an unbounded conductor Σ in the complex plane in the presence of an external field Q defined on Σ. It is well known that if Q is admissible, that is, lower semi–continuous, such that the logarithmic capacity of $\{z \in \Sigma : Q(z) < \infty\}$ is positive, and Q satisfies the condition

$$\lim_{|z|\to\infty,\, z\in\Sigma} (Q(z) - \log|z|) = +\infty, \tag{1.2}$$

then the existence of the equilibrium measure and the compactness of its support hold true, see [29, Theorem I.1.3]. Furthermore, in the last 15 years a class of external fields satisfying a weaker growth condition than (1.2), namely,

$$\lim_{|z|\to\infty,\, z\in\Sigma}(Q(z)-\log|z|)=M\in(-\infty,\infty],$$

has received a growing interest. Under this condition it can be proven that there still exists an equilibrium measure, though the compactness of its support is, in general, no longer valid (see [5], [17] and [30], among others); however, recently a broad class of "weakly admissible" external fields with compactly supported equilibrium measures was found in [26].

The study of Riesz s-potentials was initiated in [27, 28]. Related minimum energy problems (or Gauss variational problems) were investigated in [25], [22], [24], and numerous contributions by N. Zorii, see the discussion before Theorem 2.1 for precise references. Recent applications of the Riesz theory often involve compact conductors such as balls, spheres or other compact manifolds in $\mathbb{R}^d$, but there are applications dealing with the case of unbounded conductors, in particular, the whole space $\mathbb{R}^d$, see e.g. [11], [21] and [23]. In these contributions the condition that

$$Q(x)\to\infty \quad \text{as} \quad |x|\to\infty, \tag{1.3}$$

is required to ensure the existence of the equilibrium measure μ_Q and the compactness of its support; see also Section 4.4 in the recent monograph [7]. As pointed out in [23], if condition (1.3) holds, it is easy to adapt the proof of [29, Theorem I.1.3] to the case of Riesz potentials for general $0<s<d$ (see also [7, Theorem 4.4.7]). However, this sufficient condition is clearly not sharp, as noted in the also recent article [2], where a simple family of external fields on the real axis for which $\lim_{|x|\to\infty}Q(x)=0$ is shown to have compactly supported equilibrium measures.

In the present paper we aim to study Riesz s-energy problems in $\mathbb{R}^d$ for general external fields Q, not necessarily satisfying (1.3). In particular, the case of the hyperplanar conductor $\mathbb{R}^d$ in the presence of external fields of the form $Q(x)=U^\nu(x)$, where ν is a signed measure compactly supported on $\mathbb{R}^{d+1}\setminus\mathbb{R}^d$, is analyzed in detail. Note that here we shall consider $\mathbb{R}^d$ as embedded in $\mathbb{R}^{d+1}$; that is, $\mathbb{R}^d=\mathbb{R}^d\times\{0\}\subset\mathbb{R}^{d+1}$.

The outline of the paper is as follows. In Section 2 we state and comment on our main results, first regarding general external fields Q and second, relating to the particular case where Q equals the potential U^ν, with ν as above. Section 3 is devoted to some definitions, properties and auxiliary results useful for the proofs of the main results. In particular we prove a version for Riesz potentials of a classical result by de La Vallée Poussin for Newton potentials. We also show the monotonicity of the Robin constant with respect to the external field. These results may be of independent interest. In Section 4 the simplest cases corresponding to discrete measures ν are analyzed with more detail. Finally, Section 5 is devoted to the proofs of our theorems.

2 Riesz energy problems in $\mathbb{R}^d$ and main results

Hereafter, all the positive measures μ considered are finite on compact sets and, when they have unbounded supports, they integrate the Riesz kernel at infinity, namely

$$\int_{|t|>1} \frac{d\mu(t)}{|t|^s} < \infty. \tag{2.1}$$

Let

- Σ be a closed (possibly) unbounded conductor on $\mathbb{R}^d$,
- $\mathcal{P}(\Sigma)$ be the set of probability measures on Σ,
- $Q(x)$ be a lower-semicontinuous function on Σ, lower-bounded if Σ is unbounded, such that $\{x \in \Sigma,\ Q(x) < \infty\}$ has positive capacity (see below for the definition of capacity).

Now, we consider the following energy problem for Riesz s-potentials : find a measure $\mu_Q \in \mathcal{P}(\Sigma)$ that minimizes the weighted energy

$$I_Q(\sigma) := I(\sigma) + 2\int Q(x)d\sigma(x) = \iint \frac{d\sigma(x)\,d\sigma(t)}{|x-t|^s} + 2\int Q(x)d\sigma(x), \tag{2.2}$$

among all measures in $\mathcal{P}(\Sigma)$. We denote by $W_Q(\Sigma)$ this infimum. When $Q = 0$ we obtain the energy of Σ, that we simply denote by $W(\Sigma)$. The capacity $\text{cap}(K)$ of a compact set K is the inverse of its energy and for a Borel set B,

$$\text{cap}(B) := \sup\{\text{cap}(K),\ K \subset B,\ K \text{ compact}\}.$$

As usual, we say that an inequality holds quasi-everywhere (abbreviated q.e.) on a set if the set of exceptional points is of capacity zero or, equivalently, has infinite energy.

Our first Theorem is a reminder of basic properties about the Riesz energy problem. These properties are well-known in the case of the logarithmic kernel. For the Riesz kernel, they are particular cases of more general results, obtained by N. Zorii, about energy problems for families of noncompact conductors on a locally compact space, with general kernels and external fields (also in the form of a potential of a signed measure of finite energy), see [32, Theorems 1-4], [33, Theorem 1.2], [34, Theorems 7.2, 7.3]. The results stated in the next theorem can also be found in [11, Theorem 1.2] or [7, Theorem 4.4.14], when (1.3) is satisfied. One of the main properties is a characterization of a minimizing measure μ_Q (also called equilibrium measure), whenever it exists, in terms of the so-called Frostman inequalities. Throughout, we will denote by S_μ the support of a measure μ.

Theorem 2.1. *Assume $0 < s < d$. Then,*
(i) $-\infty < W_Q(\Sigma) < \infty$;
(ii) if a minimizing measure μ_Q exists, it is unique;
(iii) if μ_Q is a minimizing measure, the Frostman inequalities

$$U^{\mu_Q}(x) + Q(x) \geq F_Q, \quad q.e.\ x \in \Sigma, \tag{2.3}$$
$$U^{\mu_Q}(x) + Q(x) \leq F_Q, \quad x \in S_{\mu_Q}, \tag{2.4}$$

hold true, where the constant

$$F_Q = I(\mu_Q) + \int Q d\mu_Q,$$

is referred to as the equilibrium constant or the (modified) Robin constant.
(iv) Conversely, if $\mu \in \mathcal{P}(\Sigma)$ is a measure with $I_Q(\mu) < \infty$ satisfying

$$U^\mu(x) + Q(x) \geq F, \quad q.e.\ x \in \Sigma, \tag{2.5}$$
$$U^\mu(x) + Q(x) \leq F, \quad q.e.\ x \in S_\mu, \tag{2.6}$$

with some constant F, then μ is a minimizing measure and $F = F_Q$. Moreover, if μ_Q exists, (2.5) may be replaced with the weaker inequality

$$U^\mu(x) + Q(x) \geq F, \quad q.e.\ x \in S_{\mu_Q}. \tag{2.7}$$

For the sake of completeness, we will provide a proof of Theorem 2.1, see Section 5.

For the next results, we will need the notion of thinness of a set at infinity, due, in the newtonian case, to M. Brelot [9] and J.L. Doob [13, pp. 175-176]. This notion has been extended to the Riesz case ($0 < s < d$) by Kurokawa and Mizuta in [20], and since we will need one result of [20], see Lemma 2.7 below, we will stick to the definition given there, namely that a Borel set Σ is thin at infinity if it satisfies the following Wiener condition :

$$\sum_{k=1}^{\infty} 2^{-ks} \operatorname{cap}\left(\Sigma^{(k)}\right) < \infty,$$

where $\Sigma^{(k)} = \left\{x \in \Sigma,\ 2^k \leq |x| < 2^{k+1}\right\}$. Denoting by Φ the inversion $\Phi(x) = x/|x|^2$ with respect to the unit sphere of $\mathbb{R}^d$, the previous inequality is equivalent to (see [22, p.287]),

$$\sum_{k=1}^{\infty} 2^{ks} \operatorname{cap}\left(\Phi(\Sigma^{(k)})\right) < \infty.$$

When $d-2 \leq s < d$, this is a Wiener necessary and sufficient condition for the point 0 to be an irregular point of (the complement of) $\Phi(\Sigma)$. It is known from [22, Theorem 5.10] that a point x in the closure of a set E is irregular if and only if E is thin at x. Hence, with the definition above and when $d-2 \leq s < d$, the set Σ is thin at infinity if and only if $\Phi(\Sigma)$ is thin at 0. For more details about the notions of thinness of a set at a point and the fine topology, see e.g. [22, Chapter V, §3]. For a recent study of thinness of a set at infinity in the theory of Riesz potentials, in particular how the notions introduced by Brelot and Doob extend to this framework, see [35].

In the sequel, when Σ is unbounded, we moreover assume that

A1) If the point at infinity belongs to the boundary of Σ, then Σ is non thin at infinity.

A2) Q is continuous at ∞, i.e.

$$\lim_{|x|\to\infty,\, x\in\Sigma} Q(x) = Q(\infty) \in (-\infty, \infty].$$

Remark 2.2. The assumptions A1-A2 imply that, when Σ is unbounded, (2.3) is satisfied at $x = \infty$, namely $Q(\infty) \geq F_Q$. Indeed, recall that a potential of a measure μ that satisfies condition (2.1) tends to 0 at infinity, up to a set thin at infinity, see Lemma 2.7 below. Also, inequality (2.1) is a necessary and sufficient condition for finiteness almost everywhere of the potential U^{μ}, see [22, p. 61]. It follows in particular that a minimizing measure μ_Q satisfies (2.1).

Our next result concerns the existence of a minimizing measure on an unbounded set Σ, in a general external field Q, depending on its behavior at infinity. In that connection, see also [34, Theorem 8.1].

Theorem 2.3. *Let Σ be an unbounded conductor on $\mathbb{R}^d$ and Q an external field satisfying conditions A1–A2. Then, for $0 < s < d$,*

(i) sufficient conditions for the existence of a solution μ_Q to the Riesz energy problem are the following growth conditions on Q at infinity:

$$|x|^s(Q(x) - Q(\infty)) \leq -1, \qquad \text{for large } x \text{ in } \Sigma; \tag{2.8}$$

or

$$\lim_{|x|\to\infty,\ x\in\Sigma\setminus P} |x|^s(Q(x) - Q(\infty)) \leq -1, \tag{2.9}$$

for some set P thin at infinity.

(ii) if a minimizing measure μ_Q exists on Σ, a necessary condition for μ_Q to have unbounded support is that $Q(\infty)$ is finite and

$$\liminf_{|x|\to\infty,\ x\in\Sigma\setminus P} |x|^s\ (Q(x) - Q(\infty)) = -1, \tag{2.10}$$

for some set P thin at infinity.

(iii) if $d - 2 \leq s < d$ and there exists $c < 1$ such that

$$-\frac{c}{|x|^s} \leq Q(x) - Q(\infty), \qquad x \in \Sigma\,, \tag{2.11}$$

then the Riesz energy problem has no solution.

Remark 2.4. Note that if $Q(\infty) = \infty$, condition (2.8) is trivially true. As to be expected, if Q equals a constant, condition (2.8) is not satisfied.

Theorem 2.3 applies to the particular case where the external field Q is due to the action of an attractive mass placed at a point outside of the conductor Σ (so that Q is lower-semicontinuous on Σ). For instance, assuming $0 \notin \Sigma$, we have the following.

Corollary 2.5. *Under the assumptions in Theorem 2.3, if $0 \notin \Sigma$ and for some $c > 0$,*

$$Q(x) = -\frac{c}{|x|^s}, \qquad x \in \Sigma\,,$$

then
(i) for $0 < s < d$ and $c \geq 1$, the minimizing measure μ_Q exists and, for $c > 1$, it has bounded support;
(ii) for $d - 2 \leq s < d$ and $c < 1$, a minimizing measure μ_Q does not exist.

Remark 2.6. Corollary 2.5 extends for a general dimension $d \geq 2$ the results obtained in [2, Theorem 2.1] for the case where $d = 2$, $\Sigma = \mathbb{R}$ and Q is created by a single attractive charge placed at a point in $\mathbb{R}^2 \setminus \mathbb{R}$. When $c = 1$, the support of the weighted equilibrium measure can be unbounded, as, for instance, in the above case, or more generally, when $\Sigma = \mathbb{R}^d$ and Q is created by an attractive charge placed at a point in $\mathbb{R}^{d+1} \setminus \mathbb{R}^d$, see Section 4.2, case (i). It can also be bounded, see Remark 5.3 for more details.

In the current paper, we will also consider in detail the case where the conductor $\Sigma = \mathbb{R}^d$ and Q is the potential U^ν of a signed measure ν supported in $\mathbb{R}^{d+1} \setminus \mathbb{R}^d$,

$$Q(x) = U^\nu(x) = \int \frac{d\nu(t)}{|x-t|^s}.$$

An auxiliary result about Riesz potentials will be useful.

Lemma 2.7 (cf. [20, Theorem 3.3]). *Let μ be a positive measure in $\mathbb{R}^d$. Then, there exists a Borel set P thin at infinity such that*

$$\lim_{|x|\to\infty,\ x\in\mathbb{R}^d\setminus P} U^\mu(x) = 0, \qquad \text{if} \int_{|t|>1} \frac{d\mu(t)}{|t|^s} < \infty, \tag{2.12}$$

$$\lim_{|x|\to\infty,\ x\in\mathbb{R}^d\setminus P} |x|^s U^\mu(x) = \mu(\mathbb{R}^d), \qquad \text{if } \mu(\mathbb{R}^d) < \infty. \tag{2.13}$$

From this lemma, we see that condition (2.9) of assertion (i) and assertion (ii) of Theorem 2.3 imply the following.

Corollary 2.8. *Let $0 < s < d$. Let ν be a signed measure of finite mass (both, ν^+ and ν^- in the Jordan decomposition $\nu = \nu^+ - \nu^-$ have finite mass) with support in $\mathbb{R}^{d+1} \setminus \mathbb{R}^d$ (not necessarily compact), and let $Q = U^\nu$ be as above. If $\nu(\mathbb{R}^{d+1}) \leq -1$, then there exists a unique equilibrium measure μ_Q. Moreover, if $\nu(\mathbb{R}^{d+1}) < -1$, the support S_{μ_Q} is a compact subset of $\mathbb{R}^d$.*

Indeed, we have $Q = U^\nu = U^{\nu^w}$ with ν^w the weak balayage of ν onto $\mathbb{R}^d$ and $\nu(\mathbb{R}^{d+1}) = \nu^w(\mathbb{R}^d)$, see Lemma 3.11 for details. Moreover (2.13) can be applied to each terms in the Jordan decomposition of the signed measure $\nu^w = \nu^w_+ - \nu^w_-$, and the union of two sets thin at infinity is also thin at infinity.

The situation where Q and ν are as above, and $\nu(\mathbb{R}^{d+1}) = -1$, will be referred to as a *weakly admissible setting*, using the parallelism with the logarithmic potential scenario. For this purpose, we denote by $y = (y_1, \ldots, y_d, y_{d+1})$ an arbitrary point in $\mathbb{R}^{d+1}$ so that the distance from y to the conductor $\mathbb{R}^d$ equals $|y_{d+1}|$.

Our final main result is the following one.

Theorem 2.9. *Let $d-2 \leq s < d$ and $Q = U^\nu$, $\nu(\mathbb{R}^{d+1}) = -1$. If*

$$\int |y_{d+1}|^{d-s} d\nu(y) > 0\,, \tag{2.14}$$

then S_{μ_Q} is a compact subset of $\mathbb{R}^d$.

Remark 2.10. Theorem 2.9 extends to Riesz potentials what was established in [26] for the logarithmic kernel in $\mathbb{R}^2$ with $\Sigma = \mathbb{R}$ in the weakly admissible setting. Here, when the total attractive charge of the external field equals in size the charge to be spread on $\mathbb{R}^d$ it is possible to get a compactly supported equilibrium measure. However, in this border case not just the sizes of the charges are important but also their distances to the conductor. This very interesting phenomenon may be illustrated with a simple example. Namely, if the external field consists just of an attractor of unit charge, then the equilibrium measure exists but its support will be unbounded (the whole conductor $\mathbb{R}^d$ in the current setting); but if this simple configuration is slightly modified by adding a negative charge $-\varepsilon$ to this attractor, it is possible to place a repellent of positive charge ε sufficiently far from the conductor so that the equilibrium measure becomes compactly supported (observe that the total attractive charge remains equal to one).

With the level of generality considered so far, not much more can be said about the equilibrium measure μ_Q and its support S_{μ_Q}. In Section 4 we will discuss some particular situations where the simplicity of the external field allows for a more detailed analysis.

3 Auxiliary results

In this section properties and auxiliary results related to Riesz energy problem are proved. Also, we recall the notions of Kelvin transform, balayage and signed equilibrium measures, which will be useful in the sequel. Throughout, it will also be convenient to introduce the parameter α, $0 < \alpha < d$, given by

$$\alpha := d - s.$$

3.1 A de La Vallée Poussin theorem for Riesz kernels

In this subsection, we give a version for Riesz potentials of the following result by de La Vallée–Poussin for Newtonian potentials, or more generally, for superharmonic functions, see [12, p. 21]:

Theorem 3.1. *Let u and v be two superharmonic functions defined in an open subset Ω of $\mathbb{R}^d$, with respective Riesz measures μ and ν. Assume $u \geq v$ in Ω. Then the restriction of the signed measure $\mu - \nu$ to the set*

$$E = \{x \in \Omega,\ u(x) = v(x) < \infty\}$$

is a negative measure.

See also [29, Theorem IV.4.5] for the logarithmic case in the complex plane. A similar result was proved by Janssen [18, Theorem 2.5] in a general potential theoretic setting, see also Fuglede [15, Theorem 1.1] for the Newton case.

In this subsection it is assumed that $d - 2 < s < d$ or equivalently $0 < \alpha < 2$.

We can now state the main result of this subsection.

Theorem 3.2. *Let μ and ν be two positive measures such that the potential U^μ is finite μ-a.e. (for instance μ has finite energy). Assume*

$$U^\nu \leq U^\mu + C, \quad \nu\text{-almost everywhere,} \tag{3.1}$$

where $C \geq 0$. Then the restriction of the signed measure $\mu - \nu$ to the set

$$E = \{x \in \mathbb{R}^d,\ U^\nu(x) = U^\mu(x) + C < \infty\}$$

is a negative measure.

Remark 3.3. Notice that the statement of Theorem 3.1 does not hold for Riesz subharmonic kernels $|x|^{-s}$ with $d-2 < s < d$. Indeed, considering the equilibrium measure ω of the unit ball $B \subset \mathbb{R}^d$, we know that

$$U^\omega(x) = W(B), \qquad x \in B,$$

where $W(B)$ denotes the Riesz energy of B. Applying Theorem 3.1 with U^ω and the constant function equal to $W(B)$, the inequality $W(B) \leq U^\omega(x)$ would imply that $\omega \leq 0$ on any open subset Ω of B, which is not true as the support of ω is the entire ball B when $d-2 < s < d$.

Let us first recall the notion of *balayage*, see [22, Chapter IV, §5], in particular p.264. Given a closed set $F \subset \mathbb{R}^d$ of positive capacity and a measure σ satisfying (2.1), there exists a unique measure

$$\widehat{\sigma} := Bal(\sigma, F)$$

called the Riesz s-balayage of σ onto F satisfying the following:
- $S_{\widehat{\sigma}} \subseteq F$,
- $\widehat{\sigma}$ is zero on the set of irregular points of F, and

$$U^{\widehat{\sigma}}(x) = U^\sigma(x) \text{ q.e. on } F, \qquad U^{\widehat{\sigma}}(x) \leq U^\sigma(x) \text{ on } \mathbb{R}^d. \tag{3.2}$$

If $d = 2$ and $s = 0$, the usual logarithmic case, the balayage $\widehat{\sigma}_0$ preserves the total mass: $\|\widehat{\sigma}_0\| = \|\sigma\|$, but this is not true for general $d-2 \leq s < d$. Actually, $\|\widehat{\sigma}\| = \|\sigma\|$ for all measures σ if and only if F is not thin at infinity, see [16, Theorem 3.22]. In the logarithmic case, (3.2) is modified in the sense that

$$U^{\widehat{\sigma}_0}(z) = U^\sigma(z) + C \quad \text{q.e. on } F, \qquad U^{\widehat{\sigma}_0}(z) \leq U^\sigma(z) + C \ \text{ on } \mathbb{C},$$

where $C = 0$ if, for example, $\mathbb{C} \setminus F$ is a bounded set.

We remark that for $0 < s < d-2$ we shall utilize a measure σ^w, which we call *weak balayage*, for which only the first part of (3.2) holds true, namely $U^{\sigma^w}(x) = U^\sigma(x)$ q.e. on F. For signed measure ν, we define $\nu^w := (\nu^+)^w - (\nu^-)^w$, where $\nu = \nu^+ - \nu^-$ is the Jordan decomposition of ν.

Now we consider the family of balayages $\widehat{\delta}_{0,r}$, $r > 0$, of δ_0 (the unit point mass at $x = 0$) outside of the balls $|x| < r$. The balayage $\widehat{\delta}_{0,r}$ is a probability measure supported on the complement of $B(0, r)$. Its potential satisfies

$$U^{\widehat{\delta}_{0,r}}(x) = \begin{cases} U^{\delta_0}(x) = |x|^{-s}, & |x| \geq r, \\ U^{\widehat{\delta}_{0,r}}(x) \leq |x|^{-s}, & |x| \leq r. \end{cases}$$

For $x_0 \in \mathbb{R}^d$, we set $\widehat{\delta}_{x_0,r}(x) := \widehat{\delta}_{0,r}(x - x_0)$ and, for u a potential, we will consider its average with respect to $\widehat{\delta}_{x_0,r}$:

$$L_u(x_0, r) = \int u(x) d\widehat{\delta}_{x_0,r}(x), \qquad r > 0.$$

Note that, when u is the potential of a measure μ,

$$u(x) = \int \frac{d\mu(y)}{|x-y|^s},$$

$L_u(x_0, r)$ is a finite number. Indeed,

$$L_u(x_0, r) = \int u(x) d\widehat{\delta}_{x_0,r}(x) = \int U^{\widehat{\delta}_{x_0,r}}(x) d\mu(x),$$

and for $0 < r_1 < r$, we have

$$\int_{r_1 \le |x-x_0|} U^{\widehat{\delta}_{x_0,r}}(x) d\mu(x) \le \int_{r_1 \le |x-x_0|} \frac{d\mu(x)}{|x-x_0|^s} < \infty,$$

where the last inequality follows from (2.1), and

$$\int_{|x-x_0|<r_1} U^{\widehat{\delta}_{x_0,r}}(x) d\mu(x) \le \int_{|x-x_0|<r_1} \frac{d\mu(x)}{|r-r_1|^s} < \infty,$$

where the first inequality uses the fact that the mass of $\widehat{\delta}_{x_0,r}$ is outside of the ball $B(x_0, r)$.

For the proof of Theorem 3.2, we will need several lemmas.

Lemma 3.4. *Let*

$$u(x) = \int \frac{d\mu(y)}{|x-y|^s},$$

with μ a positive measure, finite on compact sets. Let $x_0 \in \mathbb{R}^d$. The function $r \to L_u(x_0, r)$ satisfies

$$\lim_{r\to 0} L_u(x_0, r) = u(x_0), \tag{3.3}$$

and is non-increasing with r.

Moreover, the function $r \mapsto L_u(x_0, r)$ is absolutely continuous on any closed subinterval of $(0, \infty)$ and, almost everywhere, one has

$$L_u'(x_0, r) = -\frac{2c_{d,\alpha}}{r^{d-1}} \int_{|x-x_0|\le r} (r^2 - |x-x_0|^2)^{\alpha/2-1} d\mu(x), \tag{3.4}$$

where

$$c_{d,\alpha} = \frac{\Gamma(d/2)}{2^\alpha \pi^{d/2} \Gamma^2(\alpha/2)}. \tag{3.5}$$

Remark 3.5. Note that (3.4) is finite for almost every $r > 0$. Note also that in the newtonian case $\alpha = 2$, (3.4) simplifies to

$$-r^{d-1} L_u'(x_0, r) = 2c_{d,2}\mu(B(x_0, r)). \tag{3.6}$$

In view of 3.3 we will define $L_u(x_0, 0)$ as $u(x_0)$.

Proof. Equality (3.3) and the fact that $r \mapsto L_u(x_0, r)$ is non-increasing are shown in [22] p. 114 and p. 126 respectively.

In the sequel, we choose $x_0 = 0$, the proof being identical at any other point $x_0 \in \mathbb{R}^d$. Let r_1 be a fixed positive number with $0 < r_1 < r$. We write

$$L(r) - L(r_1) := L_u(0,r) - L_u(0,r_1)$$
$$= \int_{|x|\le r_1} (U^{\widehat{\delta}_{0,r}}(x) - U^{\widehat{\delta}_{0,r_1}}(x))d\mu(x) + \int_{r_1<|x|\le r} (U^{\widehat{\delta}_{0,r}}(x) - U^{\delta_0}(x))d\mu(x). \tag{3.7}$$

To reexpress the last integrals, we will use the following expression for the Green function of the ball of radius r, see e.g. [10, Theorem 3.1]

$$G_r(x,y) = U^{\delta_y}(x) - U^{\widehat{\delta}_{y,r}}(x) = \frac{c_{d,\alpha}}{|x-y|^{d-\alpha}} \int_0^a s^{\alpha/2-1}(1+s)^{-d/2}ds, \qquad x,y \in B(0,r),$$

where $c_{d,\alpha}$ is the constant (3.5) and

$$a = \frac{(r^2-|x|^2)(r^2-|y|^2)}{r^2|x-y|^2}.$$

When $y = 0$, we get

$$G_r(x,0) = U^{\delta_0}(x) - U^{\widehat{\delta}_{0,r}}(x) = \frac{c_{d,\alpha}}{|x|^{d-\alpha}} \int_0^a s^{\alpha/2-1}(1+s)^{-d/2}ds, \qquad a = \frac{r^2-|x|^2}{|x|^2}.$$

Making use of the above expression, (3.7) becomes

$$-\int_{|x|\le r_1} \frac{c_{d,\alpha}}{|x|^{d-\alpha}} \int_{a_1}^a s^{\alpha/2-1}(1+s)^{-d/2}ds\ d\mu(x)$$
$$- \int_{r_1<|x|\le r} \frac{c_{d,\alpha}}{|x|^{d-\alpha}} \int_0^a s^{\alpha/2-1}(1+s)^{-d/2}ds\ d\mu(x).$$

Performing the change of variable $t = |x|\sqrt{s+1}$, we get

$$L(r) - L(r_1) = -2c_{d,\alpha} \int_{|x|\le r_1} \left(\int_{t=r_1}^r (t^2-|x|^2)^{\alpha/2-1} t^{1-d} dt \right) d\mu(x)$$
$$- 2c_{d,\alpha} \int_{r_1<|x|\le r} \left(\int_{t=|x|}^r (t^2-|x|^2)^{\alpha/2-1} t^{1-d} dt \right) d\mu(x).$$

Since the integrand in the double integrals is positive, we may interchange the order of integrations which leads to

$$L(r) - L(r_1) = -2c_{d,\alpha} \int_{t=r_1}^r \left(\int_{|x|=0}^t (t^2-|x|^2)^{\alpha/2-1} t^{1-d} d\mu(x) \right) dt.$$

Finally, since both $L(r)$ and $L(r_1)$ are finite numbers, the double integral is also finite which implies, by Fubini theorem, that the function

$$t \mapsto \int_{|x|=0}^t (t^2-|x|^2)^{\alpha/2-1} t^{1-d} d\mu(x)$$

is integrable. Hence, $L(r) - L(r_1)$ is the indefinite integral of an integrable function, from which follows that the function $r \mapsto L(r)$ is absolutely continuous, differentiable almost everywhere, with derivative equal to

$$-2c_{d,\alpha} r^{1-d} \int_{|x|=0}^{r} (r^2 - |x|^2)^{\alpha/2-1} d\mu(x).$$

□

The following lemma is a consequence of the previous one.

Lemma 3.6. *Assume $L_u(x_0, r)$ is differentiable at $r > 0$. Then,*

$$-\, r^d L_u'(x_0, r) = 2c_{d,\alpha} \frac{d}{dr} \int_{t=0}^{r} t(r^2 - t^2)^{\alpha/2-1} \mu(B(x_0, t))dt. \tag{3.8}$$

Remark 3.7. Note that in the newtonian case $\alpha = 2$, (3.8) just gives (3.6) again.

Proof. We have

$$\begin{aligned}\int_{t=0}^{r} t(r^2 - t^2)^{\alpha/2-1} \int_{|x-x_0|=0}^{t} d\mu(x)dt &= \int_{|x-x_0|=0}^{r} \int_{t=|x-x_0|}^{r} t(r^2 - t^2)^{\alpha/2-1} dt d\mu(x) \\ &= \frac{1}{2} \int_{|x-x_0|=0}^{r} (r^2 - |x - x_0|^2)^{\alpha/2} \int_{s=0}^{1} (1 - s)^{\alpha/2-1} ds d\mu(x) \\ &= \frac{1}{\alpha} \int_{|x-x_0|=0}^{r} (r^2 - |x - x_0|^2)^{\alpha/2} d\mu(x),\end{aligned}$$

where the variable s in the second equality is such that $t^2 - |x - x_0|^2 = s(r^2 - |x - x_0|^2)$. Hence, the right-hand side of (3.8) is equal to

$$\begin{aligned}\frac{2c_{d,\alpha}}{\alpha} \frac{d}{dr} \int_{|x-x_0|=0}^{r} (r^2 - |x - x_0|^2)^{\alpha/2} d\mu(x) &= 2c_{d,\alpha} \int_{|x-x_0|=0}^{r} r(r^2 - |x - x_0|^2)^{\alpha/2-1} d\mu(x) \\ &= -r^d L_u'(x_0, r).\end{aligned}$$

□

Finally we recall a result from measure theory. Its proof uses the Vitali covering theorem, see [31] for details.

Lemma 3.8. *Let A be a Borel set, and let ν be a signed measure in $\mathbb{R}^d$. Suppose that for each $x \in A$ there is a sequence $t_n \downarrow 0$ with $\nu(B(x, t_n)) \geq 0$. Then $\nu|_A$ is a non-negative measure.*

Proof of Theorem 3.2. From the domination principle, see [22, p. 115], we know that (3.1) actually holds everywhere on $\mathbb{R}^d$. Hence, with $u = U^\mu$ and $v = U^\nu$, and for $R > 0$,

$$L_v(x, R) \leq L_u(x, R) + C, \qquad x \in \mathbb{R}^d.$$

Consider a point $x_0 \in E$. Then $v(x_0) = u(x_0) + C < \infty$ and thus

$$L_v(x_0, R) - v(x_0) \leq L_u(x_0, R) - u(x_0).$$

Since $r \mapsto L_v(x_0, r)$ is an absolutely continuous function, we have, for any $\epsilon > 0$,

$$\int_\epsilon^R L_v'(x_0, r)dr = L_v(x_0, R) - L_v(x_0, \epsilon).$$

From the non-positivity of $L_v'(x_0, r)$ and the monotone convergence theorem, we get by letting ϵ tend to 0,

$$\int_0^R L_v'(x_0, r)dr = L_v(x_0, R) - v(x_0),$$

and the same holds true for the function L_u. Consequently,

$$\int_0^R L_v'(x_0, r)dr \leq \int_0^R L_u'(x_0, r)dr.$$

Next, considering equation (3.8) applied to u and v, taking the difference between the two and then the antiderivative, we get

$$-\int_{r=0}^R r^d L_{u-v}'(x_0, r)dr = 2c_{d,\alpha} \int_{t=0}^R t(R^2 - t^2)^{\alpha/2-1}(\mu - \nu)(B(x_0, t))dt, \quad R > 0. \tag{3.9}$$

In the sequel, we fix an $R > 0$ and show that there exists an $0 < r \leq R$ for which the inequality $\mu(B(x_0, r)) \leq \nu(B(x_0, r))$ holds true. We proceed by contradiction. Assume that, for all $0 < r \leq R$, we have $\mu(B(x_0, r)) > \nu(B(x_0, r))$. Then, the right-hand side of (3.9) is a positive number. On the other hand, consider a fixed $\delta > 0$. For all $\epsilon > 0$ sufficiently small, we have

$$-\delta \leq \int_\epsilon^R L_{u-v}'(x_0, r)dr.$$

Then, for some $R' \in (0, R]$,

$$-\delta \leq \int_\epsilon^R L_{u-v}'(x_0, r)dr = \int_\epsilon^R r^{-d}(r^d L_{u-v}'(x_0, r))dr = \epsilon^{-d} \int_\epsilon^{R'} r^d L_{u-v}'(x_0, r)dr,$$

where in the last equality we have used the Bonnet's mean value theorem for integrals, cf. [1, Section 12.6] (note that r^{-d} is a positive decreasing function). Observing that

$$0 = \lim_{\epsilon \to 0}(-\delta\epsilon^d) \leq \lim_{\epsilon \to 0} \int_\epsilon^{R'} r^d L_{u-v}'(x_0, r)dr = \int_0^{R'} r^d L_{u-v}'(x_0, r)dr,$$

we get a contradiction since by (3.9), the above integral is a negative number.

Repeating the argument with a smaller value of R, we get a sequence $r_n \downarrow 0$ such that $\mu(B(x_0, r_n)) \leq \nu(B(x_0, r_n))$. It then suffices to apply Lemma 3.8 to conclude the proof. $\square$

3.2 Monotonicity and continuity of the equilibrium constant

In this section, we always assume that a minimizing measure exists. We derive monotonicity and continuity of the equilibrium constant with respect to the external field. This result, which may be of independent interest, is an analog of [29, Corollary I.4.2] for the logarithmic kernel. It will be used in the proof Theorem 2.3.

Let us start with a lemma.

Lemma 3.9. *For $d-2 \leq s < d$, and ν a probability measure on $\mathbb{R}^d$ satisfying (2.1), the following inequality holds*

$$\text{“}\inf_{x\in S_{\mu_Q}}\text{”}(U^{\nu}(x)+Q(x)) \leq F_Q, \tag{3.10}$$

*where “*inf*” denotes the essential inf in the sense of [29, p. 43].*

Proof. We essentially follow the proof of [14, Theorem 1.3], where (3.10) was proved in the case of an external field Q on the d-dimensional sphere $S^d \subset \mathbb{R}^{d+1}$ and $d-1 \leq s < d$. Let L be some constant such that $L \leq U^{\nu}+Q$, q.e. on S_{μ_Q} or

$$U^{\nu} - U^{\mu_Q} \geq L - Q - U^{\mu_Q}, \quad \text{q.e on } S_{\mu_Q}.$$

From the second Frostman inequality, $\inf_{S_{\mu_Q}}(-U^{\mu_Q}-Q) \geq -F_Q$; hence

$$U^{\nu} - U^{\mu_Q} \geq L - F_Q, \quad \text{q.e. on } S_{\mu_Q} \implies \mu_Q\text{-a.e.}$$

Assume now that $L > F_Q$ and consider any bounded non polar subset A of S_{μ_Q}. If S_{μ_Q} is bounded one can take $A = S_{\mu_Q}$. Let ω_A be the equilibrium measure of A and let $\sigma = \mathrm{cap}(A)\omega_A = W^{-1}(A)\omega_A$, where $\mathrm{cap}(A)$ and $W(A)$ respectively denote the Riesz capacity and Riesz energy of A. Then $U^{\sigma} \leq 1$ on $\mathbb{R}^d$ and in particular on S_{μ_Q}. Thus we have

$$U^{\mu_Q+(L-F_Q)\sigma} \leq U^{\nu}, \quad \text{q.e. on } S_{\mu_Q}.$$

By the domination principles [22, Theorems 1.27 and 1.29], the inequality is satisfied everywhere, and then, by the principle of positivity of mass, cf. [16, Theorem 3.11] or [24, Theorem 1.7 p. 62], we get $1+(L-F_Q)\,\mathrm{cap}(A) \leq 1$, which is a contradiction. □

Proposition 3.10. *For $d-2 \leq s < d$, the Robin constant F_Q is a non-decreasing, continuous function of Q. More precisely, let $C, \epsilon \geq 0$ be some nonnegative constants. Then*

$$C \leq Q_2 - Q_1 \text{ on } S_{\mu_{Q_2}} \implies \begin{cases} C \leq (U^{\mu_{Q_1}}(x)-F_1)-(U^{\mu_{Q_2}}(x)-F_2), & x \in \mathbb{R}^d, \\ C \leq F_2 - F_1, \end{cases} \tag{3.11}$$

and

$$|Q_2 - Q_1| \leq \epsilon \text{ on } S_{\mu_{Q_1}} \cup S_{\mu_{Q_2}} \implies \begin{cases} |(U^{\mu_{Q_1}}(x)-F_1)-(U^{\mu_{Q_2}}(x)-F_2)| \leq \epsilon, & x \in \mathbb{R}^d, \\ |F_2 - F_1| \leq \epsilon. \end{cases} \tag{3.12}$$

Proof. The proof is based on the following identity:

$$U^{\mu_Q}(x) - F_Q = \inf\{U^\nu(x) - \text{“}\inf_{x\in\Sigma}\text{”}(U^\nu + Q),\ \nu \in \mathcal{P}(\Sigma)\}, \quad x \in \mathbb{R}^d, \tag{3.13}$$

where the essential “inf” can also be taken on S_{μ_Q} instead of Σ. The left-hand side is no-smaller than the infimum because $\mu_Q \in \mathcal{P}(\Sigma)$ and “$\inf_{x\in\Sigma}$”$(U^{\mu_Q} + Q) = F_Q$. For the converse inequality, using (2.4), note that,

$$U^{\mu_Q}(x) + Q(x) - F_Q + \text{“}\inf_{x\in\Sigma}\text{”}(U^\nu + Q) \le U^\nu(x) + Q(x), \quad x \in S_{\mu_Q},$$

or equivalently

$$U^{\mu_Q}(x) \le U^\nu(x) + (F_Q - \text{“}\inf_{x\in\Sigma}\text{”}(U^\nu + Q)), \quad x \in S_{\mu_Q},$$

where the constant term $F_Q -$ “$\inf_{x\in\Sigma}$”$(U^\nu + Q)$ is non-negative by (3.10). Then, by the domination principles in the preceding proof, the inequality holds everywhere. Thus the left-hand side in (3.13) is also no-larger than the right-hand side.

Now, if $C \le Q_2 - Q_1$ on $S_{\mu_{Q_2}}$, we have, for a given $\nu \in \mathcal{P}(\Sigma)$,

$$\text{“}\inf_{x\in\Sigma}\text{”}(U^\nu + Q_1) \le \text{“}\inf_{x\in S_{\mu_{Q_2}}}\text{”}(U^\nu + Q_1) \le \text{“}\inf_{x\in S_{\mu_{Q_2}}}\text{”}(U^\nu + Q_2) - C;$$

hence

$$U^\nu(x) - \text{“}\inf_{x\in S_{\mu_{Q_2}}}\text{”}(U^\nu + Q_2) + C \le U^\nu(x) - \text{“}\inf_{x\in\Sigma}\text{”}(U^\nu + Q_1), \qquad x \in \mathbb{R}^d.$$

Taking the infimum over ν and making use of (3.13), one gets the first inequality in (3.11); the second inequality $C \le F_2 - F_1$ simply follows by letting x go to infinity (here we note that a potential that integrates the Riesz kernel at infinity, tends to 0 at infinity, up to a set thin at infinity, see Lemma 2.7. The second implication (3.12) can be proved in the same way. $\square$

3.3 Kelvin transform and formulas for the sphere and the ball

In the sequel, we will use the Kelvin transform, see e.g. [22, Chapter IV.5]. We denote by T the Kelvin transform in $\mathbb{R}^{d+1}$, with respect to the point $y = (y_1, \dots, y_{d+1})$, $y_{d+1} > 0$ and with radius $\sqrt{2y_{d+1}}$. It maps $\mathbb{R}^d$ onto the sphere S_y^d of radius 1, having y as its north pole. This transformation was used in [3] and, recently, in [2]. Then, with $x^* = T(x)$, $t^* = T(t)$, we have

$$|x - y||x^* - y| = 2y_{d+1}, \qquad |t^* - x^*| = \frac{2y_{d+1}|x - t|}{|x - y|\,|t - y|}.$$

For the Riesz potentials and energies, we have the following relations:

$$U^{\mu^*}(x^*) = \frac{|x - y|^s}{(2y_{d+1})^{s/2}} U^\mu(x), \qquad I_s(\mu^*) = I_s(\mu), \tag{3.14}$$

where

$$d\mu^*(t^*) = \frac{(2y_{d+1})^{s/2}}{|t - y|^s} d\mu(t) = \frac{|t^* - y|^s}{(2y_{d+1})^{s/2}} d\mu(t). \tag{3.15}$$

Note, in particular, that

$$\mu(\mathbb{R}^{d+1}) = (2y_{d+1})^{s/2} U^{\mu^*}(y). \tag{3.16}$$

We also recall that the surface area of the d-dimensional unit sphere $S^d \subset \mathbb{R}^{d+1}$ equals

$$\omega_d = \frac{2\pi^{(d+1)/2}}{\Gamma\left((d+1)/2\right)}. \tag{3.17}$$

The equilibrium measure of the d-dimensional unit sphere S^d is the normalized surface measure, denoted by σ_d. Its energy $W(S^d)$ is given by

$$W(S^d) = \begin{cases} \dfrac{\Gamma\left(\frac{d+1}{2}\right)\Gamma(\alpha)}{\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma\left(d-\frac{s}{2}\right)}, & 0 < s < d,\, d \geq 3, \\ 2^{1-s}/(2-s), & 0 < s < 2,\, d = 2, \end{cases} \tag{3.18}$$

see [7, Formula (4.6.5)]. For $d-2 < s < d$ and $d \geq 2$, the equilibrium measure ω_R of the closed ball B_R of radius R in $\mathbb{R}^d$ is absolutely continuous with respect to Lebesgue measure, with density

$$\omega_R'(x) = \frac{c_R}{(R^2 - |x|^2)^{\alpha/2}}, \qquad c_R = \frac{\pi^{-d/2}\Gamma(1+s/2)}{R^s \Gamma(1-\alpha/2)}, \tag{3.19}$$

see [22, p. 163], [7, Eq.(4.6.12)]. Its potential at the point $y = (0, y_{d+1})$ equals

$$U^{\omega_R}(y) = \int_{\mathbb{R}^d} \frac{d\omega_R(x)}{|x-y|^s} = y_{d+1}^{-s}\ {}_2F_1\left(\frac{s}{2}, \frac{d}{2}, 1+\frac{s}{2}, -\frac{R^2}{y_{d+1}^2}\right), \tag{3.20}$$

which is easily checked from the Euler integral formula for hypergeometric functions. Finally, the s-energy of B_R is

$$W(B_R) = \frac{s}{2R^s} B\left(\frac{s}{2}, \frac{\alpha}{2}\right), \tag{3.21}$$

where $B(x,y)$ denotes the Beta function, see [7, Section 4.6, p. 183].

3.4 Weak balayage and signed equilibrium measures

The following result plays an important role for our analysis.

Lemma 3.11. *Let $0 < s < d$ and $y = (y_1, \ldots, y_{d+1}) \in \mathbb{R}^{d+1} \setminus \mathbb{R}^d$ with $y_{d+1} \neq 0$. The weak balayage δ_y^w of δ_y onto $\mathbb{R}^d$ is given by*

$$d\delta_y^w(x) = \frac{(2|y_{d+1}|)^{d-s}}{\omega_d W(S^d)|x-y|^{2d-s}} dx, \tag{3.22}$$

where dx denotes the Lebesgue measure in $\mathbb{R}^d$. Moreover, there is no mass loss in this case; that is, $\|\delta_y^w\| = \|\delta_y\| = 1$. Furthermore, if ν is a signed measure of finite mass with support in $\mathbb{R}^{d+1} \setminus \mathbb{R}^d$ (not necessarily compact), its weak balayage ν^w is given by the superposition

$$d\nu^w(x) = \left(\int_{S_\nu} \delta_y^w(x)\, d\nu(y)\right) dx. \tag{3.23}$$

Remark 3.12. We note that for $d-2 \leq s < d$ the weak balayage measures δ_y^w and ν^w coincide with the respective s-balayage measures $\widehat{\delta}_y$ and $\widehat{\nu}$ defined in (3.2).

Proof. Without loss of generality assume $y_{d+1} > 0$. To prove (3.22), it suffices to verify that

$$U^{\delta_y^w}(z) = \int_{\mathbb{R}^d} \frac{(2y_{d+1})^{d-s}}{\omega_d W(S^d)|z-x|^s|x-y|^{2d-s}} dx = \frac{1}{|z-y|^s}, \quad z \in \mathbb{R}^d.$$

We shall utilize the Kelvin transform T from the previous subsection. The relation between the measure normalized unit surface measure $d\sigma_d$ on S_y^d and the Lebesgue measure on $\mathbb{R}^d$ is

$$\omega_d d\sigma_d(x^*) = \frac{|x^*-y|^d}{|x-y|^d} dx = \frac{(2y_{d+1})^d \, dx}{|x-y|^{2d}},$$

which yields

$$U^{\delta_y^w}(z) = \frac{1}{W(S^d)|z-y|^s} \int_{S_y^d} \frac{1}{|z^*-x^*|^s} d\sigma_d(x^*), \quad z^* \in S_y^d.$$

As $\sigma_d(x^*)$ is the equilibrium measure on S_y^d, the integral above equals $W(S^d)$ for any $z^* \in S_y^d$, equation (3.22) follows. The total mass of δ_y^w is computed as

$$\|\delta_y^w\| = \int_{\mathbb{R}^d} \frac{(2y_{d+1})^{d-s}}{\omega_d W(S^d)|x-y|^{2d-s}} dx = \frac{1}{W(S^d)} \int_{S_y^d} \frac{1}{|x^*-y|^s} d\sigma_d(x^*) = 1.$$

The equation (3.23) for any $z \in \mathbb{R}^d$ can be derived as in [22, Section IV.5, (4.5.5)], which completes the proof. $\square$

For our analysis the notion of signed equilibrium measure will be very important.

Definition 3.13. Let Σ be a closed subset of $\mathbb{R}^d$. A *signed equilibrium measure* for Σ in the external field Q is a signed measure $\eta_{Q,\Sigma}$ supported on Σ such that $\eta_Q(\Sigma) = 1$ and there exists a finite constant $C_{Q,\Sigma}$ such that

$$U^{\eta_{Q,\Sigma}}(x) + Q(x) = C_{Q,\Sigma} \quad \text{q.e. on } \Sigma. \tag{3.24}$$

Remark 3.14. If this signed equilibrium measure exists, then it is unique, see [8, Lemma 23]. Note also that $\eta_{Q,\Sigma}$ has finite energy (and finite weighted energy) as can be seen from integrating (3.24) with respect to $\eta_{Q,\Sigma}$.

Our next result describes the main properties of the signed equilibrium measure. This result corresponds to [19, Lemma 3], where it was established for the logarithmic kernel in the complex plane, when S_{μ_Q} is a compact set.

Lemma 3.15. *Let $d-2 \leq s < d$ and let Σ be a closed subset of $\mathbb{R}^d$. Assume an equilibrium measure μ_Q and a signed equilibrium measure $\eta_{Q,\Sigma}$ exist. Denote by $\eta_{Q,\Sigma}^+$ the positive part in the Jordan decomposition of $\eta_{Q,\Sigma}$. Then,*

(i) one has

$$\mu_Q \leq \eta_{Q,\Sigma}^+. \tag{3.25}$$

In particular,

$$S_{\mu_Q} \subseteq S_{\eta_{Q,\Sigma}^+}.$$

(ii) Let $\Sigma_1 \subset \Sigma$ be such that $S_{\mu_Q} \subset \Sigma_1$. If η_{Q,Σ_1} is a positive measure, then $\mu_Q = \eta_{Q,\Sigma_1}$.

(iii) Let $\Sigma = \mathbb{R}^d$ and let Q be an external field such that S_{μ_Q} is compact. Assume that there exists an $R_0 > 0$ such that, for each R larger than R_0, the signed measure η_{Q,B_R} is negative near the boundary of B_R. Then $S_{\mu_Q} \subset B_{R_0}$.

Proof. (i) From (3.24) and (2.3)–(2.4) we have that

$$U^{\eta^+_{Q,\Sigma}}(x) \leq U^{\eta^-_{Q,\Sigma}+\mu_Q}(x) + C - F_Q, \quad \text{q.e. on } \Sigma, \tag{3.26}$$

$$U^{\eta^+_{Q,\Sigma}}(x) = U^{\eta^-_{Q,\Sigma}+\mu_Q}(x) + C - F_Q, \quad \text{q.e. on } S_{\mu_Q}. \tag{3.27}$$

The assumptions in Theorem 3.2 are satisfied. Indeed, by the domination principle, (3.26) holds everywhere in $\mathbb{R}^d$, which implies that $C - F_Q \geq 0$ by letting x tends to infinity. From Theorem 3.2 and (3.27) we thus derive that $\mu_Q + \eta^-_{Q,\Sigma} \leq \eta^+_{Q,\Sigma}$ on S_{μ_Q} and consequently $\mu_Q \leq \eta^+_{Q,\Sigma}$.
(ii) The fact that $\mu_Q = \eta_{Q,\Sigma_1}$ is a consequence of (3.24) since, under this hypothesis, the inequalities (2.6) and (2.7) characterizing the equilibrium measure are satisfied by the positive measure η_{Q,Σ_1}.
(iii) Assume S_{μ_Q} is not included in B_{R_0}. Consider the smallest ball B_R, $R > R_0$, that contains S_{μ_Q}. Since μ_Q satisfies (2.5)–(2.6) on B_R, it holds that $\mu_Q = \mu_{Q,R}$, the weighted equilibrium measure of B_R. Now pick some $x \in \partial B_R \cap S_{\mu_Q}$. In a small neighborhood V of x, we have $\eta_{Q,B_R}(V) < 0$ while $\mu_{Q,R}(V) > 0$ which contradicts (3.25). □

4 External fields created by pointwise charges

Now, we are concerned with the particular case of external fields as in Corollary 2.8 and Theorem 2.9, created by signed discrete measures ν supported in $\mathbb{R}^{d+1} \setminus \mathbb{R}^d$, of the form

$$Q(x) := \sum_{j=1}^{k} \gamma_j \, U^{\delta_{y_j}}(x) = \sum_{j=1}^{k} \gamma_j \, |x - y_j|^{-s} = \sum_{j=1}^{k} \gamma_j \, \left(|x|^2 + y_{j,d+1}^2\right)^{-s/2}, \tag{4.1}$$

where $y_j = (0, \ldots, 0, y_{j,d+1})$, with $y_{j,d+1} \neq 0$. Without loss of generality, we assume that $y_{j,d+1} > 0$, $j = 1, \ldots, k$. For the charges, we assume that

$$\Gamma := \sum_{j=1}^{k} \gamma_j \leq -1.$$

In the sequel, the case $\Gamma < -1$, where the compactness of the support of μ_Q is guaranteed by Corollary 2.8, will be called admissible (using the analogy with the logarithmic potential setting, see [29, Chapter I]), and the case $\Gamma = -1$ will be called weakly admissible (here, only the existence of μ_Q is ensured a priori by Theorem 2.3).

4.1 Admissible setting

We will focus on the case of a single "attractor", that is $k = 1$, and

$$\gamma = \gamma_1 < -1.$$

We suppose, without loss of generality, that $y = (0; y_{d+1})$, with $y_{d+1} > 0$. Then, the external field acting on the hyperplanar conductor $\mathbb{R}^d$ is given by

$$Q(x) = \frac{\gamma}{(|x|^2 + y_{d+1}^2)^{s/2}}\,,\ x \in \mathbb{R}^d. \tag{4.2}$$

From Theorem 2.3 we know that the equilibrium measure μ_Q has compact support, extending the result in [2, Theorem 2.1] for the case $d = 1$. Since the external field is radial, the support S_{μ_Q} has circular symmetry, but it may be, in principle, a ball, a sphere or several spheres, or even several shells. Observe that Q is convex in the closed ball

$$|x| \leq \frac{y_{d+1}}{\sqrt{s+1}},$$

(but not in the whole $\mathbb{R}^d$); hence we can only assert that the intersection of S_{μ_Q} with that ball is a convex set and, in our case, it is a ball.

We now state our main result in this section.

Theorem 4.1. *Let $d - 2 \leq s < d$. The equilibrium problem in $\mathbb{R}^d$ in the external field* (4.2) *satisfies the following.*

(i) The support S_{μ_Q} of the equilibrium measure μ_Q is a closed ball B_{R_0}.

(ii) The density of μ_Q is given by

$$\mu_Q'(x) = -\gamma H_{y,R_0}(x), \quad H_{y,R_0}(x) = \frac{(2y_{d+1})^{\alpha}}{W(S^d)\omega_d}\left(\frac{1}{|x-y|^{2d-s}} - \frac{\sin(\alpha\pi/2)}{\pi}J(x,y)\right), \tag{4.3}$$

where $W(S^d)$ and ω_d are given in (3.18) *and* (3.17)*, respectively, and*

$$J(x,y) := \int_0^{\infty} \frac{u^{\alpha/2-1}du}{(u+1)\left((R_0^2 - |x|^2)u + R_0^2 + y_{d+1}^2\right)^{d-s/2}}. \tag{4.4}$$

The density of μ_Q vanishes on the boundary of B_{R_0}.

(iii) The radius R_0 is equal to $R_0 = y_{d+1}\sqrt{z}$ where z is the unique positive solution of the equation

$$z^{s/2+1}{}_2F_1\left(1+\frac{s}{2}, 1+\frac{d}{2}, 2+\frac{s}{2}, -z\right) = -\frac{\Gamma(\alpha/2)\Gamma(2+s/2)}{\gamma\Gamma(1+d/2)}. \tag{4.5}$$

In particular, R_0 is a linear non-decreasing function of the distance y_{d+1}.

Remark 4.2. Theorem 4.1 extends for general dimension d and $d-2 \leq s < d$ the results in [2, Theorems 2.1 and 2.3] for $d = 1$ and $0 \leq s < 1$.

Remark 4.3. When $s = d - 2$ (Newton–Coulomb case), with $d \geq 3$, [23, Proposition 2.13], which in turn extends [29, Sec. IV.6], yields the explicit expression of S_{μ_Q} and the density of μ_Q. Observe that in [23] the growth condition (1.3) is required, but it is just to ensure the existence of μ_Q and the compactness of S_{μ_Q}; the proof of that result easily follows without assuming (1.3). Indeed, [23, Proposition 2.13] shows that if $d \geq 3$

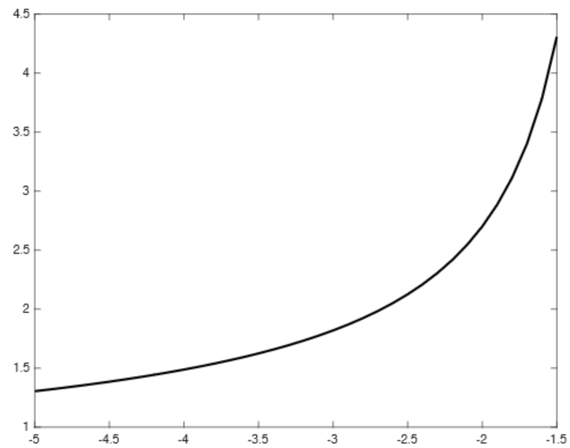

Figure 1: The radius R_0 of the support S_{μ_Q} as a function of the attractive charge γ $(d = 3, s = 2, y_4 = 1)$

and $Q(x) = Q(|x|) = Q(r)$, with $r^{d-1}\, Q'(r)$ being a non-decreasing function on $[0, \infty)$, then $S_{\mu_Q} = \{x \in \mathbb{R}^d \, : \, r_0 \leq |x| \leq R_0\}$, where r is the smallest value of $r > 0$ for which $r^{d-1}\, Q'(r) > 0$ for $r > r_0$, and R_0 is the smallest positive solution of the equation

$$R^{d-1}\, Q'(R) = d - 2. \tag{4.6}$$

Since for our external field (4.2) we have that $r^{d-1}\, Q'(r) = -\gamma s r^d\, (r^2 + y_{d+1}^2)^{-1-s/2} > 0\,$, for each $r > 0$, and

$$(r^{d-1}\, Q'(r))' = -\gamma s d r^{d-1} y_{d+1}^2\, (r^2 + y_{d+1}^2)^{-2-s/2} > 0\,,$$

we get $r_0 = 0$ and, consequently, S_{μ_Q} is a closed ball. To determine the radius R of the ball, (4.6) may be used. Namely, for $d = 3, s = 1, y_4 = 1$ and different values of γ, the following values of R are obtained in Table 1.

γ	-10.0	-2.5	-1.1	-1.001
R	0.524	1.090	3.90	38.73

Table 1: The radius R of S_{μ_Q} for different values of the charge γ $(d = 3, s = 1, y_4 = 1)$

It is easy to check that, as is natural, the higher the size of the attractive charge, the lowest the radius of the ball. Moreover, the result in [23] provides the expression for the density of μ_Q, namely, in our case,

$$d\mu_Q(x) = \frac{-\gamma s r^{d+1}}{(r^2 + y_{d+1}^2)^{2+s/2}}\, dr\, d\sigma_{d-1}(u)\,,\; x = ru\,,\; r = |x|\,,$$

with $d\sigma_{d-1}$ denoting the normalized surface measure of the unit sphere S^{d-1} in $\mathbb{R}^d$.

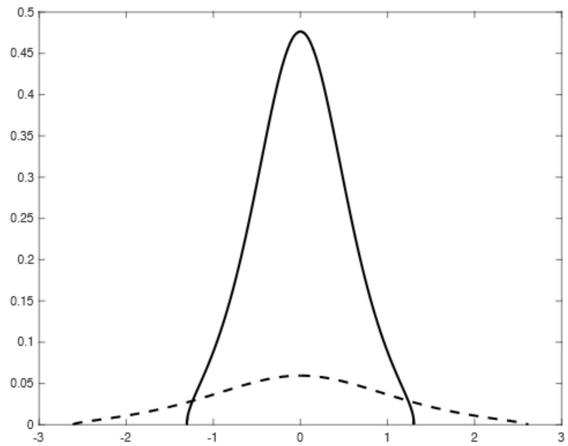

Figure 2: Density of the equilibrium measure as a function of r for $d = 3$, $s = 2$, $\gamma = -5$, $y_4 = 1$ (solid line) and $y_4 = 2$ (dashed line)

Remark 4.4. The results in [23] were partially extended in the recent paper [4, Theorem I.1] for more general values of s, in particular for $d-2 < s < d$; but there the conductor is the unit ball in $\mathbb{R}^d$ and the external field Q is assumed to be convex. Furthermore, in [4, Theorem I.2] the author obtains the expression of the density of the equilibrium measure, provided that its support S_{μ_Q} is a ball in $\mathbb{R}^d$. This last result also applies to our Theorem 4.1, in particular to the expression for the density of the equilibrium measure in (4.3), and one can check the coincidence of (4.3)–(4.4) with [4, Eq. (8)]. However, there is an important difference in the method of proof which makes this part of Theorem 4.1 of interest itself: while we need to prove first that S_{μ_Q} is a ball, and take advantage of this proof to find the density of the equilibrium measure, for the proof of [4, Theorem I.2] the author obtain that density by handling the Fredholm integral equation of the second kind derived from the Frostman's identity in S_{μ_Q} (see (2.3)–(2.4)). Furthermore, though our external field is not convex in the whole $\mathbb{R}^d$ it would be possible to adapt the method of proof of [4, Theorem I.1] to prove that S_{μ_Q} is a ball centered at the origin in our case.

4.2 Weakly admissible setting

If $\Gamma = -1$, assertion (i) of Theorem 2.3 guarantees the existence of the equilibrium measure μ_Q, and condition (2.14) applied to the external field (4.1) provides

$$\sum_{j=1}^{k} \gamma_j y_{j,d+1}^{\alpha} > 0,$$

as a sufficient condition for the compactness of S_{μ_Q}. Thus, we get infinitely many configurations (indeed, a continuum) of charges and distances $\{(\gamma_j, y_{j,d+1}), j = 1, \ldots, k\}$ producing compactly supported equilibrium measures.

Case (i). A single attractor at $y = (0, y_{d+1})$, $y_{d+1} > 0$ and $\gamma = -1$.

In this case, $\mu_Q = \widehat{\delta}_y$, the balayage of δ_y on $\mathbb{R}^d$, and from Lemma 3.11, we know that its support is the entire space $\mathbb{R}^d$ (see Remark 2.10 above).

Let us focus now in the simplest non–trivial case of two points, that is, an "attractor–repellent" pair.

Case (ii). An "attractor–repellent" pair

$$y_1 = (0; y_{1,d+1}), \quad y_2 = (0; y_{2,d+1}) \quad \text{with} \quad \gamma_1 = -1-\gamma, \quad \gamma_2 = \gamma > 0.$$

Now, the external field is given by

$$Q(x) = -\frac{1+\gamma}{(|x|^2 + y_{1,d+1}^2)^{s/2}} + \frac{\gamma}{(|x|^2 + y_{2,d+1}^2)^{s/2}}, \tag{4.7}$$

and, by Lemma 3.11, the density of the signed equilibrium measure on $\mathbb{R}^d$ is given by

$$\eta'_{Q,\mathbb{R}^d}(x) = \frac{2^\alpha}{\omega_d W(S^d)} \left(\frac{(1+\gamma) y_{1,d+1}^\alpha}{|x-y_1|^{2d-s}} - \frac{\gamma y_{2,d+1}^\alpha}{|x-y_2|^{2d-s}} \right). \tag{4.8}$$

We set

$$g = \frac{\gamma}{1+\gamma}, \qquad \rho = \left[g \left(\frac{y_{2,d+1}}{y_{1,d+1}} \right)^\alpha \right]^{2/(2d-s)}, \qquad R = \left(\frac{\rho y_{1,d+1}^2 - y_{2,d+1}^2}{1-\rho} \right)^{1/2}. \tag{4.9}$$

where, in the definition of R, it is assumed that the argument of the square root is nonnegative.

Theorem 4.5. *Let $d-2 \le s < d$. Regarding the equilibrium problem in $\mathbb{R}^d$ in the weakly admissible external field (4.7) three different cases arise according to the value of the quotient $y_{2,d+1}/y_{1,d+1}$:*

(i) If $y_{2,d+1}/y_{1,d+1} \in (0, g^{1/d})$, $S_{\mu_Q} \subset B_R^c$, where B_R^c denotes the complement of the ball B_R, and the radius R is given by (4.9).

(ii) If $y_{2,d+1}/y_{1,d+1} \in [g^{1/d}, g^{-1/\alpha}]$, then $S_{\mu_Q} = \mathbb{R}^d$ and $\mu_Q = \eta_{Q,\mathbb{R}^d}$.

(iii) If $y_{2,d+1}/y_{1,d+1} \in (g^{-1/\alpha}, \infty)$, $S_{\mu_Q} \subset B_R$, with R also given by (4.9).

Remark 4.6. We conjecture that in case (i), S_{μ_Q} is the complement of a ball of radius larger than R, and that in case (iii), S_{μ_Q} is a ball of radius smaller than R. Note that the case where $y_{2,d+1}/y_{1,d+1} = 1$, that is, $y_{2,d+1} = y_{1,d+1}$, is included in part (ii) of the previous theorem. In this case, part of the negative charge cancels with the positive one and it results in a single attractor of charge -1 as in case (i) at the beginning of this section. Furthermore, the conclusion is the same if the attractor and the repellent are placed at the same distance of the hyperplanar conductor $\mathbb{R}^d$ but on different half-hyperplanes of $\mathbb{R}^{d+1}$.

For an illustrative purpose, in Figure 3 the density of the positive part of the signed equilibrium measure of $\mathbb{R}^2$, with $s = 1$, in the presence of the external field created by an attractor of charge -2 placed at $(0,0,1) \in \mathbb{R}^3$ and a repellent of charge 1 located at $(0,0,3)$ is shown; also, in Figure 4 it is plotted that density as a function of $r = \sqrt{x^2+y^2}$. It is clear that the support of the positive part of η_Q is a compact subset of $\mathbb{R}^2$. Thus, Lemma 3.15 implies that S_{μ_Q} is also a compact subset of $\mathbb{R}^2$. In this case, part (iii) in Theorem 4.5 applies and we have that $S_{\mu_Q} \subset B_R$, with $R = 4.978$.

Numerically one can check the validity of the conjectures made after Theorem 4.5. In that connection, Figure 5 shows the radius R_0 of the ball S_{μ_Q} (case (iii) of Theorem 4.5) as a function of the charge γ and as a function of $y_{2,4}$ with $d = 3$, $s = 2$ and $y_{1,4}$=1.

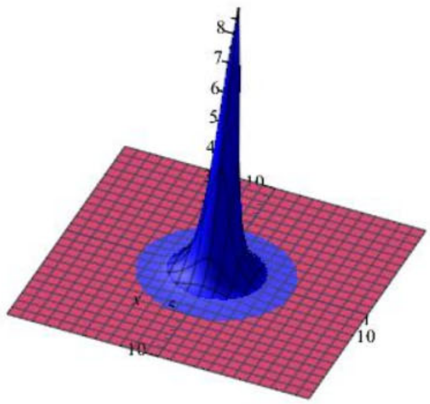

Figure 3: Density of the positive part of the signed equilibrium measure (blue). In red, the plane $\mathbb{R}^2$.

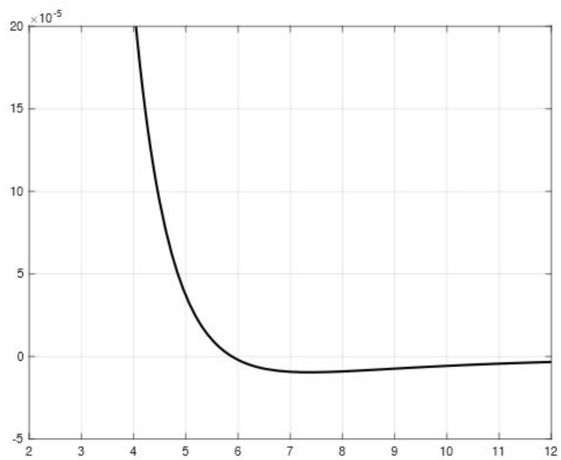

Figure 4: Density of the signed equilibrium measure $\eta_{Q,\mathbb{R}^d}$ as a function of r, near the point where it vanishes.

5 Proofs

5.1 Proofs of Theorems 2.1, 2.3 and Corollary 2.5

We start with the proof of Theorem 2.1.

Proof of Theorem 2.1. **(i)** The inequality $-\infty < W_Q(\Sigma)$ holds because the unweighted energy is positive and Q is lower-bounded on $\mathbb{R}^d$. Moreover, since $\{x \in \Sigma,\ Q(x) < \infty\}$

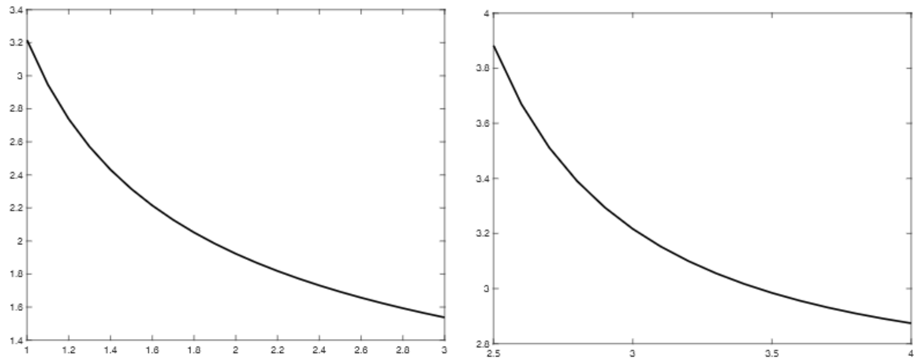

Figure 5: The radius R_0 of the support S_{μ_Q} as a function of γ (left) and as a function of $y_{2,4}$ (right) $(d = 3, s = 2, y_{1,4} = 1)$

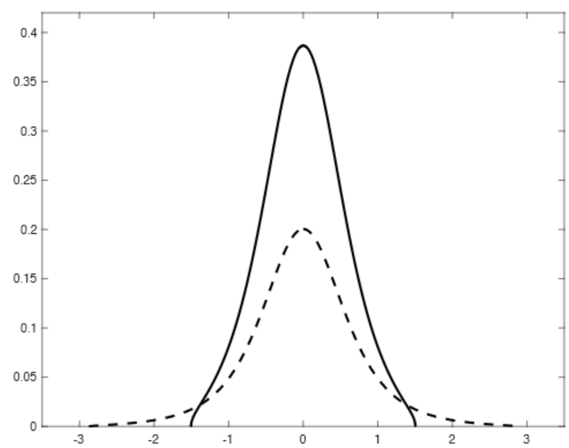

Figure 6: Density of the equilibrium measure as a function of r for $d = 3$, $s = 2$, $y_{1,4} = 1$, $y_{2,4} = 4$, $\gamma = 3$ (solid line) and $\gamma = 1$ (dashed line)

has positive capacity, there exists an $n \in \mathbb{N}$ such that $\Sigma_n := \{x \in \Sigma,\ Q(x) < n\}$ has the same property. Hence, there exists $\mu_n \in \mathcal{P}(\Sigma_n)$ such that $I_Q(\mu_n) < \infty$.
(ii) The uniqueness of a minimizing measure can be proved as usual, see e.g. [29, Theorem I.1.3], based on the fact that for two measures μ and ν, $I(\mu - \nu) = 0$ if and only if $\mu = \nu$, see [22, Theorem 1.15].
(iii) The Frostman inequalities can be proved by following the arguments of [29, Theorem I.1.3] for weighted logarithmic potentials.
(iv) For the characterization of the equilibrium measure, we follow the proof of [6, Proposition 2.6] : assume μ satisfies (2.5)-(2.6). Pick any $\nu \in \mathcal{P}(\Sigma)$, with $I_Q(\nu) < \infty$. Writing $\nu = \mu + (\nu - \mu)$, we have

$$I_Q(\nu) = I_Q(\mu) + I(\nu - \mu) + 2 \int_\Sigma (U^\mu + Q)(d\nu - d\mu), \tag{5.1}$$

where the right-hand side is well-defined since Q is lower-bounded on $\mathbb{R}^d$, the energies and weighted energies of μ and ν are finite, and the mixed energy satisfies $I(\mu, \nu) \leq I(\mu)I(\nu)$, see [22, p. 82]. Making use of the Frostman inequalities for μ, we obtain

$$\int (U^\mu + Q)(d\nu - d\mu) \geq F \int d\nu - F \int d\mu = F(\nu - \mu)(\mathbb{R}^d) = 0. \tag{5.2}$$

Moreover, by [22, Theorem 1.15], $I(\nu - \mu) \geq 0$. Hence, from (5.1) we derive that $I_Q(\nu) \geq I_Q(\mu)$. We conclude that μ is a minimizing measure, and, finally, $F = F_Q$ by uniqueness. For the last assertion, one may restrict ν to be in $\mathcal{P}(S_{\mu_Q})$. Then, (5.2) still holds and the end of the argument remains the same. □

We now proceed with preparations for the proof of Theorem 2.3. We will make use of the Kelvin transform T as defined in Section 3.3. Here, we choose T of radius $\sqrt{2}$ and center $y = (0, 0, \ldots, 1)$ in $\mathbb{R}^{d+1}$, which sends $\mathbb{R}^d$ onto the d-dimensional sphere S in $\mathbb{R}^{d+1}$, centered at 0 of radius 1. We keep the notation $x^* = T(x)$ so that, in particular,

$$|x - y||x^* - y| = 2. \tag{5.3}$$

Note that $\Sigma^* = T(\Sigma)$ is a closed subset of S, hence a compact set in $\mathbb{R}^{d+1}$.

We first check how the minimization problem (2.2) and the Frostman inequalities (2.3)-(2.4) translate when we apply the Kelvin transform. We know from (3.14)-(3.15) that for Riesz potentials and energies, we have the following relations,

$$U^{\mu^*}(x^*) = 2^{-s/2}|x - y|^s U^\mu(x), \quad I_s(\mu^*) = I_s(\mu),$$

where

$$d\mu^*(t^*) = 2^{s/2}\frac{d\mu(t)}{|t - y|^s} = 2^{-s/2}|t^* - y|^s d\mu(t).$$

Note that, by (3.16), the condition $\mu \in \mathcal{P}(\Sigma)$ translates into $2^{s/2}U^{\mu^*}(y) = 1$.

Then, with $Q^*(x) = Q(x^*)$, the minimization problem (2.2) becomes

$$\min\left(I_s(\mu^*) + 2\int \frac{2^{s/2}Q^*(t)}{|t - y|^s} d\mu^*(t)\right),$$

where the minimum is taken over all measures μ^* supported on Σ^* such that $2^{s/2}U^{\mu^*}(y) = 1$. The Frostman inequalities (2.3)-(2.4) on Σ become the following ones on Σ^*,

$$U^{\mu_Q^*}(x) + 2^{s/2}\frac{Q^*(x) - F}{|x - y|^s} \geq 0, \quad \text{q.e. on } \Sigma^*, \tag{5.4}$$

$$U^{\mu_Q^*}(x) + 2^{s/2}\frac{Q^*(x) - F}{|x - y|^s} \leq 0, \quad x \in S^*_{\mu_Q} = S_{\mu_Q^*}. \tag{5.5}$$

As a preliminary result, we show the existence, under some assumptions on Q^*, of a measure μ^* on Σ^* satisfying $2^{s/2}U^{\mu^*}(y) = 1$ and the above Frostman inequalities.

Lemma 5.1. *Assume*

$$Q^*(x) \leq Q^*(y) - 2^{-s}|x - y|^s, \quad \textit{in a neighborhood of } y, \tag{5.6}$$

or

$$\lim_{x \to y,\ x \in \Sigma^* \setminus P} 2^s \frac{Q^*(x) - Q^*(y)}{|x - y|^s} \leq -1, \tag{5.7}$$

where P is thin at y. Then there exists a constant $F \leq Q^*(y)$ and a measure μ^* on the compact set Σ^* with $2^{s/2}U^{\mu^*}(y) = 1$ such that

$$U^{\mu^*}(x) + 2^{s/2}\frac{Q^*(x) - F}{|x-y|^s} \geq 0, \quad q.e. \ on \ \Sigma^*, \tag{5.8}$$

$$U^{\mu^*}(x) + 2^{s/2}\frac{Q^*(x) - F}{|x-y|^s} \leq 0, \quad x \in S_{\mu^*}. \tag{5.9}$$

Remark 5.2. Recall that (5.8)-(5.9) on Σ^* correspond, via the Kelvin transform, to (2.3)-(2.4) on Σ.

Proof of Lemma 5.1. We will obtain the measure μ^* by considering the weak-* limit of a sequence of measures μ_n^* solving the problem on the set Σ_n^*, obtained from Σ^* by removing its intersection with the open ball centered at y with radius $1/n$. The fact that there exists a measure μ_n^* on Σ_n^* with $2^{s/2}U^{\mu_n^*}(y) = 1$ satisfying for some F_n the Frostman inequalities

$$U^{\mu_n^*}(x) + 2^{s/2}\frac{Q^*(x) - F_n}{|x-y|^s} \geq 0, \quad \text{q.e. on } \Sigma_n^*, \tag{5.10}$$

$$U^{\mu_n^*}(x) + 2^{s/2}\frac{Q^*(x) - F_n}{|x-y|^s} \leq 0, \quad x \in S_{\mu_n^*}, \tag{5.11}$$

just follows by considering the corresponding problem, via the Kelvin transform, on the compact set $\Sigma_n = T(\Sigma_n^*)$ (this just uses the fact that $Q(x) = Q^*(x^*)$ is lower-semicontinuous on Σ_n). Next, since

$$\frac{\|\mu_n^*\|}{\text{diam}(\Sigma^*)^s} \leq U^{\mu_n^*}(y) = 2^{-s/2},$$

the masses of the μ_n^* are uniformly bounded, and we may consider a subsequence (still denoted by μ_n^*) which converges weak-* to some measure μ^*. The F_n's are lower bounded since, from (5.11) and the fact that a Riesz potential is positive, we have, for $x \in S_{\mu_n^*}$,

$$-\infty < \inf_{x\in\Sigma^*} Q^*(x) \leq Q^*(x) \leq F_n \,.$$

They are also upper bounded. Indeed, we know that U^{μ^*} is finite q.e. and, by the lower envelope theorem,

$$\liminf_n U^{\mu_n^*}(x) = U^{\mu^*}(x) \quad \text{q.e. } x \in \mathbb{R}^d.$$

Thus, in view of (5.10), and possibly by considering a subsequence, we can find some $x_0 \neq y \in \Sigma^*$ such that

$$\forall n, \quad \frac{F_n}{|x_0 - y|^s} \leq 2^{-s/2}U^{\mu^*}(x_0) + \frac{Q^*(x_0)}{|x_0-y|^s} + 1 < \infty,$$

(here we also use that $\{x \in \Sigma^*, \ Q^*(x) < \infty\}$ is of positive capacity). Consequently, we may again consider a subsequence so that F_n tends to some constant F as n goes large. Taking the limit in (5.10) and making use of the lower envelope theorem, we get

$$U^{\mu^*}(x) + 2^{s/2}\frac{Q^*(x) - F}{|x-y|^s} \geq 0, \quad \text{q.e. on } \Sigma^*.$$

For the second inequality (5.11), recall from [22, Eq.(0.1.10)] that

$$S_{\mu^*} \subset \bigcap_{N=1}^{\infty} \overline{\bigcup_{n=N}^{\infty} S_{\mu_n^*}}.$$

Hence, for $x \in S_{\mu^*}$, we can find a sequence $x_n \to x$ with $x_n \in S_{\mu_n^*}$. Making use of the principle of descent, we get

$$U^{\mu^*}(x) + 2^{s/2}\frac{Q^*(x) - F}{|x-y|^s} \le \liminf_n U^{\mu_n^*}(x_n) + \liminf_n 2^{s/2}\frac{Q^*(x_n) - F_n}{|x_n - y|^s} \le 0.$$

Next, we show that $F \le Q^*(y)$. For some set A_0 thin at y, we have

$$\lim_{x\to y,\ x\notin A_0} U^{\mu^*}(x) = U^{\mu^*}(y) \le \liminf_n U^{\mu_n^*}(y) = 2^{-s/2},$$

where we refer to [24, Theorem 5.1, p. 79] for the equality and to the principle of descent for the inequality. Now, rewriting (5.8) as

$$U^{\mu^*}(x) + 2^{s/2}\frac{Q^*(x) - Q^*(y)}{|x-y|^s} + 2^{s/2}\frac{Q^*(y) - F}{|x-y|^s} \ge 0, \quad \text{q.e. on } \Sigma^*,$$

we get, together with (5.6) or (5.7), that the sum of the first two terms is less than or equal to 0 q.e. near y. The inequality $Q^*(y) \ge F$ follows.

Finally we show that $2^{s/2}U^{\mu^*}(y) = 1$. Assume to the contrary that $2^{s/2}U^{\mu^*}(y) < 1$. Then, the previous reasoning shows that $Q^*(y) > F$. But (5.9) entails that

$$\frac{Q^*(x) - F}{|x-y|^s} \le 0, \quad x \in S_{\mu^*},$$

which shows that $y \notin S_{\mu^*}$. Hence U^{μ^*} is continuous at y and $2^{s/2}U^{\mu^*}(y) = 1$, a contradiction. □

Proof of Theorem 2.3. **(i)** The Riesz energy problem on Σ and the associated Frostman inequalities are equivalent, via the Kelvin transform, to the Frostman inequalities on Σ^*, considered in Lemma 5.1. Moreover, from (5.3), we see that assumptions (5.6) or (5.7) are respectively equivalent, under the Kelvin transform, to

$$Q(x) \le Q(\infty) - |x-y|^{-s} = Q(\infty) - (|x|^2+1)^{-s/2}, \quad \text{in a neighborhood of } \infty,$$

or

$$\lim_{|x|\to\infty,\ x\in\Sigma\setminus P} |x-y|^s(Q(x) - Q(\infty)) = \lim_{|x|\to\infty,\ x\in\Sigma\setminus P} (|x|^2+1)^{s/2}(Q(x) - Q(\infty)) \le -1,$$

where P is thin at ∞. Since these conditions are slightly weaker than (2.8) and (2.9) , i) follows from Lemma 5.1 and assertion iv) in Theorem 2.1.

(ii) Assume that μ_Q satisfies (2.3)-(2.4) and has unbounded support, that is $\infty \in S_{\mu_Q}$. As $\lim_{|x|\to\infty} U^{\mu_Q}(x) = 0$ (up to a set thin at infinity, see Lemma 2.7), we already get from (2.3)-(2.4) evaluated at infinity that $Q(\infty)$ is finite and equals the Robin constant F_Q. Now, multiplying both inequalities (2.3)-(2.4) by $|x|^s$ and making use of the fact that, outside of a set thin at infinity, $\lim_{|x|\to\infty}|x|^s U^{\mu_Q}(x) = 1$, see Lemma 2.7, we get (2.10).

(iii) Assume now that $d-2 \le s < d$ and that a probability measure μ_Q exists that satisfies the Frostman inequalities on Σ, where we substract $Q(\infty)$ on both sides:

$$U^{\mu_Q}(x) + Q(x) - Q(\infty) \ge F_Q - Q(\infty), \quad \text{q.e. on } \Sigma,$$
$$U^{\mu_Q}(x) + Q(x) - Q(\infty) \le F_Q - Q(\infty), \quad x \in S_{\mu_Q}.$$

From the first inequality evaluated in a neighborhood of infinity and the fact that both U^{μ_Q} and $Q(x) - Q(\infty)$ tend to zero at infinity (possibly outside a set thin at infinity for U^{μ_Q}, see Lemma 2.7), we derive that $0 \ge F_Q - Q(\infty)$. Then, from the second inequality, the assumption (2.11) evaluated near infinity and the fact that the potential $U^{\mu_Q}(x)$ behaves like $1/|x|^s$ near infinity, possibly outside a set thin at infinity, recall (2.13), we derive that S_{μ_Q} must be bounded. Thus, comparing external fields, we have, for some $\epsilon > 0$,

$$-\frac{1}{|x|^s} + \epsilon < -\frac{c}{|x|^s} \le Q(x) - Q(\infty), \quad x \in S_{\mu_Q}.$$

In the case of the external field $-1/|x|^s$ we know from assertion i) that a minimizing measure exists, and Corollary 2.5 shows (see below) that the corresponding Robin constant is zero. Thus, together with the property of strict monotonicity of Proposition 3.10, we obtain that $F_Q - Q(\infty) > 0$, a contradiction. Hence, no probability measure on Σ satisfies Frostman inequalities and, by the direct implication in Theorem 2.1, no measure minimizing the weighted energy I_Q exists on Σ. □

Proof of Corollary 2.5. Corollary 2.5 is a consequence of Theorem 2.3, except for the fact that the proof of assertion iii) of the theorem relies on the property that $F_Q = 0$ when $Q(x) = -1/|x|^s$. So, in that case, we give two direct proofs, of possible independent interest, of the existence of a minimizing measure μ_Q and the fact that $F_Q = 0$:

– *1st proof, on Σ using balayage* (recall that $d-2 \le s < d$): inequalities (2.3)-(2.4) become

$$U^{\mu_Q}(x) - U^{\delta_0}(x) \ge F_Q, \quad \text{q.e. on } \Sigma,$$
$$U^{\mu_Q}(x) - U^{\delta_0}(x) \le F_Q, \quad x \in S_{\mu_Q},$$

which are satisfied when $\mu_Q = \widehat{\delta}_0$, the balayage of δ_0 on Σ (see Section 3.2). Note that, by [16, Theorem 3.22] and assumption A2, we know that there is no mass loss when sweeping out δ_0 onto Σ, thus $\|\widehat{\delta}_0\| = 1$, as required. Note also that we get $F_Q = 0$ for the value of the Robin constant.

– *2nd proof, using the Kelvin transform of center 0 and radius 1* : note first, that $|x||x^*| = 1$ and for a general c, (5.4)-(5.5) translate into

$$U^{\mu_Q^*}(x) - \frac{F_Q}{|x|^s} \ge c, \quad \text{q.e. on } \Sigma^*,$$
$$U^{\mu_Q^*}(x) - \frac{F_Q}{|x|^s} \le c, \quad x \in S_{\mu_Q^*},$$

with the condition $U^{\mu_Q^*}(0) = 1$. Assume $F_Q = 0$. Then the above inequalities characterize, up to a constant, the unweighted equilibrium measure for the compact set Σ^*. Then, we know that $U^{\mu_Q^*}(x) = c$ q.e. on Σ and, in particular, because of assumptions A1-A2, equality holds at 0, which induces $c = U^{\mu_Q^*}(0) = 1$. In particular, we see that the unweighted energy problem on Σ^* corresponds to the energy problem on Σ with external field $-1/|x|^s$ and, as a consequence, we deduce again the existence of μ_Q on Σ when $c = 1$. $\square$

Remark 5.3. We just saw that the energy problem with $Q(x) = -1/|x|^s$ on Σ corresponds to the unweighted energy problem on the compact set Σ^*. When $s = d-2$, it is known that the support of the equilibrium measure is included in the outer boundary of Σ^*, and thus the support of μ_Q is included in the boundary of Σ, which may be a bounded set.

5.2 Proof of Theorem 2.9

From Lemma 3.11, the weak balayage of ν is given by (3.23)

$$d\nu^w = \frac{2^\alpha}{W(S^d)\omega_d}\left(\int \frac{|y_{d+1}|^\alpha}{|x-y|^{2d-s}}\,d\nu(y)\right)dx, \quad x \in \mathbb{R}^d.$$

Since $\nu(\mathbb{R}^{d+1}) = -1$, the signed equilibrium measure η_Q exists and agrees with $-\nu^w$, i.e.

$$d\eta_Q(x) = -\frac{2^\alpha}{W(S^d)\omega_d}\left(\int \frac{|y_{d+1}|^\alpha}{|x-y|^{2d-s}}\,d\nu(y)\right)dx\,, \quad x \in \mathbb{R}^d,$$

which implies that, for x large, its density behaves like

$$-\frac{2^\alpha}{W(S^d)\omega_d|x|^{2d-s}}\int |y_{d+1}|^\alpha\,d\nu(y).$$

From condition (2.14), this density is negative which guarantees the compactness of $S_{\eta_Q^+}$, and then Lemma 3.15 ensures the compactness of S_{μ_Q}. $\square$

5.3 Proof of Theorems 4.1 and 4.5

In this subsection we assume that $d-2 \le s < d$, therefore the weak balayage from Lemma 3.11 is regular balayage. The next lemma we study the balayage $Bal(\delta_y, B_R)$ of δ_y onto the ball $B_R \subset \mathbb{R}^d$ of radius R. By symmetry, its density $Bal'(\delta_y, B_R)(x)$ is a radial function of x. As we are about to see, it behaves like $(R^2 - |x|^2)^{-\alpha/2}$ near the boundary of B_R. Hence, it will be convenient to introduce the following notation,

$$\Lambda_R(x) = (R^2 - |x|^2)^{\alpha/2} Bal'(\delta_y, B_R)(x), \qquad |x| < R,$$

$$\Lambda_R^* = \lim_{|x|\to R_-}(R^2 - |x|^2)^{\alpha/2} Bal'(\delta_y, B_R)(x). \tag{5.12}$$

Lemma 5.4. *Let $y = (0; y_{d+1})$ with $y_{d+1} > 0$. The following holds true:*
(i) The density $Bal'(\delta_y, B_R)(x)$, $|x| < R$, is given by

$$Bal'(\delta_y, B_R)(x) = \frac{(2y_{d+1})^\alpha}{W(S^d)\omega_d}\left(\frac{1}{(|x|^2 + y_{d+1}^2)^{d-s/2}} + \frac{\sin(\alpha\frac{\pi}{2})I(x)}{\pi(R^2 - |x|^2)^{\alpha/2}}\right), \tag{5.13}$$

where

$$I(x) = \int_0^\infty \frac{v^{\alpha/2}\,dv}{(v + R^2 + y_{d+1}^2)^{d-s/2}(v + R^2 - |x|^2)}.$$

(ii) Let m_R be the mass of $Bal(\delta_y, B_R)$. Then

$$m_R = \frac{U^{\omega_R}(y)}{W(B_R)} \in (0,1). \tag{5.14}$$

The mass m_R is a non-decreasing function of R and a decreasing function of y_{d+1}. Moreover

$$\lim_{R\to 0} m_R = 0, \qquad \lim_{R\to\infty} m_R = 1. \tag{5.15}$$

The second limit can be made more precise, namely, as $R \to \infty$,

$$m_R = 1 - \frac{2}{\alpha B(s/2, \alpha/2)} \left(\frac{y_{d+1}}{R}\right)^{\alpha} + o\left(\left(\frac{y_{d+1}}{R}\right)^{\alpha}\right) \quad \text{as } R \to \infty. \tag{5.16}$$

(iii) With the notation introduced in (5.12), we have

$$\Lambda_R^* = \frac{y_{d+1}^{\alpha} K_{s,d}^{(1)}}{(R^2 + y_{d+1}^2)^{d/2}}, \qquad K_{s,d}^{(1)} = \frac{2^{\alpha}\sin(\alpha\frac{\pi}{2}) B(d/2, \alpha/2)}{\pi\omega_d W(S^d)}. \tag{5.17}$$

(iv) For $x \in B_R$, we have

$$\Lambda_R(x) - \Lambda_R^* = \frac{(2y_{d+1})^{\alpha}(R^2 - |x|^2)}{W(S^d)\omega_d}$$

$$\times \left(\frac{(R^2 - |x|^2)^{\alpha/2-1}}{(|x|^2 + y_{d+1}^2)^{d-s/2}} - \frac{\sin(\alpha\frac{\pi}{2})}{\pi} \int_0^{\infty} \frac{v^{\alpha/2-1} dv}{(v + R^2 + y_{d+1}^2)^{d-s/2}(v + R^2 - |x|^2)} \right) \geq 0. \tag{5.18}$$

Proof. **(i)** From Lemma 3.11, the *superposition principle*, [22, Eq.(4.5.6)], and the expression for the Poisson kernel of a ball, [22, Eq.(1.6.11), p. 121], we derive that the density of the balayage of δ_y onto B_R is given by

$$Bal'(\delta_y, B_R)(x) = \frac{(2y_{d+1})^{\alpha}}{W(S^d)\,\omega_d} \left(\frac{1}{(|x|^2 + y_{d+1}^2)^{d-s/2}} + \frac{\Gamma(d/2)\sin(\alpha\pi/2) A}{\pi^{d/2+1}(R^2 - |x|^2)^{\alpha/2}} \right), \tag{5.19}$$

where

$$A = \int_{|t|\geq R} \frac{(|t|^2 - R^2)^{\alpha/2}}{(|t|^2 + y_{d+1}^2)^{d-s/2}} \, \frac{dt}{|x-t|^d}.$$

Using polar coordinates in $\mathbb{R}^d$, A may be rewritten as

$$2\pi \prod_{k=1}^{d-3} \int_0^{\pi} \sin^k\theta\, d\theta \int_R^{\infty} \int_0^{\pi} \frac{(\rho^2 - R^2)^{\alpha/2}\,\rho^{d-1}\sin^{d-2}\theta\, d\rho\, d\theta}{(\rho^2 + y_{d+1}^2)^{d-s/2}\,(\rho^2 + |x|^2 - 2\rho|x|\cos\theta)^{d/2}}.$$

Setting $\rho = \rho_1|x|$, we have

$$\int_0^{\pi} \frac{\sin^{d-2}\theta\, d\theta}{(\rho^2 + |x|^2 - 2\rho|x|\cos\theta)^{d/2}} = \frac{1}{|x|^d} \int_0^{\pi} \frac{\sin^{d-2}\theta\, d\theta}{(\rho_1^2 + 1 - 2\rho_1\cos\theta)^{d/2}}.$$

From [22, p. 400], the above expression equals

$$\frac{1}{\rho^{d-2}(\rho^2 - |x|^2)} \int_0^{\pi} \sin^{d-2}\theta d\theta,$$

which allows us to write A as

$$A = 2\pi \prod_{k=1}^{d-2} \int_0^{\pi} \sin^k\theta\, d\theta \int_R^{\infty} \frac{(\rho^2 - R^2)^{\alpha/2}\,\rho\, d\rho}{(\rho^2 + y_{d+1}^2)^{d-s/2}\,(\rho^2 - |x|^2)}$$

$$= \frac{\pi^{d/2}}{\Gamma(d/2)} \int_0^{\infty} \frac{v^{\alpha/2}\, dv}{(v + R^2 + y_{d+1}^2)^{d-s/2}(v + R^2 - |x|^2)}.$$

This, along with (5.19), complete the proof of (5.13).
(ii) Formula (5.14) follows from [22, Eq.(4.5.6')]. From the second Frostman inequality and the domination principle, $U^{\omega_R}(y) \leq W(B_R)$ and thus, $m_R \leq 1$. Since the mass of a measure can only decrease when performing a balayage, and since, for $R < R'$, the balayage onto B_R can be obtained by the balayage onto $B_{R'}$ followed by a balayage onto B_R, we derive that m_R increases with R. From (5.14) it is immediate that m_R is a decreasing function of y_{d+1}.

When $R \to 0$, $U^{\omega_R}(y) \to |y|^{-s}$ while $W(B_R) \to \infty$ which implies the first limit in (5.15). The behavior (5.16) of m_R at infinity is derived from (5.14), (3.21), and an expansion of (3.20) when $R \to \infty$.
(iii) The behavior of the density of the balayage near the boundary of B_R simply follows from (5.13) and the easily checked equality

$$I(R) = (R^2 + y_{d+1}^2)^{-d/2} B(d/2, \alpha/2).$$

(iv) The expression (5.18) for $\Lambda_R(x) - \Lambda_R^*$ follows from the definition of $\Lambda_R(x)$, Λ_R^*, and the expression (5.13) for the balayage. The inequality is equivalent to

$$\frac{(R^2 - |x|^2)^{\alpha/2-1}}{(|x|^2 + y_{d+1}^2)^{d-s/2}} \geq \frac{1}{\pi}\sin(\alpha\frac{\pi}{2}) \int_0^{\infty} \frac{v^{\alpha/2-1}\, dv}{(v + R^2 + y_{d+1}^2)^{d-s/2}(v + R^2 - |x|^2)}.$$

But the last integral is smaller than

$$\frac{1}{(R^2 + y_{d+1}^2)^{d-s/2}} \int_0^{\infty} \frac{v^{\alpha/2-1}\, dv}{(v + R^2 - |x|^2)} = \frac{(R^2 - |x|^2)^{\alpha/2-1}}{(R^2 + y_{d+1}^2)^{d-s/2}} \int_0^{\infty} \frac{v^{\alpha/2-1}\, dv}{(v+1)}$$
$$= \frac{(R^2 - |x|^2)^{\alpha/2-1}}{(R^2 + y_{d+1}^2)^{d-s/2}} \frac{\pi}{\sin(\alpha\frac{\pi}{2})},$$

which proves the inequality since $(R^2 + y_{d+1}^2)^{d-s/2} \geq (|x|^2 + y_{d+1}^2)^{d-s/2}$. □

Proof of Theorem 4.1 (i) The proof is based on the signed equilibrium measures, as explained in Section 3.2. Since the support S_{μ_Q} of the equilibrium measure μ_Q is a compact set, it is a subset of a ball B_R, for R large enough. Pick such an $R > 0$. The signed equilibrium measure $\eta_{Q,R}$ of the ball B_R exists and can be expressed in terms of the balayage $Bal(\delta_y, B_R)$ and the equilibrium measure ω_R,

$$\eta_{Q,R} = -\gamma Bal(\delta_y, B_R) + (1 + \gamma m_R)\omega_R. \tag{5.20}$$

Indeed, the right-hand side satisfies (3.24) and has total mass 1. In view of (3.19), (5.17) and (5.20), the density of $\eta_{Q,R}$ near the sphere $|x| = R$ satisfies

$$H(R) := \lim_{|x|\to R}(R^2 - |x|^2)^{\alpha/2}\eta'_{Q,R}(x) = -\gamma\Lambda_R^* + (1 + \gamma m_R)c_R$$

$$= -\frac{\gamma y_{d+1}^{\alpha} K_{s,d}^{(1)}}{(R^2 + y_{d+1}^2)^{d/2}} + \frac{\Gamma(1 + s/2)}{\pi^{d/2}\Gamma(1 - \alpha/2)} \frac{(1 + \gamma m_R)}{R^s}.$$

Observe that

$$\lim_{R\to 0} H(R) = +\infty, \qquad \lim_{R\to\infty} H(R) = 0_-,$$

where, for the first limit, we note that $m_R \to 0$ as $R \to 0$, and, for the second limit, we note that, as $R \to \infty$, the second term is dominant (since $s < d$) and that $1+\gamma m_R \to 1+\gamma < 0$. Hence, by continuity, there exists values of R for which $H(R) = 0$. We denote by R_0 the largest such value (which exists).

We next check that η_{Q,R_0} is nonnegative on the closed ball B_{R_0}. Indeed, from (5.20), and the definition of R_0, we have

$$-\gamma\Lambda^*_{R_0} + (1+\gamma m_{R_0})c_{R_0} = 0. \tag{5.21}$$

Hence, for $|x| \le R_0$,

$$(R^2 - |x|^2)^{\alpha/2}\eta'_{Q,R}(x) = -\gamma\Lambda_{R_0}(x) + (1+\gamma m_{R_0})c_{R_0} = -\gamma(\Lambda_{R_0}(x) - \Lambda^*_{R_0}) \ge 0,$$

where, for the last inequality, we use (5.18).

Now, for $R > R_0$, we know that $\eta'_{Q,R}(R) < 0$. Hence by iii) of Lemma 3.15, S_{μ_Q} is included in B_{R_0}, and by ii) of that same lemma, $\mu_Q = \eta_{Q,R_0}$. In particular, the support of μ_Q is the ball of radius R_0.

(ii) By combining (5.20) with the explicit expressions (3.19) and (5.13) of ω_{R_0} and $Bal(\delta_y, B_{R_0})$, together with (5.21), we may derive that

$$\mu'_Q(x) = -\gamma\frac{(2y_{d+1})^\alpha}{W(S^d)\omega_d}\left(\frac{1}{(|x|^2+y_{d+1}^2)^{d-s/2}} - \frac{\sin(\alpha\pi/2)(I(x)-I(R_0))}{\pi(R_0^2-|x|^2)^{\alpha/2}}\right).$$

Performing the change of variable $v = (R_0^2 - |x|^2)u$ in $I(x)$ and $I(R_0)$ leads to (4.3). Finally, when $|x| = R_0$, the second factor of (4.3) reduces to

$$\frac{1}{(R_0^2+y_{d+1}^2)^{d-s/2}}\left(1 - \frac{\sin(\alpha\pi/2)}{\pi}\int_0^\infty (u+1)^{-1}u^{\alpha/2-1}du\right) = 0,$$

which shows the vanishing of μ'_Q on the boundary of the ball.

(iii) Recall that the radius R_0 of the ball S_{μ_Q} is a solution of (5.21), that is

$$-\gamma\Lambda^*_{R_0} + (1+\gamma m_{R_0})c_{R_0} = 0.$$

Plugging the explicit expressions of $\Lambda^*_{R_0}$, m_{R_0} and c_{R_0} given respectively by (5.17), (5.14), and (3.19), and making also use of (3.20) and (3.21), one may check, after some computations, that the following equation is obtained,

$$z^{s/2}\left({}_2F_1\left(\frac{s}{2},\frac{d}{2},1+\frac{s}{2},-z\right) - (1+z)^{-d/2}\right) = -\frac{\Gamma(\alpha/2)\Gamma(1+s/2)}{\gamma\Gamma(d/2)}, \tag{5.22}$$

where $z = (R_0/y_{d+1})^2$. The left-hand side, that we denote by $G(z)$, is equal to

$$\begin{aligned} G(z) &= z^{s/2}\left({}_2F_1\left(\frac{s}{2},\frac{d}{2},1+\frac{s}{2},-z\right) - {}_2F_1\left(1+\frac{s}{2},\frac{d}{2},1+\frac{s}{2},-z\right)\right) \\ &= \frac{\Gamma(1+s/2)}{\Gamma(d/2)\Gamma(1-\alpha/2)}z^{s/2}\int_0^1 x^{d/2-1}(1-x)^{-\alpha/2}(1+zx)^{-s/2-1}(zx)dx \\ &= \frac{d}{s+2}z^{s/2+1}{}_2F_1\left(1+\frac{s}{2},1+\frac{d}{2},2+\frac{s}{2},-z\right). \end{aligned}$$

This gives (4.5). Finally, we show that a solution to equation (5.22) exists and is unique. First, we notice that $G(z)$ is a non-decreasing function of $z \in [0, \infty)$. Indeed,

$$G(z) = \frac{\Gamma(1+s/2)}{\Gamma(d/2)\Gamma(1-\alpha/2)} \int_0^1 x^{d/2}(1-x)^{-\alpha/2}(x+1/z)^{-s/2-1}dx, \tag{5.23}$$

and it is clear that, for any $x \in (0,1)$, $(x+1/z)^{-s/2-1}$ is a non-decreasing function of z. Second, $G(z)$ satisfies $\lim_{z\to 0} G(z) = 0$ and

$$\lim_{z\to\infty} G(z) = \frac{\Gamma(\alpha/2)\Gamma(1+s/2)}{\Gamma(d/2)},$$

where the above limit follows, e.g., from (5.23). Third the right hand-side of (5.22) is positive and always smaller than the above limit because $\gamma < -1$. Actually it tends to that limit as γ tends to -1 from the left. □

Proof of Theorem 4.5 The density (4.8) of the signed equilibrium measure $\eta_{Q,\mathbb{R}^d}$ vanishes when

$$(1+\gamma)y_{1,d+1}^\alpha(|x|^2 + y_{2,d+1}^2)^{d-s/2} - \gamma y_{2,d+1}^\alpha(|x|^2 + y_{1,d+1}^2)^{d-s/2} = 0.$$

Solving for $|x|^2$, we get

$$|x|^2 = \frac{\rho y_{1,d+1}^2 - y_{2,d+1}^2}{1-\rho} = R^2,$$

where ρ and R are defined in (4.9). Hence, $\eta_{Q,\mathbb{R}^d}$ is a positive measure if and only if

$$\frac{\rho y_{1,d+1}^2 - y_{2,d+1}^2}{1-\rho} \le 0 \qquad \text{and} \qquad (1+\gamma)y_{1,d+1}^\alpha - \gamma y_{2,d+1}^\alpha \ge 0,$$

where the second inequality expresses the fact that the density of $\eta_{Q,\mathbb{R}^d}$ is positive at infinity. It is then straightforward to see that the above inequalities are equivalent to $y_{2,d+1}/y_{1,d+1} \in [g^{1/d}, g^{-1/\alpha}]$. Finally, if the first inequality is not met, $\eta_{Q,\mathbb{R}^d}$ is a true signed measure and one checks that

$$\begin{aligned}
&\eta_{Q,\mathbb{R}^d}(x) < 0,\ |x| \in [0,R), \quad \eta_{Q,\mathbb{R}^d}(x) \ge 0,\ |x| \in [R,\infty), \quad \text{when } y_{2,d+1}/y_{1,d+1} \in [0, g^{1/d}],\\
&\eta_{Q,\mathbb{R}^d}(x) \ge 0,\ |x| \in [0,R], \quad \eta_{Q,\mathbb{R}^d}(x) < 0,\ |x| \in (R,\infty), \quad \text{when } y_{2,d+1}/y_{1,d+1} \in [g^{-1/\alpha}, \infty).
\end{aligned}$$

Together with (i) of Lemma 3.15, it implies the statements (i) and (iii) and finishes the proof of the theorem. □

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