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Unconstrained Polarization (Chebyshev) Problems: Basic Properties and Riesz Kernel Asymptotics

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Abstract

We introduce and study the unconstrained polarization (or Chebyshev) problem which requires to find an N-point configuration that maximizes the minimum value of its potential over a set A in p-dimensional Euclidean space. This problem is compared to the constrained problem in which the points are required to belong to the set A. We find that for Riesz kernels $1/|x-y|^s$ with $s > p-2$ the optimum unconstrained configurations concentrate close to the set A and based on this fundamental fact we recover the same asymptotic value of the polarization as for the more classical constrained problem on a class of d-rectifiable sets. We also investigate the new unconstrained problem in special cases such as for spheres and balls. In the last section we formulate some natural open problems and conjectures.

Keywords Maximal Riesz polarization · Unconstrained polarization · Chebyshev constant · Riesz potential

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1 Introduction and Statement of Main Results

Let A, B be two non-empty sets, and $K : B \times A \to (-\infty, +\infty]$ be a kernel (or pairwise potential). For $N \in \mathbb{N}$ we consider the max-min optimization problem

$$P_K(A, \omega_N) := \inf_{y \in A} \sum_{i=1}^{N} K(x_i, y), \quad \mathcal{P}_K(A, B, N) := \sup_{\omega_N \subset B} P_K(A, \omega_N), \tag{1.1}$$

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where the maximum is taken over N-point multisets $\omega_N = \{x_1, \ldots, x_N\} \subset B$. (Note that a multiset is a list where elements can be repeated.) The determination of Eq. 1.1 is called the *two-plate polarization (or Chebyshev) problem* (see Proposition 1.4 below for the link to the theory of Chebyshev polynomials, justifying this name). For background and motivation of the study of polarization problems, see [9, Chapter 14]. If $A' \subset A$ and $B' \subset B$ we note the basic monotonicity properties

$$\mathcal{P}_K(A', B, N) \geqslant \mathcal{P}_K(A, B, N), \qquad \mathcal{P}_K(A, B', N) \leqslant \mathcal{P}_K(A, B, N). \tag{1.2}$$

The case $A = B$ of Eq. 1.1, also known as *the single-plate polarization (or Chebyshev) problem* for A, has been the more studied so far (see [9, 16, 29]); and for it we introduce the notation

$$\mathcal{P}_K(A, N) := \mathcal{P}_K(A, A, N). \tag{1.3}$$

A related quantity is the value of the *minimum N-point K-energy*,[1] given by

$$\mathcal{E}_K(A, N) := \inf_{\omega_N \subset A} \sum_{i=1}^{N} \sum_{\substack{j:j\neq i \\ j=1}}^{N} K(x_i, x_j). \tag{1.4}$$

If $N \geqslant 2$, $A \subset B$ are compact sets and $K : B \times B \to (-\infty, +\infty]$ is a symmetric function, we have the following relation between the above quantities (see [9, Prop. 14.1.1], [16, Thm. 2.3])

$$\mathcal{P}_K(A, B, N) \geqslant \mathcal{P}_K(A, N) \geqslant \frac{\mathcal{E}_K(A, N+1)}{N+1} \geqslant \frac{\mathcal{E}_K(A, N)}{N-1}. \tag{1.5}$$

The goal of this article is to study the case $A \subset B = \mathbb{R}^p$ of Eq. 1.1, in which the configurations ω_N are *unconstrained*, and we use the notation

$$\mathcal{P}_K^*(A, N) := \mathcal{P}_K(A, \mathbb{R}^p, N) = \sup_{\omega_N \subset \mathbb{R}^p} P_K(A, \omega_N). \tag{1.6}$$

Directly from Eq. 1.2 and from the definitions Eq. 1.3 and Eq. 1.6, we find that

$$\mathcal{P}_K^*(A', N) \geqslant \mathcal{P}_K^*(A, N) \quad \text{whenever} \quad A' \subset A, \tag{1.7}$$

and, for all $A \subset \mathbb{R}^p$,

$$\mathcal{P}_K(A, N) \leqslant \mathcal{P}_K^*(A, N). \tag{1.8}$$

Our results are motivated by the study of the important class of kernels called *Riesz s-potentials*:

$$K_s(x, y) := \begin{cases} |x-y|^{-s} & \text{if } s > 0\,, \\ -\log|x-y| & \text{if } s = 0\,, \\ -|x-y|^{-s} & \text{if } s < 0\,. \end{cases} \tag{1.9}$$

In Eq. 1.9, we define $K_s(x, x) = +\infty$ if $s \geqslant 0$. For brevity we set

$$P_s(A, \omega_N) := P_{K_s}(A, \omega_N), \quad \mathcal{P}_s(A, N) := \mathcal{P}_{K_s}(A, N), \quad \mathcal{P}_s^*(A, N) := \mathcal{P}_{K_s}^*(A, N). \tag{1.10}$$

Note that the monotonicity property Eq. 1.7 is not true for $\mathcal{P}_s(A, N)$ (see [9, Sec. 14.2]) which, in some cases, may make the problem $\mathcal{P}_s^*(A, N)$ more tractable than $\mathcal{P}_s(A, N)$.

[1] See, e.g., the recent book [9] and recent articles [5], [32]

We shall refer to Eq. 1.3 as the *constrained polarization* problem and to Eq. 1.6 as the *unconstrained* problem.

The above definition Eq. 1.9 for $s = 0$ is justified by the results of Propositions 1.2 and 1.3, which say that optimal configurations for $\mathcal{P}_0$ are the limits as $s \downarrow 0$ of optimal configurations for the problems $\mathcal{P}_s$. The study of s-polarization for large values of s is related to *best-covering problems*; the limits of Eq. 1.10 as $s \to \infty$ yield best-covering constants, also treated in Propositions 1.2 and 1.3. In preparation for these propositions, we give the following definitions:

Definition 1.1 If $A \subset \mathbb{R}^p$ is a non-empty set, then the *covering radius* of a configuration $\omega_N = \{x_1, \ldots, x_N\}$ with respect to the set A is

$$\eta(\omega_N, A) = \sup_{x\in A} \min_{1\leqslant i\leqslant N} |x - x_i| \, . \tag{1.11}$$

The *minimal N-point covering radius of a set A relative to the set B* is defined as

$$\eta_N(A, B) := \inf\{\eta(\omega_N, A) : \ \omega_N \subset B, \#\omega_N = N\} \ . \tag{1.12}$$

The *minimal N-point covering radius $\eta_N(A)$ of A* and the *minimal N-point unconstrained covering radius $\eta_N^*(A)$ of A* are given by:

$$\eta_N(A) := \eta_N(A, A) \, , \qquad \eta_N^*(A) := \eta_N(A, \mathbb{R}^d) \, . \tag{1.13}$$

Proposition 1.2 ([12, Thm. III.2.1, Thm. III.2.2]) *If $N \in \mathbb{N}$ is fixed and A is an infinite compact subset of $\mathbb{R}^p$, then*

$$\lim_{s\to\infty} (\mathcal{P}_s(A, N))^{1/s} = \frac{1}{\eta_N(A)} \, , \tag{1.14}$$

$$\lim_{s\to 0^+} \frac{\mathcal{P}_s(A, N) - N}{s} = \mathcal{P}_0(A, N) \, . \tag{1.15}$$

Moreover, every cluster point of $\mathcal{P}_s(A, N)$-optimizers in Eq. 1.14 *is an optimal configuration for $\eta_N(A)$ and every cluster point of $\mathcal{P}_s(A, N)$-optimizers in* Eq. 1.15 *is an optimal configuration for $\mathcal{P}_0(A, N)$.*

The above results have their basis in the observations:

$$\lim_{s\to\infty} \left(\sum_{j=1}^{N} |x_j - a|^{-s} \right)^{1/s} = \left(\min_j (|x_j - a| \right)^{-1} \text{ and } \lim_{s\to 0^+} \frac{|x_j - a|^{-s} - 1}{s} = -\log|x_j - a| .$$

A generalization of Eq. 1.14 for the problem $\mathcal{P}_s(A, B, N)$ is presented in [9, §14.4]. By the same proof as in [12], we find the analogous asymptotics for the $\mathcal{P}_s^*(A, N)$ problem:

Proposition 1.3 *The assertions of Proposition 1.2 hold if we replace $\mathcal{P}_s(A, N)$ and $\eta_N(A)$, respectively, by $\mathcal{P}_s^*(A, N)$ and $\eta_N^*(A)$.*

An important case, which justifies the alternative name "Chebyshev problem", is the setting of $s = 0$, $p = 2$, namely the study of polarization problems for the kernel $K(x, y) = -\log|x - y|$ in $\mathbb{R}^2$, here identified with $\mathbb{C}$. Indeed let $A \subset \mathbb{C}$ be an infinite compact set. A monic complex polynomial T_N^A of degree N is called *the Chebyshev polynomial of degree N corresponding to the set A* if $\|T_N^A\|_A \leqslant \|p\|_A$ for any monic complex polynomial p of degree N, where

$$\|p\|_A := \max_{z\in A} |p(z)| \tag{1.16}$$

is the max norm on the set A. Then denoting by $z_1, \dots, z_N$ the zeros of the polynomial p repeated according to their multiplicity and using an algebraic manipulation, we rewrite Eq. 1.16 in the equivalent form

$$\log\frac{1}{\|p\|_A} = \log\frac{1}{\max_{z\in A}\prod_{j=1}^N |z - z_j|} = \min_{z\in A}\sum_{j=1}^N \log\frac{1}{|z - z_j|}, \tag{1.17}$$

which directly gives a proof of the following well-known result:

Proposition 1.4 *Let $A \subset \mathbb{C}$ be an infinite compact set. A multiset $\omega_N^* = \{z_1, \dots, z_N\}$ is optimal for the maximal unconstrained polarization problem on A with respect to the logarithmic potential if and only if $T(z) = (z - z_1)\cdots(z - z_N)$ is the Chebyshev polynomial for A. If T_N^A is such a polynomial, then $\mathcal{P}_0^*(A, N) = \log(1/\|T_N^A\|_A)$.*

It is well known that T_N^A is unique (see [39, Thm. III.23]), and that by a classical result of Fejér [20] the zeros of T_N^A lie in the convex hull of A; but need not lie on A. For example, an application of the maximum modulus principle shows that $T_N^A(z) = z^N$ is the unique Chebyshev polynomial for the unit circle $A = \mathbb{S}^1$. This observation generalizes to the principle that optimal unconstrained polarization configurations may accumulate away from the set A if $K(x, y)$ is superharmonic in y (see Proposition 2.2 below).

We recall (see [31]) that in a very general setting we may relate the two-plate polarization problem to the so-called *continuous two-plate polarization (Chebyshev) constant $T_K(A, B)$* defined in Eq. 1.19 below. The next theorem describes the large N limit of discrete two-plate polarization.

Theorem 1.5 ([31]) *Let X, Y be locally compact nonempty Hausdorff spaces, $A \subset X$ be compact nonempty and $B \subset Y$ be nonempty, and the kernel $K : X \times Y \to (-\infty, +\infty]$ be a lower semi-continuous function. Then*

$$\lim_{N\to\infty}\frac{\mathcal{P}_K(A, B, N)}{N} = T_K(A, B), \tag{1.18}$$

where

$$T_K(A, B) := \sup_{\mu\in\mathcal{M}_1(B)}\inf_{x\in A}\int K(x, y)d\mu(y) \in (-\infty, +\infty], \tag{1.19}$$

and $\mathcal{M}_1(B)$ is the set of all probability measures with compact support contained in B.

The integral $\int K(x, y)d\mu(y)$ in Eq. 1.19 is called the *K-potential of μ*.

Known results relating the discrete and continuous one-plate ($A = B$) polarization problems for a *continuous* kernel directly extend to the two-plate case:

Theorem 1.6 ([9, Prop. 14.6.6]) *Let A and B be two nonempty compact metric spaces, and $K \in C(A \times B)$. A sequence $\{\omega_N\}_{N=1}^{\infty}$ of N-point configurations on B satisfies*

$$\lim_{N\to\infty} \frac{P_K(A, \omega_N)}{N} = T_K(A, B) \tag{1.20}$$

if and only if every weak-$*$ limit measure μ of the sequence of the normalized counting measures

$$\left\{ \nu(\omega_N) := \frac{1}{N} \sum_{x\in\omega_N} \delta_x \right\}_{N=1}^{\infty}$$

is an extremal measure for the continuous 2-plate polarization problem; i.e., it satisfies

$$T_K(A, B) = \inf_{x\in A} \int K(x, y)d\mu(y). \tag{1.21}$$

We remark that there are few results regarding uniqueness of the extremal measure for the above problem (e.g., see [16], [34], and [37]).

1.1 Main Results

Our first important property can be viewed as a generalization of the aforementioned result of Fejér [20] for zeros of Chebyshev polynomials. It asserts that for a large class of kernels, the two problems $P_K^*(A, N)$ and $P_K(A, N)$ are equivalent when A is convex. Hereafter, we always assume $A, B \subset \mathbb{R}^p$, $A \neq \emptyset$ and let $\mathrm{conv}(A)$ denote the *convex hull* of A.

Proposition 1.7 *Let $f : [0, +\infty) \to (-\infty, +\infty]$ be a strictly decreasing function and let $K(x, y) := f(|x - y|)$. If $A \subset \mathbb{R}^p$ is a compact set, then any configuration $\omega_N^* = \{x_1, \ldots, x_N\}$ such that $P_K(A, \omega_N^*) = \mathcal{P}_K^*(A, N) < +\infty$ has the property that $x_i \in conv(A)$ for each $1 \leqslant i \leqslant N$. In particular, if A is convex and $\mathcal{P}_K^*(A, N) < +\infty$ then $\omega_N^* \subset A$ and $\mathcal{P}_K(A, N) = \mathcal{P}_K^*(A, N)$.*

Proof Assume to the contrary that some point of ω_N^*, say x_1, satisfies $x_1 \notin \mathrm{conv}(A)$. Then after replacing x_1 by the nearest-point projection $\pi_{\mathrm{conv}(A)}(x_1)$, the sum $\sum_{i=1}^N K(x_i, y)$ strictly increases, contradicting the optimality of ω_N^*. □

In the following result and hereafter we denote by $\#\omega$ the cardinality of a multiset ω including repetitions. We utilize the following notation for the ϵ-neighborhood of a set:

$$A_\epsilon := \{x \in \mathbb{R}^p : \ \mathrm{dist}(x, A) < \epsilon\}. \tag{1.22}$$

Our next result is also a generalization of a well known property of the zeros of Chebyshev polynomials T_N^A that lie outside the polynomial convex hull of the set $A \subset \mathbb{C}$. Namely, for any compact subset F of the unbounded component of the complement of A, there is a number $M = M_F$ depending only on F such that each T_N^A has at most M zeros in F (see e.g. [35, Theorem III.3.4]). The proof of the following theorem which concerns Riesz kernels will be given in Section 3.

Theorem 1.8 *Let $A \subset \mathbb{R}^p$ be a compact set and assume $s > p - 2$, $p \geqslant 2$. There exist $\kappa_{s,p}, c_{s,p} > 0$ depending only on p and s, such that for every $\epsilon > 0$, if $\mathcal{P}_s^*(A, N) < +\infty$ and $\omega_N^* = \{x_1, \ldots, x_N\}$ satisfies*

$$P_s(A, \omega_N^*) = \mathcal{P}_s^*(A, N), \tag{1.23}$$

(i.e., ω_N^* is an unconstrained N-point maximal s-Riesz polarization configuration), then

$$\#\left(\omega_N^* \setminus A_\epsilon\right) \leqslant \kappa_{s,p} \frac{\mathcal{L}_p\left((conv(A))_{\epsilon c_{s,p}}\right)}{\epsilon^p}, \tag{1.24}$$

where $\mathcal{L}_p$ denotes p-dimensional Lebesgue measure on $\mathbb{R}^p$.

As a direct consequence of Theorem 1.8, for any compact set B in the complement of A the number of points of ω_N^* in B is uniformly bounded in N. In particular, we get the following:

Corollary 1.9 *Under the hypotheses of Theorem 1.8, if $\left(\omega_N^*\right)_{N\in\mathbb{N}}$ is a sequence of unconstrained N-point maximal s-Riesz polarization configurations, then any weak* cluster point of the sequence*

$$\nu(\omega_N^*) := \frac{1}{N} \sum_{x\in\omega_N^*} \delta_x, \qquad N = 1, 2, 3, \ldots, \tag{1.25}$$

is a probability measure supported on A.

We now describe new asymptotic results for $\mathcal{P}_s^*(A, N)$ for fixed s and $N \to \infty$ and their connection to the previously known asymptotics for $\mathcal{P}_s(A, N)$. We define the *renormalization factors* and relevant asymptotic quantities as follows:

$$\tau_{s,d}(N) := \begin{cases} N^{s/d}, & s > d, \\ N \log N, & s = d, \\ N, & s < d, \end{cases} \tag{1.26}$$

and

$$\underline{h}_{s,d}^*(A) := \liminf_{N\to\infty} \frac{\mathcal{P}_s^*(A, N)}{\tau_{s,d}(N)}, \quad \overline{h}_{s,d}^*(A) := \limsup_{N\to\infty} \frac{\mathcal{P}_s^*(A, N)}{\tau_{s,d}(N)}. \tag{1.27a}$$

If $\underline{h}_{s,d}^*(A) = \overline{h}_{s,d}^*(A)$, we set

$$h_{s,d}^*(A) := \lim_{N\to\infty} \frac{\mathcal{P}_s^*(A, N)}{\tau_{s,d}(N)}. \tag{1.27b}$$

For the problem $\mathcal{P}_s(A, N)$, the quantities $\underline{h}_{s,d}(A)$, $\overline{h}_{s,d}(A)$ and $h_{s,d}(A)$ were analogously defined in [10] and [11].

As a consequence of Theorem 1.8 in combination with a new geometric deformation technique for optimizers of $\mathcal{P}_s^*(A, N)$ given in Proposition 4.6, we provide conditions that the asymptotics of $\mathcal{P}_s^*(A, N)$ are equal to those of $\mathcal{P}_s(A, N)$. For this purpose, we make use of the following definition.

Definition 1.10 Let $d' > 0$ and let $p > 0$ be an integer. A compact set $A \subset \mathbb{R}^p$ is d'-*regular* if there exists a measure λ supported on A and a positive constant C such that for any $x \in A$ and $r < \operatorname{diam}(A)$ there holds

$$C^{-1} r^{d'} \leqslant \lambda(B(x, r)) \leqslant C r^{d'} . \tag{1.28}$$

A measure μ is called *upper-d-regular* at x if for some constant $c(x)$ and any $r > 0$ there holds

$$\mu(B(x, r)) \leqslant c(x) r^d . \tag{1.29}$$

Hereafter, we denote by $\mathcal{H}_d$ the d-dimensional Hausdorff measure on $\mathbb{R}^p$, $d \leqslant p$, normalized so that the $\mathcal{H}_d$-measure of a d-dimensional unit cube embedded in $\mathbb{R}^p$ is 1. Furthermore, for a compact set $A \subset \mathbb{R}^p$ with $0 \leqslant s < \dim_{\mathcal{H}}(A)$ (where $\dim_{\mathcal{H}}$ denotes the Hausdorff dimension), the *equilibrium measure* $\mu_{s,A}$ is the unique probability measure supported on A that minimizes

$$\int\int K_s(x, y) d\mu(x)\, d\mu(y)$$

over all probability measures supported on A.

Theorem 1.11 *For integers p, d such that $p \geqslant 2$, $1 \leqslant d \leqslant p$ and $A \subset \mathbb{R}^p$ compact, suppose that one of the following conditions holds:*

(i) $s > \max\{d, p-2\}$ *and* $\mathcal{H}_d(A) > 0$,
(ii) $p-2 \leqslant s < d$ *and for some* $d \leqslant d' \leqslant p$, *the set* A *is* d'*-regular and the equilibrium measure* $\mu_{s,A}$ *on* A *is upper* d*-regular at every point* $x \in A$.

If the limit $h^*_{s,d}(A)$ exists as an extended real number, then the limit $h_{s,d}(A)$ also exists and

$$h_{s,d}(A) = h^*_{s,d}(A). \tag{1.30}$$

Furthermore, if (ii) holds, then $h^*_{s,d}(A)$ exists and is finite; consequently Eq. 1.30 *holds.*

We remark that our method of proof of Theorem 1.11 given in Section 4 requires the weak-separation result from Proposition 4.2 which makes use of the assumptions (i) and (ii) above.

Concerning the actual values of the quantities in Eq. 1.30, it is established in [11] (also, see [9, Chapter 14]) that, for A equal the unit cube in $\mathbb{R}^p$ and $s \geqslant p$, the limit

$$\sigma_{s,p} := h_{s,p}([0,1]^p) \tag{1.31}$$

exists as a finite and positive number.

In preparation for the following theorems, we say that a sequence of N-point configurations, denoted $\Omega := \{\omega_N\}_{N\geqslant 1}$, is *asymptotically extremal for the unconstrained problem* if $\lim_{N\to\infty} P_s(A, \omega_N)/\mathcal{P}^*_s(A, N) = 1$, with a similar definition for the constrained problem.

Theorem 1.12 *If $A \subset \mathbb{R}^p$ is a compact set and $s \geqslant p$, then*

$$h^*_{s,p}(A) = h_{s,p}(A) = \frac{\sigma_{s,p}}{\mathcal{L}_p(A)^{s/p}}. \tag{1.32}$$

Moreover, if $\mathcal{L}_p(A) > 0$, then for any asymptotically extremal sequence $\Omega = \{\omega_N\}_{N\geqslant 1}$ (for either the constrained or unconstrained polarization problem) we have the weak-$*$ convergence

$$\frac{1}{N}\sum_{x_i\in\omega_N} \delta_{x_i} \xrightarrow{*} \frac{\mathcal{L}_p|_A}{\mathcal{L}_p(A)} \quad as \quad N \to \infty, \tag{1.33}$$

where $\mathcal{L}_p|_A := \mathcal{L}_p(\cdot \cap A)$ is the restriction to A of $\mathcal{L}_p$.

We further note the following:

- The second equality in Eq. 1.32 for $s > p$ improves upon the corresponding constrained result in [11] which required the additional assumption that $\mathcal{L}_p(\partial A) = 0$.
- For $s = p$, the second equality in Eq. 1.32 was proved in [10].

- Our next result, Theorem 1.14, is a generalization of Theorem 1.12 since the hypotheses of the former are trivially satisfied for $d = p$. Theorem 1.12 is stated separately because it plays an essential role in the proof of the more general theorem and is of independent interest.
- As shown in [16], it is known that $\sigma_{p,p} = \beta_p$, which is the volume of the p-dimensional unit ball.
- As follows by the result of [24] for $\mathcal{P}_s(\mathbb{S}^1, N)$, $\sigma_{s,1} = 2\zeta(s)(2^s - 1)$ for $s > 1$.
- For $p = 2, s > 2$, the conjecture in [11, §2] for $\sigma_{s,2}$ is equivalent to the conjecture that $\sigma_{s,2} = (3^{s/2} - 1)\zeta_\Lambda(s)/2$, where

$$\zeta_\Lambda(s) = \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{\left((n + m/2)^2 + 3m^2/4\right)^{s/2}} \tag{1.34}$$

is the Epstein zeta-function for the hexagonal lattice $\Lambda \subset \mathbb{R}^2$.

Theorem 1.12 provides asymptotics for compact sets of full dimension. We next consider embedded sets for which it is necessary to consider certain geometric constraints. Following [21], we say that a set $A \subset \mathbb{R}^p$ is d*-rectifiable* if it can be written as $\phi(K)$ for $K \subset \mathbb{R}^d$ bounded and $\phi : K \to \mathbb{R}^p$ Lipschitz. We note the following two important properties of a d-rectifiable set A (see [21, Theorems 3.2.18 & 3.2.39]): (a) if A is closed then $\mathcal{H}_d(A)$ equals the d-dimensional Minkowski content $\mathcal{M}_d(A)$ (see Definition 6.1 for the definition of Minkowski content) and (b) for each $\epsilon > 0$, A can be written as the disjoint union

$$A = A_0 \cup \bigcup_{j=1}^{\infty} \varphi_j(K_j), \tag{1.35}$$

where $\mathcal{H}_d(A_0) = 0$, the maps $\varphi_j : K_j \to \varphi_j(K_j)$ are $(1+\epsilon)$-biLipschitz and $K_j \subset \mathbb{R}^d$ are compact sets. We remark that any set $A \subset \mathbb{R}^p$ of the form Eq. 1.35 is called $(\mathcal{H}_d, d)$*-rectifiable*. Setting

$$R_k := A_0 \cup \bigcup_{j=k+1}^{\infty} \varphi_j(K_j), \tag{1.36}$$

we may take k large enough so that $\mathcal{H}_d(R_k) < \epsilon$.

For $\epsilon > 0$ and positive integers $d \leqslant p$, we say that $\mathcal{G}$ is a d*-dimensional* ϵ*-Lipschitz graph in* $\mathbb{R}^p$ if there is a d-dimensional subspace $H \subset \mathbb{R}^p$ and an ϵ-Lipschitz mapping $\psi : H \to H^\perp$ such that $\mathcal{G} = \{h + \psi(h) : h \in H\}$. It is useful to note that given an isometry $\iota : \mathbb{R}^d \to H$, the mapping $\varphi : \mathbb{R}^d \to \mathcal{G}$ defined by $\varphi(x) = \iota(x) + \psi(\iota(x))$ is a $(1+\epsilon)$-biLipschitz mapping. Now we introduce the following stronger requirement, needed below.

Definition 1.13 We say that $A \subset \mathbb{R}^p$ is *strongly* $(\mathcal{H}_d, d)$*-rectifiable* if for each $\epsilon > 0$ there exist a compact set $R_\epsilon \subset \mathbb{R}^p$ with $\overline{\mathcal{M}}_d(R_\epsilon) < \epsilon$ and finitely many compact, pairwise disjoint sets $\widetilde{K}_1, \ldots, \widetilde{K}_k \subset \mathbb{R}^p$ such that

$$A = R_\epsilon \cup \bigcup_{j=1}^{k} \widetilde{K}_j, \tag{1.37}$$

where each $\widetilde{K}_j$ is contained in some d-dimensional ϵ-Lipschitz graph $\mathcal{G}_j$ in $\mathbb{R}^p$.

As shown in Lemma 7.4, a compact subset of a C^1-manifold of dimension d contained in $\mathbb{R}^p$, $d \leqslant p$, is strongly $(\mathcal{H}_d, d)$-rectifiable. It can be shown that any strongly $(\mathcal{H}_d, d)$-rectifiable set is $(\mathcal{H}_d, d)$-rectifiable.

Theorem 1.14 *Let d and p be positive integers with $d \leqslant p$, and $A \subset \mathbb{R}^p$ be a compact strongly $(\mathcal{H}_d, d)$-rectifiable set. If $s > d$, then*

$$h_{s,d}(A) = h^*_{s,d}(A) = \frac{\sigma_{s,d}}{\left[\mathcal{H}_d(A)\right]^{s/d}}. \tag{1.38}$$

Moreover, if $\mathcal{H}_d(A) > 0$, then for any asymptotically K_s-extremal sequence $\Omega = \{\omega_N\}_{N\geqslant 1}$ (for either the constrained or unconstrained polarization problem) we have the weak-$*$ convergence

$$\frac{1}{N}\sum_{x_i\in\omega_N}\delta_{x_i} \xrightarrow{*} \frac{\mathcal{H}_d|_A}{\mathcal{H}_d(A)} \quad \text{as} \quad N\to\infty, \tag{1.39}$$

where $\mathcal{H}_d|_A := \mathcal{H}_d(\cdot \cap A)$ is the restriction to A of the Hausdorff measure $\mathcal{H}_d$.

1.2 Outline of the Paper

Section 2 includes some results and conjectures for unconstrained polarization on the circle and on higher dimensional spheres. Sections 3, 4, 5 and 7 are dedicated to auxiliary results and proofs of the main results stated in the introduction. Section 6 contains a bound of independent interest, stated in Proposition 6.2, and later used in the proof of Theorem 1.14 in Section 7. Finally, in Section 8 we discuss some open problems related to polarization.

2 Results for the Case of Spheres $\mathbb{S}^{p-1} \subset \mathbb{R}^p$

This section is dedicated to results for an important special case, the unconstrained polarization on the unit sphere $\mathbb{S}^{p-1} \subset \mathbb{R}^p$. We start with the following simple result, valid for rather general kernels.

Proposition 2.1 *Let $f : [0, +\infty) \to (-\infty, +\infty]$ be a strictly decreasing function and $K(x, y) := f(|x-y|)$. If $p \geqslant 2$, $1 \leqslant N \leqslant p$ and $\omega^*_N = \{x_1, \ldots, x_N\}$ satisfies*

$$\mathcal{P}^*_K(\mathbb{S}^{p-1}, N) = \min_{y\in A}\sum_{i=1}^{N} K(x_i, y)\ , \tag{2.1}$$

then $x_j = 0$ for all $1 \leqslant j \leqslant N$.

Proof Let $N' \geqslant 1$ be the smallest natural number such that there exists an N'-point configuration $\omega^*_{N'} = \{x_1, \ldots, x_{N'}\}$ satisfying Eq. 2.1 such that for some $1 \leqslant j \leqslant N'$ there holds $x_j \neq 0$. We will prove by contradiction that $N' \geqslant d+1$, which is equivalent to our statement.

Up to reordering the points, there exists $k \in \{0, \ldots, N'\}$ such that

$$x_j \neq 0 \text{ for } j = 1, \ldots, k\ , \qquad x_j = 0 \text{ for } j = k+1, \ldots, N'\ .$$

Let $\omega_{N'}^0 := \{0, \ldots, 0\}$ the configuration composed of N' instances of the origin. Then

$$0 \leqslant P_K(\mathbb{S}^{p-1}, \omega_{N'}) - P_K(\mathbb{S}^{p-1}, \omega_{N'}^0) = P_K(\mathbb{S}^{p-1}, \{x_1, \ldots, x_k\}) - P_K(\mathbb{S}^{p-1}, \omega_k^0) \ . \quad (2.2)$$

thus by the minimality of N' we obtain $k = 0$, and *all* points in $\omega_{N'}^*$ are away from the origin.

As f is decreasing, for each $x \in \mathbb{R}^p$ the set S_x composed of all points y at which the potential generated by 0 is higher than that generated by x is a half-space containing the origin. More precisely,

$$S_x := \{y \in \mathbb{R}^p : \ K(0, y) > K(x, y)\} = \{y \in \mathbb{R}^p : \ \langle x, y\rangle < |y|/2\} \ . \quad (2.3)$$

The intersection of N' half-spaces $S_{x_1}, \ldots, S_{x_{N'}}$ is a convex set containing the origin. If $N' < d + 1$ this intersection is also unbounded, and thus it intersects $\mathbb{S}^{p-1}$ at some point y_0. Therefore, using Eq. 2.3, for $N' < d + 1$ we find

$$\forall 1 \leqslant i \leqslant N' , \quad K(0, y_0) > K(x_j, y_0) \ . \quad (2.4)$$

Summing up the inequalities Eq. 2.4, we find a contradiction to Eq. 2.2, and thus $N' \geqslant d + 1$, as desired. □

By Theorem 1.8, for subharmonic Riesz kernels (i.e. for $s > p - 2 \geqslant 0$), points do not accumulate away from A. In contrast, the following result demonstrates that the opposite property can hold for superharmonic Riesz potentials. This proposition generalizes a result from [16].

Proposition 2.2 *Fix $p \geqslant 2$ and $s \in (-\infty, p - 2]$. If the compact set $A \subset \mathbb{R}^p$ is such that*

$$\mathbb{S}^{p-1} \subset A \subset \mathbb{B}^p \ , \quad (2.5)$$

where $\mathbb{B}^p$ denotes the unit ball in $\mathbb{R}^p$ centered at the origin, then a multiset $\omega_N^* = \{x_1, \ldots, x_N\} \subset \mathbb{R}^p$ satisfies

$$P_s(A, \omega_N^*) = \mathcal{P}_s^*(A, N) \ , \quad (2.6)$$

if and only if $x_i = 0$ for all $i \in \{1, \ldots, N\}$.

Proof We first note that for all $-\infty < s \leqslant p - 2$ the function $K_s(x, y) = f_s(|x - y|)$ as defined in Eq. 1.9 is superharmonic in x and in y separately.

Step 1. *The case $A = \mathbb{S}^{p-1}$.* We consider the case $A = \mathbb{S}^{p-1}$ first. Let $\omega_N^* = \{x_1, \ldots, x_N\}$ and $y^* \in \mathbb{S}^{p-1}$ are such that there holds

$$\mathcal{P}_s^*(\mathbb{S}^{p-1}, N) = P_s^*(\mathbb{S}^{p-1}, \omega_N^*) = \sum_{i=1}^{N} K_s(x_i, y^*) \ . \quad (2.7)$$

We assume that the points composing ω_N^* are ordered so that for some $0 \leqslant N_0 \leqslant N$ there holds $x_1, \ldots, x_{N_0} \neq 0$ and $x_{N_0+1} = \cdots = x_N = 0$. For any choice of $y_0 \in \mathbb{S}^{p-1}$ and denoting $\mu_{SO(p)}$ the right-invariant Haar measure on $SO(p)$, there holds

$$
\begin{aligned}
P_s^*(\mathbb{S}^{p-1}, \omega_N^*) - N f_s(1) &= \sum_{i=1}^{N_0} K_s(x_i, y^*) - N_0 f_s(1) = \min_{y \in \mathbb{S}^{p-1}} \sum_{i=1}^{N_0} f_s(|x_i - y|) - N_0 f_s(1) \\
&\leqslant \int_{SO(p)} \sum_{i=1}^{N_0} f_s(|x_i - R y_0|) d\mu_{SO(p)}(R) - N_0 f_s(1) && (2.8) \\
&= \int_{SO(p)} \sum_{i=1}^{N_0} f_s(|R^{-1} x_i - y_0|) d\mu_{SO(p)}(R) - N_0 f_s(1) && (2.9) \\
&= \sum_{i=1}^{N_0} \frac{1}{\mathcal{H}_{p-1}(\partial B(0, |x_i|))} \int_{\partial B(0,|x_i|)} f_s(|x - y_0|) d\mathcal{H}_{p-1}(x) - N_0 f_s(1) \\
&\leqslant 0\,, && (2.10)
\end{aligned}
$$

where in Eq. 2.9 we used the fact that rotations $R \in SO(p)$ preserve distances and in Eq. 2.10 we used the fact that $f_s(|x-y|)$ is superharmonic in x. By Eq. 2.7 this shows that the choice $x_i = 0$ for all $1 \leqslant i \leqslant N$, realizes the optimum in $\mathcal{P}_s^*(\mathbb{S}^{p-1}, \omega_N^*)$. On the other hand, in order for ω_N^* to be an optimizer, inequalities Eq. 2.8 and Eq. 2.10 must become equalities, thus the value $\sum_{i=1}^{N_0} f_s(|x_i - y|)$ is constant in $y \in \mathbb{S}^{p-1}$ and hence the multiset ω_N^* is invariant under rotation. This in turn is possible only if all the points x_i are at the origin, as desired.

Step 2. *The case* $A = \mathbb{B}^p$. In this case by Proposition 1.7 we have that the problem reduces to the classical constrained polarization, and the statement was proved in [16].

Step 3. *General case* $\mathbb{S}^{p-1} \subset A \subset \mathbb{B}^p$. Due to Eq. 1.7, Eq. 2.10 and Step 2, we have $\mathcal{P}_s^*(A, N) = N f_s(1)$ as well. For any multiset $\{x_1, \ldots, x_N\}$ there holds

$$
\min_{y \in \mathbb{B}^p} \sum_{i=1}^{N} K_s(x_i, y) \leqslant \min_{y \in A} \sum_{i=1}^{N} K_s(x_i, y) \leqslant \min_{y \in \mathbb{S}^{p-1}} \sum_{i=1}^{N} K_s(x_i, y)\,, \tag{2.11}
$$

which implies that the multiset with $x_i = 0$ for all $1 \leqslant i \leqslant N$ is an optimizer for $\mathcal{P}_s^*(A, N)$. If by contradiction a distinct optimizer would exist, then it would be an optimizer also for $\mathcal{P}_s^*(\mathbb{S}^{p-1}, N)$, which is excluded by Step 1. This concludes the proof of Proposition 2.2. □

The following proposition describes the case where $s > p-2$, $s \neq p-1$, and establishes that $\mathcal{P}_s^*$-optimal configurations ω_N^* for $\mathbb{S}^{p-1}$ lie at a positive distance from $\mathbb{S}^{p-1}$. We conjecture that the result continues to hold for $s = p-1$, but the proof is left to future work.

Proposition 2.3 *Let* $p \geqslant 2$ *and* $s > p-2$, $s \neq p-1$. *Then there exists a constant* $C > 0$ *depending only on* s *and* p, *such that for any* N*-point multiset* ω_N^* *satisfying* $P_s(\mathbb{S}^{p-1}, \omega_N^*) = \mathcal{P}_s^*(\mathbb{S}^{p-1}, N)$ *there holds*

$$
dist(\omega_N^*, \mathbb{S}^{p-1}) \geqslant C N^{-2/(p-1)}\,. \tag{2.12}
$$

Proof For the proof, we will use Proposition 4.5 below, with s, p as in the statement of the lemma and $A = \mathbb{S}^{p-1}$. We need to verify that the hypotheses of Theorem 1.11 (which are inherited by Proposition 4.5) hold. Note that by Proposition 3.1, the equilibrium measure $\mu_{s,\mathbb{S}^{p-1}}$ is the uniform measure on $\mathbb{S}^{p-1}$ and that $\mathcal{H}_{p-1}(\mathbb{S}^{p-1}) > 0$. Therefore the hypotheses of point (i) from Theorem 1.11 hold if $p-2 < s < p-1$ and the conditions of point (ii) from the same theorem hold if $s > p-1$.

By Eq. 4.4 of Proposition 4.5 there exists a constant $C_1 > 0$ independent of N such that the minimum value

$$\mathcal{P}_s^*(\mathbb{S}^{p-1}, N) = \min_{y\in\mathbb{S}^{p-1}} \sum_{x\in\omega_N^*} |x-y|^{-s}$$

is not achieved at $y \in B(x_0, C_1 N^{-1/(p-1)})$. Since the function $|x-y|^{-s}$ is continuous away from the diagonal, there exists $\epsilon > 0$ such that

$$\min_{y\in B(x_0, C_1 N^{-1/(p-1)})} \sum_{x\in\omega_N^*} |x-y|^{-s} \geqslant \mathcal{P}_s^*(\mathbb{S}^{p-1}, N) + \epsilon\,. \tag{2.13}$$

We claim that

$$x_0 \in \operatorname{conv}(\mathbb{S}^{p-1} \setminus B(x_0, C_1 N^{-1/(p-1)})). \tag{2.14}$$

Assume by contradiction that this is not true and that

$$\operatorname{dist}\Big(x_0, \operatorname{conv}(\mathbb{S}^{p-1} \setminus B(x_0, C_1 N^{-1/(p-1)}))\Big) > 0. \tag{2.15}$$

Let $x_0' \neq x_0$ be the projection of x_0 on $\operatorname{conv}(\mathbb{S}^{p-1} \setminus B(x_0, C_1 N^{-1/(p-1)}))$. Since $|x-y|^{-s}$ is decreasing in $|x-y|$, we find that for any x_0'' in the segment $(x_0, x_0']$ there holds

$$\forall y \in \mathbb{S}^{p-1} \setminus B(x_0, C_1 N^{-1/(p-1)})\,, \quad \sum_{x\in\omega_N^*} |x-y|^{-s} < \sum_{x\in\omega_N^*\setminus\{x_0\}} |x-y|^{-s} + |x_0''-y|^{-s}\,. \tag{2.16}$$

On the other hand, by Eq. 2.13, Eq. 2.15 and by the continuity of $|x-y|^{-s}$ for $x \neq y$, there exists an open neighborhood U_ϵ of x_0 in the segment $[x_0, x_0']$ such that for $x_0'' \in U_\epsilon$ there holds

$$\min_{y\in B(x_0, C_1 N^{-1/(p-1)})} \sum_{x\in\omega_N^*\setminus\{x_0\}} |x-y|^{-s} + |x_0''-y|^{-s} \geqslant \mathcal{P}_s^*(\mathbb{S}^{p-1}, N) + \epsilon/2\,. \tag{2.17}$$

By Eq. 2.16 and Eq. 2.17 there holds $P_s(\mathbb{S}^{p-1}, (\omega_N^* \setminus \{x_0\}) \cup \{x_0''\}) \geqslant \mathcal{P}_s^*(\mathbb{S}^{p-1}, N) + \epsilon/2$, giving the desired contradiction to Eq. 2.15. Thus Eq. 2.14 holds.

It follows that for any point $y \in \partial B(x_0, C_1 N^{-1/(p-1)}) \cap \mathbb{S}^{p-1}$ there holds $\langle x_0 - y, x_0\rangle \leqslant 0$ and thus $1 = |y| \leqslant |x_0|^2 + |x_0-y|^2 = |x_0|^2 + C_1^2 N^{-2/(p-1)}$, from which it follows that $|x_0| \leqslant \sqrt{1 - C_1^2 N^{-2/(p-1)}} \leqslant 1 - \frac{C_1^2}{2} N^{-2/(p-1)}$ and the thesis follows with $C = C_1^2/2$. $\square$

The next result states the equivalence of constrained and unconstrained covering problems, which is a well-known property of spherical coverings:

Proposition 2.4 *Let $p \geqslant 2$ and $N \in \mathbb{N}$.*

- *If $N \leqslant p$, then a configuration realizing the infimum $\eta_N^*(\mathbb{S}^{p-1})$ in* Eq. 1.13 *is given by taking all the N points at the origin of $\mathbb{R}^p$.*

- *If $N \geqslant p+1$, then for every configuration $\omega_N = \{x_1, \dots, x_N\} \subset \mathbb{S}^{p-1}$ that realizes the infimum $\eta_N(\mathbb{S}^{p-1})$ in* Eq. 1.13*, then $\omega_N^* := \{r_N x_1, \dots, r_N x_N\}$ realizes the infimum $\eta_N^*(S^{p-1})$ in* Eq. 1.13 *for $r_N = \sqrt{1-(\eta_N^*)^2(\mathbb{S}^{p-1})}$. Furthermore,*

$$(\eta_N^*)^2(\mathbb{S}^{p-1}) = \eta_N^2(\mathbb{S}^{p-1}) - \frac{1}{4}\eta_N^4(\mathbb{S}^{p-1}). \tag{2.18}$$

Proof It is not difficult to verify that $\mathbb{S}^{p-1}$ cannot be covered by p balls of radius less than 1. This can be proved by induction on the dimension using the fact that if $r \in (0,1)$, $y \in \mathbb{R}^p$, then $\mathbb{S}^{p-1} \setminus B(y,r)$ contains a congruent copy of $\mathbb{S}^{p-2}$ as long as $p \geqslant 2$, and the observation that $\mathbb{S}^0 = \{\pm 1\}$ requires at least two balls of radius r to be covered. It follows that $\eta_N^*(\mathbb{S}^{p-1}) = 1$: a configuration realizing this infimum is given by the case when all the N points are at the origin of $\mathbb{R}^p$. This proves the first item of the proposition.

If $N \geqslant p+1$ then $\eta_N(\mathbb{S}^{p-1}) < \sqrt{2}$ and $\eta_N^*(\mathbb{S}^{p-1}) < 1$, a bound shown by rough estimates for competitor configurations for $\eta_N(\mathbb{S}^{p-1})$ where $p+1$ of the points form a regular simplex, and those for $\eta_N^*(\mathbb{S}^{p-1})$ sit at centroids of the faces of such simplex.

In order to compare the two covering problems with minima $\eta_N(\mathbb{S}^{p-1})$ and $\eta_N^*(\mathbb{S}^{p-1})$ for $N \geqslant p+1$, we introduce some notations, as follows. For $y \in \mathbb{B}^p \setminus \{0\}$ the set $\partial B(y,\rho) \cap \mathbb{S}^{p-1}$ is nonempty if and only if $\rho \in [1-|y|, 1+|y|]$. For these choices of y, ρ, we note the following:

- $\partial B(y,\rho)) \cap \mathbb{S}^{p-1}$ is a congruent copy of a $(p-2)$-dimensional sphere of radius given by $f(|y|,\rho) = \frac{\sqrt{4|y|^2-(|y|^2+1-\rho^2)^2}}{2|y|}$. Moreover, for fixed $\rho \in (0,1)$ the function $(1-\rho, 1) \ni |y| \mapsto f(|y|,\rho)$ achieves its unique maximum, equal to ρ, at $|y| = \sqrt{1-\rho^2}$.
- There holds $B(y,\rho) \cap \mathbb{S}^{p-1} = B(y/|y|, \bar\rho) \cap \mathbb{S}^{p-1}$, with $\bar\rho = \bar\rho(|y|,\rho) := \sqrt{\frac{\rho^2-(1-|y|)^2}{|y|}}$. Moreover, we note that in the above range of ρ, for $|y| = \sqrt{1-\rho^2}$ we get $\bar\rho = \sqrt{2}\sqrt{1-\sqrt{1-\rho^2}}$, which is increasing in ρ.

Now for $N \geqslant p+1$ let $\omega_N = \{x_1, \dots, x_N\} \subset \mathbb{S}^{p-1}$ be at optimizer configuration for $\eta_N(\mathbb{S}^{p-1})$, in particular

$$\bigcup_{j=1}^N B(x_j, \eta_N(\mathbb{S}^{p-1})) \supset \mathbb{S}^{p-1}, \tag{2.19}$$

and define simlarly to the claim of the second bullet in the proposition

$$\rho' := \eta_N^2(\mathbb{S}^{p-1}) - \frac{1}{4}\eta_N^4(\mathbb{S}^{p-1}) \quad \text{and} \quad r' := \sqrt{1-(\rho')^2}. \tag{2.20}$$

With this notation the balls $B(r'x_j, \rho'), 1 \leqslant j \leqslant N$ cover $\mathbb{S}^{p-1}$. Indeed, $B(x_j, \eta_N(\mathbb{S}^{p-1})) \cap \mathbb{S}^{p-1} = B(r'x_j, \rho') \cap \mathbb{S}^{p-1}$ and the claim follows from Eq. 2.19. This implies that $\rho' \geqslant \eta_N^*(\mathbb{S}^{p-1})$ as well. Also note that

$$f(r',\rho') = \rho' \quad \text{and} \quad \bar\rho(r',\rho') = \bar\rho(\sqrt{1-(\rho')^2}, \rho') = \eta_N(\mathbb{S}^{p-1}). \tag{2.21}$$

We next prove that $\rho' = \eta_N^*(\mathbb{S}^{p-1})$, which directly implies Eq. 2.18. Assume that this were not true, and that we had $\rho' > \eta_N^*(\mathbb{S}^{p-1})$. Then there would exist $\widetilde{\rho} < \rho'$ and a configuration $\{y_1, \ldots, y_N\} \subset \mathbb{B}^p \setminus \{0\}$ such that

$$\bigcup_{j=1}^{N} B(y_j, \widetilde{\rho}) \supset \mathbb{S}^{p-1}. \tag{2.22}$$

Up to moving some of the points radially, we may suppose that $|y_j|$ are all equal to $\sqrt{1-\widetilde{\rho}^2}$ at which the function $f(\cdot, \widetilde{\rho})$ achieves its maximum. Since $\rho \mapsto \bar{\rho}(\sqrt{1-\rho^2}, \rho)$ is increasing, we find that

$$\bar{\rho}(\sqrt{1-\widetilde{\rho}^2}, \widetilde{\rho}) < \bar{\rho}(\sqrt{1-(\rho')^2}, \rho') = \eta_N(\mathbb{S}^{p-1}), \tag{2.23}$$

where we used Eq. 2.21. But due to the geometric interpretation of $\bar{\rho}$ and to Eq. 2.22, we also have that

$$\bigcup_{j=1}^{N} B\left(\frac{y_j}{|y_j|}, \bar{\rho}\left(\sqrt{1-\widetilde{\rho}^2}, \widetilde{\rho}\right)\right) \supset \mathbb{S}^{p-1},$$

which implies $\bar{\rho}(\sqrt{1-\widetilde{\rho}^2}, \widetilde{\rho}) \geqslant \eta_N(\mathbb{S}^{p-1})$, which as desired contradicts Eq. 2.23. Therefore we have $\rho' = \eta_N^*(\mathbb{S}^{p-1})$, and the second bullet of the proposition follows. $\square$

2.1 Results for $\mathbb{S}^1 \subset \mathbb{R}^2$

For $\mathbb{S}^1$ it is easily seen that the minimal N-point covering optimal configurations constrained to $\mathbb{S}^1$ are given by the vertices of the inscribed regular N-gon. In the case of the minimal N-point *unconstrained* covering we prove the following more precise version of Proposition 2.4:

Proposition 2.5 *The configurations ω_N^* realizing the infimum in the definition of $\eta_N^*(\mathbb{S}^1)$ are, up to rotation, the following.*

- *For $N = 1$, $\omega_1^* = \{0\}$.*
- *For $N = 2$, $\omega_2^* = \{0, 0\}$.*
- *For $N \geqslant 3$, ω_N^* consists of the midpoints of the sides of the regular N-gon inscribed in $\mathbb{S}^1$.*

Proof The cases $N < 2$ of the statement follows directly from Proposition 2.1 by using Proposition 1.3. Therefore we assume $N \geqslant 3$ for the rest of the proof.

The midpoints of the sides of an inscribed regular N-gon are given by

$$p_j := (\cos(\pi/N)\cos(\theta+2\pi j/N), \cos(\pi/N)\sin(\theta+2\pi j/N)) \in \mathbb{R}^2, \quad \text{for } j \in \{0, \ldots, N-1\}, \tag{2.24}$$

where $\theta \in [0, 2\pi/N)$ gives the orientation of our N-gon. A closed disk of radius $\sin(\pi/N)$ centered at p_j covers the interval $I_j := \{(\cos\phi, \sin\phi) : \phi \in [\theta + (2j-1)\pi/N, \theta + (2j+1)\pi/N]\}$ inside $\mathbb{S}^1$, thus the union of all such disks covers the unit circle $\mathbb{S}^1$. Note that $\mathbb{S}^1 \cap B(x, r)$ is always an arc of the form

$$I(\theta_0, \rho) := \{(\cos\phi, \sin\phi) : \phi \in [\theta_0 - \rho, \theta_0 + \rho]\}. \tag{2.25}$$

By direct computation of the local minimum, we find for $0 \leqslant \rho < \pi/2$

$$\min\{r > 0 :\ \exists x \in \mathbb{R}^2,\ I(\theta_0, \rho) = \mathbb{S}^1 \cap B(x, r)\} = \sin(\rho)\ , \tag{2.26}$$

and the unique x realizing the above minimum is the point $(\cos(\rho)\cos(\theta_0), \cos(\rho)\sin(\theta_0))$. Further, we have that for fixed N if N arcs $I(\theta_j, \rho_j)$, $j = 1, \ldots, N$ cover $\mathbb{S}^1$ then

$$\sum_{j=1}^{N} 2\rho_j \geqslant 2\pi \quad \text{and} \quad \max_{1\leqslant j\leqslant N} \rho_j \geqslant \pi/N\ ,$$

and thus

$$\min\left\{\rho > 0 :\ \exists\theta_1, \rho_1, \ldots, \theta_N, \rho_N,\ \bigcup_{j=1}^{N} I(\theta_j, \rho_j) = \mathbb{S}^1,\ \max_{1\leqslant j\leqslant N} \rho_j \leqslant \rho\right\} = \pi/N\ , \tag{2.27}$$

and the minimum is realized by a collection of equal intervals. Noting that for $\rho \leqslant \pi/N, N \geqslant 3$ the function $\rho \mapsto \sin(\rho)$ is increasing, we find that as a consequence of Eq. 2.27 and Eq. 2.26, there holds

$$\min\left\{r > 0 :\ \exists\omega_N = \{x_1, \ldots, x_N\},\ \bigcup_{j=1}^{N} B(x_i, r_j) \supset \mathbb{S}^1,\ \max_{1\leqslant j\leqslant N} r_j \leqslant r\right\} = \sin(\pi/N)\ ,$$

and the minimum is realized by the points p_j from Eq. 2.24. This completes the proof of Proposition 2.5. □

For N-point constrained Riesz s-polarization on $\mathbb{S}^1$, it is proved in [24] that optimal configurations are again equally spaced points on $\mathbb{S}^1$, for each $0 < s < \infty$. The proof of Proposition 2.6 below follows the strategy of [24]. For related results, see also [1, 2, 16] and [18]. For the unconstrained s-polarization we have not yet determined the precise optimizers $\omega^*_{N,s}$. However, numerical evidence (see Fig. 1) strongly suggests that for $N \geqslant 3$ the configurations form a regular N-gon inscribed in a circle of radius $\bar{r}_{N,s} < 1$, where

$$\bar{r}_{N,s} := \arg\max_{r\in[0,1]} \sum_{j=1}^{N} \left(r^2 + 1 - 2\cos\frac{(2j+1)\pi}{2N}\right)^{-s/2}. \tag{2.28}$$

We remark that for fixed $N \geqslant 3$, Propositions 1.3 and 2.5 imply that as $s \to \infty$ maximal N-point unconstrained polarization configurations $\omega^*_{N,s}$ (with one of the points fixed at 1) approach the midpoints of the sides of a regular N-gon in $\mathbb{S}^1$.

Under the extra assumption that the optimal configuration $\omega^*_{N,s}$ lies on a concentric circle with radius r satisfying Eq. 2.38, we are able to establish the above conjecture, based on the following result, which is of independent interest and improves the main result of [24] by removing the convexity condition for f on the interval $(0, \pi/N]$.

If we take $t \in [-\pi, \pi]$ to parametrize the counterclockwise signed angle between two points $x, y \in \mathbb{S}^1$, then the *geodesic distance* between x and y is given by $\mathrm{dist}_{\mathbb{S}^1}(x, y) := \min\{|t|, |2\pi - t|\}$.

Proposition 2.6 *For $x, y \in \mathbb{S}^1$ let $\mathrm{dist}_{\mathbb{S}^1}(x, y) \in [0, \pi]$ be the geodesic distance (or smallest angle) between x, y, and set $K(x, y) := f(\mathrm{dist}_{\mathbb{S}^1}(x, y))$, for $f : [0, \pi] \to (-\infty, +\infty]$, and assume that the following hypotheses hold:*

(i) *the function f is strictly decreasing on $(0, \pi]$ and convex on $(\frac{\pi}{N}, \pi]$;*

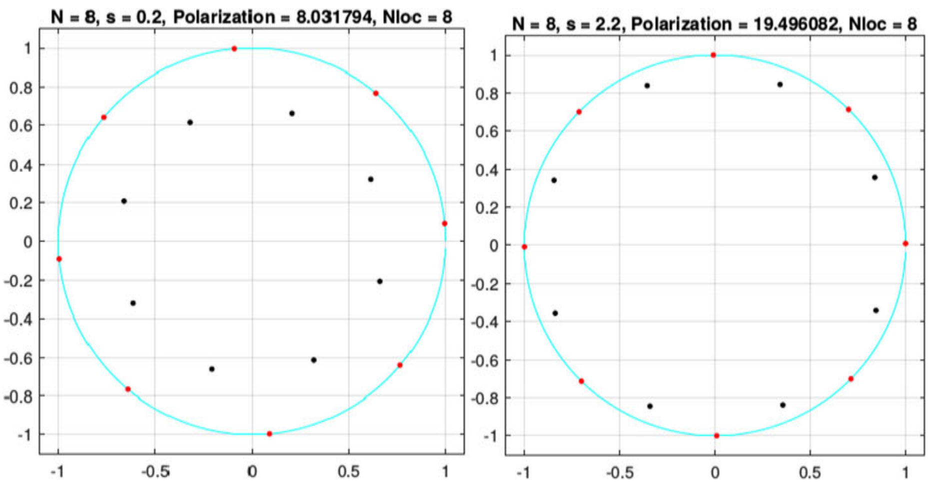

Fig. 1 Numerically computed optimizers ω_N^* (black points) for unconstrained Riesz s-polarization for $\mathbb{S}^1$ and points on $\mathbb{S}^1$ where the minimum of the s-potential is attained (red points) for $N = 8$ and the values $s = 0.2$ and $s = 2.2$

(ii) *for the configuration* $\omega_{N,\text{eq}} \subset \mathbb{S}^1$ *given by* $x_k = e^{i\frac{2\pi k}{N}}$ *for* $k = 1, \ldots, N$, *the minimum value* $P_K(\omega_{N,\text{eq}})$ *is achieved at the midpoints of the arcs between successive points* x_k, x_{k+1}.

Then any configuration $\omega_N^* \subset \mathbb{S}^1$ *that satisfies* $P_K(\mathbb{S}^1, \omega_N^*) = \mathcal{P}_K(\mathbb{S}^1, N)$ *equals* $\omega_{N,\text{eq}}$, *up to rotation.*

Proof We recall that the proof in [24, Thm. 1] consisted of starting from a general N-point configuration $x_1, \ldots, x_N \in \mathbb{S}^1$, initially ordered in counterclockwise manner, and applying a sequence of N elementary moves to the points (see [24, Lem. 5]). The elementary moves are denoted $\tau_{\Delta_k^*}$, with $1 \leqslant k \leqslant N$, $\Delta_k^* \in \mathbb{R}$. The move $\tau_{\Delta_k^*}$ leaves the positions of $x_1, \ldots, x_{k-1}, x_{k+2}, \ldots, x_N$ unchanged, and replaces the points x_k and x_{k+1} (with indices taken modulo N) by new points $x_k' := x_k e^{-i\Delta_k^*}$ and $x_{k+1}' := x_{k+1} e^{i\Delta_k^*}$, respectively. A simple linear algebra argument shows (see [24, Lem. 5]) that there is a sequence of elementary moves such that:

(a) $\Delta_k^* \geqslant 0$ for $k = 1, \ldots, N$,
(b) There exists $1 \leqslant j \leqslant N$ such that $\Delta_j^* = 0$,
(c) The composition $\tau_{\Delta^*} := \tau_{\Delta_N^*} \circ \cdots \circ \tau_{\Delta_1^*}$ sends ω_N to a rotation of the configuration $\omega_{N,\text{eq}}$.

We first assume that none of the elementary moves change the counterclockwise ordering of the points. Let x^* denote the midpoint of the arc between $\tau_{\Delta^*}(x_j), \tau_{\Delta^*}(x_{j+1})$ for j as in (b). By the above properties, we can prove by backwards induction on k that

$$\min_{y \in \tau_{\Delta_k^*} \circ \cdots \circ \tau_{\Delta_1^*}(\omega_N)} \text{dist}_{\mathbb{S}^1}(y, x^*) \geqslant \frac{\pi}{N}. \tag{2.29}$$

Indeed, this is true for $k = N$ due to item (c) above; furthermore, if it is true for $k = n$ for some $2 \leqslant n \leqslant N$ then due to items (a), (b) then it also holds for $k = n - 1$.

Next, as in [24, Lem. 4], we prove that the potential generated by the points *increases* on the arc $\gamma_{k,N}$ going from $x'_{k+1}e^{i\pi/N}$ to $x'_k e^{-i\pi/N}$ in the counterclockwise direction, during the move $\tau_{\Delta_k^*}$. Towards this end, let $x \in \gamma_{k,N}$ and consider $\ell_k = \mathrm{dist}_{\mathbb{S}^1}(x_k, x)$, $\ell'_k = \mathrm{dist}_{\mathbb{S}^1}(x'_k, x)$, $\ell_{k+1} = \mathrm{dist}_{\mathbb{S}^1}(x_{k+1}, x)$, and $\ell'_{k+1} = \mathrm{dist}_{\mathbb{S}^1}(x'_{k+1}, x)$. Without loss of generality we may assume $\ell_k \leqslant \ell_{k+1}$, in which case we note that

$$\ell'_k = \ell_k - \Delta_k^* \text{ and } \ell'_{k+1} \leqslant \ell_{k+1} + \Delta_k^*. \tag{2.30}$$

Extending f as a decreasing convex function on $[\pi/N, \infty)$ and using $\ell_k \leqslant \ell_{k+1}$, it follows that

$$\begin{aligned} &\left[f(\ell'_{k+1}) - f(\ell_{k+1})\right] + \left[f(\ell'_k) - f(\ell_k)\right] \geqslant \left[f(\ell_{k+1} + \Delta_k^*) - f(\ell_{k+1})\right] \\ &+ \left[f(\ell_k - \Delta_k^*) - f(\ell_k)\right] \geqslant 0. \end{aligned} \tag{2.31}$$

Due to Eq. 2.29, x^* belongs to all the intervals $\gamma_{k,N}$ as above, for $k = 1, \dots, N, k \neq j$. As a consequence of the inequality Eq. 2.31, during the sequence of moves as in the above steps (a),(b),(c) the value of the polarization potential at x^* increases. Thus we have

$$P_K(\omega_N) \leqslant \sum_{x \in \omega_N} f(\mathrm{dist}_{\mathbb{S}^1}(x, x^*)) \leqslant \sum_{x \in \omega_{N,\mathrm{eq}}} f(\mathrm{dist}_{\mathbb{S}^1}(x, x^*)) = P_K(\omega_{N,\mathrm{eq}}), \tag{2.32}$$

where for the last equality we used hypothesis (ii). This shows that $\omega_{N,\mathrm{eq}}$ is an optimal configuration, as desired.

If not all of the elementary moves preserve the counterclockwise ordering, then we modify the above argument by considering compositions of moves $\tau_{t_k\Delta^*} := \tau_{t_k\Delta_N^*} \circ \cdots \circ \tau_{t_k\Delta_1^*}$, $k = 1, \dots, n$, for $t_k > 0$ sufficiently small so that the ordering is preserved (see [24, Lem. 6]) at each step and such that $\sum_{k=1}^n t_k = 1$.

If f is strictly convex on $[\pi/N, \pi]$, then the fact that in the middle inequality in Eq. 2.32 the equality holds, implies that during all the moves all the terms as in Eq. 2.32 are zero, which can only be true if $\Delta_k^* = 0$ for all k, showing that $\omega_N = \omega_{N,\mathrm{eq}}$ up to rotation in this case. □

The following lemma gives two important cases in which the hypothesis (ii) from Proposition 2.6 holds, the second of which is due to Nikolov and Rafailov [30, Thm. 1.2 (1)].

Lemma 2.7 *Let $K : \mathbb{S}^1 \times \mathbb{S}^1 \to (-\infty, +\infty]$ be given by $K(x, y) = f(\mathrm{dist}_{\mathbb{S}^1}(x, y))$ for a fixed function $f : [0, \pi] \to (-\infty, +\infty]$. Assume that we are in one of the following cases:*

(i) *the function f satisfies the hypothesis (i) of Proposition 2.6 and furthermore*

$$\min_{\theta \in [0, \pi/N]} \left(f(\theta) + f\left(\frac{2\pi}{N} - \theta\right)\right) = 2f\left(\frac{\pi}{N}\right);$$

(ii) *there exist $R, s > 0$ such that $f(t) = (R^2 + 1 - 2R\cos(t))^{-s/2}$.*

Then hypothesis (ii) of Proposition 2.6 holds, namely

$$\min_{y \in \mathbb{S}^1} \sum_{k=1}^N K\left(e^{i\frac{2\pi k}{N}}, y\right) = \sum_{k=1}^N K\left(e^{i\frac{2\pi k}{N}}, e^{i\frac{\pi}{N}}\right). \tag{2.33}$$

Proof The proof of the claim in the case (ii) is precisely [30, Thm. 1.2 (1)], therefore we need to prove the claim in the case (i) only.

Let $x_k = e^{i\frac{2\pi k}{N}}$ for $k = 1, \ldots, N$. By symmetry, we consider the values of $\sum_{k=1}^{N} f(\mathrm{dist}_{\mathbb{S}^1}(x_k, y))$ only for $y = e^{i\theta}$ with $\theta \in [0, \pi/N]$. We split $\omega_{N,\mathrm{eq}}$ into pairs of points x_k, x_{N-k}, for $k = 1, \ldots, \lfloor N/2 \rfloor$, to which we add, if $N = 2n-1$ is odd, the potential $f(\mathrm{dist}_{\mathbb{S}^1}(x_n, y)) = \min\{|\pi - \theta|, |\pi + \theta|\}$. The latter potential has a minimum at $\theta = 0$ due to the decreasing nature of f. For the remaining pairs of points, we claim that the potential of each pair has a minimum at $\theta = 0$ as well, and by superposition this will prove the claim.

The points x_k, x_{N-k} generate at $e^{i\theta}$ the joint potential equal to

$$f\left(\mathrm{dist}_{\mathbb{S}^1}(x_k, e^{i\theta})\right) + f\left(\mathrm{dist}_{\mathbb{S}^1}(x_{N-k}, e^{i\theta})\right) = f\left(\frac{2k-1}{N}\pi - \theta\right) + f\left(\frac{2k+1}{N}\pi + \theta\right). \tag{2.34}$$

For $k = 1$ this is minimized at $\theta = 0$ by the second hypothesis on f from the statement of the proposition, whereas for $k > 1$ we may use the convexity of f to obtain that $(f(a) + f(b))/2 \geqslant f((a+b)/2)$ for $a = (2k-1)\pi/N - \theta$ and $b = (2k+1)\pi/N + \theta$, in order to show again, using the symmetry of the configuration that the minimum is achieved at $\theta = 0$, as desired. □

Corollary 2.8 *Let $s, r > 0$, $N \in \mathbb{N}$, $A = \mathbb{S}^1$, $B = r\mathbb{S}^1$, and define*

$$x_{r,s} := \frac{-(1+r^2) + \sqrt{(1+r^2)^2 + 4r^2 s(2+s)}}{2rs} \tag{2.35}$$

and

$$R_{N,s} := \frac{1}{2}\sec(\pi/N)\left(s\sin^2(\pi/N) + 2 + \sqrt{\sin^2(\pi/N)\left((s+2)^2 - s^2\cos^2(\pi/N)\right)}\right). \tag{2.36}$$

If N, r, and s satisfy

$$\cos(\pi/N) \leqslant x_{r,s}, \tag{2.37}$$

or

$$R_{N,s}^{-1} \leqslant r \leqslant R_{N,s}, \tag{2.38}$$

then any $\omega_N^* \subset B$ such that $P_s(A, \omega_N^*) = \mathcal{P}_s(A, B, N)$ equals, up to rotation, the regular N-gon inscribed in the circle B.

Proof The function $f_s(\theta) := (1 + r^2 - 2r\cos\theta)^{-s/2}$ is decreasing for $\theta \in [0, \pi]$. Differentiating f_s twice gives

$$f_s''(\theta) = -\frac{rs}{2}\,\frac{2(1+r^2)\cos\theta + r(-4 + 2s(\cos^2\theta - 1))}{(1 + r^2 - 2r\cos\theta)^{2+s/2}}.$$

Letting $g(r, s, x) := 2(1+r^2)x + r(-4+2s(x^2-1))$ then $f_s''(\theta)$ is positive on any interval where $g(r, s, \cos\theta)$ is negative. Noting that $g(r, s, x)$ is an increasing function of x (with r and s fixed) for $x > 0$ and that $g(r, s, x_{r,s}) = 0$ shows that $g(r, s, x) \leqslant 0$ if and only if $x \in [-1, x_{r,s}]$. Hence, if Eq. 2.37 holds, then f_s is convex on $[\pi/N, \pi] \subset [\arccos x_{r,s}, \pi]$ and so we may use Proposition 2.6 to prove that any $\omega_N^* \subset B$ such that $P_s(A, \omega_N^*) = \mathcal{P}_s(A, B, N)$ must consist of N equally spaced points in the circle B.

To complete the proof we show that Eq. 2.38 implies that Eq. 2.37 holds. Towards this end, let

$$r_{x,s}^{\pm} := \frac{s + 2 - sx^2 \pm \sqrt{(1-x^2)\left((s+2)^2 - s^2x^2\right)}}{2x},$$

denote the solutions to $g(r,s,x)=0$ for fixed $x>0$ and s and note that $g(r,s,x)<0$ for $r^-_{x,s}<r<r^+_{x,s}$ and $r^-_{x,s}r^+_{x,s}=1$. Observe that $R_{N,s}=r^+_{\cos(\pi/N),s}$ and $R^{-1}_{N,s}=r^-_{\cos(\pi/N),s}$. Therefore, if $R^{-1}_{N,s}\leqslant r\leqslant R_{N,s}$, we have $g(r,s,\cos(\pi/N))\leqslant 0$ and so it follows that $\cos(\pi/N)\leqslant x_{r,s}$; i.e., that Eq. 2.37 holds. □

Note that, due to the fact that $|x-y|^{-s}$ is symmetric, up to inverting the roles of A, B we can restrict to the case $r\in[\bar r_{N,s},1]$, where $\bar r_{N,s}$ is as in Eq. 2.28. We found good numerical evidence (as shown in special cases in Fig. 2) that for $s>0$ there exists $N_0(s)\in\mathbb{N}$ such that for all $N\geqslant N_0(s)$ there holds $R^{-1}_{N,s}<\bar r_{N,s}<1<R_{N,s}$, therefore the range Eq. 2.38 of r in which Corollary 2.8 applies includes the expected radius $\bar r_{N,s}$ from Eq. 2.28. We found numerically that $N_0(s)=2$ for $s\geqslant 0.7$.

3 Proof of Theorem 1.8

We first prove an auxiliary result, Lemma 3.3, and then proceed to the proof of Theorem 1.8. This result in turn uses the result stated in Remark 3.2, a special case of Proposition 3.1. A result similar to Lemma 3.3, with a non-sharp version of bound Eq. 3.6 below, and with an additional convexity requirement on the set A, appears in [33, Thm. 2.3]. What allows us to obtain a stronger result are two ingredients: (a) the precise statement on homogeneous spaces of Proposition 3.1 and, in particular, the study of the case of spheres described in Remark 3.2; and (b) the fact that we don't need to restrict to convex sets A simplifies our constructions.

Let G be a locally compact topological group. We recall that a metric space X is a *homogeneous space with group* G if there exists a transitive G-action on X, i.e., for each $x,y\in X$ there exists $g\in G$ such that $g(x)=y$. In this case we may assume that there exists a subgroup $H\subset G$ such that $X=G/H$, endowed with the canonical multiplication action of G (see [28]). In this case G acts on X transitively. If G is compact, then we denote by $\mathcal{H}_{X,G}$ the unique probability measure on X that is invariant under each $g\in G$, which is the projection of the Haar measure of G.

Proposition 3.1 *Let G be a locally compact topological group and X be a compact homogeneous space with group G and let $K:X\times X\to(-\infty,+\infty]$ be a lower semicontinuous*

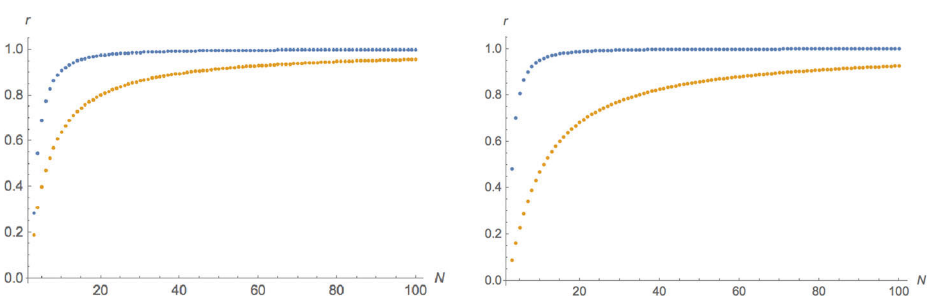

Fig. 2 Graphs of $\bar r_{N,s}$ from Eq. 2.28 (blue dots) and $R^{-1}_{N,s}$ from Eq. 2.36 (orange dots) for $s=1$ (left) and $s=5$ (right) as N ranges from 3 to 100. In both cases, $R^{-1}_{N,s}<\bar r_{N,s}$, therefore the range Eq. 2.38 of r in which Corollary 2.8 applies includes the expected radius $\bar r_{N,s}$ from Eq. 2.28

kernel that satisfies $K(g(x), g(y)) = K(x, y)$ *for every* $x, y \in X$ *and for every* $g \in G$. *Then the continuous single-plate polarization problem*

$$T_K(X) := \max_{\mu\in\mathcal{M}_1(X)} \min_{y\in X} \int K(x, y)d\mu(x) \tag{3.1}$$

is realized by $\mathcal{H}_{X,G}$. *Moreover, a probability measure* $\mu \in \mathcal{M}_1(X)$ *is an optimizer of* Eq. 3.1 *if and only if the* K*-potential of* μ *is constant on* X, *and we have*

$$\int K(x, y)d\mu(x) = \int K(x, y)d\mathcal{H}_{X,G}(x) = T_K(X) \quad \textit{for all } y \in X. \tag{3.2}$$

As emphasized in Remark 3.2, polarization-optimizing measures need not be unique.

For use in the following proof, we introduce the notation $f_{\#}\mu \in \mathcal{M}(Y)$ to denote the *pushforward* of a Radon measure $\mu \in \mathcal{M}(X)$ by the measurable function $f : X \to Y$, and is defined by requiring that, for every test function $g \in C^0(Y)$, there holds

$$\int g(y)df_{\#}\mu(y) = \int g(f(x))d\mu(x).$$

Proof Using the fact that $\mathcal{H}_{X,G}$ and K are G-invariant and G acts transitively on X, we find that for any $x, x_0 \in X$, there exists $g_{x,x_0} \in G$ such that $g_{x,x_0}(x) = x_0$, $(g_{x,x_0})_{\#}\mathcal{H}_{X,G} = \mathcal{H}_{X,G}$ and for any $x' \in X$, there holds $K(x, x') = K(x_0, g_{x,x_0}(x'))$. This allows us to write

$$\begin{aligned}\int K(x, x')d\mathcal{H}_{X,G}(x') &= \int K(x_0, g_{x,x_0}(x'))d\mathcal{H}_{X,G}(x')\\ &= \int K(x_0, x')d\left((g_{x,x_0})_{\#}\mathcal{H}_{X,G}\right)(x') = \int K(x_0, x')d\mathcal{H}_{X,G}(x').\end{aligned} \tag{3.3}$$

Using Eq. 3.3 and the fact that $\mathcal{H}_{X,G}$ and μ are probability measures, we may compare the minima of the potentials generated by μ and $\mathcal{H}_{X,G}$ as follows:

$$\begin{aligned}\min_{y\in X}\int K(x, y)d\mu(x) &\leqslant \iint K(x, x')d\mu(x)d\mathcal{H}_{X,G}(x') = \iint K(x, x')d\mathcal{H}_{X,G}(x')d\mu(x)\\ &\overset{\text{Eq. 3.3}}{=} \int K(x_0, x')d\mathcal{H}_{X,G}(x') = \min_{y\in X}\int K(y, x')d\mathcal{H}_{X,G}(x').\end{aligned} \tag{3.4}$$

This shows that $\mathcal{H}_{X,G}$ realizes the maximum in Eq. 3.1, and thus Eq. 3.2 holds. If the minimum in Eq. 3.4 is not achieved at all points $y \in X$, then a strict inequality holds in Eq. 3.4 implying that μ is not a maximizer. □

Remark 3.2 We note, as a special case of the above, that we could take $K(x, y) := \langle x, y\rangle^k$ with $k \in \mathbb{N}$ an even integer, $X = \mathbb{S}^{p-1}$ and $G = O(p)$, where $O(p) := \{M \in \mathbb{R}^{p\times p} : M^t = M^{-1}\}$ is the group of orthogonal matrices, acting on X by $M(x) := M \cdot x$. In this case the optimal K-polarization can be explicitly computed. Denoting by $\bar{\sigma}$ the uniform measure on $\mathbb{S}^{p-1}$, we have

$$T_{\langle\cdot,\cdot\rangle^k}(\mathbb{S}^{p-1}) = \int_{\mathbb{S}^{p-1}} \langle x_0, x'\rangle^k d\bar{\sigma}(x') = \frac{|\mathbb{S}^{p-2}|}{|\mathbb{S}^{p-1}|} B\left(\frac{k+1}{2}, \frac{p-1}{2}\right) = \frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{p+k}{2}\right)}, \tag{3.5}$$

where $B(\cdot, \cdot)$ denotes the Beta function and $\left|\mathbb{S}^d\right|$ is the surface area of $\mathbb{S}^d$.

As a special case which will be used in the proof of the next lemma, (see Fig. 3) we note that for $k = 2$ the above expression gives $T_{\langle\cdot,\cdot\rangle^2}(\mathbb{S}^{p-1}) = 1/p$, and this value is also achieved as the continuous single-plate polarization of the measure $\mu_p := (p+1)^{-1}\sum_{i=0}^{p}\delta_{v_i}$, where $\omega_{\triangle_p} := \{v_0, \dots, v_p\}$ is the (multi)set of vertices of a regular simplex inscribed in $\mathbb{S}^{p-1}$. This fact is a consequence of the property that $\omega_{\triangle_p}$ is a spherical 2-design, see [15].

Lemma 3.3 *Let $p \geqslant 2$ and $A \subset \mathbb{R}^p$ be a compact set. Then for each $s > p-2$, there exists a constant $0 < c_{s,p} < 1/2$ depending only on p and s such that if N is an integer such that $\mathcal{P}_s^*(A, N) < +\infty$, then for any $\mathcal{P}_s^*(A, N)$-optimizing multiset ω_N^* and any $\hat{x} \in \mathbb{R}^p \setminus A$ there holds*

$$\#\left[\omega_N^* \cap B\left(\hat{x}, c_{s,p} dist(\hat{x}, A)\right)\right] \leqslant p\,. \tag{3.6}$$

Remark 3.4 Regarding the sharpness of the lemma, Propositions 2.1 and 2.2 show that the bound in Eq. 3.6 cannot be replaced by $p-1$ when $A = \mathbb{S}^{p-1}$.

Proof of Lemma 3.3: **Step 1.** To simplify notation, we write

$$r := \operatorname{dist}(\hat{x}, A) > 0. \tag{3.7}$$

For $c_1 \in (0, 1/2)$, assume that ω_N^* contains $p+1$ points inside $B(\hat{x}, c_1 r)$, say

$$x_0, \dots, x_p \in B(\hat{x}, c_1 r)\,. \tag{3.8}$$

Our goal is to prove that there exists a constant $c_{s,p} \leqslant 1/2$ such that $c_1 < c_{s,p}$ gives a contradiction to the minimality of ω_N^*.

Step 2. For $c_2 > 0$ consider the new configuration

$$\widetilde{\omega}_N := \omega_N^* \setminus \{x_0, \dots, x_p\} \bigcup \left(\bar{x} + c_2 r \omega_{\triangle_p}\right), \quad \text{where} \quad \bar{x} := \frac{1}{p+1}\sum_{i=0}^{p} x_j, \tag{3.9}$$

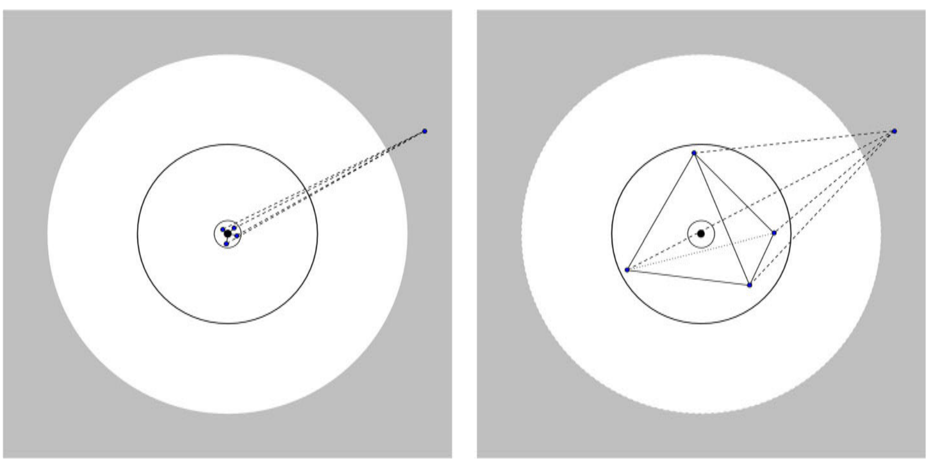

Fig. 3 Idea for the proof of Lemma 3.3, for $p = 3$: (left) the four points in the small sphere, create, in the grey area, a potential close to 4 times the one of a charge in the center; (right) after moving the points out to form a regular simplex, the potential in the grey area is then increased, as can be shown by using the Taylor expansion of the potential, using the assumption that $s > p-2$

and $\omega_{\triangle_p}$ is as in Remark 3.2, the set of vertices of a regular simplex inscribed in $\mathbb{S}^{p-1}$.

For $y \in \mathbb{R}^p$, define

$$f_y(x) := \frac{1}{|y-x|^s}.$$

We will consider the following Taylor expansions of f_y around $\bar{x}$ under the condition that $|x| \leqslant \frac{1}{4}|\bar{x}-y|$:

$$\begin{aligned} f_y(\bar{x}+x) &= f_y(\bar{x}) - s\frac{\langle x, \bar{x}-y\rangle}{|\bar{x}-y|^{s+2}} + \frac{\mathcal{R}_2(x)}{|\bar{x}-y|^{s+2}} && (3.10)\\ &= f_y(\bar{x}) - s\frac{\langle x, \bar{x}-y\rangle}{|\bar{x}-y|^{s+2}} + \frac{s}{2}\frac{(s+2)\langle x, \bar{x}-y\rangle^2 - |x|^2\,|\bar{x}-y|^2}{|\bar{x}-y|^{s+4}} + \frac{\mathcal{R}_3(x)}{|\bar{x}-y|^{s+3}}, && (3.11)\end{aligned}$$

where for some constants $\gamma_{s,p} > 0$ depending only on p, s we have

$$|\mathcal{R}_2(x)| \leqslant \gamma_{s,p}|x|^2 \quad \text{and} \quad |\mathcal{R}_3(x)| \leqslant \gamma_{s,p}|x|^3. \tag{3.12}$$

Step 3. As discussed in Remark 3.2, for $k = 2$ we have $T_{\langle\cdot,\cdot\rangle^2}(\mathbb{S}^{p-1}) = 1/p$, which is attained by the regular simplex $\omega_{\triangle_p}$. Therefore the condition $s > p-2$ can be rewritten as $(s+2)T_{\langle\cdot,\cdot\rangle^2}(\mathbb{S}^{p-1}) > 1$. Thus there exists a positive number $\varepsilon_{s,p} > 0$ depending only on s, p such that if v_i denote, as in Remark 3.2, the vertices of a regular simplex inscribed in $\mathbb{S}^{p-1}$, then

$$\frac{s+2}{p+1}\sum_{i=0}^{p}\langle v_i, y\rangle^2 - 1 = (s+2)T_{\langle\cdot,\cdot\rangle^2}(\mathbb{S}^{p-1}) - 1 > 2\varepsilon_{s,p}, \quad \text{for all} \quad y \in \mathbb{S}^{p-1}. \tag{3.13}$$

Now note that for any $v \in \mathbb{R}^p$ there holds

$$\sum_{i=0}^{p} v_i = 0, \quad \frac{s+2}{p+1}\sum_{i=0}^{p}\langle c_2 r v_i, v\rangle^2 \geqslant (1+2\varepsilon_{s,p})c_2^2 r^2\,|v|^2, \quad \frac{1}{p+1}\sum_{i=0}^{p}|c_2 r v_i|^2 = c_2^2 r^2, \tag{3.14}$$

where for the middle inequality we used Eq. 3.13. From the assumption Eq. 3.8, since $c_1 < c_{s,p} \leqslant 1/2$ and $\bar{x} \in \mathrm{conv}\{x_0, \ldots, x_p\} \subset B(\hat{x}, c_1 r)$, we obtain

$$\min_{y\in A}|\bar{x}-y| \geqslant (1-c_1)r \geqslant \frac{r}{2} \quad \text{and} \quad \max_{0\leqslant j\leqslant p}|x_j - \bar{x}| \leqslant 2c_1 r. \tag{3.15}$$

Conditions Eq. 3.15 and the fact that $\omega_{\triangle_p} \subset \mathbb{S}^{p-1}$ allow to obtain that for $c_1 \leqslant 1/16$ and $c_2 \leqslant 1/4$ the conditions $|x| \leqslant \frac{1}{4}|\bar{x}-y|$ required for Eq. 3.11 and Eq. 3.12 to hold are satisfied for $x = x_j - \bar{x}$ and for $x \in c_2 r\omega_{\triangle_p}$.

We now sum Eq. 3.11 over $x \in c_2 r\omega_{\triangle_p}$. Using Eq. 3.12, Eq. 3.14 and the first bound in Eq. 3.15, we can then estimate

$$\begin{aligned}\frac{1}{p+1}\sum_{i=0}^{p} f_y(\bar{x}+c_2 r v_i) &\geqslant f_y(\bar{x}) + c_2^2 r^2 \frac{s\,\varepsilon_{s,p}}{|\bar{x}-y|^{s+2}} - c_2^3\gamma_{s,p}\frac{r^3}{|\bar{x}-y|^{s+3}}\\ &\geqslant f_y(\bar{x}) + c_2^2\left(s\,\varepsilon_{s,p} - 2c_2\gamma_{s,p}\right)\frac{r^2}{|\bar{x}-y|^{s+2}}. \end{aligned} \tag{3.16}$$

Step 4. By writing the expansion Eq. 3.10 at $x = x_j - \bar{x}$, for $j \in \{0, \ldots, p\}$ we find

$$f_y(x_j) = f_y(\bar{x}) - s\frac{\langle x_j - \bar{x}, \bar{x}-y\rangle}{|\bar{x}-y|^{s+2}} + \frac{\mathcal{R}_2(x_j-\bar{x})}{|\bar{x}-y|^{s+2}}.$$

We now sum the above equation over $j = 0, \dots, p$, and divide by $p+1$, and get

$$
\begin{aligned}
\frac{1}{p+1}\sum_{j=0}^{p} f_y(x_j) &= f_y(\bar{x}) - \frac{s}{p+1}\frac{1}{|\bar{x}-y|^{s+2}}\left\langle \sum_{j=1}^{p}(x_j-\bar{x}), \bar{x}-y \right\rangle \\
&\quad + \frac{1}{p+1}\frac{1}{|\bar{x}-y|^{s+2}}\sum_{j=0}^{p}\mathcal{R}_2(x_j-\bar{x}) \\
&= f_y(\bar{x}) + \frac{1}{p+1}\frac{1}{|\bar{x}-y|^{s+2}}\sum_{j=0}^{p}\mathcal{R}_2(x_j-\bar{x}) \\
&\leqslant f_y(\bar{x}) + 4c_1^2\gamma_{s,p}\frac{r^2}{|\bar{x}-y|^{s+2}}\,,
\end{aligned}
\tag{3.17}
$$

where to obtain the second line we note that the first term on the right in the first line vanishes due to the definition of $\bar{x}$ from Eq. 3.9, and for obtaining the inequality in the last line we use the first bound in Eq. 3.12 together with the second bound from Eq. 3.15.

Step 5. Now, using Eq. 3.9, we find that the bounds Eq. 3.16 and Eq. 3.17 give

$$
\begin{aligned}
\sum_{x\in\widetilde{\omega}_N}\frac{1}{|x-y|^s} - \sum_{x\in\omega_N^*}\frac{1}{|x-y|^s} &= \sum_{i=0}^{p} f_y(\bar{x}+c_2 r v_i) - \sum_{j=0}^{p} f_y(x_j) \\
&= \sum_{i=0}^{p}\big[f_y(\bar{x}+c_2 r v_i) - f_y(\bar{x})\big] - \sum_{j=0}^{p}\big[f_y(x_j) - f_y(\bar{x})\big] \\
&\geqslant (p+1)\Big[c_2^2(s\,\varepsilon_{s,p} - 2c_2\gamma_{s,p}) - 4c_1^2\gamma_{s,p}\Big]\frac{r^2}{|\bar{x}-y|^{s+2}}
\end{aligned}
\tag{3.18a}
$$

$$
:= E_{s,p}(c_1,c_2)\frac{r^2}{|\bar{x}-y|^{s+2}}\,, \tag{3.18b}
$$

for any $y \in A$. As a function of c_2, the value of $c_2^2(s\ \varepsilon_{s,p} - 2c_2\gamma_{s,p})$ in Eq. 3.18a is positive and increasing for $c_2 \in (0, s\varepsilon_{s,p}/(3\gamma_{s,p})]$, and we will take $c_2 = \bar{c}_2 := \min\{1/4,\ s^3\varepsilon_{s,p}^3/(27\gamma_{s,p}^2)\} > 0$. By comparing this value with the term $4c_1^2\gamma_{s,p}$ from Eq. 3.18a, we find that if c_1 satisfies

$$
c_1 < \min\left\{\sqrt{\bar{c}_2^2(s\ \varepsilon_{s,p} - 2\bar{c}_2\gamma_{s,p})}\,,\ \frac{1}{16}\right\} := c_{s,p}, \tag{3.19}
$$

then the expression defined in Eq. 3.18b satisfies $E_{s,p}\,(c_1,\bar{c}_2) > 0$. If $\widetilde{y} \in A$ achieves the minimum of $P_s(\widetilde{\omega}_N, y)$ in Eq. 3.18a and $c_1 < c_{s,p}$, then from Eq. 3.18 we obtain

$$
P_s(\widetilde{\omega}_N, A) = \sum_{x\in\widetilde{\omega}_N}\frac{1}{|x-\widetilde{y}|^s} \geqslant E_{s,p}\,(c_1,\bar{c}_2)\frac{r^2}{|\bar{x}-\widetilde{y}|^{s+2}} + \sum_{x\in\omega_N^*}\frac{1}{|x-\widetilde{y}|^s} > P_s(\omega_N^*, A), \tag{3.20}
$$

which contradicts the optimality of ω_N^*. Therefore the value $c_{s,p}$ defined in Eq. 3.19 is as required in Step 1, and this concludes the proof of the lemma. □

Completion of Proof of Theorem 1.8: We first note that as a consequence of Proposition 1.7, for each N, optimal configurations ω_N^* are contained in the convex hull $\operatorname{conv}(A)$, which has diameter equal to $\operatorname{diam}(A) < \infty$. We then note that, for $c_{s,p}$ chosen according to Lemma 3.3,

$$\operatorname{conv}(A) \setminus A_\epsilon \subset \bigcup_{x \in \operatorname{conv}(A) \setminus A_\epsilon} B\left(x, \epsilon c_{s,p}\right) \subset (\operatorname{conv}(A))_{\epsilon c_{s,p}} . \tag{3.21}$$

We then apply Besicovitch's covering lemma and find a finite subcover of $\operatorname{conv}(A) \setminus A_\epsilon$ by at most $N_{\mathrm{Bes},p}$ families of disjoint balls. Note that, in particular, we have for all $x \notin A_\epsilon$ that $B(x, \epsilon c_{s,p}) \subset B(x, \operatorname{dist}(x, A))$. We may thus apply the bound Eq. 3.6 to each one of the above $N_{\mathrm{Bes},p}$ families, and then sum the bounds. Thus we find that via a direct volume bound

$$\#\left(\omega_N^* \setminus A_\epsilon\right) \leqslant p\, N_{\mathrm{Bes},p} \frac{\mathcal{L}_p\left((\operatorname{conv}(A))_{\epsilon c_{s,p}}\right)}{\mathcal{L}_p(B(0, \epsilon c_{s,p}))},$$

which concludes the proof of the theorem. □

4 Weak Separation and Proof of Theorem 1.11

In this section we first present Proposition 4.2 on the weakly well-separated property of maximal unconstrained polarization configurations and its consequences in Proposition 4.5. Then we prove the general point replacement result of Proposition 4.6. Finally, Propositions 4.5 and 4.6 together with Theorem 1.8 allow us to prove Theorem 1.11 in Section 4.2.

4.1 Weakly Well-Separated Families of Configurations

The results on the asymptotics of $\mathcal{P}_s^*(A, N)$ presented in this section are set in a framework similar to the one for $\mathcal{P}_s(A, N)$ from [11].

Recall the following definition from [33] and [27]:

Definition 4.1 Let $0 < d \leqslant p$ be integers. A family Ω of multisets $\omega \subset \mathbb{R}^p$ is called *weakly well-separated for dimension d and parameter $\eta > 0$* if there exists a number $M > 0$ such that for each $\omega \in \Omega$ and each $x \in \mathbb{R}^p$, there holds

$$\#\left(\omega \cap B(x, \eta \cdot (\#\omega)^{-1/d})\right) \leqslant M , \tag{4.1}$$

where $B(x, r)$ denotes the p-dimensional open ball with center x and radius r.

Proposition 4.2 *Under the same conditions as in Theorem 1.11, there exists a constant $\eta > 0$ depending on s, d and A such that the family of all optimal configurations*

$$\Omega_s := \left\{\omega \subset \mathbb{R}^p : \ P_s(A, \omega) = \mathcal{P}_s^*(A, \#\omega)\right\} \tag{4.2}$$

is weakly well-separated for dimension d and parameter η with $M = p$.

Remark 4.3 Note that the value $M = p$ in the above proposition is optimal, as a consequence of Proposition 2.1. The proof in [33] is done for $M = 2p - 1$ but can be modified along the lines of the proof of our Lemma 3.3 (applying the perturbation as in Fig. 3) in order to achieve the value $M = p$ as stated in Proposition 4.2.

Proposition 4.4 below follows as in [16, Thm. 2.4] (simply note that the restriction $\omega_N \subset A$ for finite-N configurations is never used in the proof from [16]).

Proposition 4.4 *Let $p \geqslant 2$ be an integer and let $1 \leqslant d \leqslant p$ be a real number.*

(i) For $s \geqslant d$ there exists a constant $c_s > 0$ depending only on s, such that if $A \subset \mathbb{R}^p$ is a compact set such that $\mathcal{H}_d(A) > 0$, then for $N \geqslant 2$ there holds

$$\mathcal{P}_s^*(A, N) \leqslant \frac{c_s}{s-d} N^{s/d}, \quad \text{if} \quad s > d, \tag{4.3a}$$

$$\mathcal{P}_d^*(A, N) \leqslant c_d N \log N, \quad \text{if} \quad s = d. \tag{4.3b}$$

(ii) If $A \subset \mathbb{R}^p$ is a compact set then there exists a probability measure μ_A supported on A and a constant $C_A \in (0, \infty)$ such that

$$\int_A \frac{1}{|x-y|^s} d\mu_A(y) \leqslant C_A, \quad \text{for} \quad x \in \mathbb{R}^p,$$

then, for all $N \geqslant 1$,

$$\mathcal{P}_s^*(A, N) \leqslant N C_A, \quad \text{for} \quad s > 0. \tag{4.3c}$$

The next result is proved as in [33, Prop. 1.5] for the case $s > d$ and as in [27, Thm. 2.5] for the cases $p-2 < s < d$. Indeed, in the proofs of those results the fact that the configurations are constrained to the set A is not used; furthermore, Proposition 4.4 precisely replaces the use of results from [16] in those proofs.

Proposition 4.5 *Under the same hypotheses on p, d, d', s, A and $\mu_{s,A}$ as in Proposition 4.2, let $\omega_N \subset \mathbb{R}^p$ be an N-point configuration and $y^* \in A$ be a point such that*

$$\sum_{x \in \omega_N} |x - y^*|^{-s} = \min_{y \in A} \sum_{x \in \omega_N} |x-y|^{-s}$$

is achieved. There exists a constant $C > 0$, which depends only on s if $s > d$ and only on s and on the upper d-regularity constant of $\mu_{s,A}$ if $p-2 < s < d \leqslant d' \leqslant p$, but is in either case independent of N, of ω_N and of the choice of y^*, such that

$$\min_{x \in \omega_N} |x - y^*| \geqslant C N^{-1/d}. \tag{4.4}$$

With Proposition 4.5 at hand, Proposition 4.2 follows by a modification of the proof of Lemma 3.3. These results allows us to proceed with the same overall strategy as for the analogous result for constrained polarization $\mathcal{P}_s(N, A)$, see [33, Thm. 2.3] and [27, Thm. 2.3].

Proof of Proposition 4.2 For any N-point configuration ω_N, set

$$S_s(A, \omega_N) := \left\{ y \in A : \sum_{x \in \omega_N} K_s(x, y) = \min_{y' \in A} \sum_{x \in \omega_N} K_s(x, y') \right\}.$$

Then, by the bound Eq. 4.4, for $C > 0$ as in Proposition 4.5,

$$\operatorname{dist}(\omega_N, S_s(A, \omega_N)) \geqslant C N^{-1/d}. \tag{4.5}$$

Now assume that for a radius $R > 0$ and for some $x \in \mathbb{R}^p$ and some optimal s-polarization configuration ω_N^* there exist $p+1$ distinct points

$$x_0, \ldots, x_p \in \omega_N^* \cap B(x, R). \tag{4.6}$$

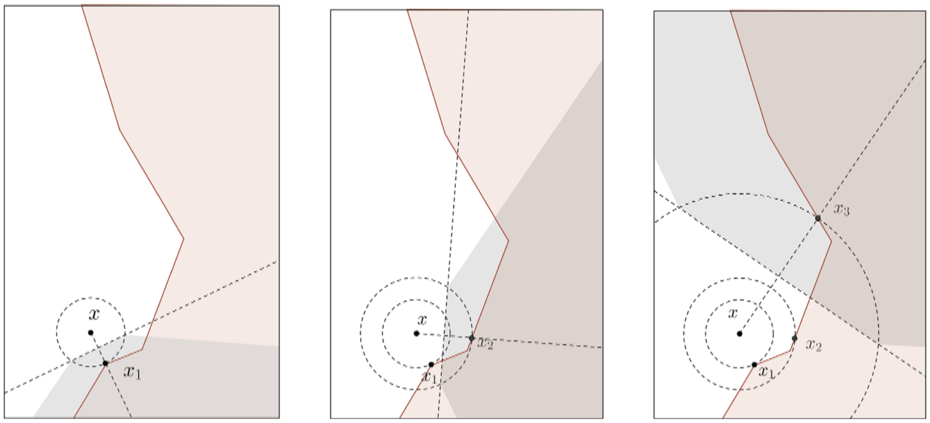

Fig. 4 The construction from Proposition 4.6, for $p = 2$. The set A is shaded in brown. Iteratively we select points $x_j \in A$ such that a charge positioned at x_j creates a higher potential than x (at least) on the intersection of the shaded region (which itself is the intersection of a cone from x and a hyperplane) with A. The union of such regions eventually covers A. Further, any two of the so-constructed points x_j, viewed from x, form angles of at least $\pi/3$; thus, by a simple best-packing upper bound on $\mathbb{S}^{p-1}$, the necessary number of points can be controlled, depending only on the dimension

By using the hypothesis that $s > p - 2$, we will proceed along the same lines as in the proof of Lemma 3.3 in order to reach a contradiction if

$$R < \eta N^{-1/d}, \quad \text{where} \quad \eta := \frac{1}{2} c_{s,p} C > 0, \tag{4.7}$$

where C is as in Proposition 4.5 and $c_{s,p}$ is as in Lemma 3.3. Indeed, set $R = \frac{1}{2} c_1 r$, where $r := C N^{-1/d}$ and $0 < c_1 < c_{s,p}$. Then, with these values of c_1 and r, and for a choice of $c_2 > 0$ to be determined, we can use the same formulas Eq. 3.9 as in Step 2 of the proof of Lemma 3.3 to define $\bar{x}$ and $\widetilde{\omega}_N$. Due to Eq. 4.5, to Eq. 4.6 and to the choice of c_1, we verify that for any $y = \tilde{y} \in S_s(A, \widetilde{\omega}_N)$ the bounds Eq. 3.15 hold. Then the estimates of the proof of Lemma 3.3 continue to hold, and we determine with the same choice of c_2 as in Step 5 that Eq. 3.18 and Eq. 3.20 hold for $y = \tilde{y}$. As a consequence of Eq. 3.20, and of the assumed optimality of ω_N^*, we have

$$\mathcal{P}_s(A, N) \geqslant P_s(A, \widetilde{\omega}_N) > P_s(A, \omega_N^*) = \mathcal{P}_s(A, N), \tag{4.8}$$

which is a contradiction. It follows that under condition Eq. 4.7 there cannot exist $p + 1$ points such that Eq. 4.6 holds, which concludes the proof of the proposition. $\square$

4.2 Proof of Theorem 1.11

The main new tool that we will use in the proof of Theorem 1.11 is the geometric result of Proposition 4.6 below, which holds for a very general class of kernels. It allows us to replace a charge x positioned at positive distance from A by a *bounded number* of charges in A, without decreasing the polarization value on A. The principle underlying this proposition is illustrated in Fig. 4.

Proposition 4.6 *For each $p \geqslant 2$, let $n_{\pi/6,p} > 0$ be the cardinality of the best packing of $\mathbb{S}^{p-1}$ by spherical caps of angle $\pi/6$. Let $A \subset \mathbb{R}^p$ be a compact set, and let $x \notin A$.*

Then there exist points $x_1, \ldots, x_n \in A$ *with* $n \leqslant n_{\pi/6,p}$, *such that for all decreasing* $f : \mathbb{R}^+ \to (-\infty, +\infty]$ *there holds*

$$\forall y \in A, \quad f(|x-y|) \leqslant \max_{1 \leqslant j \leqslant n} f\left(|x_j - y|\right). \tag{4.9}$$

Proof Set

$$\operatorname{rad}(A, x) := \{y \in A : \forall \lambda \in [0,1),\ x + \lambda(y-x) \notin A\}. \tag{4.10}$$

In other words, $\operatorname{rad}(A, x)$ contains the first contact point with A of each ray starting from x that intersects A. Also set

$$\operatorname{rad}_1(A, x) := \{(y-x)/|y-x| : \ y \in \operatorname{rad}(A, x)\}.$$

Note that the projection

$$\pi_{1,x} : \mathbb{R}^p \setminus \{x\} \to \mathbb{S}^{p-1}, \qquad \pi_{1,x}(y) := \frac{y-x}{|y-x|} \tag{4.11}$$

induces a bijection between $\operatorname{rad}(A, x)$ and $\operatorname{rad}_1(A, x)$.

We now iteratively construct the set $x_1, \ldots, x_n$ as required in the statement of the proposition.

Step 1. Fix a point $x_1 \in \operatorname{rad}(A, x)$ such that

$$|x_1 - x| = \min\{|x' - x| : x' \in \operatorname{rad}(A, x)\}. \tag{4.12}$$

As f is decreasing, $f(|x-y|) \leqslant f(|x_1 - y|)$ for all y belonging to the half-space $H(x, x_1)$, where for $a \neq b \in \mathbb{R}^p$ we set

$$H(a, b) := \{y \in \mathbb{R}^p : |y-a| \geqslant |y-b|\} = \left\{y \in \mathbb{R}^p : \ \langle y-a, b-a\rangle \geqslant \frac{1}{2}|b-a|^2\right\}. \tag{4.13}$$

We next let $\mathcal{K}(x_1) \subset \mathbb{S}^{p-1}$ be the spherical cap of angle $\pi/3$ centered at $\pi_{1,x}(x_1)$. Then

$$\mathcal{K}(x_1) = \pi_{1,x}\left\{u \in B(x, |x-x_1|) : \langle \pi_{1,x}(u), \pi_{1,x}(x_1)\rangle \geqslant \frac{1}{2}\right\} = \pi_{1,x}\left(H(x, x_1) \cap B(x, |x-x_1|)\right),$$

and by Eq. 4.12 and Eq. 4.10, we obtain

$$A \cap \pi_{1,x}^{-1}(\mathcal{K}(x_1)) \subset H(x, x_1). \tag{4.14}$$

Step $k+1$**.** For $k \geqslant 1$, suppose that the points $x_1, \ldots, x_k$ have already been chosen such that

$$\pi_{1,x}(x_1), \ldots, \pi_{1,x}(x_k) \in \mathbb{S}^{p-1} \quad \text{form a } \pi/3\text{-separated set,} \tag{4.15a}$$

with respect to the geodesic distance on $\mathbb{S}^{p-1}$ and such that

$$A \cap \pi_{1,x}^{-1}\left(\bigcup_{j=1}^{k} \mathcal{K}(x_j)\right) \subset A \cap \bigcup_{j=1}^{k} H(x, x_j). \tag{4.15b}$$

If we next choose $x_{k+1} \in \operatorname{rad}(A, x) \setminus \pi_{1,x}^{-1}\left(\bigcup_{j=1}^{k} \mathcal{K}(x_j)\right)$ such that

$$|x_{k+1} - x| = \min\left\{|x' - x| : \ x \in \operatorname{rad}(A, x) \setminus \pi_{1,x}^{-1}\left(\bigcup_{j=1}^{k} \mathcal{K}(x_j)\right)\right\},$$

then automatically $\pi_{1,x}(x_{k+1})$ is $\pi/3$-separated from $\pi_{1,x}(x_1), \ldots, \pi_{1,x}(x_k)$. Combining this with the bound Eq. 4.14 for the point x_{k+1}, conditions Eq. 4.15 now hold with k

replaced by $k+1$. Directly from the definition of $n_{\pi/6,p}$, we see that the above iterative construction must stop at step n for some $n \leqslant n_{\pi/6,p}$. After step n we have

$$\mathrm{rad}_1(A,x) = \bigcup_{j=1}^{n} \mathcal{K}(x_j) \tag{4.16}$$

and by Eq. 4.15b,

$$f(|x-y|) \leqslant \max_{1\leqslant j\leqslant n} f\left(|x_j - y|\right) \quad \text{for} \quad y \in A \cap \pi_{1,x}^{-1}\left(\bigcup_{j=1}^{n} \mathcal{K}(x_j)\right) = A. \tag{4.17}$$

The last inclusion in Eq. 4.17 follows from Eq. 4.16. The claim Eq. 4.9 now follows from Eq. 4.17. □

Completion of Proof of Theorem 1.11: The statement follows from the two inequalities

$$\limsup_{N\to\infty} \frac{\mathcal{P}_s(A,N)}{\tau_{s,d}(N)} \leqslant \lim_{N\to\infty} \frac{\mathcal{P}_s^*(A,N)}{\tau_{s,d}(N)} \leqslant \liminf_{N\to\infty} \frac{\mathcal{P}_s(A,N)}{\tau_{s,d}(N)}. \tag{4.18}$$

The first inequality follows directly from the simple bound Eq. 1.8, so we only need to prove the second inequality. For this purpose, fix $\epsilon > 0$ and consider for fixed N a configuration ω_N^* optimizing $\mathcal{P}_s^*(A,N)$. By Eq. 1.24 of Theorem 1.8 we have

$$\#\left(\omega_N^* \setminus A_\epsilon\right) \leqslant \kappa_{s,p} \frac{\mathcal{L}_p\left((\mathrm{conv}(A))_{\epsilon c_{s,p}}\right)}{\epsilon^p} := C_0(\epsilon). \tag{4.19}$$

Next, for $\eta > 0$ depending on s, d, A as in Proposition 4.2 we use the Besicovitch covering theorem in order to cover $A_\epsilon \setminus A$ by a finite collection of balls of radius $\eta N^{-1/p}$ which is the union of at most $N_{\mathrm{Bes},p}$ collections of disjoint balls, where $N_{\mathrm{Bes},p}$ depends only on p. In particular, all balls in the cover are then contained in $(A_\epsilon \setminus A)_\epsilon$ if $N > (\eta/\epsilon)^p$.

By the weak point separation bound of Proposition 4.2 combined with a volume comparison argument, for $N > (\eta/\epsilon)^p$ we have

$$\#\left(\omega_N^* \cap (A_\epsilon \setminus A)\right) \leqslant \frac{p\, N_{\mathrm{Bes},p}}{\beta_p \eta^p} \mathcal{L}_p\left((A_\epsilon \setminus A)_\epsilon\right) N =: C_1(\epsilon) N \quad \text{and} \quad \lim_{\epsilon\to 0} C_1(\epsilon) = 0, \tag{4.20}$$

where β_p is volume of the p-dimensional unit ball $B_p(0,1)$ and where in the last part we used the regularity of the $\mathcal{L}_p$-measures and the fact that A being compact implies $\mathcal{L}_p(A) < \infty$.

By Proposition 4.6 for each $x \in \omega_N \cap (A_\epsilon \setminus A)$ there exists a configuration $\omega_x \subset A$ such that

$$1 \leqslant \#\omega_x \leqslant n_{\pi/6,p}, \quad \text{and} \quad \forall y \in A,\ \frac{1}{|x-y|^s} \leqslant \max_{x'\in\omega_x} \frac{1}{|x'-y|^s} \leqslant \sum_{x'\in\omega_x} \frac{1}{|x'-y|^s}. \tag{4.21}$$

We then define a new configuration $\omega_{M_N} \subset A$ of cardinality M_N by

$$\omega_{M_N} := \left(\omega_N^* \cap A\right) \cup \bigcup_{x\in\omega_N^*\cap(A_\epsilon\setminus A)} \omega_x, \tag{4.22}$$

where by Eq. 4.19, Eq. 4.20 and the first part of Eq. 4.21 we have

$$N - C_0(\epsilon) \leqslant M_N \leqslant \#(\omega_N^*\cap A) + n_{\pi/6,p}\#\left(\omega_N^* \cap (A_\epsilon \setminus A)\right) \leqslant (1 + n_{\pi/6,p} C_1(\epsilon))N. \tag{4.23}$$

Then by the bounds Eq. 4.19 and the second part of Eq. 4.21, we find that

$$\begin{aligned}\mathcal{P}_s(A, M_N) &\geqslant P_s(A, \omega_{M_N}) \\ &\geqslant P_s(A, \omega_N^* \cap A_\epsilon) \\ &\geqslant P_s(A, \omega_N^*) - \max_{y\in A} \sum_{x\in\omega_N^*\setminus A_\epsilon} \frac{1}{|x-y|^s} \\ &\geqslant P_s(A, \omega_N^*) - \frac{C_0(\epsilon)}{\epsilon^s} =: P_s(A, \omega_N^*) - C_2(\epsilon),\end{aligned} \tag{4.24}$$

where $C_2(\epsilon)$ depends only on ϵ, p, s, A; in particular $C_2(\epsilon)$ is independent of N.

Let now $\{N_k\}_{k\in\mathbb{N}}$ be a strictly increasing subsequence that realizes the limit inferior in Eq. 4.18 and let the sequence $\overline{N}_k$ be such that, for each $k \in \mathbb{N}$,

$$\frac{N_k}{1+n_{\pi/6,p}C_1(\epsilon)} \in \left[\overline{N}_k, \overline{N}_k + 1\right). \tag{4.25}$$

Note that $\overline{N}_k \to \infty$ as $k \to \infty$. Using the fact that $\mathcal{P}_s(A, N)$ is increasing in N, Eq. 4.25, Eq. 4.23 and Eq. 4.24 give for $s > d$ the bounds

$$\begin{aligned}\frac{\mathcal{P}_s(A, N_k)}{N_k^{s/d}} &\geqslant \frac{\mathcal{P}_s(A, M_{\overline{N}_k})}{N_k^{s/d}} \geqslant \frac{\mathcal{P}_s^*(A, \overline{N}_k) - C_2(\epsilon)}{N_k^{s/d}} \\ &\geqslant \frac{\mathcal{P}_s^*(A, \overline{N}_k) - C_2(\epsilon)}{\left((1+n_{\pi/6,p}C_1(\epsilon))(\overline{N}_k+1)\right)^{s/d}} \\ &= \frac{1}{\left(1+n_{\pi/6,p}C_1(\epsilon)\right)^{s/d}} \frac{\mathcal{P}_s^*(A, \overline{N}_k)}{\overline{N}_k^{s/d}} \frac{\mathcal{P}_s^*(A, \overline{N}_k) - C_2(\epsilon)}{\mathcal{P}_s^*(A, \overline{N}_k)} \left(\frac{\overline{N}_k}{\overline{N}_k+1}\right)^{s/d}.\end{aligned} \tag{4.26}$$

Due to the fact that $\mathcal{P}_s^*(A, N) \to \infty$ as $N \to \infty$ by compactness of A and to the fact that $\overline{N}_k \to \infty$ as $k \to \infty$, using Eq. 4.20 we find

$$\lim_{k\to\infty} \frac{\mathcal{P}_s^*(A, \overline{N}_k) - C_2(\epsilon)}{\mathcal{P}_s^*(A, \overline{N}_k)} \left(\frac{\overline{N}_k}{\overline{N}_k+1}\right)^{s/d} = 1. \tag{4.27}$$

By Eq. 4.27 and Eq. 4.26, we thus find

$$\liminf_{N\to\infty} \frac{\mathcal{P}_s(A, N)}{N^{s/d}} = \lim_{k\to\infty} \frac{\mathcal{P}_s(A, N_k)}{N_k^{s/d}} \geqslant \left(1+n_{\pi/6,p}C_1(\epsilon)\right)^{-s/d} \lim_{k\to\infty} \frac{\mathcal{P}_s^*(A, \overline{N}_k)}{\overline{N}_k^{s/d}}. \tag{4.28}$$

In Eq. 4.28 we use the hypothesis that the limit of $\mathcal{P}_s^*(A, N)/N^{s/d}$ exists as an extended real number. By now taking $\epsilon \to 0$ and using Eq. 4.20, the desired second inequality in Eq. 4.18 follows, and this completes the proof of the theorem as well for the case $s > d$. The remaining range of exponents $p-2 < s < d$ is treated as above, with the difference that the function $\tau_{s,d}(N) = N^{s/d}$ is replaced according to the definition Eq. 1.26. We leave the verifications to the reader.

Finally, suppose the hypotheses of case (ii) of Proposition 4.2 hold. Theorem 1.5 and Proposition 4.4(ii) imply the limit $h_{s,d}^*(A)$ exists and is finite. □

5 Proof of Theorem 1.12

We begin with a known lemma for constrained polarization.

Lemma 5.1 ([11] or [9]) *Let $1 \leqslant d \leqslant p$, $s \geqslant d$, and $A, B \subset \mathbb{R}^p$ be nonempty sets. Then*

$$\underline{h}_{s,d}(A \cup B)^{-d/s} \leqslant \underline{h}_{s,d}(A)^{-d/s} + \underline{h}_{s,d}(B)^{-d/s}. \tag{5.1}$$

We remark that the analogous subadditivity result holds with $\underline{h}_{s,d}$ replaced by $\underline{h}^*_{s,d}$ in Eq. 5.1, but we will not need that result in this paper. However, the two related results given in the next lemma do play an essential role in the proofs of part (ii) of Theorem 1.12 and of Theorem 1.14. This lemma is proved using similar arguments as in [11, Sec. 6, 7] and [9, Sec. 14.7] for the one-plate polarization problem $\mathcal{P}_s$. We provide a sketch of the proof for the convenience of the reader.

Lemma 5.2 *Let $1 \leqslant d \leqslant p$, $s \geqslant d$, and $A, B \subset \mathbb{R}^p$ be nonempty sets.*

(i) *If the limits $h^*_{s,d}(A \cup B)$ and $h^*_{s,d}(B)$ exist, then*

$$h^*_{s,d}(A \cup B)^{-d/s} \leqslant \overline{h}^*_{s,d}(A)^{-d/s} + h^*_{s,d}(B)^{-d/s}. \tag{5.2}$$

(ii) *If dist$(A, B) > 0$, then*

$$\overline{h}^*_{s,d}(A \cup B)^{-d/s} \geqslant \overline{h}^*_{s,d}(A)^{-d/s} + \overline{h}^*_{s,d}(B)^{-d/s}. \tag{5.3}$$

(iii) *If $A \subset \mathbb{R}^p$ is such that $0 < \overline{h}^*_{s,d}(A) < \infty$, $\mathcal{N} \subset \mathbb{N}$ is any sequence and $\{\widetilde{\omega}_N\}_{N\in\mathcal{N}}$ are N-point configurations in $\mathbb{R}^p$ such that*

$$\lim_{\substack{N\to\infty\\ N\in\mathcal{N}}} \frac{P_s(A, \widetilde{\omega}_N)}{N^{s/d}} = \overline{h}^*_{s,d}(A), \tag{5.4}$$

then for any $B \subset A$, $B \neq \emptyset$ and any $\epsilon > 0$,

$$\liminf_{\substack{N\to\infty\\ N\in\mathcal{N}}} \frac{\#(\widetilde{\omega}_N \cap B_\epsilon)}{N} \geqslant \left(\frac{\overline{h}^*_{s,d}(A)}{\overline{h}^*_{s,d}(B)}\right)^{d/s}. \tag{5.5}$$

We remark that assertion (iii) above with $B = A$ shows that if $\{\widetilde{\omega}_N\}_{N\in\mathcal{N}}$ satisfies Eq. 5.4, then any weak-$*$ limit measure of the normalized counting measures $\{\nu(\widetilde{\omega}_N)\}_{N\in\mathcal{N}}$ is supported on the closure of A.

The following elementary result (whose proof is omitted) will be useful in the proof of Lemma 5.2.

Lemma 5.3 *Let $s \geqslant d > 0$ and $b, c \geq 0$. Then the function $f(t) := \min\left\{t^{s/d}b, (1-t)^{s/d}c\right\}$ has maximum value $\left(b^{-d/s}+c^{-d/s}\right)^{-s/d}$ on the interval* $[0, 1]$. *If both numbers b and c are positive, the maximum is attained at the unique point*

$$t^* := \frac{c^{d/s}}{b^{d/s}+c^{d/s}}.$$

Proof of Lemma 5.2 We leave it to the reader to verify that the inequalities in Lemma 5.2 hold if any of its terms are 0 or ∞. Thus, hereafter, we assume the terms appearing in these inequalities are positive and finite.

We first establish the inequality Eq. 5.2. Let $N, N_1, N_2 \in \mathbb{N}$ be such that $N_1 + N_2 = N$. Let $\omega^*_{N_1} \subset \mathbb{R}^p$ be an N_1-point configuration such that $P_s(A, \omega^*_{N_1}) = \mathcal{P}^*_s(A, N_1)$ and let $\omega^*_{N_2} \subset \mathbb{R}^p$ be an N_2-point configuration such that $P_s(B, \omega^*_{N_2}) = \mathcal{P}^*_s(B, N_2)$. Then, with $\widetilde{\omega}_N := \omega^*_{N_1} \cup \omega^*_{N_2}$, we have

$$\begin{aligned}\mathcal{P}^*_s(A\cup B, N) &\geqslant P_s(A\cup B, \widetilde{\omega}_N)\\ &= \min\{P_s(A, \widetilde{\omega}_N), P_s(B, \widetilde{\omega}_N)\}\\ &\geqslant \min\left\{P_s(A, \omega^*_{N_1}), P_s(B, \omega^*_{N_2})\right\}\\ &= \min\left\{\mathcal{P}^*_s(A, N_1), \mathcal{P}^*_s(B, N_2))\right\},\end{aligned} \tag{5.6}$$

and so,

$$\frac{\mathcal{P}^*_s(A\cup B, N)}{\tau_{s,d}(N)} \geqslant \min\left\{\left(\frac{\tau_{s,d}(N_1)}{\tau_{s,d}(N)}\right)\frac{\mathcal{P}^*_s(A, N_1)}{\tau_{s,d}(N_1)}, \left(\frac{\tau_{s,d}(N_2)}{\tau_{s,d}(N)}\right)\frac{\mathcal{P}^*_s(B, N_2)}{\tau_{s,d}(N_2)}\right\}. \tag{5.7}$$

Suppose that both $h^*_{s,d}(A\cup B)$ and $h^*_{s,d}(B)$ exist and define

$$\alpha := \frac{h^*_{s,d}(B)^{d/s}}{\overline{h}^*_{s,d}(A)^{d/s}+h^*_{s,d}(B)^{d/s}}. \tag{5.8}$$

For $N_1 \in \mathbb{N}$, let $N = \lfloor N_1/\alpha \rfloor$ and $N_2 = N - N_1$ so that $N_1 + N_2 = N$ as above. Let $\mathcal{N}_1 \subset \mathbb{N}$ be such that

$$\lim_{\substack{N_1\to\infty\\ N_1\in\mathcal{N}_1}} \frac{\mathcal{P}^*_s(A, N_1)}{\tau_{s,d}(N_1)} = \overline{h}^*_{s,d}(A)^{d/s}.$$

Note that $\alpha \in (0, 1)$ due to our hypothesis on the terms in the lemma not being 0 or ∞, and in this case we have for $s \geqslant d$,

$$\lim_{N\to\infty} \frac{\tau_{s,d}(N_1)}{\tau_{s,d}(N)} = \alpha^{s/d} \quad \text{and} \quad \lim_{N\to\infty} \frac{\tau_{s,d}(N_2)}{\tau_{s,d}(N)} = (1-\alpha)^{s/d}. \tag{5.9}$$

Then, taking the limit as $N_1 \to \infty$, $N_1 \in \mathcal{N}_1$, of Eq. 5.7, using Eq. 5.9 and Lemma 5.3 we obtain

$$h^*_{s,d}(A\cup B) \geq \min\left\{\alpha^{s/d}\overline{h}^*_{s,d}(A), (1-\alpha)^{s/d}h^*_{s,d}(B)\right\} = \left(\overline{h}^*_{s,d}(A)^{-d/s}+h^*_{s,d}(B)^{-d/s}\right)^{-s/d}, \tag{5.10}$$

which proves assertion (i).

To prove Eq. 5.3, let $\operatorname{dist}(A, B) > 0$ and $\{\widetilde{\omega}_N\}_{N\in\mathcal{N}_0}$ be any sequence of N-point configurations in $\mathbb{R}^p$ such that

$$\lim_{\substack{N\to\infty\\ N\in\mathcal{N}_0}} \frac{P_s(A\cup B, \widetilde{\omega}_N)}{N^{s/d}} = \overline{h}^*_{s,d}(A\cup B). \tag{5.11}$$

Then for any $N \in \mathcal{N}_0$ and $\epsilon > 0$,

$$\begin{aligned} P_s(A \cup B, \widetilde{\omega}_N) &= \min\{P_s(A, \widetilde{\omega}_N), P_s(B, \widetilde{\omega}_N)\} \\ &\leqslant \min\{P_s(A, \widetilde{\omega}_N \cap A_\epsilon), P_s(B, \widetilde{\omega}_N \cap B_\epsilon)\} + N\epsilon^{-s} \\ &\leqslant \min\left\{\mathcal{P}_s^*(A, N_{A,\epsilon}), \mathcal{P}_s^*(B, N_{B,\epsilon})\right\} + N\epsilon^{-s}, \end{aligned} \tag{5.12}$$

where

$$N_{A,\epsilon} := \#(\widetilde{\omega}_N \cap A_\epsilon) \quad \text{and} \quad N_{B,\epsilon} := \#(\widetilde{\omega}_N \cap B_\epsilon).$$

Let $\mathcal{N}_1 \subset \mathcal{N}_0$ be any infinite subset such that the limit

$$\alpha := \lim_{\substack{N\to\infty \\ N\in\mathcal{N}_1}} \frac{N_{A,\epsilon}}{N}$$

exists and belongs to $(0, 1)$, leaving the cases $\alpha = 0$ and $\alpha = 1$ to the reader. Then from (5.12), we have

$$\begin{aligned} \overline{h}^*_{s,d}(A \cup B) &= \lim_{\substack{N\to\infty \\ N\in\mathcal{N}_1}} \frac{P_s(A\cup B, \widetilde{\omega}_N)}{\tau_{s,d}(N)} \\ &\leq \limsup_{\substack{N\to\infty \\ N\in\mathcal{N}_1}} \min\left\{\left(\frac{\tau_{s,d}(N_{A,\epsilon})}{\tau_{s,d}(N)}\right) \cdot \frac{\mathcal{P}_s^*(A, N_{A,\epsilon})}{\tau_{s,d}(N_{A,\epsilon})}, \left(\frac{\tau_{s,d}(N_{B,\epsilon})}{\tau_{s,d}(N)}\right) \cdot \frac{\mathcal{P}_s^*(B, N_{B,\epsilon})}{\tau_{s,d}(N_{B,\epsilon})}\right\}. \end{aligned} \tag{5.13}$$

If $\epsilon < \frac{1}{2}\mathrm{dist}(A, B)$ then A_ϵ and B_ϵ are disjoint, therefore $N_{A,\epsilon} + N_{B,\epsilon} \leqslant N$. Using this and the fact that $\alpha \in (0, 1)$, we obtain

$$\limsup_{\substack{N\to\infty \\ N\in\mathcal{N}_1}} \frac{\log N_{A,\epsilon}}{\log N} = 1 \quad \text{and} \quad \limsup_{\substack{N\to\infty \\ N\in\mathcal{N}_1}} \frac{\log N_{B,\epsilon}}{\log N} \leqslant 1,$$

and thus for all $s \geqslant d$ there holds

$$\limsup_{\substack{N\to\infty \\ N\in\mathcal{N}_1}} \frac{\tau_{s,d}(N_{A,\epsilon})}{\tau_{s,d}(N)} = \alpha^{s/d} \quad \text{and} \quad \limsup_{\substack{N\to\infty \\ N\in\mathcal{N}_1}} \frac{\tau_{s,d}(N_{B,\epsilon})}{\tau_{s,d}(N)} \leqslant (1-\alpha)^{s/d}.$$

Plugging the above into Eq. 5.13 we get

$$\overline{h}^*_{s,d}(A \cup B) \leq \min\left\{\alpha^{s/d}\overline{h}^*_{s,d}(A), (1-\alpha)^{s/d}\overline{h}^*_{s,d}(B)\right\}. \tag{5.14}$$

Appealing to Lemma 5.3, it follows that

$$\overline{h}^*_{s,d}(A \cup B) \leq \left(\overline{h}^*_{s,d}(A)^{-d/s} + \overline{h}^*_{s,d}(B)^{-d/s}\right)^{-s/d},$$

which proves assertion (ii).

Finally, suppose $B \subset A$ and $\{\widetilde{\omega}_N\}_{N\in\mathcal{N}}$ is such that Eq. 5.4 holds. The inequality (5.12) with $B \subset A$ gives

$$\frac{P_s(A, \widetilde{\omega}_N)}{\tau_{s,d}(N)} \leqslant \frac{\mathcal{P}_s^*(B, N_{B,\epsilon})}{\tau_{s,d}(N_{B,\epsilon})} \frac{\tau_{s,d}(N_{B,\epsilon})}{\tau_{s,d}(N)} + \frac{N\epsilon^{-s}}{\tau_{s,d}(N)}, \tag{5.15}$$

Taking the limit inferior as $N \to \infty$ with $N \in \mathcal{N}$

$$\overline{h}^*_{s,p}(A) = \liminf_{\substack{N\to\infty \\ N\in\mathcal{N}}} \frac{P_s(A, \widetilde{\omega}_N)}{\tau_{s,d}(N)} \leqslant \liminf_{\substack{N\to\infty \\ N\in\mathcal{N}}} \frac{\mathcal{P}_s^*(B, N_{B,\epsilon})}{\tau_{s,d}(N_{B,\epsilon})} \frac{\tau_{s,d}(N_{B,\epsilon})}{\tau_{s,d}(N)} \leqslant \overline{h}^*_{s,p}(B) \left(\liminf_{\substack{N\to\infty \\ N\in\mathcal{N}}} \frac{N_{B,\epsilon}}{N}\right)^{s/d}, \tag{5.16}$$

which proves assertion (iii). ☐

Completion of Proof of Theorem 1.12.
As mentioned in the remarks following the statement of Theorem 1.12, it is proved in [11] that for $s > p$ the second equality in Eq. 1.32 holds for compact sets in $\mathbb{R}^p$ with boundary of $\mathcal{L}_p$ measure zero and is known from [10] that this equality holds for arbitrary compact sets when $s = p$. We will make use of these facts in our proof.

Let $A \subset \mathbb{R}^p$ be compact. We will separately establish for $s \geqslant p$ the following two inequalities:

$$\underline{h}^*_{s,p}(A) \geqslant \frac{\sigma_{s,p}}{\mathcal{L}_p(A)^{s/p}} \tag{5.17}$$

and

$$\overline{h}^*_{s,p}(A) \leqslant \frac{\sigma_{s,p}}{\mathcal{L}_p(A)^{s/p}}. \tag{5.18}$$

To prove Eq. 5.17, let $\epsilon > 0$ and select a set $G \subset \mathbb{R}^p$ such that $\mathcal{L}_p(\partial G) = 0$, $A \subset G$ and $\mathcal{L}_p(G \setminus A) < \epsilon$. Then using Eq. 1.7, we find

$$\underline{h}^*_{s,p}(A) \geqslant \underline{h}^*_{s,p}(G) \geqslant h_{s,p}(G) = \frac{\sigma_{s,p}}{\mathcal{L}_p(G)^{s/p}} \geqslant \frac{\sigma_{s,p}}{\left(\mathcal{L}_p(A) + \epsilon\right)^{s/p}}. \tag{5.19}$$

Letting $\epsilon \downarrow 0$, we obtain Eq. 5.17 for $\mathcal{L}_p(A) > 0$ and also that $h^*_{s,p}(A) = h_{s,p}(A) = \infty$ if $\mathcal{L}_p(A) = 0$ which establishes Eq. 1.32 when $\mathcal{L}_p(A) = 0$.

Hereafter we assume $\mathcal{L}_p(A) > 0$. To prove Eq. 5.18, let

$$A^* := \left\{ x \in A : \limsup_{r \to 0^+} \frac{\mathcal{L}_p(\overline{B(x,r)} \cap A)}{\mathcal{L}_p(\overline{B(x,r)})} = 1 \right\}. \tag{5.20}$$

Then by the Lebesgue density theorem there holds $\mathcal{L}_p(A \backslash A^*) = 0$. By an iterative covering argument using Besicovitch's covering theorem, we can find a finite collection of disjoint closed balls B_i, $i \in \{1, \ldots, n\}$ of radii $r_i \in (0,1)$, such that

$$\forall i \in \{1, \ldots, n\}, \qquad \frac{\mathcal{L}_p(A \cap B_i)}{\mathcal{L}_p(B_i)} \geqslant 1 - \epsilon, \tag{5.21a}$$

$$\mathcal{L}_p\left(\bigcup_{i=1}^n (A \cap B_i)\right) = \sum_{i=1}^n \mathcal{L}_p(A \cap B_i) \geqslant (1-\epsilon)\mathcal{L}_p(A). \tag{5.21b}$$

Now Eq. 1.7 together with Eq. 5.3 of Lemma 5.2 gives

$$\overline{h}^*_{s,p}(A) \leqslant \overline{h}^*_{s,p}\left(\bigcup_{i=1}^n (A \cap B_i)\right) \leqslant \left(\sum_{i=1}^n \overline{h}^*_{s,p}(A \cap B_i)^{-p/s}\right)^{-s/p}. \tag{5.22}$$

Due to Eq. 5.21a, and to the regularity of the Radon measure $\mathcal{L}_p$, there exist sets $G_i \subset B_i$ such that $\mathcal{L}_p(\partial G_i) = 0$ and $B_i \setminus A \subset G_i$ and $\mathcal{L}_p(G_i) < 2\epsilon \mathcal{L}_p(B_i)$. Now we use Eq. 5.2 of Lemma 5.2, with the choices $d \mapsto p$, $A \mapsto A \cap B_i$ and $B \mapsto G_i$, obtaining

$$\begin{aligned} \overline{h}^*_{s,p}(A \cap B_i)^{-p/s} &\overset{\text{Eq. 5.2}}{\geqslant} h_{s,p}(B_i)^{-p/s} - h_{s,p}(G_i)^{-p/s} \\ &= \left(\sigma_{s,p}\right)^{-p/s} \left(\mathcal{L}_p(B_i) - \mathcal{L}_p(G_i)\right) \geqslant (1-2\epsilon)\left(\sigma_{s,p}\right)^{-p/s} \mathcal{L}_p(B_i). \end{aligned} \tag{5.23}$$

By Eq. 5.21b, Eq. 5.22 and Eq. 5.23 we obtain

$$\begin{aligned}\overline{h}^*_{s,p}(A) &\leqslant \frac{\sigma_{s,p}}{(1-2\epsilon)^{s/p}}\left(\sum_{i=1}^N \mathcal{L}_p(B_i)\right)^{-s/p} \leqslant \frac{\sigma_{s,p}}{(1-2\epsilon)^{s/p}}\left(\sum_{i=1}^N \mathcal{L}_p(A\cap B_i)\right)^{-s/p}\\ &\leqslant \frac{\sigma_{s,p}}{((1-2\epsilon)(1-\epsilon))^{-s/p}}\mathcal{L}_p(A)^{-s/p}. \end{aligned}\tag{5.24}$$

By taking the limit $\epsilon \downarrow 0$ in Eq. 5.24 we obtain Eq. 5.18, as desired. Combining Eq. 5.17 and Eq. 5.18 proves

$$h^*_{s,p}(A) = \frac{\sigma_{s,p}}{\mathcal{L}_p(A)^{s/p}},$$

for $s \geqslant p$. The fact that the same equality holds for the constrained case follows from Theorem 1.11 in the case $s > p$ since $\mathcal{L}_p(A) > 0$. For $s = p$ the constrained equality is proved in [10]. □

6 A General Lower Bound Via Minkowski Content

The main result of this section, Proposition 6.2, is the analogue for the case of polarization problems of the rough bound [25, Lemma 8] in the setting of energy minimization problems.

We start by recalling the definition of Minkowski content:

Definition 6.1 The *upper* and *lower Minkowski contents* of A, denoted respectively by $\overline{\mathcal{M}}_d(A)$, $\underline{\mathcal{M}}_d(A)$ are respectively defined as

$$\overline{\mathcal{M}}_d(A) := \limsup_{r\downarrow 0} \frac{\mathcal{L}^p(A_r)}{\beta_{p-d}r^{p-d}}, \quad \underline{\mathcal{M}}_d(A) := \liminf_{r\downarrow 0} \frac{\mathcal{L}^p(A_r)}{\beta_{p-d}r^{p-d}},$$

where A_r is as defined in Eq. 1.22 and $\beta_k > 0$ is for $k \in \mathbb{N}, k \geqslant 1$ the volume of the k-dimensional unit ball

$$\beta_k := \frac{\pi^{k/2}}{\Gamma\left(\frac{k}{2}+1\right)}. \tag{6.1}$$

If $\overline{\mathcal{M}}_d(A) = \underline{\mathcal{M}}_d(A)$, their common value is called the *Minkowski content* of A, and denoted by $\mathcal{M}_d(A)$.

The next proposition is essentially a generalization of [11, Lemma 8.2] and will be used in the proof of Theorem 1.14 given in Section 7.2.

Proposition 6.2 *Let $p \geqslant d \geqslant 1$ be natural numbers and let $s > d$. Then there exists a constant $\mathcal{C}_{p,d,s}$ depending only on p, d, s such that the following holds. Let $A \subset \mathbb{R}^p$ be a set such that $\overline{\mathcal{M}}_d(A) < \infty$. Then*

$$\underline{h}_{s,d}(A) \geqslant \frac{\mathcal{C}_{p,d,s}}{\overline{\mathcal{M}}_d(A)^{s/d}}. \tag{6.2}$$

The above proposition follows from Lemmas 6.3 and 6.4 below. Lemma 6.3 says that Minkowski content controls best-covering at all scales, and this will enable us to bound the polarization constant from below by covering constants in Lemma 6.4.

Lemma 6.3 *Let $p \geqslant d \geqslant 1$ be natural numbers. Then there exists a constant $\mathcal{C}_{p,d} > 0$ depending only on p, d such that for any set $A \subset \mathbb{R}^p$ with $0 < \overline{\mathcal{M}}_d(A) < \infty$, for all $r > 0$ sufficiently small there exists $W_r \subset \mathbb{R}^p$ such that*

$$A \subset \bigcup_{x \in W_r} B(x, r)$$

and

$$\#W_r \leqslant \mathcal{C}_{p,d} \frac{\overline{\mathcal{M}}_d(A)}{r^d}. \tag{6.3}$$

If $A \subset \mathbb{R}^p$ is such that $\overline{\mathcal{M}}_d(A) = 0$, then for any $\epsilon > 0$ there is some $r_0 > 0$ such that for all $r \in (0, r_0)$,

$$\#W_r \leqslant \frac{\epsilon}{r^d}. \tag{6.4}$$

Lemma 6.4 *Let p, d, s, A be fixed as in Proposition 6.2. If there exists $C > 0$ such that for every sufficiently small $r > 0$ there exists a covering of A by balls of radius r of cardinality at most C/r^d, then*

$$\underline{h}_{s,d}(A) \geqslant \left(\widetilde{\mathcal{C}}_{p,d} C\right)^{-s/d}, \tag{6.5}$$

where $\widetilde{\mathcal{C}}_{p,d} > 0$ is a constant depending only on p, d.

Proof of Proposition 6.2: By Lemma 6.3, the hypotheses of Lemma 6.4 hold for the choice $C = \mathcal{C}_{p,d}\overline{\mathcal{M}}_d(A)$ when $\overline{\mathcal{M}}_d(A) > 0$ and for any $C > 0$ when $\overline{\mathcal{M}}_d(A) = 0$, where $\mathcal{C}_{p,d} > 0$ is the constant from Lemma 6.3. Then the inequality Eq. 6.5 directly gives Eq. 6.2 for the choice

$$\mathcal{C}_{p,d,s} := \left(\widetilde{\mathcal{C}}_{p,d}\mathcal{C}_{p,d}\right)^{-s/d}. \tag{6.6}$$

□

We now provide the proofs for the above lemmas.

Proof of Lemma 6.3: Let $\epsilon > 0$ be as in the statement of the lemma and let $\tilde{\epsilon} > 0$ be a constant which will be fixed below depending only on A, p, d, ϵ. If $r > 0$ is small enough (depending on A and $\tilde{\epsilon}$), there holds

$$\frac{\mathcal{L}_p(A_{2r})}{2^{p-d}\beta_{p-d} r^{p-d}} \leqslant \overline{\mathcal{M}}_d(A) + \tilde{\epsilon}. \tag{6.7}$$

There exists $N_{\mathrm{Cov},p} \in \mathbb{N}$, depending only on p such that for any $r > 0$, there are points $x_1, \ldots, x_{N_{\mathrm{Cov},p}} \in [0, 2r)^p$ such that the open r-balls with centers in $\{x_j + y : \ y \in (2r\mathbb{Z})^p, 1 \leqslant j \leqslant N_{\mathrm{Cov},p}\}$ cover $\mathbb{R}^p$. Then, for each $j \in \{1, \ldots, N_{\mathrm{Cov},p}\}$, the r-balls with centers in $W_r^j := A_r \cap (x_j + (2r\mathbb{Z})^p)$ are disjoint and the set $W_r := \bigcup_{j=1}^{N_{\mathrm{Cov},p}} W_r^j$ satisfies

$$A \subset \bigcup_{x \in W_r} B(x, r) \subset A_{2r}. \tag{6.8}$$

Due to Eq. 6.8 and Eq. 6.7, for $j = 1, \ldots, N_{\mathrm{Cov},p}$, we have

$$\#\left(W_r^j\right) \cdot \beta_p r^p = \left|\bigcup_{x \in W_r^j} B(x, r)\right| \leqslant |A_{2r}| \leqslant 2^{p-d}\beta_{p-d} r^{p-d}(\overline{\mathcal{M}}_d(A) + \tilde{\epsilon}),$$

where $|\cdot|$ denotes the $\mathcal{L}_p$-measure of a set. By summing over j we obtain

$$\#W_r \leqslant N_{\mathrm{Cov},p} \frac{\beta_{p-d} 2^{p-d}}{\beta_p} \frac{\overline{\mathcal{M}}_d(A) + \tilde{\epsilon}}{r^d}.$$

If $\overline{\mathcal{M}}_d(A) > 0$, then choosing $\tilde{\epsilon} = \overline{\mathcal{M}}_d(A)$ shows that Eq. 6.3 holds with $\mathcal{C}_{p,d} := \frac{\beta_{p-d} 2^{p-d+1}}{\beta_p}$. If $\overline{\mathcal{M}}_d(A) = 0$, then choosing $\tilde{\epsilon} = \epsilon / \mathcal{C}_{p,d}$ proves Eq. 6.4. $\square$

Proof of Lemma 6.4: Let $\mathcal{B} = \{B(x_i, r) : i = 1, \ldots, N_r\}$ be a minimum-cardinality covering of A by r-balls and for each $i = 1, \ldots, M$, choose $\tilde{x}_i \in A \cap B(x_i, r)$. Setting $W_r = \{\tilde{x}_i : i = 1, \ldots, N_r\}$, we have

$$N_r = \#W_r \leqslant \frac{C}{r^d}, \tag{6.9}$$

due to the hypothesis of the lemma. Since for each point in A there exists a point in W_r at distance at most $2r$ from A, we have

$$\mathcal{P}_s(A, N_r) \geqslant P_s(A, W_r) \geqslant (2r)^{-s}. \tag{6.10}$$

We set $N_{\mathrm{Cov},p}$ to be the minimum number of balls of radius 1 in $\mathbb{R}^p$ required to cover a ball of radius 2. Then we have, for all $r > 0$,

$$N_r \leqslant N_{r/2} \leqslant N_{\mathrm{Cov},p} N_r.$$

Thus by Eq. 6.9, for fixed N there exists $r = r(N) > 0$ such that

$$N_r \leqslant N < N_{r/2} \leqslant N_{\mathrm{Cov},p} N_r. \tag{6.11}$$

Then we have

$$\mathcal{P}_s(A, N) \overset{\text{Eq. 6.11}}{\geqslant} \mathcal{P}_s(A, N_r) \overset{\text{Eq. 6.10}}{\geqslant} (2r)^{-s} \overset{\text{Eq. 6.9}}{\geqslant} \left(\frac{N_r}{2^d C}\right)^{s/d} \overset{\text{Eq. 6.11}}{\geqslant} \frac{N^{s/d}}{(2^d N_{\mathrm{Cov},p} C)^{s/d}}$$
$$\overset{\text{Eq. 1.26}}{=} \frac{\tau_{s,d}(N)}{(2^d N_{\mathrm{Cov},p} C)^{s/d}},$$

where we have also used the fact that the polarization value is increasing in N for the first inequality. Now by reordering the terms and by passing to the limit in N along a subsequence that realizes the value of $\underline{h}_{s,d}(A)$, the bound Eq. 6.5 follows if we set $\widetilde{\mathcal{C}}_{p,d} := 2^d N_{\mathrm{Cov},p}$. $\square$

Remark 6.5 Of course, Proposition 6.2 also provides a lower bound for $\underline{h}^*_{s,d}(A)$ since this quantity is at least as large as its constrained analog. Thanks to Proposition 6.2 the asymptotic lower bounds in [11] now follow without needing to appeal to the energy results of [8].

7 Proof of Theorem 1.14

7.1 Some Geometric Measure Theory Tools

We first quantify the increase of interpoint distances under projection on L-Lipschitz graphs:

Lemma 7.1 *For $p \in \mathbb{N}$ and $\epsilon > 0$, let $\mathcal{G}$ be a d-dimensional graph in $\mathbb{R}^p$ of an ϵ-Lipschitz function $\psi : H \to H^\perp$ over a d-dimensional subspace $H \subset \mathbb{R}^p$ having orthogonal complement $H^\perp$; i.e, $\mathcal{G} := \{h + \psi(h) : h \in H\}$. If $\pi_H : \mathbb{R}^p \to H$ is the orthogonal projection onto H and $C_\epsilon := \sqrt{1+\epsilon^2}$, then for any $x \in \mathbb{R}^p$ and any $y \in \mathcal{G}$,*

$$|\pi_H(x) + \psi(\pi_H(x)) - y| \leqslant C_\epsilon |x - y|. \tag{7.1}$$

Proof For $x \in \mathbb{R}^p$, let $x' := \pi_H(x)$, $x'' := x - y$, and note that $x'' \in H^\perp$. If $y \in \mathcal{G}$, then $y'' = \psi(y')$ and we have

$$|(x' + \psi(x')) - y|^2 = |x' - y'|^2 + |\psi(x') - \psi(y')|^2 \leqslant (1+\epsilon^2)|x' - y'|^2 \leqslant (1+\epsilon^2)|x - y|^2,$$

which proves the lemma. □

Lemma 7.1 directly implies the following rough bound for unconstrained polarization for Lipschitz graphs:

Corollary 7.2 *Under the hypotheses of Lemma 7.1, if $\widetilde{K}$ is a compact subset of $\mathcal{G}$ and $N \in \mathbb{N}$, then*

$$\mathcal{P}_s(\widetilde{K}, \mathcal{G}, N) \leqslant \mathcal{P}_s^*(\widetilde{K}, N) \leqslant C_\epsilon^s \mathcal{P}_s(\widetilde{K}, \mathcal{G}, N). \tag{7.2}$$

We also state the following simple deformation result without proof.

Lemma 7.3 *If $\Phi : \mathbb{R}^m \to \mathbb{R}^n$ is an $(1+\epsilon)$-biLipschitz map for some $\epsilon > 0$, $K \subset \mathbb{R}^m$ a compact set, $\omega \subset \mathbb{R}^m$ a finite set, and $s > 0$, then*

$$(1+\epsilon)^{-s} P_s(K, \omega) \leqslant P_s(\Phi(K), \Phi(\omega)) \leqslant (1+\epsilon)^s P_s(K, \omega), \tag{7.3a}$$

and

$$(1+\epsilon)^{-d} \mathcal{H}_d(K) \leqslant \mathcal{H}_d(\Phi(K)) \leqslant (1+\epsilon)^d \mathcal{H}_d(K). \tag{7.3b}$$

7.2 Proof of Theorem 1.14

We recall our definition of A being strongly $(\mathcal{H}_d, d)$-rectifiable: for any $\epsilon > 0$ and for $k \in \mathbb{N}$ large enough depending on ϵ we may write A as

$$A = R_\epsilon \cup \bigcup_{j=1}^k \widetilde{K}_j, \quad \text{where} \quad \begin{cases} \widetilde{K}_j \subset \mathbb{R}^p \text{ are compact and disjoint,} \\ \widetilde{K}_j \text{ are contained in } d\text{-dimensional } \epsilon\text{-Lipschitz graphs,} \\ \overline{\mathcal{M}}_d(R_\epsilon) < \epsilon. \end{cases} \tag{7.4}$$

More explicitly, for each $j = 1, \ldots, k$ there is a d-dimensional subspace $H_j \subset \mathbb{R}^p$ and an ϵ-Lipschitz map $\psi_j : H_j \to H_j^\perp$ such that $\widetilde{K}_j$ is included in the graph $\mathcal{G}_j$ of ψ_j. For each $j = 1, \ldots, k$, let $\iota_j : \mathbb{R}^d \to H_j$ be an isometry. As mentioned just before Definition 1.13, the mapping $\varphi_j : \mathbb{R}^d \to \mathcal{G}_j$ defined by $\varphi_j(x) := \iota_j(x) + \psi_j(\iota_j(x))$ is then $(1+\epsilon)$-biLipschitz for every j with $\widetilde{K}_j = \varphi_j(K_j)$, where $K_j \subset \mathbb{R}^d$ is compact.

Let $A \subset \mathbb{R}^d$ be strongly $(\mathcal{H}_d, d)$-rectifiable. We shall prove separately the inequalities

$$\underline{h}_{s,d}(A) \geqslant \frac{\sigma_{s,d}}{\mathcal{H}_d(A)^{s/d}}, \tag{7.5a}$$

$$\overline{h}^*_{s,d}(A) \leqslant \frac{\sigma_{s,d}}{\mathcal{H}_d(A)^{s/d}}, \tag{7.5b}$$

which, since $\underline{h}_{s,d}(A) \leqslant \underline{h}^*_{s,d}(A)$ and $\overline{h}_{s,d}(A) \leqslant \overline{h}^*_{s,d}(A)$, imply Eq. 1.38.

We shall show Eq. 7.5a using the decomposition Eq. 7.4. By Eq. 6.2 of Proposition 6.2, we have

$$\underline{h}_{s,d}(R_\epsilon) \geqslant \frac{\mathcal{C}_{p,d,s}}{\overline{\mathcal{M}}_d(R_\epsilon)^{s/d}} \geqslant \frac{\mathcal{C}_{p,d,s}}{\epsilon^{s/d}}. \tag{7.6}$$

By using Eq. 5.1 from Lemma 5.2 and the $(1+\epsilon)$-biLipschitz parameterizations of $\widetilde{K}_j = \varphi_j(K_j)$, we find that

$$\begin{aligned}
\underline{h}_{s,d}(A)^{-d/s} &\overset{\text{Eq. 7.4, Eq. 5.1}}{\leqslant} \underline{h}_{s,d}(R_\epsilon)^{-d/s} + \sum_{j=1}^{k} \underline{h}_{s,d}(\varphi_j(K_j))^{-d/s} \\
&\overset{\text{Eq. 7.6, Eq. 1.2}}{\leqslant} \mathcal{C}_{p,d,s}^{-d/s}\epsilon + \sum_{j=1}^{k} \left[\liminf_{N\to\infty} \frac{\mathcal{P}_s(\varphi_j(K_j), N)}{\tau_{s,d}(N)}\right]^{-d/s} \\
&\overset{\text{Eq. 7.3a}}{\leqslant} \mathcal{C}_{p,d,s}^{-d/s}\epsilon + (1+\epsilon)^d \sum_{j=1}^{k} \left[\liminf_{N\to\infty} \frac{\mathcal{P}_s(K_j, N)}{\tau_{s,d}(N)}\right]^{-d/s} \\
&\leqslant \mathcal{C}_{p,d,s}^{-d/s}\epsilon + (1+\epsilon)^d \sum_{j=1}^{k} \underline{h}_{s,d}(K_j)^{-d/s} \\
&\overset{\text{Thm. 1.12 (ii)}}{=} \mathcal{C}_{p,d,s}^{-d/s}\epsilon + (1+\epsilon)^d (\sigma_{s,d})^{-d/s} \sum_{j=1}^{k} \mathcal{H}_d(K_j) \\
&\overset{\text{Eq. 7.3b}}{\leqslant} \mathcal{C}_{p,d,s}^{-d/s}\epsilon + (1+\epsilon)^{2d} (\sigma_{s,d})^{-d/s} \sum_{j=1}^{k} \mathcal{H}_d(\varphi_j(K_j)).
\end{aligned} \tag{7.7}$$

Since $\sum_{j=1}^{k} \mathcal{H}_d(\varphi_j(K_j)) \leqslant \mathcal{H}_d(A)$, taking the limit as $\epsilon \downarrow 0$ in Eq. 7.7 yields the bound Eq. 7.5a. Note that this shows $\underline{h}_{s,d}(A) = +\infty$ if $\mathcal{H}_d(A) = 0$.

To prove Eq. 7.5b, we use the decomposition Eq. 7.4, the bound Eq. 1.7 and the bound Eq. 5.3 of Lemma 5.2, and we obtain

$$\begin{aligned}
\overline{h}_{s,d}^{*}(A)^{-d/s} &\overset{\text{Eq. 1.7}}{\geqslant} \left[\overline{h}_{s,d}^{*}\left(\bigcup_{j=1}^{k} \varphi_j(K_j)\right)\right]^{-d/s} \overset{\text{Eq. 5.3}}{\geqslant} \sum_{j=1}^{k} \overline{h}_{s,d}^{*}(\varphi_j(K_j))^{-d/s} \\
&\overset{\text{Eq. 7.2}}{\geqslant} C_\epsilon^{-d} \sum_{j=1}^{k} \left[\limsup_{N\to\infty} \frac{\mathcal{P}_s(\varphi_j(K_j), \varphi_j(\mathbb{R}^d), N)}{\tau_{s,d}(N)}\right]^{-d/s} \\
&\overset{\text{Eq. 7.3a}}{\geqslant} C_\epsilon^{-d}(1+\epsilon)^{-d} \sum_{j=1}^{k} \overline{h}_{s,d}^{*}(K_j)^{-d/s} \overset{\text{Thm. 1.12 (ii)}}{\geqslant} (\sigma_{s,d})^{-d/s} C_\epsilon^{-d}(1+\epsilon)^{-d} \sum_{j=1}^{k} \mathcal{H}_d(K_j) \\
&\overset{\text{Eq. 7.3b}}{\geqslant} (\sigma_{s,d})^{-d/s} C_\epsilon^{-d}(1+\epsilon)^{-2d} \sum_{j=1}^{k} \mathcal{H}_d(\varphi(K_j)) \\
&\overset{\text{Eq. 7.4}}{\geqslant} (\sigma_{s,d})^{-d/s} C_\epsilon^{-d}(1+\epsilon)^{-2d} (\mathcal{H}_d(A) - \epsilon).
\end{aligned} \tag{7.8}$$

Since $\lim_{\epsilon\downarrow 0} C_\epsilon = 1$, taking $\epsilon \downarrow 0$ in Eq. 7.8 we find the desired bound Eq. 7.5b.

Finally, suppose that $\mathcal{H}_d(A) > 0$ and $\{\omega_N\}_{N\in\mathcal{N}}$ satisfies

$$\lim_{\substack{N\to\infty\\ N\in\mathcal{N}}} \frac{P_s(A, \widetilde{\omega}_N)}{N^{s/d}} = h^*_{s,d}(A) = \sigma_{s,d}(\mathcal{H}_d(A))^{-s/d}. \tag{7.9}$$

For $N \in \mathcal{N}$, let $\nu_N := \nu(\widetilde{\omega}_N)$ denote the normalized counting measure associated with $\widetilde{\omega}_N$ and let μ_A denote the measure $\frac{\mathcal{H}_d|_A}{\mathcal{H}_d(A)}$. Let $G \subset \mathbb{R}^p$ be open. For $\delta > 0$, let B be a closed subset of $A \cap G$ such that $\mu_A(B) \geqslant (1-\delta)\mu_A(G)$. Since A is compact, there is some $\epsilon > 0$ such that $B_\epsilon \subset G$. Since A is strongly $(\mathcal{H}_d, d)$-rectifiable, B is also strongly $(\mathcal{H}_d, d)$-rectifiable and so $h^*_{s,d}(B) = \sigma_{s,d}(\mathcal{H}_d(B))^{-s/d}$. Using Eq. 5.5 gives

$$\liminf_{\substack{N\to\infty\\ N\in\mathcal{N}}} \nu_N(G) \geqslant \liminf_{\substack{N\to\infty\\ N\in\mathcal{N}}} \frac{\#(\widetilde{\omega}_N \cap B_\epsilon)}{N} \geqslant \left(\frac{h^*_{s,d}(A)}{h^*_{s,d}(B)}\right)^{d/s} = \mu_A(B) \geqslant (1-\delta)\mu_A(G),$$

and since $\delta > 0$ is arbitrary,

$$\liminf_{\substack{N\to\infty\\ N\in\mathcal{N}}} \nu_N(G) \geqslant \mu_A(G).$$

The Portmanteau Theorem (e.g., see [6]) then implies that ν_N converges in the weak* topology to μ_A.
□

We conclude this section by showing that compact subsets of C^1-embedded manifolds are strongly $(\mathcal{H}_d, d)$-rectifiable:

Lemma 7.4 *Let $M \subset \mathbb{R}^p$ be a C^1-embedded submanifold of dimension d and let $A \subset M$ be a compact set. Then A is strongly $(\mathcal{H}_d, d)$-rectifiable.*

Proof As M is a C^1-embedded submanifold, for each $\epsilon > 0$ there exists a radius $\rho = \rho(\epsilon) > 0$ such that for every $x \in A$ the intersection $M \cap \overline{B(x,\rho)}$ is an ϵ-Lipschitz graph over the tangent subspace $T_x M$ of M at x. As A is compact, we can find a cover by balls $B(x_i, \rho)$, $i = 1, \ldots, k_0$ with $x_i \in M$. We will introduce a small parameter $\epsilon_1 \in (0,1)$ to be appropriately restricted later. We define the sets

$$\widetilde{K}_1 := A \cap \overline{B(x_1, (1-\epsilon_1)\rho)} \quad \text{and} \quad \widetilde{K}_{k+1} := \left(A \cap \overline{B(x_{k+1}, (1-\epsilon_1)\rho)}\right)$$
$$\setminus \bigcup_{j=1}^{k} B(x_j, \rho) \quad \text{for } 1 \leqslant k \leqslant k_0 - 1.$$

Each $\widetilde{K}_j$ is compact and contained in an ϵ-Lipschitz graph over the tangent space $T_{x_j} M \subset \mathbb{R}^p$. The sets $\widetilde{K}_j$ are at distance at least $\epsilon_1 > 0$ from each other and the points of A not covered by any of the $\widetilde{K}_j$ are contained in the set

$$R_k := A \cap \bigcup_{i=1}^{k_0} \left(\overline{B(x_i, \rho) \setminus B(x_i, (1-\epsilon_1)\rho)}\right).$$

In order to prove Eq. 7.4 it remains to prove that for $\epsilon_1 \in (0,1)$ small enough, R_k has $\mathcal{M}_d(R_k) < \epsilon$. Indeed, R_k is a compact subset of M and thus $\mathcal{M}_d(R_k) = \mathcal{H}_d(R_k)$ by a known result valid for closed subsets of d-rectifiable sets, see [21, Thm. 3.2.39]. By Eq. 7.3b we then bound

$$\begin{aligned}\mathcal{H}_d(R_k) &\leqslant \sum_{i=1}^{k_0} \mathcal{H}_d\left(M \cap \overline{B(x_i,\rho) \setminus B(x_i,(1-\epsilon_1)\rho)}\right)\\ &\leqslant k_0(1+\epsilon)^d \mathcal{H}_d\left(B(0,\rho) \setminus B(0,(1-\epsilon_1)\rho)\right) = k_0(1+\epsilon)^d \rho^d |B_1|(1-(1-\epsilon_1)^d),\end{aligned}$$

where the right hand side tends to zero as $\epsilon_1 \to 0$, verifying that $\epsilon_1 > 0$ can be chosen small enough so that $\mathcal{M}_d(R_k) = \mathcal{H}_d(R_k) < \epsilon$. Therefore we have found a decomposition of A as in Eq. 7.4, as desired. $\square$

8 Some Conjectures and Open Problems

8.1 Optimal N-point Configurations for $\mathcal{P}_s^*(\mathbb{S}^{p-1}, N)$.

For conjectures regarding $\mathbb{S}^1$ see Section 2. The question of what are the N-point configurations on $\mathbb{R}^p$ that optimize $\mathcal{P}_s^*(\mathbb{S}^{p-1}, N)$ is open, except for the simple cases $N = 1, 2, 3$, in which all points sit at the center of the sphere (see Proposition 2.1). We conjecture that for $N = p + 1$ a regular simplex on a concentric sphere of smaller radius is optimal. Note that for the constrained case of $\mathcal{P}_s(\mathbb{S}^{p-1}, p+1)$, the inscribed regular simplex is known to be optimal in all dimensions, see [7] and [38] for $p = 3$.

For $N = 5$, conjectures regarding the constrained polarization $\mathcal{P}_s(\mathbb{S}^2, 5)$ are discussed in [9, Chapter 14]. Concerning the problem $\mathcal{P}_s^*(\mathbb{S}^2, 5)$, based on numerical experiments optimal configurations do not seem to lie on a concentric sphere and in this case it is an open problem to find the geometric structure of optimal configurations.

As mentioned in Proposition 1.3, the limit of the maximal polarization problem on the sphere for $s \to \infty$ is the question of best *unconstrained covering*. For the sphere, due to Proposition 2.4, the one-plate and unconstrained best covering problems are equivalent, and thus the former gives information on the latter, and produces useful candidates for the configurations optimizing $\mathcal{P}_s^*(\mathbb{S}^{p-1}, N)$ for very large s. Optimal configurations for the constrained covering of $\mathbb{S}^2$ were determined for $N = 4, 6, 12$ by L. Fejes Tóth (see [22]), for $N = 5$ and 7 by Schütte [36], for $N = 8$ by L. Wimmer [40] and for $N = 10$ and 14 by G. Fejes Tóth [23].

8.2 The Large N Limit of Optimal Polarization Configurations

If K is a lower semicontinuous integrable kernel on $A \times A$ and for each $N \geqslant 1$ we choose an optimal multiset $\omega_N^* \subset \mathbb{R}^p$ that realizes the maximum in the definition of $\mathcal{P}_K^*(A, N)$, where $A \subset \mathbb{R}^p$ is a compact set of positive K-capacity (i.e., there exists some probability measure μ supported on A whose K-potential is μ integrable), then is it true that every weak-$*$ limit μ of the sequence

$$\left\{\frac{1}{N}\sum_{x_j \in \omega_N^*}^{N} \delta_{x_j}\right\}_{N=1}^{\infty}$$

satisfies

$$\min_{y\in A}\int K(x,y)d\mu(X)=T_K^*(A),$$

where $T_K^*(A):=T_K(A,\mathbb{R}^p)$?

8.3 Polarization for Lattices in $\mathbb{R}^2$

A natural question is the following. Assume $f:(0,\infty)\to(0,\infty)$ is a decreasing convex function and let $K(x,y)=f(|x-y|^2)$. Which lattices $\Lambda\subset\mathbb{R}^2$ of determinant 1 maximize the polarization value

$$\min_{y\in\mathbb{R}^2}\sum_{x\in\Lambda\setminus\{0\}} f(|x-y|^2) \quad ? \tag{8.1}$$

We note that under rapid decay conditions on f that ensure that the sum in Eq. 8.1 converges, there exist optimizers Λ, y that realize the above value. In dimension $d=2$ we conjecture that for completely monotone f, the optimizer of Eq. 8.1 is the hexagonal lattice A_2. In [38] it is shown that the minimum in Eq. 8.1 for such f and $\Lambda=A_2$ occurs at the centroids of the equilateral triangles that divide each fundamental domain in half.

8.4 Optimal Infinite Configurations in $\mathbb{R}^d$

Related to the conjecture for $\sigma_{s,2}$ presented in the introduction, it is interesting to explore the generalization of the maximization of Eq. 8.1 for infinite configurations in $\mathbb{R}^d$. If $\omega_\infty\subset\mathbb{R}^d$ is a countable configuration such that

$$\limsup_{R\to\infty}\frac{\#(\omega_\infty\cap[-R/2,R/2]^d)}{R^d}=1, \tag{8.2}$$

then we define as in [11] for $K(x,y)=f(|x-y|^2)$ the polarization constant

$$P_K(\omega_\infty):=\limsup_{R\to\infty}P_K([-R/2,R/2]^d,\omega_\infty\cap[-R/2,R/2]^d). \tag{8.3}$$

Is it true that under suitable conditions on f the supremum of Eq. 8.3 among $\omega_\infty\subset\mathbb{R}^d$ satisfying Eq. 8.2 equals the maximum of Eq. 8.1 over unit density lattices in low dimensions?

8.5 Weighted Unconstrained Polarization

Part (ii) of Theorem 1.12 can be extended to the case of weighted kernels. This procedure represents a setup, or modification, of the theory presented so far, which allows us to prescribe, or to control, the asymptotic distribution of polarization points at the expense of modifying the kernels $K_s(x,y)=|x-y|^{-s}$ by a suitable weight; i.e. working with $K_s^w(x,y):=w(x,y)|x-y|^{-s}$ where $w(x,y)$ a CPD-weight as defined in [11, Def. 2.3]. Under these conditions, analogues of Theorems 1.12, 1.11 and 1.14 are expected to hold for K_s^w for the cases $s\geqslant d$, allowing to relax the hypotheses of [11, Thm. 2.3, Thm. 3.1] and to formulate analogues for the unconstrained polarization. We leave this endeavor to future work.

8.6 Point Separation for Maximum-Polarization Configurations

Is it true that, for $s > p-2$, there exists a constant $c_{s,p} > 0$ independent of N such that for any optimizer $\omega_N^* = \{x_{N,1}^*, \ldots, x_{N,N}^*\} \subset \mathbb{R}^p$ for the problem $\mathcal{P}_s^*(\mathbb{S}^{p-1}, N)$ we have

$$\min_{1\leqslant i\neq j\leqslant N} |x_{N,i}^* - x_{N,j}^*| \geqslant c_{s,p} N^{-1/p} \quad \text{for} \quad N = 1, 2, \ldots \quad ?$$

The weak separation analogue of the above, giving rise to this question in the constrained polarization problem, has been considered in [27].

9 Glossary of Notation

$\omega_N = \{x_1, \ldots, x_N\}$	-	an N-point configuration (multiset) in $\mathbb{R}^p$
$\nu(\omega_N) = \frac{1}{N}\sum_{x\in\omega_N}\delta_x$	-	probability measure associated to a point configuration
$A_r := \{x \in \mathbb{R}^p : \operatorname{dist}(x, A) < r\}$	-	r-neighborhood of a set, for $A \subset \mathbb{R}^p$
$\operatorname{conv}(A)$	-	convex hull of a set A
$\operatorname{dist}_{\mathbb{S}^1}(x, y) = \min\{\lvert t\rvert, \lvert 2\pi - t\rvert\}$	-	geodesic distance between $x, y \in \mathbb{S}^1$ such that $y = e^{it}x$.
$\mathcal{L}_p(A)$	-	p-dimensional Lebesgue measure of set A
$\mathcal{H}_d(A)$	-	d-dimensional Hausdorff measure of a set A
$\beta_k := \frac{\pi^{k/2}}{\Gamma\left(\frac{k}{2}+1\right)}$	-	volume of the k-dimensional Euclidean ball
$P_K(A, \omega_N), \mathcal{P}_N(A, B, N)$	-	polarization of a configuration, two-plate polarization, Eq. 1.1.
$\mathcal{P}_K(A, N)$	-	constrained best N-point polarization (single-plate problem) Eq. 1.3
$\mathcal{P}_K^*(A, N)$	-	unconstrained best N-point polarization Eq. 1.6
$K_s(x, y)$	-	inverse-power kernel Eq. 1.9
$P_s(A, \omega_N), \mathcal{P}_s(A, N), \mathcal{P}_s^*(A, N)$	-	see Eq. 1.10
$\eta_N(A, B), \eta_N(A), \eta_N^*(A)$	-	two-plate/constrained/unconstrained covering radii, Eq. 1.12, Eq. 1.13
$T_K(A, B), T_K(X)$	-	continuum polarization problems Eq. 1.19, Eq. 3.1
$\tau_{s,d}(N)$	-	scaling factor for the optimal polarization, Eq. 1.26
$\underline{h}_{s,d}^*(A), \overline{h}_{s,d}^*(A), h_{s,d}^*(A)$	-	asymptotic values of rescaled optimal polarization, Eq. 1.27
$n_{\pi/6,p}$	-	maximum number of balls with angular radius $\frac{\pi}{6}$ that pack $\mathbb{S}^{p-1}$
$\overline{\mathcal{M}}_d(A), \underline{\mathcal{M}}_d(A), \mathcal{M}_d(A)$	-	Minkowski contents defined in Definition 6.1

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References

1. Ambrus, G.: Analytic and Probabilistic problems in discrete geometry. Ph.D thesis University College London (2009)
2. Ambrus, G., Ball, K.M., Erdélyi, T.: Chebyshev constants for the unit circle. Bull. Lond. Math Soc. **45**(2), 236–248 (2013)
3. Bernstein, S.: Sur les Fonctions Absolument Monotones. Acta Math. **52**, 1–66 (1929)
4. Bétermin, L., Petrache, M.: Dimension reduction techniques for the minimization of theta functions on lattices. J. Math Phys. **58**, 071902 (2017)
5. Bétermin, L., Sandier, E.: Renormalized energy and asymptotic expansion of optimal logarithmic energy on the sphere. Constr. Approx. **47**(1), 39–74 (2018)
6. Billingsley, P.: Convergence of Probability Measures, 2nd Edition Wiley (1999)
7. Borodachov, S.V.: Polarization problem on a high-dimensional sphere for a simplex (submitted) (2019)
8. Borodachov, S.V., Hardin, D.P., Saff, E.B.: Asymptotics of weighted best-packing on rectifiable sets. Mat Sb. **199**(11), 1579–1595 (2008)
9. Borodachov, S.V., Hardin, D.P., Saff, E.B.: Discrete Energy on Rectifiable Sets. Springer Monographs in Mathematics, Springer Nature (2019)
10. Borodachov, S.V., Bosuwan, N.: Asymptotics of discrete Riesz d-polarization on subsets of d-dimensional manifolds. Potential Anal. **41**(1), 35–49 (2014)
11. Borodachov, S.V., Hardin, D.P., Reznikov, A., Saff, E.B.: Optimal discrete measures for Riesz potentials. Trans. Amer. Math Soc. **370**(10), 6973–6993 (2018)
12. Bosuwan, N.: Two Problems in Asymptotic Analysis Padé-Orthogonal Approximation and Riesz Polarization Constants and Configurations. Ph.D. Thesis, Vanderbilt University (2013)
13. Breger, A., Ehler, M., Graef, M.: Points on manifolds with asymptotically optimal covering radius. J Complexity **48**, 1–14 (2018)
14. Conway, J.H., Sloane, N.J.A.: Sphere Packings, Lattices and Groups Volume 290 of Grundlehren Der Mathematischen Wissenschaften. Springer-Verlag, New York, third edition (1999)
15. Delsarte, P., Goethals, J.-M., Seidel, J.J.: Spherical codes and designs. Geom. Dedicata **6**, 363–388 (1977)
16. Erdélyi, T., Saff, E.B.: Riesz polarization inequalities in higher dimensions. J Approx. Theory **171**, 128–147 (2013)
17. Farkas, B., Nagy, B.: Transfinite diameter, Chebyshev constant and energy on locally compact spaces. Potential Anal **28**, 241–260 (2008)
18. Farkas, B., Nagy, B., Révész, S.G.: A potential theoretic minimax problem on the torus. Trans. London Math Soc. (2018)
19. Farkas, B., Révész, S.G.: Potential theoretic approach to rendezvous numbers. Monatsh. Math. **148**(4), 309–331 (2006)
20. Fejér, L.: Über die Lage der Nullstellen von Polynomen, die aus Minimumforderungen gewisser Art entspringen. (German) Math. Ann. **85**(1), 41–48 (1922)
21. Federer, H.: Geometric Measure Theory Springer-Verlag, Berlin (1969)
22. Fejes Tóth, L.: Regular Figures A. Pergamon Press Book, The Macmillan Co., New York (1964)
23. Fejes Tóth, G.: Kreisüberdeckungen der sphäre. Studia Sci Math. Hungar. **4**, 225–247 (1969)
24. Hardin, D.P., Kendall, A.P., Saff, E.B.: Polarization optimality of equally spaced points on the circle for discrete potentials. Discrete Comput Geom. **50**(1), 236–243 (2013)
25. Hardin, D.P., Saff, E.B., Whitehouse, J.T.: Quasi-uniformity of minimal weighted energy points on compact metric spaces. J Complexity **28**(2), 177–191 (2012)
26. Hardin, D.P., Saff, E.B., Simanek, B.Z., Su, Y.: Next order energy asymptotics for Riesz potentials on flat tori. Int. Math. Res. Not. IMRN **12**(16), 3529–3556 (2019)
27. Hardin, D.P., Reznikov, A., Saff, E.B., Volberg, A.: Local properties of Riesz minimal energy configurations and equilibrium measures. Int. Math. Res. Not. IMRN **16**, 5066–5086 (2019)
28. Helgason, S.: Geometric Analysis on Symmetric Spaces. American Mathematical Society, Providence, RI (1994)

29. Kershner, R.: The number of circles covering a set. Amer. J. Math. **61**, 665–671 (1939)
30. Nikolov, N., Rafailov, R.: On the sum of powered distances to certain sets of points on the circle. Pacific J. Math. **253**(1), 157–168 (2011)
31. Ohtsuka, M.: On various definitions of capacity and related notions. Nagoya Math. J. **30**, 121–127 (1967)
32. Petrache, M., Rota Nodari, S.: Equidistribution of jellium energy for Coulomb and Riesz interactions. Constr. Approx. **47**(1), 163–210 (2018)
33. Reznikov, A., Saff, E.B., Volberg, A.: Covering and separation of Chebyshev points for non-integrable Riesz potentials. J. Complexity **46**, 19–44 (2018)
34. Reznikov, A., Saff, E.B., Vlasiuk, O.V.: A minimum principle for potentials with application to Chebyshev constants. Potential Anal. **47**(2), 235–244 (2017)
35. Saff, E.B., Totik, V.: Logarithmic Potentials with External Fields. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 316,. Springer-Verlag, Berlin (1997). ISBN: 3-540-57078-0
36. Schütte, K.: Überdeckungen der Kugel mit höchstens acht Kreisen. Math. Ann. **129**(1), 181–186 (1955)
37. Simanek, B.: Asymptotically optimal configurations for Chebyshev constants with an integrable kernel. New York J. Math. **22**, 667–675 (2016)
38. Su, Y.: Discrete minimal energy on flat tori and four-point maximal polarization on $\mathbb{S}^2$. Ph.D. Thesis, Vanderbilt University (2015)
39. Tsuji, M. Potential Theory in Modern Function Theory, 2nd edn. Chelsea Publ. Co., New York (1975)
40. Wimmer, L.: Covering the sphere with equal circles. Discrete Comput. Geom. **57**(4), 763–781 (2017)