On Two Problems Concerning Universal Bounds for Codes

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***Abstract*—We consider two problems related to proofs and technologies of obtaining linear programming bounds for codes (spherical and in Hamming spaces). We develop a verification technique for a conjecture concerning the optimality of the Levenshtein bounds for spherical codes and prove that the conjecture holds true under certain mild assumptions. We investigate recent conditions which are sufficient for the validity of Levenshtein-type bounds for q-ary codes with given minimum and maximum distances. We provide description of all cases for lengths $n \leq 36$ and alphabet sizes $2 \leq q \leq 4$ such that our conditions are fulfilled.**

***Index terms*—linear programming, minimum energy problems, bounds for codes.**

I. Introduction

We are interested in universal upper bounds for the maximal cardinality of a spherical code C on $\mathbb{S}^{n-1}$ of prescribed maximal inner product $s(C) := \max\{\langle x, y\rangle \,:\, x, y \in C, x \neq y\}$

$$\mathcal{A}(n,s) := \max\{|C| \colon C \subset \mathbb{S}^{n-1}, s(C) = s\} \qquad (1)$$

and closely related universal lower bounds on the *minimum h-energy*

$$\mathcal{E}_h(n,M) := \inf_{|C|=M}\{E_h(n,C)\}, \qquad (2)$$

where, for a given function $h : [-1,1] \to [0,+\infty]$, the *$h$-energy* (or the potential energy) of C is defined by

$$E_h(n,C) := \sum_{x,y\in C, x\neq y} h(\langle x,y\rangle)$$

(here $\langle x,y\rangle$ denotes the inner product of x and y). Universal linear programming (LP) bounds for $\mathcal{A}(n,s)$ were described in detail in [12] and the interplay between the above problems is explained in [4], [2, Chapter 5].

The research of the first author was supported, in part, by the Bulgarian Ministry of Education and Science by Grant No. DO1-221/03.12.2018 for NCHDC, a part of the Bulgarian National Roadmap on RIs. The research of the second author was supported, in part, by a Simons Foundation grant no. 282207. The research of the third and fourth authors was supported, in part, by the U. S. National Science Foundation under grant DMS-1516400. The research of the fifth author was supported, in part, by a Bulgarian NSF contract DN2-02/2016.

Similarly, we are interested in universal LP bounds that were derived in [5] for codes in q-ary Hamming spaces with given minimum and maximum distances. We leave the details for Section V.

In this paper we investigate conditions for optimality and derivation, respectively, of universal LP bounds for $\mathcal{A}(n,s)$ and $\mathcal{E}_h(n,M)$ and their counterparts in q-ary Hamming spaces. Sections 2-4 are devoted to a conjecture about LP for spherical codes and in Sections 5-6 we consider a necessary condition for validity of Levenshtein-type bounds for q-ary codes of prescribed minimum and maximum distances.

For spherical codes, we consider a conjecture concerning the existence of better than the bounds (7) and (8) below. We prove that the conjecture holds true in all dimensions $5 \leq n \leq 24$ and in many cases in dimensions 3 and 4.

For codes in q-ary Hamming spaces, we present investigations of a condition (called (k,ℓ)-strengthened Krein condition) which extends what Levenshtein called strengthened Krein condition. Complete investigation is implemented for lengths $n \leq 36$ for $q \in \{2,3,\dots,10\}$ as all pairs $(k(\ell),\ell)$ for which the $(k(\ell),\ell)$-strengthened Krein condition is fullfilled are found.

II. Universal linear programming bounds for $\mathcal{A}(n,s)$ and $\mathcal{E}_h(n,M)$

Let $\{P_i^{(n)}(t)\}_{i=0}^{\infty}$ be the Gegenbauer polynomials [14], normalized by $P_i^{(n)}(1) = 1$. Every real polynomial $f(t)$ has its unique Gegenbauer expansion

$$f(t) = \sum_{i=0}^{n} f_i P_i^{(n)}(t).$$

Let $F_{\geq} := \{f(t) \,:\, f_0 > 0, f_i \geq 0, i = 1,2,\dots,\deg(f)\}$,

$$A_{n,s} := \{f \in F_{\geq} : f(t) \leq 0,\ t \in [-1,s]\}, \qquad (3)$$

$$E_{n,h} := \{f \in F_{\geq} : f(t) \leq h(t),\ t \in [-1,1]\}, \tag{4}$$

where s and h are fixed as in (1) and (2), respectively.

Linear programming bounds on $\mathcal{A}(n,s)$ and $\mathcal{E}_h(n,M)$ are obtained by polynomials as feasible solutions in the following two problems

$$\mathcal{A}(n,s) \leq \min_{f \in A_{n,s}} \frac{f(1)}{f_0}, \tag{5}$$

$$\mathcal{E}_h(n,M) \geq \max_{f \in E_{n,h}} M f_0 \left(M - \frac{f(1)}{f_0} \right). \tag{6}$$

We describe now two universal LP bounds we are interested in. For $\varepsilon \in \{0,1\}$ let $t_i^{n,\varepsilon}$ be the largest root of the Jacobi polynomial $P_i^{(n-1)/2,(n-1-2\varepsilon)/2)}(t) =: P_i^{n,\varepsilon}(t)$ [14]. Let

$$I_m^{(n)} := \left[t_{k-1+\varepsilon}^{n,1-\varepsilon}, t_k^{n,\varepsilon} \right],$$

where $m = 2k-1+\varepsilon$ and $t_0^{1,1} := -1$. Hereafter we use the parameter ε to distinguish between the odd and even m's.

The intervals $\{I_m^{(n)}\}_{m=1}^{\infty}$ constitute a partition of $[-1,1)$ and in each interval $I_m^{(n)}$ the Levenshtein bound (see, for example [12] for details)

$$\mathcal{A}(n,s) \leq L_m(n,s),\ s \in I_m^{(n)}, \tag{7}$$

holds. In 2016, the authors obtained a closely related universal lower bound (ULB; see [4] and [2, Chapter 5] for details)

$$\mathcal{E}_h(n,M) \geq Q_h(n,M),\ M \in J_m^{(n)}, \tag{8}$$

where

$$J_m^{(n)} := L_m(n, I_m^{(n)})$$

is the image of the interval $I_m^{(n)}$. Spherical codes which attain both (7) and (8) (the attaining is simultaneous only) are called sharp (see [1], [7]).

Denote

$$T_i^{n,\varepsilon}(x,y) := \sum_{j=0}^{i} r_j^{n,\varepsilon} P_j^{n,\varepsilon}(x) P_j^{n,\varepsilon}(y), \tag{9}$$

where

$$r_j^{n,\varepsilon} := ||P_i^{n,\varepsilon}(t)||^{-1},$$

the norm is taken with respect to $(1-t)(1+t)^{\varepsilon}(1-t^2)^{\frac{n-3}{2}}dt$.

The Levenshtein quadratures described in the next theorem are major deriving and relating ingredients of the bounds (7) and (8).

Theorem 1: [12, Theorem 5.39] For any $s \in I_m^{(n)}$ the polynomial

$$(t-s)(t+1)^{\varepsilon}\, T_{k-1}^{n,\varepsilon}(t,s)$$

has $k+\varepsilon$ simple roots $\alpha_0 < \alpha_1 < \cdots < \alpha_{k-1+\varepsilon}$, where $\alpha_{k-1+\varepsilon} = s$ and $\alpha_0 \geq -1$ with equality holding if and only if $\varepsilon = 1$ or $\varepsilon = 0$ and $s = t_{k-1}^{1,1}$. Moreover, for any polynomial $f(t)$ of degree at most $m = 2k-1+\varepsilon$ the following equality holds

$$f_0 = \frac{f(1)}{L_m(n,s)} + \sum_{i=0}^{k-1+\varepsilon} \rho_i f(\alpha_i), \tag{10}$$

where the coefficients $\rho_i = \rho_i(n,s)$, $i = 1, \ldots, k-1+\varepsilon$, are positive, and $\rho_0 = \rho_0(n,s) \geq 0$ with equality holding if and only if $s = t_k^{1,0}$.

In terms of the parameters, just introduced by Theorem 1, the RHS of the ULB (8) is written as

$$Q_h(n,M) = M^2 \sum_{i=0}^{k-1+\varepsilon} \rho_i h(\alpha_i),$$

where the parameters $\alpha_0 < \alpha_1 < \cdots < \alpha_{k-1+\varepsilon} = s$ come as roots of the equation $M = L_m(n,s)$ and $\rho_0, \rho_1, \ldots, \rho_{k-1+\varepsilon}$ are their weights as in Theorem 1. Equivalently, one can construct the same parameters by taking the largest root s of $M = L_m(n,s)$ and constructing the Levenshtein polynomial

$$f_m^{(n,s)}(t) := (t-s)(t+1)^{\varepsilon} \left(T_{k-1}^{n,\varepsilon}(t,s) \right)^2.$$

Then the numbers $\alpha_0 < \alpha_1 < \cdots < \alpha_{k-1+\varepsilon} = s$ are the distinct roots of $f_m^{(n,s)}(t)$ in increasing order. In both approaches the weights $\rho_0, \rho_1, \ldots, \rho_{k-1+\varepsilon}$ are computed via the Lagrange basis in the Levenshtein quadrature (10) as in [12].

III. A CONJECTURE ABOUT TEST FUNCTIONS

The bounds (7) and (8) were obtained by polynomials which solve the problems (5) and (6), respectively, in the class of polynomials of degree at most m. Thus, (7) and (8) cannot be improved in the framework of (5)-(6) by using polynomials of degree at most m. On the other hand, improvements by higher degree polynomials are known (see, for example, [3], [5], [13]). Such improvements can be justified by the so-called test functions introduced in [3] for the Levenshtein bound and re-involved for the ULB in [4].

For fixed n and m, for any real number $s \in I_m^{(n)}$ and any positive integer $j \geq m+1$, define

$$R_j^{(n)}(s) := \frac{1}{L_m(n,s)} + \sum_{i=0}^{k-1+\varepsilon} \rho_i P_j^{(n)}(\alpha_i),$$

where the parameters (ρ_i, α_i) are as in Theorem 1. Note that, since $P_j^{(n)}(1) = 1$, this definition reproduces the RHS of the Levenshtein quadrature from Theorem 1, written for the polynomial $f(t) = P_j^{(n)}(t)$ (not necessarily of degree at most m).

Similarly, for fixed n and m, for any positive integers $M \in J_m^{(n)}$ and $j \geq m+1$, define

$$S_j^{(n)}(M) := \frac{1}{M} + \sum_{i=0}^{k-1+\varepsilon} \rho_i P_j^{(n)}(\alpha_i),$$

where the parameters $(\rho_i, \alpha_i)_{i=0}^{k-1+\varepsilon}$ come from the equation $M = L_m(n,s)$ as explained in the end of the previous section. Note that the values of $S_j^{(n)}(M)$ are particular values of $R_j^{(n)}(s)$.

The test functions $R_j^{(n)}(s)$ and $S_j^{(n)}(M)$ are utilized in the next theorem to give necessary and sufficient conditions for optimality[1] of the bounds (7) and (8).

[1]In other words, for existence of better LP bounds.

Theorem 2: (a) ([3, Theorem 3.1], see also [12, Theorem 5.47]) Given n, m, and $s \in I_m^{(n)}$, the bound (7) can be improved by a polynomial from $A_{n,s}$ of degree at most $m+1$ if and only if there exists positive integer $j \geq m+1$ such that $R_j^{(n)}(s) < 0$.

(b) ([4, Theorem 5.1]) Given n, m, $M \in J_m^{(n)}$, the bound (8) can be improved by a polynomial form $E_{n,h}$ of degree at most $m+1$ if and only if there exists positive integer $j \geq m+1$ such that $S_j^{(n)}(M) < 0$.

Since $R_{m+1}^{(n)}(s) \geq 0$ and $R_{m+2}^{(n)}(s) \geq 0$ for every $s \in I_m^{(n)}$, negative test functions could only exist for $j \geq m+3$. Moreover, in [3] the following was conjectured.

Conjecture 3: [3, Conjecture 5.1] Let $R_j^{(n)}(s)$, $j \geq m+1$, be the test functions for the Levenshtein bound $L_m(n,s)$ for spherical codes. Then exactly one of the following two happens:

(1) at least one of $R_{m+3}^{(n)}(s)$ and $R_{m+4}^{(n)}(s)$ is negative,

(2) $R_j^{(n)}(s) \geq 0$ for every $j \geq m+3$.

The corresponding conjecture about the ULB as a particular case follows.

Conjecture 4: Let $S_j^{(n)}(M)$, $j \geq m+1$, be the test functions for the ULB bound $Q_h(n,M)$ for minimum energy of spherical codes. Then exactly one of the following two happens:

(1) at least one of $S_{m+3}^{(n)}(M)$ and $S_{m+4}^{(n)}(M)$ is negative,

(2) $S_j^{(n)}(M) \geq 0$ for every $j \geq m+3$.

We have not found any counterexample neither to Conjectures 3 and 4 despite checking great amount of numerical data. In the next section, we present a methodology for verification of Conjecture 4 in a fixed dimension, implying verification of a weaker version of Conjecture 3.

IV. On verification of Conjectures 3 and 4

We first consider Conjecture 4. For fixed dimension $n \geq 5$ we reduce the verification to finitely many computational tasks.

As mentioned above, the values of the "energy" test functions $S_j^{(n)}(M)$ are just specific values of $R_j^{(n)}(s)$ for some $s \in I_m^{(n)}$ as the correspondence between M and s is bijective because of the strict monotonicity of the Levenshtein bound. Therefore we need to consider only these s for which the value of $L_m(n,s)$ is an integer (i.e., equal to M). Such integers are finitely many for fixed m – there are exactly

$$L_m(n, t_k^{n,\varepsilon}) - L_m(n, t_{k-1+\varepsilon}^{n,1-\varepsilon}) + 1$$

cases for consideration. Moreover, it follows from [3, Theorems 4.9 and 4.10] that for $n \geq 5$ only finitely many m are of interest, because there exist positive integers

$$m_\varepsilon = m_\varepsilon(n)$$

such that $R_{m+4-\varepsilon}^{(n)}(s) < 0$ for every $m \geq m_\varepsilon$ and every $s \in I_m^{(n)}$. Therefore we need to consider only test functions $S_j^{(n)}(M)$ corresponding to positive integers $m < m_\varepsilon$, which leaves for consideration only finitely many cases of M. This makes the verification of Conjecture 4 finite with respect to M for fixed dimension n. The problem is not finite yet since we still have to consider infinitely many test functions $S_j^{(n)}(M)$ as only j varies now.

We proceed as follows. For fixed M, if (1) happens[2], then we conclude that Conjecture 4 holds true for this value of M. Otherwise, n and M are already fixed and we are interested only in $S_j^{(n)}(M)$ for $j \geq m+5$. We apply now a technique from [4, Section 4] to prove that (2) happens. That technique requires finding of a positive integer

$$j_0 = j_0(n, M)$$

(in fact, $m_1(n) = 2[\sqrt{n-2}]$ while the expression for $m_0(n)$ is more complicated) such that the inequalities $S_j^{(n)}(M) > 0$ follow for all $j \geq j_0$ from known (not necessarily the best) estimations for Gegenbauer polynomials. Afterwards we have only to check the signs of the test functions $S_j^{(n)}(M)$ with $m+5 \leq j \leq j_0 - 1$.

Remark 5: In our computational conclusions we need to decide only if a given real number is positive or negative. This approach is subject to good precision and can fail only in case that the corresponding number is equal (or very close) to zero. Fortunately, such cases seem to be quite rare. In all cases we have considered, zero test functions $R_j^{(n)}(s)$ with $j \geq m+5$ appear only when $s = t_k^{n,1}$ is the right end of the even interval $I_{2k}^{(n)}$. This do not impact negatively our proof since $R_j^{(n)}(t_k^{n,1}) = 0$ holds (non-computationally) true because of the symmetry of the system $(\rho_i, \alpha_i)_{i=0}^{k-1+\varepsilon}$ (see, for example, [4, Section 2.4]).

As an example, we describe in detail the above verification process in dimension $n = 7$. We have

$$\begin{aligned} m_0(7) &= 17 \text{ (corresponds to } M \geq 6006), \\ m_1(7) &= 6 \text{ (corresponds to } M \geq 112), \end{aligned}$$

and it remains to consider the odd $m \leq 15$ and even $m \leq 4$.

For even $m \in \{2, 4\}$ we have to investigate the cases

$$\begin{aligned} M &\in [D(7,2)+1, D(7,3)] \cup [D(7,4)+1, D(7,5)] \\ &= [9,14] \cup [36,56]. \end{aligned}$$

Among these, the alternative (1) of Conjecture 4 happens in all cases $42 \leq M \leq 55$ with $S_7^{(7)}(M) < 0$. In all other 12 cases we prove that the alternative (2) happens by explicitly finding $j_0 = j_0(7, M)$. For example, $j_0(7,M) = 19$, 15, 14, 13, 13, and 13 for $M = 15$, 16, 17, 18, 19, and 20, respectively.

For odd $m \in \{3, 5, \ldots, 15\}$ we have to consider

$$\begin{aligned} M &\in [D(7,3)+1, D(7,4)] \cup [D(7,5)+1, D(7,6)] \cup \cdots \\ &= [15,35] \cup [57,112] \cup \cdots \end{aligned}$$

Now (1) of Conjecture 4 happens for $60 \leq M \leq 72$ with $S_8^{(7)}(M) < 0$ and for all $M \geq 75$ (recall that we check all odd intervals until we reach $M = 6006$) with the same dynamics

[2] It may still happen for $m < m_\varepsilon$ in the whole interval $I_m^{(n)}$ or in some of its parts.

of the test function in all intervals. For example, when $m = 9$, we have $M \in [421, 672]$ and $S_{12}^{(7)} < 0$ for $M \in [421, 450]$, $S_{12}^{(7)} < 0$ and $S_{13}^{(7)} < 0$ for $M \in [451, 555]$, and $S_{13}^{(7)} < 0$ for $M \in [556, 672]$. In all remaining 39 cases (2) happens and we are able to present j_0 in every separate case.

Theorem 6: Conjecture 4 holds true in all dimensions $5 \leq n \leq 24$.

The above technique works in dimensions $n = 3$ and 4 for all even m. If m is odd, then there are small (containing just a few integers) subintervals of $J_m^{(n)}$, where $S_{m+4}^{(n)}(M) > 0$. This prevents us making the verification finite. However, we conjecture that $S_{m+3}^{(n)}(M) < 0$ in all these cases which gives the alternative (1) of Conjecture 4. Our verification in dimensions 3 and 4 reached $M = 2400$ (corresponding to $m = 95$) and $M = 21320$ (corresponding to $m = 77$), respectively.

Theorem 7: Conjecture 4 holds true in dimension $n = 3$ for every $M \in \{2, 3, \ldots, 2400\}$ and in dimension $n = 4$ for every $M \in \{2, 3, \ldots, 21320\}$.

To verify cases of Conjecture 3 we use the following mild assumption. Since improvements of bounds for $\mathcal{A}(n, s)$ make sense in the integer part only, we may restrict ourselves on values of $L_m(n, s)$ which are equal to an integer minus some small positive number. This argument is based on the idea that if we improve

$$L_m(n, s) = N - \epsilon$$

in its integer part (so our new bound is $N - 1$ or less) then we can improve (again in the integral part) all bounds $L_m(n, s')$ from the interval $[N - 1, N - \epsilon]$. This modification, together with the above reduction arguments makes the verification of Conjecture 3 finite for fixed dimension n. Moreover, by continuity, the computational conclusions about the signs of the test function $S_j^{(n)}(M)$ suffice (see Remark 5) for the corresponding values of $R_j^{(n)}(s)$.

Theorem 8: Under the above assumptions, Conjecture 3 holds true in all cases described in Theorems 6 and 7.

All numerical data from the computations described in this section is available upon request and will be published openly somewhere in internet.

V. The (k, ℓ)-strengthened Krein condition

For fixed n and q, the (normalized) Krawtchouk polynomials are defined by

$$Q_i^{(n)}(t) := \frac{1}{r_i} K_i^{(n,q)}(z),\ i = 0, 1, \ldots, n,$$

where

$$r_i := (q-1)^i \binom{n}{i},\ i = 0, 1, \ldots, n,$$

$z = n(1-t)/2$, and

$$K_i^{(n,q)}(z) := \sum_{j=0}^{i} (-1)^j (q-1)^{i-j} \binom{z}{j} \binom{n-z}{i-j},$$

$i = 0, 1, \ldots, n$, are the (usual) Krawtchouk polynomials.

Denote

$$T_i(x, y) := \sum_{j=0}^{i} r_j Q_j^{(n)}(x) Q_j^{(n)}(y) \tag{11}$$

and utilize (as in [11]; see also [12, Eq. (5.65)]) the kernels $T_i(x, y)$ in the definition of $(1, 0)$-adjacent polynomials as follows

$$Q_i^{1,0}(t) := \frac{T_i(t, 1)}{T_i(1, 1)},\quad i = 0, 1, \ldots, n-1. \tag{12}$$

With a next step, consider the $(1, 0)$-kernel

$$T_i^{1,0}(x, y) := \sum_{j=0}^{i} r_j^{1,0} Q_j^{1,0}(x) Q_j^{1,0}(y), \tag{13}$$

where

$$r_j^{1,0} = \left(\sum_{u=0}^{j} r_u \right)^2 / \binom{n-1}{j} (q-1)^j$$

are the $(1, 0)$ counterparts of r_j. Then $(1, \ell)$-adjacent polynomials are defined (see [6]) by

$$Q_i^{1,\ell}(t) := \frac{T_i^{1,0}(t, \ell)}{T_i^{1,0}(1, \ell)},\quad i = 0, 1, \ldots, n-2. \tag{14}$$

The polynomials $Q_i^{1,\ell}(t)$ are, in a sense, generalization of the Levenshtein polynomials $Q_i^{1,1}(t)$ (see [12, Eq. (5.68)]).

For any real polynomial $g(t)$ of degree at most n we consider its Krawtchouk expansion

$$g(t) = \sum_{i=0}^{n} g_i Q_i^{(n)}(t)$$

and set

$$G_> := \{g(t) \,:\, g_i > 0, i = 0, 1, \ldots, \deg(g)\}.$$

In the proof of the positive definiteness of his polynomials Levenshtein used (see [12, (3.88) and (3.92)]) what he called the strengthened Krein condition

$$(t+1) Q_i^{1,1}(t) Q_j^{1,1}(t) \in G_>,$$

where the polynomials $Q_i^{1,1}(t)$ can be obtained from (14) for $\ell = -1$ [12, Eq. (5.68)]. The strengthened Krein condition holds true for every $i, j \in \{0, 1, \ldots, n-3\}$ (see [12, Lemma 3.25]). We need to consider the following modification.

Definition 9: We say that the polynomials $\{Q_i^{1,\ell}(t)\}_{i=0}^{k}$ satisfy the (k, ℓ)-strengthened Krein condition if

$$(t - \ell) Q_i^{1,\ell}(t) Q_j^{1,\ell}(t) \in G_> \tag{15}$$

for every $i, j \in \{0, 1, \ldots, k\}$ except possibly for $i = j = k$.

The (k, ℓ)-strengthened Krein condition is crucial in our proofs [6] of a Levenshtein-type bound for the quantity

$$\mathcal{A}_q(n, \ell, s) := \max\{|C| : \ell(C) = \ell, s(C) = s\},$$

the maximum cardinality of a code $C \subset F_q^n$ with prescribed minimum and maximum distances, $d = n(1-s)/2$ and $D = n(1-\ell)/2$, respectively [6, Theorem 5.2]. Explicitly, we have

$$\mathcal{A}_q(n,\ell,s) \leq \frac{S_k\left(Q_{k-1}^{1,\ell}(s) - Q_k^{1,\ell}(s)\right)}{\frac{r_{k+1}Q_{k+1}^{(n)}(\ell)Q_{k-1}^{1,\ell}(s)}{S_{k+1}\left(Q_{k+1}^{1,0}(\ell)-Q_k^{1,0}(\ell)\right)} - \frac{r_k Q_k^{(n)}(\ell)Q_k^{1,\ell}(s)}{S_{k-1}\left(Q_k^{1,0}(\ell)-Q_{k-1}^{1,0}(\ell)\right)}},$$

where

$$S_j = \sum_{i=0}^{j} r_i, \quad j \in \{k-1,k,k+1\},$$

subject to the (k,ℓ)-strengthened Krein condition and two further conditions, namely

$$\ell < t_{k,1}^{1,0}, \tag{16}$$

where $t_{k,1}^{1,0}$ is the smallest root of $Q_k^{1,0}(t)$, and

$$\frac{Q_{k+1}^{1,0}(\ell)}{Q_k^{1,0}(\ell)} < 1. \tag{17}$$

Similarly, the fulfillment of the conditions (15)-(17) implies [6, Theorem 5.2]the validity of a universal lower bound on the quantity

$$\mathcal{E}_h(n,M,\ell) := \min\{E_h(C) : |C| = M, \ell(C) = \ell\},$$

the smallest possible h-energy of a code $C \subset F_q^n$ with prescribed cardinality M and maximum distance $D = n(1-\ell)/2$. Here h is an absolutely monotone potential and

$$E_h(C) := \sum_{x,y\in C, x\neq y} h(\langle x,y\rangle)$$

is the h-energy of C, $\langle x,y\rangle = 1 - 2d(x,y)/n$.

As it might be expected, the (k,ℓ)-strengthened Krein condition is not true for every ℓ, and for fixed ℓ, it is not true for every k. Lemma 4.3 from [6] says that the condition is satisfied for all pairs $(i,0)$, $i = 0,1,\ldots,k-1$, provided certain conditions on ℓ and s are satisfied. On the other hand, for fixed n, all relevant pairs (k,ℓ) are finitely many and can be therefore subject to computational checks.

The case $k = 1$ was considered by Helleseth, Kløve, and Levenshtein [9] where conditions of validity were stated. We found it inappropriate to explain the bounds by fixing $k > 1$. Instead, we fix ℓ (starting from $\ell = t_1 = -1 + 2/n$) and then the explanation goes on with varying s or, equivalently, varying k. It is clear that validity of the conditions (15)-(17) for some pair (k,ℓ) implies their validity for all pairs (k',ℓ) with $1 \leq k' \leq k$. Therefore, we are able to built a system of bounds similar to the system of Levenshtein bounds which appears when $\ell = -1$. In fact, our system is a mix of Levenshtein bounds obtained by odd degrees polynomials and our bounds obtained by even degrees polynomials.

Examples of codes attaining the discussed bounds can be found in [9] for $k = 1$ and in [6]. In fact, in [6] the distance distributions of all feasible attaing codes are found as functions of the weights ρ_i (i.e., functions of n, q, ℓ, k, and s).

VI. On verification of the (k,ℓ)-strengthened Krein condition

We describe our computational effort in the verification of the (k,ℓ)-strengthened Krein condition to be satisfied simultaneously with the conditions (16) and (17). For fixed length n we are interested in the determination of all pairs (k,ℓ) such that (15)-(17) hold true. It is clear that the problem is finite. Moreover, strictly positive Krawtchouk coefficients can be verified easily by setting enough precision of the computation.

In the next three tables we summarize our results for alphabet sizes $q = 2$, 3, and 4 for the relevant lengths $n \leq 36$. For fixed q and n the values of k are shown in the columns corresponding to fixed $\ell = -1 + 2/n$, $-1 + 4/n$, etc., until (15)-(17) are fulfilled. The missing entries are zeros.

Similar tables for larger lengths and alphabet sizes were also computed. All numerical data from the computations described in this section is available upon request.

$q=2$					
n/ℓ	$-1+2/n$	$-1+4/n$	$-1+6/n$	$-1+8/n$	$-1+10/n$
5	1				
6	1				
7	2	1			
8	2	1			
9	3	1			
10	3	2	1		
11	4	2	1		
12	4	3	1	1	
13	5	3	2	1	
14	5	3	2	1	1
15	6	4	3	2	1
16	6	4	3	2	1
17	7	5	3	2	1
18	7	5	4	3	2
19	8	6	4	3	2
20	8	6	4	3	2
21	9	6	5	4	3
22	9	7	5	4	3
23	10	7	6	4	3
24	10	8	6	5	4
25	11	8	6	5	4
26	11	9	7	5	4
27	12	9	7	6	5
28	12	10	8	6	5
29	13	10	8	7	5
30	13	11	9	7	6
31	14	11	9	7	6
32	14	11	9	8	6
33	15	12	10	8	7
34	15	12	10	9	7
35	16	13	11	9	7
36	16	13	11	9	8

$q = 3$				
n/ℓ	$-1+2/n$	$-1+4/n$	$-1+6/n$	$-1+8/n$
7	1			
8	1			
9	1			
10	2			
11	2	1		
12	2	1		
13	3	1		
14	3	2	1	
15	3	2	1	
16	4	2	1	
17	4	2	1	
18	4	3	1	1
19	5	3	2	1
20	5	3	2	1
21	5	3	2	1
22	6	4	3	2
23	6	4	3	2
24	6	4	3	2
25	7	5	3	2
26	7	5	3	2
27	7	5	4	3
28	8	5	4	3
29	8	6	4	3
30	8	6	4	3
31	9	6	5	3
32	9	7	5	4
33	9	7	5	4
34	10	7	5	4
35	10	8	6	4
36	10	8	6	5

$q = 4$				
n/ℓ	$-1+2/n$	$-1+4/n$	$-1+6/n$	$-1+8/n$
10	1			
11	1			
12	1			
13	1			
14	2	1		
15	2	1		
16	2	1		
17	2	1		
18	3	1		
19	3	2	1	
20	3	2	1	
21	3	2	1	
22	4	2	1	
23	4	2	1	
24	4	3	1	
25	4	3	2	1
26	5	3	2	1
27	5	3	2	1
28	5	3	2	1
29	5	4	2	1
30	6	4	2	1
31	6	4	3	2
32	6	4	3	2
33	6	4	3	2
34	7	5	3	2
35	7	5	3	2
36	7	5	4	2

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