Linear Programming Bounds for Cardinality and Energy of Codes of Given Min and Max Distances

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***Abstract*—We employ signed measures that are positive definite up to certain degrees to establish Levenshtein-type upper bounds on the cardinality of codes with given minimum and maximum distance, and universal lower bounds on the potential energy (for absolutely monotone interactions) for codes with given maximum distance and fixed cardinality. In particular, we extend the framework of Levenshtein bounds for such codes.**

I. Introduction

Let F_q be an alphabet of size q. We consider codes (sets) $C \subset F_q^n = \{(x_1, \ldots, x_n) : x_i \in F_q\}$ with the Hamming distance $d(x, y)$ between words $x, y \in F_q^n$. In setting of F_q^n as a polynomial metric space [11] the following change of the variable

$$t = 1 - \frac{2d}{n} \in T_n := \{-1 + 2i/n : i = 0, 1, \ldots, n\}$$

is very convenient. For $x, y \in F_q^n$ we write $\langle x, y\rangle = 1 - 2d(x, y)/n \in T_n$.

For any code $C \subset F_q^n$ we use

$$s(C) := \max\{\langle x, y\rangle : x, y \in C, x \neq y\} \in T_n,$$

$$\ell(C) := \min\{\langle x, y\rangle : x, y \in C, x \neq y\} \in T_n,$$

to denote counterparts of the minimum and maximum distance of C, respectively. Set $C_{n,q}(s, \ell) := \{C \subset F_q^n | s(C) \leq s, \ell(C) \geq \ell\}$ and denote by

$$\mathcal{A}_q(n, s, \ell) := \max\{|C| : C \in C_{n,q}(s, \ell)\}$$

the maximum cardinality of a code in F_q^n with prescribed minimum and maximum distance $d = n(1 - s)/2$ and $D = n(1 - \ell)/2$, respectively.

Definition 1.1: Given a (potential) function $h(t) : [-1, 1] \to [0, +\infty]$ and a code $C \subset F_q^n$, the *potential energy* (also referred to as *h-energy*) of C is

$$E_h(C) := \sum_{x,y\in C, x\neq y} h(\langle x, y\rangle).$$

While we only need the values of h on the discrete set T_n for computing the h-energy, we further assume that h is *(strictly) absolutely monotone* on the interval [-1,1); that is, h and all its derivatives are defined and (positive) nonnegative on this interval. We remark that the function $F(z) = h(t)$, where $z = n(1 - t)/2$, is *completely monotone* on $(0, n]$ (i.e., $(-1)^k F^{(k)}(z) \geq 0$ for all $z \in (0, n]$) if and only if h is absolutely monotone on $[-1, 1]$.

For absolutely monotone potentials h we consider

$$\mathcal{E}_h(n, M, \ell) := \min\{E_h(C) : C \in C_{n,q}(1, \ell), |C| = M\}$$

with prescribed n, q, ℓ, and M.

In this paper we use linear programming techniques to derive explicit upper bounds for $\mathcal{A}_q(n, s, \ell)$ and $\mathcal{E}_h(n, M, \ell)$. Our bounds can be computed for all feasible values of q, n, s, and ℓ, which makes them universal in the sense of Levenshtein [11]. We are not aware of such explicit universal bounds in the existing literature (see [8] for a particular case) more than 20 years after the chapter [11] by Levenshtein and the paper [7] by Delsarte and Levenshtein.

There is an intricate interplay between the Levenshtein universal bounds for $A_q(n, s, -1)$ and universal lower bounds on $\mathcal{E}_h(n, M, -1)$ in different spaces (see [1] for the Euclidean sphere and [2] for F_q^n). We further that relationship to codes from $C_{n,q}(1, \ell)$ to obtain energy bounds. Lower bounds for $\mathcal{E}_h(n, M, \ell)$ were considered first in 2014 by Cohn and Zhao [4] and then by the authors [2].

General linear programming bounds for quantities like $\mathcal{A}_q(n, s, \ell)$ and $\mathcal{E}_h(n, M, \ell)$ are folklore [6], [7], [11]. For any real polynomial $f(t)$ of degree at most n we consider its Krawtchouk expansion $f(t) = \sum_{i=0}^n f_i K_i^{(n,q)}(t)$ and set

$$F_{\geq} := \{f(t) : f_0 > 0, f_i \geq 0, i = 1, 2, \ldots, n\}.$$

Following Delsarte [6], we have

$$\mathcal{A}_q(n, s, \ell) \leq \min_{f\in\mathcal{F}_{n,s,\ell}} f(1)/f_0, \tag{1}$$

where $\mathcal{F}_{n,s,\ell} := \{f \in F_{\geq} : f(t) \leq 0, t \in [\ell, s]\}$. Similarly,

$$\mathcal{E}_h(n, M, \ell) \geq \max_{g\in\mathcal{G}_{n,\ell;h}} M(Mg_0 - g(1)), \tag{2}$$

where $\mathcal{G}_{n,\ell;h} := \{g \in F_{\geq} : g(t) \leq h(t), t \in [\ell, 1)\}$ (see Yudin [13]). Thus, major results in this context depend on proper choice and investigation of polynomials that optimize (1) or (2).

The Levenshtein bound (see [9]–[11]) and the energy bound [2] work for $\ell = -1$ and, of course, depend on the properties of Krawtchouk polynomials and their adjacent polynomials which are orthogonal with respect to classical positive measures. The case $\ell > -1$, however, involves more challenging signed measures.

The paper is organized as follows. In Sections 2 and 3 we introduce the necessary adjacent polynomials and signed measures and establish positive definiteness of the corresponding measures up to appropriate degrees. Properties of the associated orthogonal polynomials are derived and discussed in Section 4, where we define Levenshtein-type polynomials $f_{2k}^{(n,\ell,s)}(t)$ to be used in (1). In Section 5 we obtain Levenshtein-type bounds on $\mathcal{A}_q(n,s,\ell)$. As in the case $\ell = -1$ this implies energy bounds on $\mathcal{E}_h(n,M,\ell)$. An important role in the proof is played by what we call the (k,ℓ)-strengthened Krein condition extending the Levenshtein's strengthened Krein condition.

II. Adjacent polynomials

For fixed n and q, the (normalized) Krawtchouk polynomials are defined by

$$Q_i^{(n,q)}(t) := \frac{1}{r_i} K_i^{(n,q)}(z),$$

where $z = n(1-t)/2$, $r_i := (q-1)^i \binom{n}{i}$, and

$$K_i^{(n,q)}(z) := \sum_{j=0}^{i} (-1)^j (q-1)^{i-j} \binom{z}{j}\binom{n-z}{i-j},$$

$i = 0,1,\ldots,n$, are the (usual) Krawtchouk polynomials corresponding to F_q^n.

The measure of orthogonality for the system $\{Q_i^{(n,q)}(t)\}_{i=0}^n$ is discrete and given by

$$\mu_n := q^{-n} \sum_{i=0}^{n} r_{n-i} \delta_{t_i}, \tag{3}$$

where δ_{t_i} is the Dirac-delta measure at $t_i \in T_n$. The form

$$\langle f, g\rangle = \int f(t)g(t)d\mu_n(t)$$

defines an inner product over the class of polynomials of degree at most n.

We also need *adjacent polynomials* as introduced by Levenshtein (cf. [11, Section 6.2], see also [9], [10])

$$Q_i^{(1,0,n,q)}(t) := \frac{K_i^{(n-1,q)}(z-1)}{\sum_{j=0}^{i} \binom{n}{j}(q-1)^j},$$

$$Q_i^{(1,1,n,q)}(t) := \frac{K_i^{(n-2,q)}(z-1)}{\sum_{j=0}^{i} \binom{n-1}{j}(q-1)^j},$$

where $z = n(1-t)/2$. The corresponding measures of orthogonality are, respectively,

$$c^{1,0}(1-t)d\mu_n(t), \quad c^{1,1}(1-t)(1+t)d\mu_n(t), \tag{4}$$

where $c^{a,b}$ are normalizing constants (see [11, Section 6.2]).

For $\ell = 1 - 2d/n \in T_n$ we introduce further adjacent polynomials $Q_i^{(1,\ell,n,q)}(t)$ orthogonal with respect to a signed measure $d\nu_{n,\ell}(t)$ which is defined and investigated below. We shall use $Q_i^{(1,\ell,n,q)}(t)$ to construct Levenshtein-type polynomials to be applied in (1).

For abbreviation purposes, in what follows we will omit n, q and the brackets in the indexing of the adjacent polynomials.

III. Positive definite signed measures

Signed measures were used by Cohn and Kumar in [3] in the context of linear programming bounds for energy of spherical codes.

Definition 3.1: A signed Borel measure ν on $\mathbb{R}$ for which all polynomials are integrable is called *positive definite up to degree* m if for all real polynomials $p \not\equiv 0$ of degree at most m we have $\int p(t)^2 d\nu(t) > 0$.

Let $t_{i,1}^{1,0} < t_{i,2}^{1,0} < \cdots < t_{i,i}^{1,0}$ be the zeros of the polynomial $Q_i^{1,0}(t)$. Given ℓ and s such that $\ell < t_{k,1}^{1,0} < t_{k,k}^{1,0} < s$, we define the signed measures on $[-1,1]$ (see (3) and (4))

$$d\nu_{n,\ell}(t) := c^{1,\ell}(t-\ell)(1-t)d\mu_n(t), \tag{5}$$

$$d\nu_{n,\ell,s}(t) := c^{1,\ell,s}(t-\ell)(s-t)(1-t)d\mu_n(t), \tag{6}$$

where the normalizng constants are given by

$$c^{1,\ell} = \frac{n^2 q^n}{2\sum_{i=0}^{n}(n-i)(2i-n-n\ell)r_{n-i}},$$

$$c^{1,\ell,s} = \frac{n^3 q^n}{2\sum_{i=0}^{n}(n-i)(2i-n-n\ell)(n+ns-2i)r_{n-i}}.$$

The following lemma establishes the positive definiteness of the signed measures (5) and (6) up to degrees $k-1$ and $k-2$, respectively. This will allow us to define orthogonal polynomials with respect to these signed measures providing essential ingredients for modifying Levenshtein's framework.

Lemma 3.2: For given $k > 1$, let s and ℓ satisfy $\ell < t_{k,1}^{1,0} < t_{k,k}^{1,0} < s$. Then the measures $d\nu_{n,\ell}(t)$ and $d\nu_{n,\ell,s}(t)$ are positive definite up to degree $k-1$ and $k-2$, respectively.

Proof. Modifying the classical Radau quadrature [5, Sec. 2.7] for integration with respect to discrete measures we conclude that the zeros of the corresponding discrete orthogonal polynomial, the system of $k+1$ nodes

$$\left\{t_{k,1}^{1,0} < t_{k,2}^{1,0} < \cdots < t_{k,k}^{1,0} < 1\right\},$$

defines a positive Radau quadrature with respect to μ_n,

$$f_0 := \int_{-1}^{1} f(t)d\mu_n(t) = w_{k+1}f(1) + \sum_{i=1}^{k} w_i f(t_{k,i}^{1,0}), \tag{7}$$

that is exact for all polynomials of degree at most $2k$.

We apply (7) for $q(t)$, an arbitrary polynomial of degree at most $k-1$, to see that

$$\int_{-1}^{1} q^2(t)d\nu_{n,\ell}(t) = \int_{-1}^{1} q^2(t)(1-t)(t-\ell)d\mu_n(t)$$
$$= \sum_{i=1}^{k} w_i q^2(t_{k,i}^{1,0})(1-t_{k,i}^{1,0})(t_{k,i}^{1,0}-\ell) \geq 0.$$

The equality holds only if $q(t_{k,i}^{1,0}) = 0$ for all $i = 1,\ldots,k$, which would imply that $q(t) \equiv 0$. Therefore the measure $d\nu_{n,\ell}(t)$ is positive definite up to degree $k-1$.

If $q(t) \not\equiv 0$ is of degree at most $k-2$, then we utilize (7) again to derive that $\int_{-1}^{1} q^2(t)d\nu_{n,\ell,s}(t) > 0$ as above. $\square$

Remark 3.3: It can be proved analogously that the measure $d\nu_{n,s}(t) = (s-t)(1-t)d\mu_n(t)$ is positive definite up to degree $k-1$. We will use this fact in the proof of Theorem 4.5 below.

IV. Construction of Levenshtein-type polynomials

A. Existence and uniqueness of $Q_j^{1,\ell}(t)$ and $Q_j^{1,\ell,s}(t)$

Applying Gram-Schmidt orthogonalization we derive the existence and uniqueness (for the so-chosen normalizations) of the following classes of orthogonal polynomials with respect to the signed measures (5)-(6).

Theorem 4.1: Let $\ell < t_{1,k}^{1,0} < t_{k,k}^{1,0} < s$. The following two classes of orthogonal polynomials are well-defined:

$$\{Q_j^{1,\ell}(t)\}_{j=0}^{k}, \text{ w.r.t. } d\nu_{n,\ell}(t),\ Q_j^{1,\ell}(1) = 1;$$

$$\{Q_j^{1,\ell,s}(t)\}_{j=0}^{k-1}, \text{ w.r.t. } d\nu_{n,\ell,s}(t),\ Q_j^{1,\ell,s}(1) = 1.$$

The polynomials in both classes satisfy a three-term recurrence relation and their zeros interlace.

For our purposes we shall restrict to values of ℓ such that $Q_{k+1}^{1,0}(\ell)/Q_k^{1,0}(\ell) < 1$. As shown in the proof of Theorem 4.2 below this condition is equivalent with $t_{k,k}^{1,\ell} < 1$.

B. Construction and investigation of $Q_j^{1,\ell}(t)$

Utilizing the Christoffel-Darboux formula (see, for example [12, Th. 3.2.2], [11, Eq. (5.65)]) we are able to construct the polynomials $Q_j^{1,\ell}$ explicitly. Let

$$\begin{aligned} T_i^{1,0}(x,y) &:= \sum_{j=0}^{i} r_j^{1,0} Q_j^{1,0}(x) Q_j^{1,0}(y) \qquad (8)\\ &= r_i^{1,0} b_i^{1,0} \frac{Q_{i+1}^{1,0}(x)Q_i^{1,0}(y) - Q_{i+1}^{1,0}(y)Q_i^{1,0}(x)}{x-y} \end{aligned}$$

(when $x = y$ appropriate derivatives are used).

As in [11] we utilize Christoffel-Darboux formulas to prove the interlacing properties of the zeros of $Q_j^{1,\ell}$ with respect to the zeros of $Q_i^{1,0}$. For fixed $\ell > -1$, we choose $k = k(\ell)$ to be the largest k such that the condition $\ell < t_{k,1}^{1,0}$ is satisfied.

Theorem 4.2: Let ℓ and k be such that $t_{k+1,1}^{1,0} < \ell < t_{k,1}^{1,0}$ and $Q_{k+1}^{1,0}(\ell)/Q_k^{1,0}(\ell) < 1$. Then all zeros $\left\{t_{i,j}^{1,\ell}\right\}_{j=1}^{i}$ of the polynomial $Q_i^{1,\ell}(t)$ are in the interval $[\ell, 1]$ and we have

$$Q_i^{1,\ell}(t) = \frac{T_i^{1,0}(t,\ell)}{T_i^{1,0}(1,\ell)} = \eta_i^{1,\ell} t^i + \cdots, \quad i = 0,1,\ldots,k, \qquad (9)$$

with $\eta_i^{1,\ell} > 0$ and $t_{k,k}^{1,\ell} < 1$. Finally,

$$\begin{aligned} &t_{i,j}^{1,\ell} \in \left(t_{i,j}^{1,0}, t_{i+1,j+1}^{1,0}\right),\ i = 1,\ldots,k-1, j = 1,\ldots,i;\\ &t_{k,j}^{1,\ell} \in \left(t_{k+1,j+1}^{1,0}, t_{k,j+1}^{1,0}\right),\ j = 1,\ldots,k-1. \end{aligned} \qquad (10)$$

Proof. It follows from (8) that any polynomial of degree at most i is orthogonal to the kernel $T_i^{1,0}(t,\ell)$ with respect to the measure $\nu_{n,\ell}(t)$. Hence (9) follows from the positive definiteness of $d\nu_{n,\ell}(t)$ up to degree $k-1$ and the uniqueness of the Gram-Schmidt orthogonalization process.

Next, it follows from (8) and (9) that the zeros of $Q_i^{1,\ell}(t)$ are solutions of the equation

$$\frac{Q_{i+1}^{1,0}(t)}{Q_i^{1,0}(t)} = \frac{Q_{i+1}^{1,0}(\ell)}{Q_i^{1,0}(\ell)}. \qquad (11)$$

For all $i < k$ the zeros of $Q_{i+1}^{1,0}(t)$ and $Q_i^{1,0}(t)$ are interlaced and contained in the interval $\left[t_{k,1}^{1,0}, t_{k,k}^{1,0}\right]$. Since $\operatorname{sign} Q_i^{1,0}(\ell) = (-1)^i$, we have $Q_{i+1}^{1,0}(\ell)/Q_i^{1,0}(\ell) < 0$. The rational function $Q_{i+1}^{1,0}(t)/Q_i^{1,0}(t)$ has simple poles at $t_{i,j}^{1,0}$, $j = 1,\ldots,i$, and simple zeros at $t_{i+1,j}^{1,0}$, $j = 1,\ldots,i+1$. Therefore, there is at least one solution $t_{i,j}^{1,\ell}$ of (11) on each interval $\left(t_{i,j}^{1,0}, t_{i+1,j+1}^{1,0}\right)$, $j = 1,\ldots,i$, which accounts exactly for the zeros of $Q_i^{1,\ell}(t)$.

When $i = k$ we have $Q_{k+1}^{1,0}(\ell)/Q_k^{1,0}(\ell) > 0$. Since $\ell \in \left(t_{k+1,1}^{1,0}, t_{k,1}^{1,0}\right)$, we account similarly for the first $k-1$ solutions of (11), namely $t_{k,j}^{1,\ell} \in \left(t_{k+1,j+1}^{1,0}, t_{k,j+1}^{1,0}\right)$, $j = 1,\ldots,k-1$, to establish the interlacing properties (10). For the last zero of $Q_k^{1,\ell}(t)$ we use the fact that $Q_{k+1}^{1,0}(t)/Q_k^{1,0}(t) > 0$ for $t \in (t_{k+1,k+1}^{1,0}, \infty)$. As $\lim_{t\to\infty} Q_{k+1}^{1,0}(t)/Q_k^{1,0}(t) = \infty$, we have one more solution $t_{k,k}^{1,\ell}$ of (11).

Finally, $Q_{k+1}^{1,0}(\ell)/Q_k^{1,0}(\ell) < 1$ implies that $t_{k,k}^{1,\ell} < 1$ because $Q_{k+1}^{1,0}(1)/Q_k^{1,0}(1) = 1$. Comparison of coefficients in (9) yields $\eta_k^{1,\ell} > 0$. $\square$

The positive definiteness of the measure $d\nu_{n,\ell}(t)$ implies that

$$r_i^{1,\ell} := \left(\int_{-1}^{1} \left(Q_i^{1,\ell}(t)\right)^2 d\nu_{n,\ell}(t)\right)^{-1} > 0$$

for $i = 0,1,\ldots,k-1$. The three-term recurrence relation from Theorem 4.1 can be written as

$$(t - a_i^{1,\ell})Q_i^{1,\ell}(t) = b_i^{1,\ell} Q_{i+1}^{1,\ell}(t) + c_i^{1,\ell} Q_{i-1}^{1,\ell}(t),$$

$i = 1,2,\ldots,k-1$, where

$$b_i^{1,\ell} = \frac{\eta_{i+1}^{1,\ell}}{\eta_i^{1,\ell}} > 0,\ c_i^{1,\ell} = \frac{r_{i-1}^{1,\ell} b_{i-1}^{1,\ell}}{r_i^{1,\ell}} > 0,\ a_i^{1,\ell} = 1 - b_i^{1,\ell} - c_i^{1,\ell}.$$

The initial conditions are $Q_0^{1,\ell}(t) = 1$ and

$$Q_1^{1,\ell}(t) = \frac{nq(nq\ell + nq - 2n + 2)t + A}{2B},$$

where $A = n^2(q-1)(q\ell + q - 2) + n(q\ell + 5q - 6) - 2(q-2)$ and $B = n^2(q-2)(q\ell + q - 2) + 2n(q\ell + 4q - 3) - 4(q-2)$.

By Theorem 4.1 we have that the zeros of $\{Q_i^{1,\ell}(t)\}$ interlace; i.e. $t_{j,i}^{1,\ell} < t_{j-1,i}^{1,\ell} < t_{j,i+1}^{1,\ell}$, $\ i = 1,2,\ldots,j-1$.

Lemma 4.3: If $-1 \le \ell < t_{k,1}^{1,0}$, then $(t-\ell)Q_i^{1,\ell}(t) \in F_{\geq}$ for $i = 0,1,\ldots,k-1$.

Proof. It follows from the definition (8) of the kernels $T_i^{1,0}(x,y)$ and (9) that for $i = 0,1,\ldots,k-1$ we have

$$(t-\ell)Q_i^{1,\ell}(t) = \frac{1-\ell}{1-q_i}\left(Q_{i+1}^{1,0}(t) - q_i Q_i^{1,0}(t)\right),$$

where $q_i = Q_{i+1}^{1,0}(\ell)/Q_i^{1,0}(\ell) < 0$ as in the proof of Theorem 4.2. Now $Q_i^{1,0}(t) \in F_{\geq}$ (see [11, Eq. (3.91)]) completes the proof. $\square$

C. Construction and investigation of $Q_j^{1,\ell,s}(t)$ and Levenshtein-type polynomials

We construct the polynomials $Q_i^{1,\ell,s}(t)$ using the system $\{Q_i^{1,\ell}(t)\}_{i=0}^k$ from the previous section. Consider the Christoffel-Darboux kernel associated with the polynomials $Q_j^{1,\ell}(t)$:

$$\begin{aligned} R_i(x,y;\ell) := &\ \sum_{j=0}^i r_j^{1,\ell} Q_j^{1,\ell}(x) Q_j^{1,\ell}(y) \\ = &\ r_i^{1,\ell} b_i^{1,\ell} \frac{Q_{i+1}^{1,\ell}(x)Q_i^{1,\ell}(y) - Q_{i+1}^{1,\ell}(y)Q_i^{1,\ell}(x)}{x-y}, \end{aligned}$$

for $0 \leq i \leq k-1$. Given $t_{k,k}^{1,0} \leq s \leq t_{k,k}^{1,\ell}$ we define

$$Q_{k-1}^{1,\ell,s}(t) := \frac{R_{k-1}(t,s;\ell)}{R_{k-1}(1,s;\ell)}. \tag{12}$$

The proof of the next assertion is similar to that of Theorem 4.2 and we omit it.

Theorem 4.4: Let n, ℓ, s, and k be such that $\ell \in (t_{k+1,1}^{1,0}, t_{k,1}^{1,0})$, $Q_{k+1}^{1,0}(\ell)/Q_k^{1,0}(\ell) < 1$, $s \in [t_{k,k}^{1,0}, t_{k,k}^{1,\ell}]$, and $Q_k^{1,\ell}(s)/Q_{k-1}^{1,\ell}(s) > Q_k^{1,\ell}(\ell)/Q_{k-1}^{1,\ell}(\ell)$. Then the polynomial $Q_{k-1}^{1,\ell,s}(t)$ has $k-1$ simple zeros $\beta_1 < \beta_2 < \cdots < \beta_{k-1}$ such that $\beta_1 \in (\ell, t_{k-1,1}^{1,\ell})$ and $\beta_{i+1} \in (t_{k-1,i}^{1,\ell}, t_{k-1,i+1}^{1,\ell})$, $i = 1, 2, \ldots, k-2$.

We can already define the Levenshtein-type polynomial

$$f_{2k}^{(n,\ell,s)}(t) := (t-\ell)(t-s)\left(Q_{k-1}^{1,\ell,s}(t)\right)^2 \tag{13}$$

and proceed with an investigation of its properties. The next theorem is an analog of Theorem 5.39 from [11]. It involves the zeros of $f_{2k}^{(n,\ell,s)}(t)$ to form a right end-point Radau quadrature formula with positive weights.

Theorem 4.5: Let $\beta_1 < \beta_2 < \cdots < \beta_{k-1}$ be the zeros of the polynomial $Q_{k-1}^{1,\ell,s}(t)$. Then the Radau quadrature formula

$$\begin{aligned} f_0 &= \int_{-1}^1 f(t) d\mu_n(t) \qquad (14) \\ &= \rho_0 f(\ell) + \rho_k f(s) + \rho_{k+1} f(1) + \sum_{i=1}^{k-1} \rho_i f(\beta_i) \end{aligned}$$

is exact for all polynomials of degree at most $2k$ and has positive weights ρ_i, $i = 0, \ldots, k$. If $(t-\ell)Q_k^{1,\ell}(t) \in F_{\geq}$, then $\rho_{k+1} > 0$.

Proof. Let us denote by $L_i(t)$, $i = 0, 1, \ldots, k+1$, the Lagrange basic polynomials generated by the nodes $\ell < \beta_1 < \cdots < \beta_{k-1} < s < 1$. Defining $\rho_i := \int_{-1}^1 L_i(t) d\mu_n(t)$, $i = 0, 1, \ldots, k+1$, we observe that (14) is exact for the Lagrange basis and hence for all polynomials of degree at most $k+1$. Any polynomial $f(t)$ of degree at most $2k$ can be written as

$$f(t) = q(t)(t-\ell)(t-s)(1-t)Q_{k-1}^{1,\ell,s}(t) + g(t),$$

where $\deg(q) \leq k-2$ and $\deg(g) \leq k+1$. Then the orthogonality of $Q_{k-1}^{1,\ell,s}(t)$ to all polynomials of degree at most $k-2$ with respect to the measure $d\nu_{n,\ell,s}(t)$ and the fact that the right-hand side of (14) is the same for $f(t)$ and $g(t)$ show the exactness of the quadrature formula (14) for $f(t)$.

We next show the positivity of the weights ρ_i. Using $f(t) = (s-t)(1-t)\left(Q_{k-1}^{1,\ell,s}(t)\right)^2$ in (14) we obtain

$$\rho_0 f(\ell) = \int_{-1}^1 \left(Q_{k-1}^{1,\ell,s}(t)\right)^2 d\nu_{n,s}(t) > 0,$$

whence $\rho_0 > 0$, because $f(\ell) > 0$. Similarly, with $f(t) = (1-t)(t-\ell)\left(Q_{k-1}^{1,\ell,s}(t)\right)^2$ in (14) and the positive definiteness of $d\nu_{n,\ell}(t)$ we see that $\rho_k > 0$.

To see that $\rho_i > 0$ for $i = 1, 2, \ldots, k-1$, we use the polynomial $f(t) = (1-t)(t-\ell)(s-t)u_{k-1,i}^2(t)$ in (14), where $u_{k-1,i}(t) = Q_{k-1}^{1,\ell,s}(t)/(t-\beta_i)$. Then

$$\rho_i f(\beta_i) = \int_{-1}^1 u_{k-1,i}^2(t)\, d\nu_{m,\ell,s}(t) > 0$$

and $f(\beta_i) > 0$ implies that $\rho_i > 0$.

Finally, we prove that the weight ρ_{k+1} is positive. In this case we use $f(t) = f_{2k}^{(n,\ell,s)}(t)$ in (14) and find that

$$f_0 = \rho_{k+1} f(1) = \rho_{k+1}(1-s)(1-\ell).$$

Thus it is enough to see that the coefficient f_0 of $f_{2k}^{(n,\ell,s)}(t)$ is positive. We use (12) to obtain that f_0 is equal to

$$\begin{aligned} &\int_{-1}^1 (t-\ell)(s-t)(1-t)Q_{k-1}^{1,\ell,s}(t)\frac{Q_{k-1}^{1,\ell,s}(t) - Q_{k-1}^{1,\ell,s}(1)}{t-1} d\mu_n(t) \\ &\qquad + \int_{-1}^1 (t-\ell)(t-s)Q_{k-1}^{1,\ell,s}(t) d\mu_n(t) \qquad (15) \\ &\quad = \frac{1-s}{1-p_k}\int_{-1}^1 (t-\ell)\left(Q_k^{1,\ell}(t) - p_k Q_{k-1}^{1,\ell}(t)\right) d\mu_n(t), \end{aligned}$$

where $p_k = Q_k^{1,\ell}(s)/Q_{k-1}^{1,\ell}(s) < 0$. Then the last integrand belongs to $F_{\geq}$ and in particular its zero-th coefficient is positive. This completes the proof of the theorem. $\square$

V. Bounding Cardinalities and Energies

In the proof of the positive definiteness of his polynomials Levenshtein used (see [11, (3.88) and (3.92)]) what he called the strengthened Krein condition

$$(t+1)Q_i^{1,1}(t)Q_j^{1,1}(t) \in \mathcal{F}_{\geq}.$$

We need a following modification.

Definition 5.1: We say that the polynomials $\{Q_i^{1,\ell}(t)\}_{i=0}^k$ satisfy (k,ℓ)-strengthened Krein condition if

$$(t-\ell)Q_i^{1,\ell}(t)Q_j^{1,\ell}(t) \in F_{\geq} \tag{16}$$

for every $i, j \in \{0, 1, \ldots, k\}$ except possibly for $i = j = k$.

The strengthened Krein condition holds true in F_q^n for every i and j (see [11, Lemma 3.25]). However, the (k,ℓ)-strengthened Krein condition is not true for every ℓ, and for fixed ℓ, is not true for every k.

The main result in this paper is the following. It includes the analog of Theorem 5.42 of [11].

Theorem 5.2: Let n, q, k, $\ell \in [-1, t_{k,1}^{1,0})$, and $s \in (t_{k,k}^{1,0}, t_{k,k}^{1,\ell})$ be such that the (k,ℓ)-strengthened Krein condition holds true. Let $Q_k^{1,\ell}(s)/Q_{k-1}^{1,\ell}(s) > Q_k^{1,\ell}(\ell)/Q_{k-1}^{1,\ell}(\ell)$. Then

$$\mathcal{A}_q(n,s,\ell) \leq \frac{f_{2k}^{(n,\ell,s)}(1)}{f_0} = \frac{1}{\rho_{k+1}} = L_{2k}(n,\ell,s), \qquad (17)$$

where

$$L_{2k}(n,\ell,s) = \frac{S_k\left(Q_{k-1}^{1,\ell}(s) - Q_k^{1,\ell}(s)\right)}{\frac{r_{k+1}Q_{k+1}^{(n,q)}(\ell)Q_{k-1}^{1,\ell}(s)}{S_{k+1}\left(Q_{k+1}^{1,0}(\ell)-Q_k^{1,0}(\ell)\right)} - \frac{r_k Q_k^{(n,q)}(\ell)Q_k^{1,\ell}(s)}{S_{k-1}\left(Q_k^{1,0}(\ell)-Q_{k-1}^{1,0}(\ell)\right)}},$$

and $S_j = \sum_{i=0}^j r_i$, $j \in \{k-1, k, k+1\}$.

Furthermore, for fixed ℓ, for h being an absolutely monotone function, and for M determined by $f_{2k}^{(n,\ell,s)}(1) = Mf_0$, the Hermite interpolant[1] $g(t) = H((t-s)f_{2k}^{(n,\ell,s)}(t); h)$ belongs to $\mathcal{G}_{n,\ell;h}$, and, therefore,

$$\mathcal{E}_h(n,M,\ell) \geq M(Mg_0 - g(1)) = M^2 \sum_{i=0}^{k} \rho_i h(\beta_i). \qquad (18)$$

The bounds (17) and (18) are obtained only simultaneously by codes which have all their inner products in the roots of $f_{2k}^{(n,\ell,s)}(t)$ and which are, in addition, $2k$-designs in F_q^n.

Proof. It follows from the definitions (12) and (13) that $f_{2k}^{(n,\ell,s)}(t)$ can be written as

$$c(t-\ell)\left(Q_k^{1,\ell}(t) + c_1 Q_{k-1}^{1,\ell}(t)\right)\sum_{i=0}^{k-1} r_i^{1,\ell} Q_i^{1,\ell}(t) Q_i^{1,\ell}(s),$$

where $c = (1-s)/(1+c_1)R_{k-1}(1,\ell,s) > 0$ and $c_1 = -Q_k^{1,\ell}(s)/Q_{k-1}^{1,\ell}(s) > 0$ under the assumptions for ℓ and s. Since $Q_i^{1,\ell}(s) > 0$ for $0 \leq i \leq k-1$, the polynomial $f_{2k}^{(n,\ell,s)}(t)$ becomes positive linear combination of terms like $(t-\ell)Q_i^{1,\ell}(t)Q_j^{1,\ell}(t)$, where $i \in \{k, k-1\}$ and $j \leq k-1$. Therefore $f_{2k}^{(n,\ell,s)}(t) \in F_{\geq}$. This and the obvious $f_{2k}^{(n,\ell,s)}(t) \leq 0$ for every $t \in [\ell, s]$ implies that $f_{2k}^{(n,\ell,s)}(t) \in F_{n,s,\ell}$.

Computing f_0 as in (15) and then using the representation of $(t-\ell)Q_j^{1,\ell}(t)$ by the Christoffel-Darboux formula for $j = k-1$ and k we get a linear combination of $Q_i^{1,0}(t)$, $i = k-1, k, k+1$. Since

$$\int_{-1}^{1} Q_j^{1,0}(t)d\mu_n(t) = \int_{-1}^{1} \frac{T_j(t,1)}{T_j(1,1)} d\mu_n(t) = \frac{1}{S_j},$$

where $S_j = \sum_{i=0}^j r_i$, we obtain after simplifications the explicit form of the bound (17).

We proceed with the energy bound. Denote by $t_1 \leq t_2 \leq \cdots \leq t_{2k}$ the zeros of $f_{2k}^{(n,\ell,s)}(t)$ counting their multiplicity; i.e., $t_1 = \ell$, $t_{2i} = t_{2i+1} = \beta_i$, $i = 1, \ldots, k-1$, and $t_{2k} = s$. Then $g(t)$ is a linear combination with nonnegative coefficients of the constant 1 and the partial products

$$\prod_{j=1}^{m}(t - t_j), \quad m = 1, 2, \ldots, 2k.$$

[1]The notation $g = H(f; h)$ signifies that g is the Hermite interpolant to the function h at the zeros (taken with their multiplicity) of f.

Since t_{2i}, $i = 1, \ldots, k$, are the roots of $Q_k^{1,\ell}(t) + \alpha Q_{k-1}^{1,\ell}(t)$ (see (12)) it follows from [3, Theorem 3.1] that the partial products $\prod_{j=1}^m (t - t_{2j})$, $m = 1, \ldots, k-1$, have positive coefficients when expanded in terms of the polynomials $Q_i^{1,\ell}(t)$. Then $g(t)$ is a linear combination with positive coefficients of terms $(t-\ell)Q_i^{1,\ell}(t)Q_j^{1,\ell}(t)$ and the last partial product which is in fact $f_{2k}^{(n,\ell,s)}(t)$. Now $g(t) \in F_{\geq}$ follows from the validity of the (k,ℓ)-strengthened Krein condition and $f_{2k}^{(n,\ell,s)}(t) \in F_{\geq}$.

Multiple application of the Rolle's theorem implies that $g(t) \leq h(t)$ for every $t \in [\ell, 1)$ and therefore $g(t) \in \mathcal{G}_{n,\ell;h}$. The explicit form of the bound (18) via the weights ρ_i and the nodes β_i follows from the quadrature formula (14) and the interpolation conditions. This completes the proof. $\square$

The bound (17) was obtained and investigated for (in our notations) $k = 1$ and the corresponding ℓ and s by Helleseth, Kløve and Levenshtein [8]. In that paper, comparisons with the Levenshtein bound (see [10]) obtained by polynomials of degrees 2 and 3, and detailed descriptions of all known codes attaining $L_2(n,\ell,s)$ can be found.

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