CONDENSERS WITH TOUCHING PLATES AND CONSTRAINED MINIMUM RIESZ AND GREEN ENERGY PROBLEMS

P.D. DRAGNEV, B. FUGLEDE, D.P. HARDIN, E.B. SAFF, AND N. ZORII

Abstract. We study minimum energy problems relative to the α-Riesz kernel $|x-y|^{\alpha-n}$, $\alpha \in (0,2]$, over signed Radon measures μ on $\mathbb{R}^n$, $n \geqslant 3$, associated with a generalized condenser (A_1, A_2) where A_1 is a relatively closed subset of a domain D and $A_2 = \mathbb{R}^n \setminus D$. We show that, though $A_2 \cap C\ell_{\mathbb{R}^n} A_1$ may have nonzero capacity, this minimum energy problem is uniquely solvable (even in the presence of an external field) if we restrict ourselves to μ with $\mu^+ \leqslant \xi$ where a constraint ξ is properly chosen. We establish the sharpness of the sufficient conditions on the solvability thus obtained, provide descriptions of the weighted α-Riesz potentials of the solutions, single out their characteristic properties, and analyze their supports. The approach developed is mainly based on the establishment of an intimate relationship between the constrained minimum α-Riesz energy problem over signed measures associated with (A_1, A_2) and the constrained minimum α-Green energy problem over positive measures concentrated on A_1. The results are illustrated by examples.

1. Introduction

The purpose of this paper is to study minimum energy problems with an external field (also known in the literature as weighted minimum energy problems) relative to the α-Riesz kernel $\kappa_\alpha(x,y) = |x-y|^{\alpha-n}$ of order $\alpha \in (0,2]$, where $|x-y|$ is the Euclidean distance between $x, y \in \mathbb{R}^n$, $n \geqslant 3$, and infimum is taken over classes of signed Radon measures μ on $\mathbb{R}^n$ associated with a generalized condenser $\mathbf{A} = (A_1, A_2)$. More precisely, an ordered pair $\mathbf{A} = (A_1, A_2)$ is termed a *generalized condenser* in $\mathbb{R}^n$ if A_1 is a relatively closed subset of a given (connected open) domain $D \subset \mathbb{R}^n$ and $A_2 = D^c := \mathbb{R}^n \setminus D$, while μ is said to be *associated with* $\mathbf{A}$ if the positive and the negative parts in the Hahn–Jordan decomposition of μ are concentrated on A_1 and A_2, respectively.

2010 *Mathematics Subject Classification*: 31C15.

Key words and phrases: Constrained minimum energy problems, α-Riesz kernels, α-Green kernels, external fields, condensers with touching plates.

The research of the first author was supported, in part, by a Simons Foundation grant no. 282207. The research of the third and the fourth authors was supported, in part, by the U. S. National Science Foundation under grants DMS-1516400. The research of the fifth author was supported, in part, by the Scholar-in-Residence program at IPFW and by the Department of Mathematical Sciences of the University of Copenhagen.

Note that, although $A_1 \cap A_2 = \varnothing$, the set $A_2 \cap C\ell_{\mathbb{R}^n} A_1$ may have nonzero (in particular infinite, see Example 9.2 below) α-Riesz capacity and may even coincide with the whole ∂D. Therefore the classical *condenser problem* for the generalized condenser $\mathbf{A}$, which amounts to the minimum α-Riesz energy problem over the class of all μ associated with $\mathbf{A}$ and normalized by $\mu^+(A_1) = \mu^-(A_2) = 1$, can be easily shown to have *no* solution, see Theorem 4.3. Using the electrostatic interpretation, which is possible for the Coulomb kernel $|x-y|^{-1}$ on $\mathbb{R}^3$, in the case where a minimum energy problem has no solution we say that a short-circuit occurs between the oppositely charged plates of the generalized condenser $\mathbf{A}$. It is therefore meaningful to ask what kinds of additional requirements on the objects in question will prevent this blow-up effect, and secure that a solution to the corresponding minimum α-Riesz energy problem does exist.

We show that a solution $\lambda^\xi_{\mathbf{A}}$ to the minimum α-Riesz energy problem exists (no short-circuit occurs) if we restrict ourselves to μ with $\mu^+ \leqslant \xi$ where the constraint ξ is properly chosen. More precisely, if $A_2 = D^c$ is not α-thin at infinity, then such $\lambda^\xi_{\mathbf{A}}$ exists (even in the presence of an external field) provided that ξ is a positive Radon measure concentrated on A_1 with finite α-Riesz energy $E_{\kappa_\alpha}(\xi) := \iint \kappa_\alpha(x,y)\, d\xi(x)\, d\xi(y) < \infty$ and with total mass $\xi(A_1) \in (1,\infty)$; see Theorem 6.1.[1] In particular, if the domain D is bounded, then a solution $\lambda^\xi_{\mathbf{A}}$ exists whenever $\mathbf{A} := (D, D^c)$ and $\xi := m_n|_D$, where m_n is the n-dimensional Lebesgue measure. Theorem 6.1 is sharp in the sense that it no longer holds if the requirement $\xi(A_1) < \infty$ is omitted from its hypotheses, see Theorem 6.2.

We provide descriptions of the weighted α-Riesz potentials of the solutions $\lambda^\xi_{\mathbf{A}}$, single out their characteristic properties, and analyze their supports, see Theorems 6.3, 6.4 and 6.5. The results are illustrated by Examples 9.1 and 9.2. The theory of minimum α-Riesz energy problems with a (positive) constraint ξ acting only on positive parts of measures associated with $\mathbf{A}$, thus developed, remains valid in its full generality for the signed constraint $\xi - \xi^{D^c}$ acting simultaneously on the positive and negative parts of the measures in question, see Section 6.2. (Here ξ^{D^c} is the α-Riesz balayage of ξ onto D^c.)

The approach developed is mainly based on the establishment of an intimate relationship between, on the one hand, the constrained weighted minimum α-Riesz energy problem over signed measures associated with $\mathbf{A}$ and, on the other hand, the constrained weighted minimum α-Green energy problem over positive measures concentrated on A_1 (Theorem 5.2). While proving Theorem 5.2, we have substantially used the finiteness of the α-Riesz energy of the constraint ξ. Regrettably, a similar assertion in [11], Lemma 4.2, did not require that $E_{\kappa_\alpha}(\xi) < \infty$, being based on a false statement, Lemma 2.4, that the finiteness of the α-Green energy $E_g(\mu)$ of a bounded measure μ on D implies the finiteness of its α-Riesz energy (see Example 10.1 below for a counterexample). This caused the incorrectness of the

[1] See Section 3 for the notion of α-thinness at infinity. The uniqueness of a solution $\lambda^\xi_{\mathbf{A}}$ can be established by standard methods based on the convexity of the class of admissible measures and the pre-Hilbert structure on the linear space of all signed Radon measures on $\mathbb{R}^n$ with $E_{\kappa_\alpha}(\mu) < \infty$, see Lemma 4.6.

formulations and of the proofs presented in [11]. The present paper rectifies the results on the constrained weighted α-Riesz and α-Green energy problems announced in [11].

Regarding the constrained weighted minimum α-Green energy problem over positive measures concentrated on A_1, crucial to the arguments applied in the investigation thereof is the perfectness of the α-Green kernel g on a domain D, established recently by the second and the fifth named authors [17], which amounts to the completeness in the topology defined by the energy norm $\|\nu\|_g := \sqrt{E_g(\nu)}$ of the cone of all positive Radon measures ν on D with finite α-Green energy $E_g(\nu)$.

2. Preliminaries

Let X be a locally compact (Hausdorff) space [4, Chapter I, Section 9, n° 7], to be specified below, and $\mathfrak{M}(X)$ the linear space of all real-valued Radon measures μ on X, equipped with the *vague* topology, i.e. the topology of pointwise convergence on the class $C_0(X)$ of all continuous functions on X with compact support.[2] We refer the reader to [5, 6, 13] for the theory of measures and integration on a locally compact space, to be used throughout the paper; see also [14] for a short survey.

For the purposes of the present study it is enough to assume that X is metrizable and *countable at infinity*, the latter means that X can be represented as a countable union of compact sets [4, Chapter I, Section 9, n° 9]. Then the vague topology on $\mathfrak{M}(X)$ satisfies the first axiom of countability (see Remark 2.5 in [12]), and the vague convergence is entirely determined by convergence of sequences. The vague topology on $\mathfrak{M}(X)$ is Hausdorff; hence, a vague limit of any sequence in $\mathfrak{M}(X)$ is unique provided that it exists.

Let μ^+ and μ^- denote the positive and the negative parts of a measure $\mu \in \mathfrak{M}(X)$ in the Hahn–Jordan decomposition, $|\mu| := \mu^+ + \mu^-$ its total variation, and $S(\mu) = S^\mu_X$ its support. A measure μ is said to be *bounded* if $|\mu|(X) < \infty$. Given μ and a μ-measurable function u we shall for brevity write $\langle u, \mu\rangle := \int u\, d\mu$.[3]

Let $\mathfrak{M}^+(X)$ stand for the (convex, vaguely closed) cone of all positive $\mu \in \mathfrak{M}(X)$, and let $\Psi(X)$ consist of all lower semicontinuous (l.s.c.) functions $\psi : X \to (-\infty, \infty]$, nonnegative unless X is compact. The following fact is well known, [14, Section 1.1].

Lemma 2.1. *For any $\psi \in \Psi(X)$ the mapping $\mu \mapsto \langle \psi, \mu\rangle$ is vaguely l.s.c. on $\mathfrak{M}^+(X)$.*

We define a *kernel* $\kappa(x, y)$ on X as a symmetric positive function from $\Psi(X \times X)$. Given $\mu, \nu \in \mathfrak{M}(X)$, let $E_\kappa(\mu, \nu)$ and U^μ_κ denote the *mutual energy* and the *potential* relative to

[2] When speaking of a continuous numerical function we understand that the values are *finite* real numbers.

[3] Throughout the paper the integrals are understood as *upper* integrals, see [5]. When introducing notation about numerical quantities we always assume the corresponding object on the right to be well defined — as a finite real number or $\pm\infty$.

the kernel κ, respectively, i.e.

$$E_\kappa(\mu,\nu) := \iint \kappa(x,y)\,d\mu(x)\,d\nu(y),$$
$$U_\kappa^\mu(\cdot) := \int \kappa(\cdot,y)\,d\mu(y).$$

Observe that $U_\kappa^\mu(x)$, $\mu \in \mathfrak{M}(X)$, is well defined at $x \in X$ provided that $U_\kappa^{\mu^+}(x)$ and $U_\kappa^{\mu^-}(x)$ are not both infinite, and then $U_\kappa^\mu(x) = U_\kappa^{\mu^+}(x) - U_\kappa^{\mu^-}(x)$. In particular, if $\mu \geqslant 0$, then U_κ^μ is defined everywhere on X and represents a positive l.s.c. function, see Lemma 2.1.

Also note that $E_\kappa(\mu,\nu)$, $\mu,\nu \in \mathfrak{M}(X)$, is well defined and equal to $E_\kappa(\nu,\mu)$ provided that $E_\kappa(\mu^+,\nu^+)+E_\kappa(\mu^-,\nu^-)$ or $E_\kappa(\mu^+,\nu^-)+E_\kappa(\mu^-,\nu^+)$ is finite. For $\mu=\nu$ the mutual energy $E_\kappa(\mu,\nu)$ becomes the *energy* $E_\kappa(\mu) := E_\kappa(\mu,\mu)$. Let $\mathcal{E}_\kappa(X)$ consist of all $\mu \in \mathfrak{M}(X)$ whose energy $E_\kappa(\mu)$ is finite, which by definition means that $E_\kappa(\mu^+)$, $E_\kappa(\mu^-)$, and $E_\kappa(\mu^+,\mu^-)$ are all finite, and let $\mathcal{E}_\kappa^+(X) := \mathcal{E}_\kappa(X) \cap \mathfrak{M}^+(X)$.

If $f : X \to [-\infty,\infty]$ is an *external field* then the *f-weighted potential* $W_{\kappa,f}^\mu$ and the *f-weighted energy* $G_{\kappa,f}(\mu)$ of $\mu \in \mathcal{E}_\kappa(X)$ are formally given by

$$W_{\kappa,f}^\mu := U_\kappa^\mu + f, \tag{2.1}$$

$$G_{\kappa,f}(\mu) := E_\kappa(\mu) + 2\langle f,\mu\rangle = \bigl\langle W_{\kappa,f}^\mu + f,\mu\bigr\rangle. \tag{2.2}$$

Let $\mathcal{E}_{\kappa,f}(X)$ consist of all $\mu \in \mathcal{E}_\kappa(X)$ whose f-weighted energy $G_{\kappa,f}(\mu)$ is finite.

Given a set $Q \subset X$, let $\mathfrak{M}^+(Q;X)$ consist of all $\mu \in \mathfrak{M}^+(X)$ *concentrated on* Q, which means that $X\setminus Q$ is locally μ-negligible, or equivalently that Q is μ-measurable and $\mu = \mu|_Q$, where $\mu|_Q = 1_Q \cdot \mu$ is the trace (restriction) of μ on Q [6, Section 5, n° 3, Exemple]. (Here 1_Q denotes the indicator function of Q.) If Q is closed then μ is concentrated on Q if and only if it is supported by Q, i.e. $S(\mu) \subset Q$. It follows from the countability of X at infinity that the concept of local μ-negligibility coincides with that of μ-negligibility; and hence $\mu \in \mathfrak{M}^+(Q;X)$ if and only if $\mu^*(X\setminus Q)=0$, $\mu^*(\cdot)$ being the *outer measure* of a set. Write $\mathcal{E}_\kappa^+(Q;X) := \mathcal{E}_\kappa(X) \cap \mathfrak{M}^+(Q;X)$, $\mathfrak{M}^+(Q,q;X) := \bigl\{\mu \in \mathfrak{M}^+(Q;X) :\ \mu(Q)=q\bigr\}$ and $\mathcal{E}_\kappa^+(Q,q;X) := \mathcal{E}_\kappa(X) \cap \mathfrak{M}^+(Q,q;X)$, where $q \in (0,\infty)$.

Assume for a moment that Q is *locally closed* in X. According to [4, Chapter I, Section 3, Definition 2] this means that for every $x \in Q$ there is a neighborhood V of x in X such that $V \cap Q$ is a closed subset of the subspace $Q \subset X$. Being locally closed, the set Q is universally measurable [4, Chapter I, Section 3, Proposition 5], hence $\mathfrak{M}^+(Q;X)$ consists of all the restrictions $\mu|_Q$ where μ ranges over $\mathfrak{M}^+(X)$. On the other hand, according to [4, Chapter I, Section 9, Proposition 13] the locally closed set Q itself can be thought of as a locally compact subspace of X. Thus $\mathfrak{M}^+(Q;X)$ consists, in fact, of all those $\nu \in \mathfrak{M}^+(Q)$ for each of which there exists $\widehat{\nu} \in \mathfrak{M}^+(X)$ with the property

$$\widehat{\nu}(\varphi) = \bigl\langle 1_Q\varphi,\nu\bigr\rangle \quad \text{for every} \quad \varphi \in C_0(X). \tag{2.3}$$

We say that such $\widehat{\nu}$ *extends* $\nu \in \mathfrak{M}^+(Q)$ by 0 off Q to all of X. A sufficient condition for this to happen is that ν be bounded.

In all that follows a kernel κ is assumed to be *strictly positive definite*, which means that the energy $E_\kappa(\mu)$, $\mu \in \mathfrak{M}(X)$, is nonnegative whenever defined and it equals 0 only for $\mu = 0$. Then $\mathcal{E}_\kappa(X)$ forms a pre-Hilbert space with the inner product $E_\kappa(\mu, \mu_1)$ and the energy norm $\|\mu\|_\kappa := \sqrt{E_\kappa(\mu)}$, see [14]. The (Hausdorff) topology on $\mathcal{E}_\kappa(X)$ defined by the norm $\|\cdot\|_\kappa$ is termed *strong*.

In contrast to [15, 16] where a capacity has been treated as a functional acting on positive numerical functions on X, in the present study we consider the (standard) concept of capacity as a set function. Thus the (*inner*) *capacity* of an arbitrary set $Q \subset X$ relative to the kernel κ, denoted $c_\kappa(Q)$, is defined by

$$c_\kappa(Q) := \Bigl[\inf_{\mu \in \mathcal{E}^+_\kappa(Q,1;X)} E_\kappa(\mu)\Bigr]^{-1}; \tag{2.4}$$

see e.g. [14, 21]. Then $0 \leqslant c_\kappa(Q) \leqslant \infty$. (As usual, here and in the sequel the infimum over the empty set is taken to be $+\infty$. We also put $1\big/(+\infty) = 0$ and $1\big/0 = +\infty$.) In consequence of the strict positive definiteness of the kernel κ,

$$c_\kappa(K) < \infty \quad \text{for every compact} \quad K \subset X. \tag{2.5}$$

Furthermore, by [14, p. 153, Eq. 2],

$$c_\kappa(Q) = \sup c_\kappa(K) \quad (K \subset Q, \ K \text{ compact}). \tag{2.6}$$

An assertion $\mathcal{U}(x)$ involving a variable point $x \in X$ is said to subsist c_κ-*nearly everywhere* (c_κ-*n.e.*) on Q if $c_\kappa(N) = 0$ where N consists of all $x \in Q$ for which $\mathcal{U}(x)$ fails to hold. Throughout the paper we shall often use the fact that $c_\kappa(N) = 0$ if and only if $\mu_*(N) = 0$ for every $\mu \in \mathcal{E}^+_\kappa(X)$, $\mu_*(\cdot)$ being the *inner measure* of a set; see [14, Lemma 2.3.1].

As in [19, p. 134], we call a (signed Radon) measure $\mu \in \mathfrak{M}(X)$ c_κ-*absolutely continuous* if $\mu(K) = 0$ for every compact set $K \subset X$ with $c_\kappa(K) = 0$. It follows from (2.6) that, for such a μ, $|\mu|_*(Q) = 0$ for every $Q \subset X$ with $c_\kappa(Q) = 0$. Hence every $\mu \in \mathcal{E}_\kappa(X)$ is c_κ-absolutely continuous, but not conversely, [19, pp. 134–135].

Definition 2.2. Following [14], we call a (strictly positive definite) kernel κ *perfect* if every strong Cauchy sequence in $\mathcal{E}^+_\kappa(X)$ converges strongly to any of its vague cluster points.[4]

Remark 2.3. On $X = \mathbb{R}^n$, $n \geqslant 3$, the α-Riesz kernel $\kappa_\alpha(x, y) = |x - y|^{\alpha - n}$, $\alpha \in (0, n)$, is strictly positive definite and moreover perfect, see [9, 10]; thus so is the Newtonian kernel $\kappa_2(x, y) = |x - y|^{2-n}$ [8]. Recently it has been shown that, if $X = D$ where D is an arbitrary open set in $\mathbb{R}^n$, $n \geqslant 3$, and g^α_D, $\alpha \in (0, 2]$, is the α-Green kernel on D ([19, Chapter IV, Section 5] or see Section 3 below), then $\kappa = g^\alpha_D$ is likewise strictly positive definite and perfect [17, Theorems 4.9, 4.11].

[4] It follows from Theorem 2.4 that for a perfect kernel such a vague cluster point exists and is unique.

Theorem 2.4. (see [14]) *If a kernel κ on a locally compact space X is perfect, then the cone $\mathcal{E}^+_\kappa(X)$ is strongly complete and the strong topology on $\mathcal{E}^+_\kappa(X)$ is finer than the* (*induced*) *vague topology on $\mathcal{E}^+_\kappa(X)$.*

Remark 2.5. In contrast to Theorem 2.4, for a perfect kernel κ the whole pre-Hilbert space $\mathcal{E}_\kappa(X)$ is in general strongly *incomplete*, and this is the case even for the α-Riesz kernel of order $\alpha \in (1, n)$ on $\mathbb{R}^n$, $n \geqslant 3$ (see [8] and [19, Theorem 1.19]). Compare with [22, Theorem 1] where the strong completeness has been established for the metric subspace of all (*signed*) $\nu \in \mathcal{E}_{\kappa_\alpha}(\mathbb{R}^n)$ such that ν^+ and ν^- are supported by closed nonintersecting sets in $\mathbb{R}^n$, $n \geqslant 3$. This result from [22] has been proved with the aid of Deny's theorem [9] stating that $\mathcal{E}_{\kappa_\alpha}(\mathbb{R}^n)$ can be completed by making use of tempered distributions on $\mathbb{R}^n$ with finite α-Riesz energy.

Remark 2.6. The concept of perfect kernel is an efficient tool in minimum energy problems over classes of *positive scalar* Radon measures with finite energy. Indeed, if $Q \subset X$ is closed, $c_\kappa(Q) \in (0, +\infty)$, and if κ is perfect, then the minimum energy problem in (2.4) has a unique solution λ_Q [14, Theorem 4.1]; we shall call such a λ_Q the (*inner*) *κ-capacitary measure* on Q. Later the concept of perfectness has been shown to be efficient also in minimum energy problems over classes of *vector measures* of finite or infinite dimensions associated with a standard condenser, see [23]–[26]. The approach developed in [23]–[26] substantially used the assumption of the boundedness of the kernel on the product of the oppositely charged plates of a condenser, which made it possible to extend Cartan's proof [8] of the strong completeness of the cone $\mathcal{E}^+_{\kappa_2}(\mathbb{R}^n)$ of all positive measures on $\mathbb{R}^n$ with finite Newtonian energy to an arbitrary perfect kernel κ on a locally compact space X and suitable classes of *signed* measures $\mu \in \mathcal{E}_\kappa(X)$.

3. α-Riesz balayage and α-Green function

In all that follows we fix $n \geqslant 3$, $\alpha \in (0, 2]$ and a domain $D \subset \mathbb{R}^n$ with $c_{\kappa_\alpha}(D^c) > 0$, where $D^c := \mathbb{R}^n \setminus D$, and we assume that either $\kappa(x, y) = \kappa_\alpha(x, y) := |x - y|^{\alpha-n}$ is the *α-Riesz kernel* on $X = \mathbb{R}^n$, or $\kappa(x, y) = g^\alpha_D(x, y)$ is the *α-Green kernel* on $X = D$. For the definition of g^α_D, see [19, Chapter IV, Section 5] or see below.

For given $x \in \mathbb{R}^n$ and $r \in (0, \infty)$ write $B(x, r) := \{y \in \mathbb{R}^n : \ |y - x| < r\}$, $S(x, r) := \{y \in \mathbb{R}^n : \ |y - x| = r\}$ and $\overline{B}(x, r) := B(x, r) \cup S(x, r)$. Let ∂Q denote the boundary of a set $Q \subset \mathbb{R}^n$ in the topology of $\mathbb{R}^n$.

We shall simply write α instead of κ_α if it serves as an index. When speaking of a (positive Radon) measure $\mu \in \mathfrak{M}^+(\mathbb{R}^n)$, we always assume that $U^\mu_\alpha \not\equiv +\infty$. This implies that

$$\int_{|y|>1} \frac{d\mu(y)}{|y|^{n-\alpha}} < \infty, \tag{3.1}$$

see [19, Eq. 1.3.10], and consequently that U^μ_α is finite κ_α-n.e. on $\mathbb{R}^n$, [19, Chapter III, Section 1]; these two implications can actually be reversed. We shall use the short form 'n.e.' instead of 'c_α-n.e.' if this will not cause any misunderstanding.

Definition 3.1. $\nu \in \mathfrak{M}(D)$ is called *extendible* if there exist $\widehat{\nu^+}$ and $\widehat{\nu^-}$ extending ν^+ and ν^-, respectively, to $\mathbb{R}^n$ by 0 off D, see (2.3), and if these $\widehat{\nu^+}$ and $\widehat{\nu^-}$ satisfy (3.1). We identify such a $\nu \in \mathfrak{M}(D)$ with its extension $\widehat{\nu} := \widehat{\nu^+} - \widehat{\nu^-}$, and we therefore write $\widehat{\nu} = \nu$.

Every bounded measure $\nu \in \mathfrak{M}(D)$ is extendible. The converse holds if D is bounded, but not in general (e.g. not if D^c is compact). The set of all extendible measures $\nu \in \mathfrak{M}(D)$ consists of all the restrictions $\mu|_D$ where μ ranges over $\mathfrak{M}(\mathbb{R}^n)$.

The α-*Green kernel* $g = g_D^\alpha$ on D is defined by

$$g_D^\alpha(x,y) = U_\alpha^{\varepsilon_y}(x) - U_\alpha^{\varepsilon_y^{D^c}}(x) \quad \text{for all} \quad x, y \in D,$$

where ε_y denotes the unit Dirac measure at a point y and $\varepsilon_y^{D^c}$ its α-*Riesz balayage* onto the (closed) set D^c, uniquely determined in the frames of the classical approach by [17, Theorem 3.6]. See also the book by Bliedtner and Hansen [3] where balayage is studied in the setting of balayage spaces.

We shall simply write μ' instead of μ^{D^c} when speaking of the α-Riesz balayage of $\mu \in \mathfrak{M}^+(D;\mathbb{R}^n)$ onto D^c. According to [17, Corollaries 3.19 and 3.20], for any $\mu \in \mathfrak{M}^+(D;\mathbb{R}^n)$ the balayage μ' is c_α-absolutely continuous and it is determined uniquely by relation

$$U_\alpha^{\mu'} = U_\alpha^\mu \quad \text{n.e. on} \quad D^c \tag{3.2}$$

within the c_α-absolutely continuous positive measures on $\mathbb{R}^n$ supported by D^c. Furthermore, there holds the integral representation

$$\mu' = \int \varepsilon'_y \, d\mu(y), \tag{3.3}$$

see [17, Theorem 3.17].[5] If moreover $\mu \in \mathcal{E}_\alpha^+(D;\mathbb{R}^n)$ then the balayage μ' is in fact the orthogonal projection of μ onto the convex cone $\mathcal{E}_\alpha^+(D^c;\mathbb{R}^n)$, i.e.

$$\|\mu - \theta\|_\alpha > \|\mu - \mu'\|_\alpha \quad \text{for all} \quad \theta \in \mathcal{E}_\alpha^+(D^c;\mathbb{R}^n), \quad \theta \neq \mu'; \tag{3.4}$$

see [16, Theorem 4.12] or [17, Theorem 3.1].

If now $\nu \in \mathfrak{M}(D)$ is an extendible *signed* (Radon) measure, then

$$\nu' := \nu^{D^c} := (\nu^+)^{D^c} - (\nu^-)^{D^c}$$

is a *balayage* of ν onto D^c. The balayage ν' is determined uniquely by (3.2) with ν in place of μ within the c_α-absolutely continuous signed measures on $\mathbb{R}^n$ supported by D^c.

The following definition goes back to Brelot [7]. A closed set $F \subset \mathbb{R}^n$ is said to be α-*thin at infinity* if either F is compact, or the inverse of F relative to $S(0,1)$ has $x = 0$ as an α-irregular boundary point; cf. [19, Theorem 5.10].

[5] In the literature the integral representation (3.3) seems to have been more or less taken for granted, though it has been pointed out in [6, p. 18, Remarque] that it requires that the family $(\varepsilon'_y)_{y\in D}$ is μ-*adequate* in the sense of [6, Section 3, Définition 1]; see also counterexamples (without μ-adequacy) in Exercises 1 and 2 at the end of that section. A proof of this adequacy has therefore been given in [17, Lemma 3.16].

Theorem 3.2. (see [17, Theorem 3.22]) *The set D^c is not α-thin at infinity if and only if for every bounded measure $\mu \in \mathfrak{M}^+(D)$ it holds true that*

$$\mu'(\mathbb{R}^n) = \mu(\mathbb{R}^n). \tag{3.5}$$

As noted in Remark 2.3 above, the α-Riesz kernel κ_α on $\mathbb{R}^n$ and the α-Green kernel g^α_D on D are both strictly positive definite and moreover perfect. Furthermore, the α-Riesz kernel κ_α (with $\alpha \in (0, 2]$) satisfies the complete maximum principle in the form stated in [19, Theorems 1.27, 1.29]. Regarding a similar result for the α-Green kernel g, the following assertion holds.

Theorem 3.3. (see [17, Theorem 4.6]) *Let $\mu \in \mathcal{E}^+_g(D)$, let $\nu \in \mathfrak{M}^+(D)$ be extendible, and let v be a positive α-superharmonic function on $\mathbb{R}^n$ (see [19, Chapter I, Section 5, n° 20]). If moreover $U^\mu_g \leqslant U^\nu_g + v$ μ-a.e. on D, then the same inequality holds on all of D.*

The following three lemmas establish relations between potentials and energies relative to the kernels κ_α and $g = g^\alpha_D$, respectively.

Lemma 3.4. *For any extendible measure $\mu \in \mathfrak{M}(D)$ the α-Green potential U^μ_g is well defined and finite (c_α-)n.e. on D and given by*[6]

$$U^\mu_g = U^{\mu-\mu'}_\alpha \quad \text{n.e. on} \quad D. \tag{3.6}$$

Proof. It is seen from Definition 3.1 that U^μ_α is finite n.e. on $\mathbb{R}^n$, and hence so is $U^{\mu'}_\alpha$. Applying (3.3) to $\mu^\pm$, we get by [6, Section 3, Théorème 1]

$$U^\mu_g = \int \bigl[U^{\varepsilon_y}_\alpha - U^{\varepsilon'_y}_\alpha\bigr]\, d\mu(y) = U^\mu_\alpha - U^{\mu'}_\alpha$$

n.e. on D, and the lemma follows. □

Lemma 3.5. *If $\mu \in \mathfrak{M}(D)$ is extendible and if its extension belongs to $\mathcal{E}_\alpha(\mathbb{R}^n)$, then*

$$\mu \in \mathcal{E}_g(D), \tag{3.7}$$

$$\mu - \mu' \in \mathcal{E}_\alpha(\mathbb{R}^n), \tag{3.8}$$

$$\|\mu\|^2_g = \|\mu - \mu'\|^2_\alpha = \|\mu\|^2_\alpha - \|\mu'\|^2_\alpha. \tag{3.9}$$

Proof. In view of the definition of a signed measure of finite energy, we obtain (3.7) from the inequality[7]

$$g^\alpha_D(x, y) < \kappa_\alpha(x, y) \quad \text{for all} \quad x, y \in D, \tag{3.10}$$

while (3.8) from [17, Corollary 3.7] or [17, Theorems 3.1, 3.6]. According to Lemma 3.4 and Footnote 6, U^μ_g is finite c_g-n.e. on D and given by (3.6), while by (3.7) the same holds

[6] If Q is a given subset of D, then any assertion involving a variable point holds n.e. on Q if and only if it holds c_g-n.e. on Q, see [11, Lemma 2.6].

[7] The strict inequality in (3.10) is caused by our convention that $c_\alpha(D^c) > 0$.

$|\mu|$-a.e. on D, see [14, Lemma 2.3.1]. Integrating (3.6) with respect to $\mu^{\pm}$, we therefore obtain by subtraction

$$\infty > E_g(\mu) = E_\alpha(\mu - \mu', \mu). \tag{3.11}$$

As $U_\alpha^{\mu-\mu'} = 0$ n.e. on D^c by (3.2), while μ' is c_α-absolutely continuous, we also have

$$E_\alpha(\mu - \mu', \mu') = 0, \tag{3.12}$$

which results in the former equality in (3.9) when combined with (3.11). In view of (3.8), relation (3.12) takes the form $\|\mu'\|_\alpha^2 = E_\alpha(\mu, \mu')$, and the former equality in (3.9) implies the latter. □

Lemma 3.6. *Assume that $\mu \in \mathfrak{M}(D)$ has compact support S_D^μ. Then $\mu \in \mathcal{E}_g(D)$ if and only if its extension belongs to $\mathcal{E}_\alpha(\mathbb{R}^n)$.*

Proof. According to Lemma 3.5 it is enough to establish the necessity part of the lemma. We may clearly assume that μ is positive. Since $U_\alpha^{\mu'}$ is continuous on D, hence bounded on the compact set S_D^μ, we have

$$E_\alpha(\mu, \mu') < \infty. \tag{3.13}$$

On the other hand, $E_g(\mu)$ is finite by assumption, hence likewise as in the preceding proof relation (3.11) holds. Combining (3.11) with (3.13) yields $\mu \in \mathcal{E}_\alpha(\mathbb{R}^n)$. □

Remark 3.7. The proof of Lemma 3.5 uses substantially the requirement $\mu \in \mathcal{E}_\alpha(\mathbb{R}^n)$. Being founded on the weaker assumption $\mu \in \mathcal{E}_g(D)$, the formulation and the proof of a similar assertion in [11] (see Lemma 2.4 therein) was incorrect, as shown by Example 10.1 below. The revision of [11] provided in present paper is based significantly on the current version of Lemma 3.5, as well as on the perfectness of the kernel g_D^α discovered recently in [17, Theorem 4.11].

4. Minimum α-Riesz energy problems for generalized condensers

4.1. **A generalized condenser.** Under the (permanent) assumptions stated in the beginning of Section 3, fix a (not necessarily proper) subset A_1 of D which is relatively closed in D. The pair $\mathbf{A} = (A_1, A_2)$ where $A_2 := D^c$ is said to form a *generalized condenser* in $\mathbb{R}^n$, and A_1 and A_2 are termed its *positive* and *negative plates*.[8] To avoid triviality, we shall always require that $c_\alpha(A_1) > 0$, hence

$$c_\alpha(A_i) > 0 \quad \text{for} \quad i = 1, 2. \tag{4.1}$$

The generalized condenser $\mathbf{A} = (A_1, A_2)$ is said to be *standard* if A_1 is closed in $\mathbb{R}^n$.

Example 4.1. Let $A_1 = B(0, r) = D$, $r \in (0, \infty)$. Then $\mathbf{A} = (A_1, A_2)$ is a generalized condenser in $\mathbb{R}^n$ which certainly is not a standard one. See Example 9.1 for constraints under which the constrained minimum α-Riesz energy problem (Problem 4.4) for such an $\mathbf{A}$ admits a solution (has no short-circuit) despite the fact that $A_2 \cap C\ell_{\mathbb{R}^n} A_1 = S(0, r)$.

[8] The notion of generalized condenser thus defined differs from that introduced in our recent work [12].

Unless explicitly stated otherwise, in all that follows $\mathbf{A} = (A_1, A_2)$ is assumed to be a generalized condenser in $\mathbb{R}^n$. We emphasize that, though $A_1 \cap A_2 = \varnothing$, the set $A_2 \cap C\ell_{\mathbb{R}^n} A_1$ might have nonzero α-Riesz capacity and might even coincide with the whole ∂D.

Let $\mathfrak{M}(\mathbf{A};\mathbb{R}^n)$ consist of all signed Radon measures on $\mathbb{R}^n$ whose positive and negative parts in the Hahn–Jordan decomposition are concentrated on A_1 and A_2, respectively, and let $\mathcal{E}_\alpha(\mathbf{A};\mathbb{R}^n) := \mathfrak{M}(\mathbf{A};\mathbb{R}^n) \cap \mathcal{E}_\alpha(\mathbb{R}^n)$. For any vector $\mathbf{a} = (a_1, a_2)$ with $a_1, a_2 > 0$ write

$$\mathcal{E}_\alpha(\mathbf{A},\mathbf{a};\mathbb{R}^n) := \bigl\{\mu \in \mathcal{E}_\alpha(\mathbf{A};\mathbb{R}^n) : \ \mu^+(A_1) = a_1, \ \mu^-(A_2) = a_2\bigr\}.$$

This class is nonempty, which is clear from (4.1) in view of [14, Lemma 2.3.1], and therefore it makes sense to consider the problem on the existence of $\lambda_{\mathbf{A}} \in \mathcal{E}_\alpha(\mathbf{A},\mathbf{a};\mathbb{R}^n)$ with

$$\|\lambda_{\mathbf{A}}\|^2_\alpha = w_\alpha(\mathbf{A},\mathbf{a}) := \inf_{\mu\in\mathcal{E}_\alpha(\mathbf{A},\mathbf{a};\mathbb{R}^n)} \|\mu\|^2_\alpha. \tag{4.2}$$

This problem will be referred to as the *condenser problem*. Note that, by the (strict) positive definiteness of the kernel κ_α,

$$w_\alpha(\mathbf{A},\mathbf{a}) \geqslant 0.$$

Remark 4.2. Assume for a moment that $\mathbf{A}$ is a standard condenser in $\mathbb{R}^n$. If moreover the Euclidean distance between its plates is strictly positive, i.e. if

$$\inf_{x\in A_1,\ y\in A_2} |x-y| > 0, \tag{4.3}$$

then the assumption

$$c_\alpha(A_i) < \infty \quad \text{for} \quad i = 1, 2 \tag{4.4}$$

is sufficient for the problem (4.2) to be (uniquely) solvable for every normalizing vector $\mathbf{a}$. See e.g. [25] where this result has actually been established even for infinite dimensional vector measures in the presence of a vector-valued external field and for an arbitrary perfect kernel on a locally compact space. However, if (4.4) does not hold then in general there exists a vector $\mathbf{a}'$ such that the corresponding extremal value $w_\alpha(\mathbf{A},\mathbf{a}')$ is not an actual minimum, see [25].[9] Therefore it was interesting to give a description of the set of all vectors $\mathbf{a}$ for which the condenser problem nevertheless is solvable. Such a characterization has been established in [26]. On the other hand, if assumption (4.3) is omitted then the approach developed in [25, 26] breaks down and (4.4) does not guarantee anymore the existence of a solution to the problem (4.2). This has been illustrated by [12, Theorem 4.6] pertaining to the Newtonian kernel.

The following theorem shows that for a generalized condenser $\mathbf{A}$ the condenser problem in general has no solution.

[9] In the case of the α-Riesz kernels of order $1 < \alpha \leqslant 2$ on $\mathbb{R}^3$ some of the (theoretical) results on the solvability or unsolvability of the condenser problem mentioned in [25] have been illustrated in [18, 20] by means of numerical experiments.

Theorem 4.3. *If A_2 is not α-thin at infinity and if $c_{g^\alpha_D}(A_1)=\infty$, then*

$$w_\alpha(\mathbf{A},\mathbf{1})=\bigl[c_{g^\alpha_D}(A_1)\bigr]^{-1}=0;$$

hence $w_\alpha(\mathbf{A},\mathbf{1})$ cannot be an actual minimum because $0\notin\mathcal{E}_\alpha(\mathbf{A},\mathbf{1};\mathbb{R}^n)$.

Proof. Consider compact sets $K_\ell\subset A_1$, $\ell\in\mathbb{N}$, such that $K_\ell\uparrow A_1$ as $\ell\to\infty$. By (2.6),

$$c_g(K_\ell)\uparrow c_g(A_1)=\infty \quad\text{as}\quad \ell\to\infty \tag{4.5}$$

and there is therefore no loss of generality in assuming that every $c_g(K_\ell)$ is >0. Furthermore, since the α-Green kernel g is strictly positive definite and moreover perfect (Remark 2.3), we see from (2.5) that $c_g(K_\ell)<\infty$ and hence, by Remark 2.6, there exists a (unique) g-capacitary measure λ_ℓ on K_ℓ, i.e. $\lambda_\ell\in\mathcal{E}^+_g(K_\ell,1;D)$ with

$$\|\lambda_\ell\|^2_g=1/c_g(K_\ell)<\infty.$$

According to Lemma 3.6, $E_\alpha(\lambda_\ell)$ is finite along with $E_g(\lambda_\ell)$ and hence, by Lemma 3.5,

$$\|\lambda_\ell\|^2_g=\|\lambda_\ell-\lambda'_\ell\|^2_\alpha.$$

As A_2 is not α-thin at infinity, we get from (3.5) $\lambda_\ell-\lambda'_\ell\in\mathcal{E}_\alpha(\mathbf{A},\mathbf{1};\mathbb{R}^n)$ which together with the two preceding displays gives

$$1/c_g(K_\ell)=\|\lambda_\ell\|^2_g=\|\lambda_\ell-\lambda'_\ell\|^2_\alpha\geqslant w_\alpha(\mathbf{A},\mathbf{1})\geqslant 0.$$

Letting here $\ell\to\infty$, we obtain the theorem from (4.5). □

Using the electrostatic interpretation, which is possible for the Coulomb kernel $|x-y|^{-1}$ on $\mathbb{R}^3$, we say that under the hypotheses of Theorem 4.3 a short-circuit occurs between the oppositely signed plates of the generalized condenser $\mathbf{A}$. It is therefore meaningful to ask what kinds of additional requirements on the objects in question will prevent this blow-up effect, and secure that a solution to the corresponding minimum α-Riesz energy problem does exist. To this end we have succeeded in working out a substantive theory by imposing a proper upper constraint on the measures under consideration, thereby rectifying the results on the constrained α-Riesz energy problem stated in [11], cf. Remark 3.7 above.

4.2. **A constrained f-weighted minimum α-Riesz energy problem for a generalized condenser.** In the rest of the paper we shall always require that A_2 is not α-thin at infinity and that $\mathbf{a}=\mathbf{1}$. When speaking of an external field f, see Section 2, we shall tacitly assume that either of the following Case I or Case II holds:

I. *$f\in\Psi(\mathbb{R}^n)$ and moreover*

$$f=0\quad \textit{n.e. on}\quad A_2; \tag{4.6}$$

II. *$f=U^{\zeta-\zeta'}_\alpha$ where ζ is a signed extendible Radon measure on D with $E_\alpha(\zeta)<\infty$.*

Note that relation (4.6) holds also in Case II, see (3.2). Since a set with $c_\alpha(\cdot)=0$ carries no measure with finite α-Riesz energy [14, Lemma 2.3.1], we thus see that in either Case I or Case II *no external field acts on the measures from* $\mathcal{E}^+_\alpha(A_2;\mathbb{R}^n)$. The f-weighted α-Riesz energy $G_{\alpha,f}(\mu)$, cf. (2.2), of $\mu\in\mathcal{E}_\alpha(\mathbf{A};\mathbb{R}^n)$ can therefore be defined as

$$G_{\alpha,f}(\mu)=\|\mu\|^2_\alpha+2\langle f,\mu\rangle=\|\mu\|^2_\alpha+2\langle f,\mu^+\rangle. \tag{4.7}$$

If Case II takes place, then for every $\mu\in\mathcal{E}_\alpha(\mathbf{A};\mathbb{R}^n)$ we moreover get

$$\begin{aligned}\infty>G_{\alpha,f}(\mu)&=\|\mu\|^2_\alpha+2E_\alpha(\zeta-\zeta',\mu)\\&=\|\mu+\zeta-\zeta'\|^2_\alpha-\|\zeta-\zeta'\|^2_\alpha\geqslant-\|\zeta-\zeta'\|^2_\alpha>-\infty.\end{aligned} \tag{4.8}$$

Thus in either Case I or Case II

$$G_{\alpha,f}(\mu)\geqslant -M>-\infty \quad\text{for all}\quad \mu\in\mathcal{E}_\alpha(\mathbf{A};\mathbb{R}^n). \tag{4.9}$$

Indeed, in Case I this is obvious by (4.7), while in Case II it follows from (4.8).

By a *constraint* for measures from $\mathcal{E}^+_\alpha(A_1,1;\mathbb{R}^n)$ we mean a measure ξ such that

$$\xi\in\mathcal{E}^+_\alpha(A_1;\mathbb{R}^n)\quad\text{and}\quad \xi(A_1)>1. \tag{4.10}$$

Let $\mathfrak{C}(A_1;\mathbb{R}^n)$ consist of all such constraints. Given $\xi\in\mathfrak{C}(A_1;\mathbb{R}^n)$, write

$$\mathcal{E}^\xi_\alpha(\mathbf{A},\mathbf{1};\mathbb{R}^n):=\bigl\{\mu\in\mathcal{E}_\alpha(\mathbf{A},\mathbf{1};\mathbb{R}^n):\ \mu^+\leqslant\xi\bigr\},$$

where $\mu^+\leqslant\xi$ means that $\xi-\mu^+\geqslant 0$. Note that *we do not impose any constraint on the negative parts of measures* $\mu\in\mathcal{E}_\alpha(\mathbf{A},\mathbf{1};\mathbb{R}^n)$. In the case where the class

$$\mathcal{E}^\xi_{\alpha,f}(\mathbf{A},\mathbf{1};\mathbb{R}^n):=\mathcal{E}^\xi_\alpha(\mathbf{A},\mathbf{1};\mathbb{R}^n)\cap\mathcal{E}_{\alpha,f}(\mathbb{R}^n)$$

is nonempty (see Section 2 for the definition of $\mathcal{E}_{\alpha,f}(\mathbb{R}^n)$), or equivalently if[10]

$$G^\xi_{\alpha,f}(\mathbf{A},\mathbf{1};\mathbb{R}^n):=\inf_{\mu\in\mathcal{E}^\xi_{\alpha,f}(\mathbf{A},\mathbf{1};\mathbb{R}^n)}G_{\alpha,f}(\mu)<\infty, \tag{4.11}$$

then the following *constrained f-weighted minimum α-Riesz energy problem* makes sense.

Problem 4.4. If (4.11) holds, does there exist $\lambda^\xi_{\mathbf{A}}\in\mathcal{E}^\xi_{\alpha,f}(\mathbf{A},\mathbf{1};\mathbb{R}^n)$ with

$$G_{\alpha,f}(\lambda^\xi_{\mathbf{A}})=G^\xi_{\alpha,f}(\mathbf{A},\mathbf{1};\mathbb{R}^n)\,? \tag{4.12}$$

Conditions which guarantee (4.11) are provided by the following Lemma 4.5. Write

$$A_1^\circ:=\bigl\{x\in A_1:\ |f(x)|<\infty\bigr\}. \tag{4.13}$$

Lemma 4.5. *Relation* (4.11) *holds if either Case* II *takes place, or* (*in the presence of Case* I) *if*

$$\xi(A_1^\circ)>1. \tag{4.14}$$

[10] If (4.11) is fulfilled, then $G^\xi_{\alpha,f}(\mathbf{A},\mathbf{1};\mathbb{R}^n)$ is actually finite, see (4.9).

Proof. Assume first that (4.14) holds; then there exists by (4.13) a compact set $K \subset A_1^\circ$ such that $|f| \leqslant M < \infty$ on K and that $\xi(K) > 1$. Define $\mu = \mu^+ - \mu^-$ where $\mu^+ := \xi|_K / \xi(K)$ and where μ^- is any measure from $\mathcal{E}^+_\alpha(A_2, 1; \mathbb{R}^n)$ (such μ^- exists since $c_\alpha(A_2) > 0$). Noting that $\xi|_K \in \mathcal{E}^+_\alpha(K; \mathbb{R}^n)$ by (4.10), we get $\mu \in \mathcal{E}^\xi_{\alpha,f}(\mathbf{A}, \mathbf{1}; \mathbb{R}^n)$ which yields (4.11). To complete the proof of the lemma, it is left to note that (4.14) holds automatically whenever Case II takes place, because then $U_\alpha^{\zeta-\zeta'}$ is finite n.e. on $\mathbb{R}^n$, hence ξ-a.e. by (4.10). □

Lemma 4.6. *A solution $\lambda^\xi_{\mathbf{A}}$ to Problem* 4.4 *is unique* (*whenever it exists*).

Proof. This can be established by standard methods based on the convexity of the class $\mathcal{E}^\xi_{\alpha,f}(\mathbf{A}, \mathbf{1}; \mathbb{R}^n)$ and on the pre-Hilbert structure on the space $\mathcal{E}_\alpha(\mathbb{R}^n)$. Indeed, if λ and $\breve\lambda$ are two solutions to Problem 4.4, then we obtain from the convexity of $\mathcal{E}^\xi_{\alpha,f}(\mathbf{A}, \mathbf{1}; \mathbb{R}^n)$

$$4G^\xi_{\alpha,f}(\mathbf{A}, \mathbf{1}; \mathbb{R}^n) \leqslant 4G_{\alpha,f}\Big(\frac{\lambda + \breve\lambda}{2}\Big) = \|\lambda + \breve\lambda\|^2_\alpha + 4\langle f, \lambda + \breve\lambda\rangle.$$

On the other hand, applying the parallelogram identity in $\mathcal{E}_\alpha(\mathbb{R}^n)$ to λ and $\breve\lambda$ and then adding and subtracting $4\langle f, \lambda + \breve\lambda\rangle$ we get

$$\|\lambda - \breve\lambda\|^2_\alpha = -\|\lambda + \breve\lambda\|^2_\alpha - 4\langle f, \lambda + \breve\lambda\rangle + 2G_{\alpha,f}(\lambda) + 2G_{\alpha,f}(\breve\lambda).$$

When combined with the preceding relation, this yields

$$0 \leqslant \|\lambda - \breve\lambda\|^2_\alpha \leqslant -4G^\xi_{\alpha,f}(\mathbf{A}, \mathbf{1}; \mathbb{R}^n) + 2G_{\alpha,f}(\lambda) + 2G_{\alpha,f}(\breve\lambda) = 0,$$

which establishes the lemma because of the strict positive definiteness of the kernel κ_α. □

5. Relations between minimum α-Riesz and α-Green energy problems

We are keeping the (permanent) assumptions on $\mathbf{A}$, f and ξ stated in Sections 4.1 and 4.2. Since $\mathfrak{M}^+(A_1; \mathbb{R}^n) \subset \mathfrak{M}^+(A_1; D)$, the constraint ξ can be thought of as an extendible measure from $\mathfrak{M}^+(A_1; D)$ such that its extension has finite α-Riesz energy (and total mass $\xi(A_1) > 1$). Define

$$\mathcal{E}^\xi_g(A_1, 1; D) := \big\{\mu \in \mathcal{E}^+_g(A_1, 1; D) : \ \mu \leqslant \xi\big\},$$

and let $\mathcal{E}^\xi_{g,f}(A_1, 1; D)$ consist of all $\mu \in \mathcal{E}^\xi_g(A_1, 1; D)$ such that

$$G_{g,f}(\mu) := G_{g,f|_D}(\mu) = \|\mu\|^2_g + 2\big\langle f|_D, \mu\big\rangle \tag{5.1}$$

is finite, cf. (2.2). If the class $\mathcal{E}^\xi_{g,f}(A_1, 1; D)$ is nonempty, or equivalently if

$$G^\xi_{g,f}(A_1, 1; D) := \inf_{\mu \in \mathcal{E}^\xi_{g,f}(A_1,1;D)} G_{g,f}(\mu) < \infty, \tag{5.2}$$

then the following *constrained f-weighted minimum α-Green energy problem* makes sense.

Problem 5.1. If (5.2) holds, does there exist $\lambda^\xi_{A_1} \in \mathcal{E}^\xi_{g,f}(A_1, 1; D)$ with

$$G_{g,f}(\lambda^\xi_{A_1}) = G^\xi_{g,f}(A_1, 1; D)\,? \tag{5.3}$$

Based on the convexity of $\mathcal{E}^\xi_{g,f}(A_1, 1; D)$ and on the pre-Hilbert structure on $\mathcal{E}_g(D)$, likewise as in the proof of Lemma 4.6 we see that *a solution* $\lambda^\xi_{A_1}$ *to Problem* 5.1 *is unique.*

Theorem 5.2. *Under the stated assumptions, we have*

$$G^\xi_{\alpha,f}(\mathbf{A}, \mathbf{1}; \mathbb{R}^n) = G^\xi_{g,f}(A_1, 1; D). \tag{5.4}$$

Assume moreover that either of the (equivalent) assumptions (4.11) *or* (5.2) *is fulfilled. Then the solution to Problem* 4.4 *exists if and only if so does that to Problem* 5.1*, and in the affirmative case they are related to each other by the formula*

$$\lambda^\xi_{\mathbf{A}} = \lambda^\xi_{A_1} - \bigl(\lambda^\xi_{A_1}\bigr)'. \tag{5.5}$$

Proof. We begin by establishing the inequality

$$G^\xi_{g,f}(A_1, 1; D) \geqslant G^\xi_{\alpha,f}(\mathbf{A}, \mathbf{1}; \mathbb{R}^n). \tag{5.6}$$

Assuming $G^\xi_{g,f}(A_1, 1; D) < \infty$, choose $\nu \in \mathcal{E}^\xi_{g,f}(A_1, 1; D)$. Being bounded, this ν is extendible. Furthermore, its extension has finite α-Riesz energy, for so does the extension of the constraint ξ by (4.10). Applying (3.9) and (5.1) we get

$$G_{g,f}(\nu) = \|\nu - \nu'\|^2_\alpha + 2\bigl\langle f|_D, \nu\bigr\rangle.$$

As A_2 is not α-thin at infinity, we obtain from (3.5) $\theta := \nu - \nu' \in \mathcal{E}^\xi_\alpha(\mathbf{A}, \mathbf{1}; \mathbb{R}^n)$. Furthermore, by (4.7),

$$\langle f, \theta\rangle = \bigl\langle f|_D, \nu\bigr\rangle < \infty.$$

Thus $\theta \in \mathcal{E}^\xi_{\alpha,f}(\mathbf{A}, \mathbf{1}; \mathbb{R}^n)$ and $G_{\alpha,f}(\theta) = G_{g,f}(\nu)$, the latter relation being valid according to the two preceding displays. This yields

$$G_{g,f}(\nu) = G_{\alpha,f}(\theta) \geqslant G^\xi_{\alpha,f}(\mathbf{A}, \mathbf{1}; \mathbb{R}^n),$$

which establishes (5.6) in view of the arbitrary choice of $\nu \in \mathcal{E}^\xi_{g,f}(A_1, 1; D)$.

On the other hand, for any $\mu \in \mathcal{E}^\xi_{\alpha,f}(\mathbf{A}, \mathbf{1}; \mathbb{R}^n)$ we have $\mu^+ \in \mathcal{E}^+_\alpha(\mathbb{R}^n)$ by the definition of a signed measure of finite energy, as well as $\mu^+ \in \mathcal{E}^\xi_{g,f}(A_1, 1; D)$, see (3.7) and (4.7). Because of (3.4), (3.9) and (4.7),

$$\begin{aligned} G_{\alpha,f}(\mu) &= \|\mu\|^2_\alpha + 2\langle f, \mu^+\rangle = \|\mu^+ - \mu^-\|^2_\alpha + 2\langle f, \mu^+\rangle \\ &\geqslant \|\mu^+ - (\mu^+)'\|^2_\alpha + 2\langle f, \mu^+\rangle = \|\mu^+\|^2_g + 2\langle f, \mu^+\rangle \\ &= G_{g,f}(\mu^+) \geqslant G^\xi_{g,f}(A_1, 1; D). \end{aligned} \tag{5.7}$$

As $\mu \in \mathcal{E}^\xi_{\alpha,f}(\mathbf{A}, \mathbf{1}; \mathbb{R}^n)$ has been chosen arbitrarily, this together with (5.6) proves (5.4).

Let now $\lambda^\xi_{A_1} \in \mathcal{E}^\xi_{g,f}(A_1, 1; D)$ satisfy (5.3). In the same manner as in the first paragraph of the present proof we see that $\breve\mu := \lambda^\xi_{A_1} - \bigl(\lambda^\xi_{A_1}\bigr)' \in \mathcal{E}^\xi_{\alpha,f}(\mathbf{A}, \mathbf{1}; \mathbb{R}^n)$. Substituting $\breve\mu$ instead of μ in relation (5.7), we note that all the inequalities therein are, in fact, equalities. According to (5.4) we therefore get $G_{\alpha,f}(\breve\mu) = G^\xi_{g,f}(A_1, 1; D) = G^\xi_{\alpha,f}(\mathbf{A}, \mathbf{1}; \mathbb{R}^n)$. Hence there exists the (unique) solution $\lambda^\xi_{\mathbf{A}}$ to Problem 4.4 and it is related to $\lambda^\xi_{A_1}$ by means of formula (5.5).

To complete the proof, assume further that $\lambda^\xi_{\mathbf{A}} = \lambda^+ - \lambda^- \in \mathcal{E}^\xi_{\alpha,f}(\mathbf{A}, \mathbf{1}; \mathbb{R}^n)$ satisfies (4.12). Similarly as in the second paragraph of the present proof, we have $\lambda^+ \in \mathcal{E}^\xi_{g,f}(A_1, 1; D)$. Furthermore, by (5.4) and (5.7), the latter with $\lambda^\xi_{\mathbf{A}}$ instead of μ,

$$\begin{aligned} G^\xi_{g,f}(A_1, 1; D) = G_{\alpha,f}(\lambda^\xi_{\mathbf{A}}) &\geqslant \|\lambda^+ - (\lambda^+)'\|^2_\alpha + 2\langle f, \lambda^+\rangle \\ &= \|\lambda^+\|^2_g + 2\langle f, \lambda^+\rangle = G_{g,f}(\lambda^+) \geqslant G^\xi_{g,f}(A_1, 1; D). \end{aligned}$$

Hence, all the inequalities in the last display are, in fact, equalities. This shows that $\lambda^\xi_{A_1} := \lambda^+$ solves Problem 5.1 and also, on account of (3.4), that $\lambda^- = (\lambda^+)' = \bigl(\lambda^\xi_{A_1}\bigr)'$. □

When investigating Problem 5.1 we shall need the following assertion, see [11, Lemma 4.3].

Lemma 5.3. *Assume that* (5.2) *holds. Then* $\lambda = \lambda^\xi_{A_1} \in \mathcal{E}^\xi_{g,f}(A_1, 1; D)$ *is a solution to Problem* 5.1 *if and only if*

$$\bigl\langle W^\lambda_{g,f}, \nu - \lambda\bigr\rangle \geqslant 0 \quad \text{for all} \ \ \nu \in \mathcal{E}^\xi_{g,f}(A_1, 1; D),$$

where $W^\lambda_{g,f} := W^\lambda_{g,f|_D} := U^\lambda_g + f|_D$, *cf.* (2.1).

6. Main results

We keep all the (permanent) assumptions on $\mathbf{A}$, f and ξ imposed in Sections 4.1 and 4.2.

6.1. **Formulations of the main results.** In the following Theorem 6.1 we require that relation (4.11) holds; see Lemma 4.5 providing sufficient conditions for this to occur.

Theorem 6.1. *Suppose moreover that the constraint* $\xi \in \mathfrak{C}(A_1; \mathbb{R}^n)$ *is bounded, i.e.*

$$\xi(A_1) < \infty. \tag{6.1}$$

Then in either Case I *or Case* II *Problem* 4.4 *is* (*uniquely*) *solvable.*

Theorem 6.1 is sharp in the sense that it does not remain valid if requirement (6.1) is omitted from its hypotheses (see the following Theorem 6.2).

Theorem 6.2. *Condition* (6.1) *is actually necessary* (*and sufficient*) *for the solvability of Problem* 4.4. *More precisely, suppose that* $c_\alpha(A_1) = \infty$ *and that Case* II *holds with* $\zeta \geqslant 0$. *Then there exists a constraint* $\xi \in \mathfrak{C}(A_1; \mathbb{R}^n)$ *with* $\xi(A_1) = \infty$ *such that*

$$G^\xi_{\alpha,f}(\mathbf{A}, \mathbf{1}; \mathbb{R}^n) = G^\xi_{g,f}(A_1, 1; D) = 0; \tag{6.2}$$

hence $G^{\xi}_{\alpha,f}(\mathbf{A},\mathbf{1};\mathbb{R}^n)$ *cannot be an actual minimum because* $0\notin\mathcal{E}^{\xi}_{\alpha,f}(\mathbf{A},\mathbf{1};\mathbb{R}^n)$.

In the following three assertions we provide descriptions of the f-weighted α-Riesz potential $W^{\lambda^{\xi}_{\mathbf{A}}}_{\alpha,f}$, cf. (2.1), of the solution $\lambda^{\xi}_{\mathbf{A}}$ to Problem 4.4 (provided that it exists) and single out its characteristic properties. The support of $\lambda^{\xi}_{\mathbf{A}}$ is also analyzed therein.

Theorem 6.3. *Let assumption* (4.14) *hold and let* f *be lower bounded on* A_1. *Fix an arbitrary* $\lambda\in\mathcal{E}^{\xi}_{\alpha,f}(\mathbf{A},\mathbf{1};\mathbb{R}^n)$*; such exists by Lemma* 4.5. *Then in either Case* I *or Case* II *the following two assertions are equivalent:*[11]

(i) λ *is a solution to Problem* 4.4*;*

(ii) *There exists a number* $c\in\mathbb{R}$ *possessing the properties*

$$W^{\lambda}_{\alpha,f}\geqslant c \quad (\xi-\lambda^{+})\text{-a.e.},\tag{6.3}$$

$$W^{\lambda}_{\alpha,f}\leqslant c \quad \lambda^{+}\text{-a.e.},\tag{6.4}$$

and in addition it holds true that

$$W^{\lambda}_{\alpha,f}=0 \quad \text{n.e. on } A_2.\tag{6.5}$$

If moreover Case II *holds, then relation* (6.5) *can be rewritten equivalently in the following apparently stronger form:*

$$W^{\lambda}_{\alpha,f}=0 \quad \text{on } A_2\setminus I_{\alpha,A_2},\tag{6.6}$$

where I_{α,A_2} *denotes the set of all* α*-irregular* (*boundary*) *points of* A_2.

Let $\breve{A}_2$ denote the κ_α*-reduced kernel* of A_2, [19, p. 164], namely the set of all $x\in A_2$ such that for any $r>0$ we have $c_\alpha\bigl(B(x,r)\cap A_2\bigr)>0$.

In the following Theorems 6.4 and 6.5 we suppose that there exists the solution $\lambda^{\xi}_{\mathbf{A}}=\lambda^{+}-\lambda^{-}$ to Problem 4.4. For the sake of simplicity of formulation, in Theorem 6.4 we also assume that in the case $\alpha=2$ the domain D is simply connected.

Theorem 6.4. *It holds that*

$$S^{\lambda^-}_{\mathbb{R}^n}=\begin{cases}\breve{A}_2 & \text{if} \quad \alpha<2,\\ \partial D & \text{if} \quad \alpha=2.\end{cases}\tag{6.7}$$

Theorem 6.5. *Let* $f=0$. *Then*

$$W^{\lambda^{\xi}_{\mathbf{A}}}_{\alpha,f}=U^{\lambda^{\xi}_{\mathbf{A}}}_{\alpha}=\begin{cases}U^{\lambda^+}_{g} & \text{n.e. on} \quad D,\\ 0 & \text{on} \quad D^c\setminus I_{\alpha,D^c}.\end{cases}\tag{6.8}$$

[11] In Case I the assumption of the lower boundedness of f on A_1 is automatically fulfilled. Furthermore, in Case I relation (6.4) is equivalent to the following apparently stronger assertion: $W^{\lambda}_{\alpha,f}\leqslant c$ on $S^{\lambda^+}_{D}$.

Furthermore, assertion (ii) *of Theorem* 6.3 *holds, and relations* (6.3) *and* (6.4) *now take respectively the following* (*equivalent*) *form:*

$$U_\alpha^{\lambda_{\mathbf{A}}^\xi} = c \quad (\xi - \lambda^+)\text{-a.e.}, \tag{6.9}$$

$$U_\alpha^{\lambda_{\mathbf{A}}^\xi} \leqslant c \quad \text{on} \quad \mathbb{R}^n, \tag{6.10}$$

where $0 < c < \infty$. *In addition, in the present case* $f = 0$ *relations* (6.9) *and* (6.10) *together with* $U_\alpha^{\lambda_{\mathbf{A}}^\xi} = 0$ *n.e. on* D^c *determine uniquely the solution* $\lambda_{\mathbf{A}}^\xi$ *to Problem* 4.4 *within the class of admissible measures. If moreover* U_α^ξ *is* (*finitely*) *continuous on* D, *then also*

$$U_\alpha^{\lambda_{\mathbf{A}}^\xi} = c \quad \text{on} \quad S_D^{\xi-\lambda^+}, \tag{6.11}$$

$$c_{g_D^\alpha}\bigl(S_D^{\xi-\lambda^+}\bigr) < \infty. \tag{6.12}$$

Omitting now the requirement of the continuity of U_α^ξ, *assume further that* $\alpha < 2$ *and that* $m_n(D^c) > 0$ *where* m_n *is the* n*-dimensional Lebesgue measure. Then*

$$S_D^{\lambda^+} = S_D^\xi, \tag{6.13}$$

$$U_\alpha^{\lambda_{\mathbf{A}}^\xi} < c \quad \text{on} \quad \mathbb{R}^n \setminus S_D^\xi \quad \Bigl(= \mathbb{R}^n \setminus S_D^{\lambda^+}\Bigr). \tag{6.14}$$

The proofs of Theorems 6.1, 6.2, 6.3, 6.4 and 6.5 are presented in Section 7.

6.2. **An extension of the theory.** Parallel with a constraint $\xi \in \mathfrak{C}(A_1; \mathbb{R}^n)$ given by relation (4.10) and acting only on (positive) measures from $\mathcal{E}_\alpha^+(A_1, 1; \mathbb{R}^n)$, consider also the measure $\sigma = \sigma^+ - \sigma^-$ defined as follows:

$$\sigma^+ = \xi, \quad \text{while} \quad \sigma^- = \xi'. \tag{6.15}$$

Since $\sigma \in \mathcal{E}_\alpha(\mathbf{A}; \mathbb{R}^n)$ and since $\sigma^-(\mathbb{R}^n) = \sigma^+(\mathbb{R}^n) > 1$, this σ can be thought of as a *signed constraint* acting on (signed) measures from $\mathcal{E}_\alpha(\mathbf{A}; \mathbf{1}; \mathbb{R}^n)$. Let $\mathcal{E}_\alpha^\sigma(\mathbf{A}, \mathbf{1}; \mathbb{R}^n)$ consist of all $\mu \in \mathcal{E}_\alpha(\mathbf{A}, \mathbf{1}; \mathbb{R}^n)$ such that $\mu^\pm \leqslant \sigma^\pm$, and let

$$G_{\alpha,f}^\sigma(\mathbf{A}, \mathbf{1}; \mathbb{R}^n) := \inf_{\mu \in \mathcal{E}_{\alpha,f}^\sigma(\mathbf{A}, \mathbf{1}; \mathbb{R}^n)} G_{\alpha,f}(\mu), \tag{6.16}$$

where $\mathcal{E}_{\alpha,f}^\sigma(\mathbf{A}, \mathbf{1}; \mathbb{R}^n) := \mathcal{E}_\alpha^\sigma(\mathbf{A}, \mathbf{1}; \mathbb{R}^n) \cap \mathcal{E}_{\alpha,f}(\mathbb{R}^n)$.

Theorem 6.6. *With these assumptions and notations, we have*

$$G_{\alpha,f}^\sigma(\mathbf{A}, \mathbf{1}; \mathbb{R}^n) = G_{\alpha,f}^\xi(\mathbf{A}, \mathbf{1}; \mathbb{R}^n). \tag{6.17}$$

If these (*equal*) *extremal values are finite, then Problem* 4.4 (*with the positive constraint* ξ) *is solvable if and only so is the problem* (6.16) (*with the signed constraint* σ); *and in the affirmative case their solutions coincide.*

Proof. Indeed, $G^\sigma_{\alpha,f}(\mathbf{A},\mathbf{1};\mathbb{R}^n)\geqslant G^\xi_{\alpha,f}(\mathbf{A},\mathbf{1};\mathbb{R}^n)$ follows directly from the relation

$$\mathcal{E}^\sigma_{\alpha,f}(\mathbf{A},\mathbf{1};\mathbb{R}^n)\subset\mathcal{E}^\xi_{\alpha,f}(\mathbf{A},\mathbf{1};\mathbb{R}^n). \tag{6.18}$$

To prove the converse inequality, assume $G^\xi_{\alpha,f}(\mathbf{A},\mathbf{1};\mathbb{R}^n)<\infty$ and fix $\nu\in\mathcal{E}^\xi_{\alpha,f}(\mathbf{A},\mathbf{1};\mathbb{R}^n)$. Define $\mu:=\nu^+-(\nu^+)'$. It is obvious that $\mu\in\mathcal{E}_\alpha(\mathbf{A};\mathbb{R}^n)$, while Theorem 3.2 shows that $(\nu^+)'(A_2)=\nu^+(A_1)=1$. Furthermore, $(\nu^+)'\leqslant\xi'=\sigma^-$ by the linearity of balayage, and so altogether $\mu\in\mathcal{E}^\sigma_\alpha(\mathbf{A},\mathbf{1};\mathbb{R}^n)$. According to (3.4) and (4.7), we thus have

$$\begin{aligned}G_{\alpha,f}(\nu)&=\|\nu\|^2_\alpha+2\langle f,\nu^+\rangle\geqslant\|\nu^+-(\nu^+)'\|^2_\alpha+2\langle f,\nu^+\rangle\\&=\|\mu\|^2_\alpha+2\langle f,\mu^+\rangle=G_{\alpha,f}(\mu)\geqslant G^\sigma_{\alpha,f}(\mathbf{A},\mathbf{1};\mathbb{R}^n),\end{aligned}$$

which establishes (6.17) in view of the arbitrary choice of $\nu\in\mathcal{E}^\xi_{\alpha,f}(\mathbf{A},\mathbf{1};\mathbb{R}^n)$.

Assume now that (4.11) holds. If there is a solution $\lambda^\sigma_{\mathbf{A}}$ to the problem (6.16) then this $\lambda^\sigma_{\mathbf{A}}$ also solves Problem 4.4, which is clear from (6.17) and (6.18). Conversely, if $\lambda^\xi_{\mathbf{A}}=\lambda^+-\lambda^-$ solves Problem 4.4 then by (5.5) it holds that $\lambda^-=(\lambda^+)'$, and in the same manner as in the preceding paragraph we get $\lambda^\xi_{\mathbf{A}}\in\mathcal{E}^\sigma_\alpha(\mathbf{A},\mathbf{1};\mathbb{R}^n)$. Hence $\lambda^\xi_{\mathbf{A}}$ also solves the problem (6.16) because $G_{\alpha,f}(\lambda^\xi_{\mathbf{A}})=G^\xi_{\alpha,f}(\mathbf{A},\mathbf{1};\mathbb{R}^n)=G^\sigma_{\alpha,f}(\mathbf{A},\mathbf{1};\mathbb{R}^n)$ by (6.17). □

Thus the theory of minimum α-Riesz energy problems with a (positive) constraint $\xi\in\mathfrak{C}(A_1;\mathbb{R}^n)$ acting only on positive parts of measures μ from $\mathcal{E}_\alpha(\mathbf{A},\mathbf{1};\mathbb{R}^n)$, developed in Section 6.1, remains valid in its full generality for the signed constraint σ, defined by (6.15) and acting simultaneously on positive and negative parts of $\mu\in\mathcal{E}_\alpha(\mathbf{A},\mathbf{1};\mathbb{R}^n)$.

Remark 6.7. In the case where

$$c_\alpha\bigl(C\ell_{\overline{\mathbb{R}^n}}A_1\cap C\ell_{\overline{\mathbb{R}^n}}A_2\bigr)=0, \tag{6.19}$$

Problem 4.4 has been analyzed in our recent work [12] for the α-Riesz kernel of *any* order $\alpha\in(0,n)$, for *any* normalizing vector $\mathbf{a}=(a_1,a_2)$, and for *any signed* constraint $\xi^+-\xi^-$ (not necessarily of finite α-Riesz energy); see e.g. Theorems 6.1 and 7.1 therein. (Compare with [2] where a similar problem has been analyzed for the logarithmic kernel on the plane.) However the approach developed in [12] uses substantially the requirement (6.19), and can not be adapted to the present case where (6.19) may not hold.

7. Proofs of the assertions formulated in Section 6.1

Observe that, if Case II takes place, then

$$\zeta\in\mathcal{E}_g(D), \tag{7.1}$$

$$f=U^{\zeta-\zeta'}_\alpha=U^\zeta_g\quad c_g\text{-n.e. on } D. \tag{7.2}$$

Indeed, (7.1) is obvious by (3.7), and (7.2) holds by Lemma 3.4 and Footnote 6. By (7.1) and (7.2) we get in Case II for every $\nu \in \mathcal{E}^+_g(A_1; D)$

$$G_{g,f}(\nu) = \|\nu\|^2_g + 2E_g(\zeta, \nu) = \|\nu + \zeta\|^2_g - \|\zeta\|^2_g. \tag{7.3}$$

7.1. **Proof of Theorem 6.1.** By Theorem 5.2, Theorem 6.1 will be proved once we have established the following assertion.

Theorem 7.1. *Under the assumptions of Theorem* 6.1*, Problem* 5.1 *is solvable.*

Proof. Under the assumptions of Theorem 6.1 Problem 5.1 makes sense since, by (5.4),

$$G^\xi_{g,f}(A_1, 1; D) < \infty. \tag{7.4}$$

Then $G^\xi_{g,f}(A_1, 1; D)$ is actually finite, which is clear from (5.4) and Footnote 10.

In view of (7.4), one can choose a sequence $\{\mu_k\}_{k\in\mathbb{N}} \subset \mathcal{E}^\xi_{g,f}(A_1, 1; D)$ such that

$$\lim_{k\to\infty} G_{g,f}(\mu_k) = G^\xi_{g,f}(A_1, 1; D). \tag{7.5}$$

Since $\mathcal{E}^\xi_{g,f}(A_1, 1; D)$ is a convex cone and since $\mathcal{E}_g(D)$ is a pre-Hilbert space with the inner product $E_g(\nu, \nu_1)$ and the energy norm $\|\nu\|_g = \sqrt{E_g(\nu)}$, arguments similar to those in the proof of Lemma 4.6 can be applied to the set $\{\mu_k : \ k \in \mathbb{N}\}$. This gives

$$0 \leqslant \|\mu_k - \mu_\ell\|^2_g \leqslant -4G^\xi_{g,f}(A_1, 1; D) + 2G_{g,f}(\mu_k) + 2G_{g,f}(\mu_\ell).$$

Letting here $k, \ell \to \infty$ and combining the relation thus obtained with (7.5), we see in view of the finiteness of $G^\xi_{g,f}(A_1, 1; D)$ that $\{\mu_k\}_{k\in\mathbb{N}}$ forms a strong Cauchy sequence in the metric space $\mathcal{E}^+_g(D)$. In particular, this implies

$$\sup_{k\in\mathbb{N}} \|\mu_k\|_g < \infty. \tag{7.6}$$

Since A_1 is (relatively) closed in D and since the cone $\mathfrak{M}^+(D)$ is vaguely closed in $\mathfrak{M}(D)$, so is the cone $\mathfrak{M}^\xi(A_1; D) := \bigl\{\nu \in \mathfrak{M}^+(A_1; D) : \ \nu \leqslant \xi\bigr\}$. Furthermore, $\mathfrak{M}^\xi(A_1, 1; D) := \mathfrak{M}^\xi(A_1; D) \cap \mathfrak{M}^+(A_1, 1; D)$ is vaguely bounded, hence it is vaguely relatively compact according to [5, Chapitre III, Section 2, Proposition 9]. Thus, there exists a vague cluster point μ of the sequence $\{\mu_k\}_{k\in\mathbb{N}}$ chosen above, and such a μ belongs to $\mathfrak{M}^\xi(A_1; D)$. Passing to a subsequence and changing notations, we assume that

$$\mu_k \to \mu \ \text{ vaguely in } \mathfrak{M}^+(D) \text{ as } \ k \to \infty. \tag{7.7}$$

We assert that this μ is a solution to Problem 5.1.

Applying Lemma 2.1 to $1_D \in \Psi(D)$, we conclude from (7.7) that

$$\mu(D) \leqslant \lim_{k\to\infty} \mu_k(D) = 1.$$

We proceed by showing that the inequality here is in fact an equality, hence altogether

$$\mu \in \mathfrak{M}^{\xi}(A_1, 1; D). \tag{7.8}$$

Consider an exhaustion of A_1 by an upper directed family of compact sets $K \subset A_1$. Since 1_K is upper semicontinuous on D (and of course bounded), we get from Lemma 2.1 and [14, Lemma 1.2.2]

$$\begin{aligned} 1 \geqslant \langle 1_D, \mu \rangle &= \lim_{K \uparrow A_1} \langle 1_K, \mu \rangle \geqslant \lim_{K \uparrow A_1} \limsup_{k \to \infty} \langle 1_K, \mu_k \rangle \\ &= 1 - \lim_{K \uparrow A_1} \liminf_{k \to \infty} \langle 1_{A_1 \setminus K}, \mu_k \rangle. \end{aligned}$$

Thus relation (7.8) will follow once we show that

$$\lim_{K \uparrow A_1} \liminf_{k \to \infty} \langle 1_{A_1 \setminus K}, \mu_k \rangle = 0. \tag{7.9}$$

Since by (6.1) and [14, Lemma 1.2.2] it holds that

$$\infty > \xi(D) = \lim_{K \uparrow A_1} \langle 1_K, \xi \rangle,$$

we have

$$\lim_{K \uparrow A_1} \langle 1_{A_1 \setminus K}, \xi \rangle = 0.$$

When combined with

$$\langle 1_{A_1 \setminus K}, \mu_k \rangle \leqslant \langle 1_{A_1 \setminus K}, \xi \rangle \quad \text{for every} \quad k \in \mathbb{N},$$

this implies (7.9) and consequently (7.8).

Another consequence of (7.7) is that $\mu_k \otimes \mu_k \to \mu \otimes \mu$ vaguely in $\mathfrak{M}^+(D \times D)$ [5, Chapitre III, Section 5, Exercice 5]. Applying Lemma 2.1 to $Y = D \times D$ and $\psi = g$, we thus get

$$E_g(\mu) \leqslant \liminf_{k \to \infty} \|\mu_k\|_g^2 < \infty,$$

where the latter inequality holds by (7.6). Hence $\mu \in \mathcal{E}_g^+(D)$. Combined with (7.8), this yields $\mu \in \mathcal{E}_g^{\xi}(A_1, 1; D)$. As $G_{f,g}(\mu) > -\infty$, the assertion that μ solves Problem 5.1 will therefore be established once we have shown that

$$G_{g,f}(\mu) \leqslant \lim_{k \to \infty} G_{g,f}(\mu_k). \tag{7.10}$$

Since the kernel g is perfect [17, Theorem 4.11], the sequence $\{\mu_k\}_{k \in \mathbb{N}}$, being strong Cauchy in $\mathcal{E}_g^+(D)$ and vaguely convergent to μ, converges to the same limit strongly in $\mathcal{E}_g^+(D)$, i.e.

$$\lim_{k \to \infty} \|\mu_k - \mu\|_g = 0. \tag{7.11}$$

Also note that the mapping $\nu \mapsto G_{g,f}(\nu)$ is vaguely l.s.c., resp. strongly continuous, on $\mathcal{E}_{g,f}^+(A_1; D)$ if Case I, resp. Case II, takes place. In fact, since $\|\nu\|_g$ is vaguely l.s.c., the former assertion follows from Lemma 2.1. As for the latter assertion, it is obvious by (7.3). This observation makes it possible to obtain (7.10) from (7.7) and (7.11). □

7.2. **Proof of Theorem 6.2.** In consequence of Theorem 6.1, it is enough to establish the necessity part of the theorem. Assume that the requirements of the latter part of the theorem are fulfilled. According to Theorem 5.2, the former equality in relation (6.2) holds. Furthermore, since Case II with $\zeta \geqslant 0$ takes place, we get from (7.1) and (7.2)

$$G_{g,f}(\nu) = \|\nu\|_g^2 + 2E_g(\zeta, \nu) \in [0, \infty) \quad \text{for all} \quad \nu \in \mathcal{E}_g^+(A_1; D). \tag{7.12}$$

Consider numbers $r_\ell > 0$, $\ell \in \mathbb{N}$, such that $r_\ell \uparrow \infty$ as $\ell \to \infty$, and write $B_{r_\ell} := B(0, r_\ell)$, $A_{1,r_\ell} := A_1 \cap B_{r_\ell}$. As $c_\alpha(B_{r_\ell}) < \infty$ for every $\ell \in \mathbb{N}$, while $c_\alpha(A_1) = \infty$, it follows from the subadditivity of $c_\alpha(\cdot)$ on universally measurable sets, [14, Lemma 2.3.5], that $c_\alpha(A_1 \setminus B_{r_\ell}) = \infty$. Hence for every $\ell \in \mathbb{N}$ there is $\xi_\ell \in \mathcal{E}_\alpha^+(A_1 \setminus B_{r_\ell}, 1; \mathbb{R}^n)$ of compact support $S_D^{\xi_\ell}$ such that

$$\|\xi_\ell\|_\alpha \leqslant \ell^{-2}. \tag{7.13}$$

Clearly, the r_ℓ can be chosen successively so that $A_{1,r_\ell} \cup S_D^{\xi_\ell} \subset A_{1,r_{\ell+1}}$. Any compact set $K \subset \mathbb{R}^n$ is contained in a ball $B_{r_{\ell_0}}$ with ℓ_0 large enough, and hence K has points in common with only finitely many $S_D^{\xi_\ell}$. Therefore ξ defined by the relation

$$\xi(\varphi) := \sum_{\ell \in \mathbb{N}} \xi_\ell(\varphi) \quad \text{for any} \quad \varphi \in C_0(\mathbb{R}^n)$$

is a positive Radon measure on $\mathbb{R}^n$ concentrated on A_1. Furthermore, $\xi(A_1) = \infty$ and $\xi \in \mathcal{E}_\alpha^+(\mathbb{R}^n)$. To prove the latter, note that $\eta_k := \xi_1 + \cdots + \xi_k \in \mathcal{E}_\alpha^+(\mathbb{R}^n)$ in view of (7.13) and the triangle inequality in $\mathcal{E}_\alpha(\mathbb{R}^n)$, for $L := \sum_{\ell \in \mathbb{N}} \ell^{-2} < \infty$. Also observe that $\eta_k \to \xi$ vaguely because for any $\varphi \in C_0(\mathbb{R}^n)$ there is k_0 such that $\xi(\varphi) = \eta_k(\varphi)$ for all $k \geqslant k_0$. As $\|\eta_k\|_\alpha \leqslant L$ for all $k \in \mathbb{N}$, Lemma 2.1 with $X = A_1 \times A_1$ and $\psi = \kappa_\alpha$ yields $\|\xi\|_\alpha \leqslant L$.

Each ξ_ℓ belongs to $\mathcal{E}_g^+(A_1, 1; D)$ and moreover, by (3.10) and (7.13),

$$\|\xi_\ell\|_g \leqslant \|\xi_\ell\|_\alpha \leqslant \ell^{-2}. \tag{7.14}$$

As Case II takes place, $\xi_\ell \in \mathcal{E}_{g,f}^\xi(A_1, 1; D)$ for all $\ell \in \mathbb{N}$ by (7.12). Therefore, by the Cauchy–Schwarz (Bunyakovski) inequality in the pre-Hilbert space $\mathcal{E}_g(D)$,

$$0 \leqslant G_{g,f}^\xi(A_1, 1; D) \leqslant \lim_{\ell \to \infty} \bigl[\|\xi_\ell\|_g^2 + 2E_g(\zeta, \xi_\ell)\bigr] \leqslant 2\|\zeta\|_g \lim_{\ell \to \infty} \|\xi_\ell\|_g = 0,$$

where the first and the second inequalities hold by (7.12), while the third inequality and the equality are valid by (7.14). Hence $G_{g,f}^\xi(A_1, 1; D) = 0$, and the theorem follows.

7.3. **Proof of Theorem 6.3.** Fix an arbitrary $\lambda = \lambda^+ - \lambda^- \in \mathcal{E}_{\alpha,f}^\xi(\mathbf{A}, \mathbf{1}; \mathbb{R}^n)$. We begin by observing that, since $f = 0$ n.e. on A_2, relation (6.5) can alternatively be written as $U_\alpha^\lambda = U_\alpha^{\lambda^+ - \lambda^-} = 0$ n.e. on A_2, which by (3.2) is equivalent to the equality

$$\lambda^- = (\lambda^+)'. \tag{7.15}$$

Taking Theorem 5.2 into account, we thus see that, while proving the equivalence of assertions (i) and (ii) of Theorem 6.3, there is no loss of generality in assuming that the given measure λ satisfies (7.15). By (3.6) we therefore get

$$U_\alpha^\lambda = U_\alpha^{\lambda^+ - (\lambda^+)'} = \begin{cases} U_g^{\lambda^+} & \text{n.e. on} \quad D, \\ 0 & \text{n.e. on} \quad A_2, \end{cases}$$

hence

$$W_{\alpha,f}^\lambda = \begin{cases} W_{g,f}^{\lambda^+} & \text{n.e. on} \quad D, \\ 0 & \text{n.e. on} \quad A_2, \end{cases}$$

where $W_{g,f}^{\lambda^+} := W_{g,f|_D}^{\lambda^+} = U_g^{\lambda^+} + f|_D$, cf. (2.1). If moreover Case II holds, then

$$W_{\alpha,f}^\lambda = U_\alpha^{\lambda^+ + \zeta} - U_\alpha^{(\lambda^+ + \zeta)'} \quad \text{n.e. on} \;\; \mathbb{R}^n.$$

According to [17, Corollary 3.14], the function on the right (hence that on the left) in this relation takes the value 0 at every α-regular point of A_2, which establishes (6.6).

Combined with Theorem 5.2, what has been shown just above yields that Theorem 6.3 will be proved once the following theorem has been established.

Theorem 7.2. *Under the hypotheses of Theorem* 6.3 *the following two assertions are equivalent for any* $\mu \in \mathcal{E}_{g,f}^\xi(A_1, 1; D)$*:*

(i′) μ *is a solution to Problem* 5.1*;*

(ii′) *There exists a number* $c \in \mathbb{R}$ *possessing the properties*

$$W_{g,f}^\mu \geqslant c \quad (\xi - \mu)\text{-}a.e., \tag{7.16}$$

$$W_{g,f}^\mu \leqslant c \quad \mu\text{-}a.e. \tag{7.17}$$

Proof. Throughout the proof we shall use permanently the fact that both ξ and μ have finite α-Riesz energy are hence they are c_α-absolutely continuous.

Suppose first that assertion (i′) holds. Then inequality (7.16) is valid for $c = L$, where

$$L := \sup\bigl\{q \in \mathbb{R} : \; W_{g,f}^\mu \geqslant q \quad (\xi - \mu)\text{-a.e.}\bigr\}.$$

In turn, (7.16) with $c = L$ implies that $L < \infty$ because $W_{g,f}^\mu < \infty$ holds n.e. on A_1° and hence $(\xi - \mu)$-a.e. on A_1°, while $(\xi - \mu)(A_1^\circ) > 0$ by (4.14). Also note that $L > -\infty$, for $W_{g,f}^\mu$ is lower bounded on A_1 by assumption.

We next proceed by establishing (7.17) with $c = L$. To this end, write for any $w \in \mathbb{R}$

$$A_1^+(w) := \bigl\{x \in A_1 : \; W_{g,f}^\mu(x) > w\bigr\} \quad \text{and} \quad A_1^-(w) := \bigl\{x \in A_1 : \; W_{g,f}^\mu(x) < w\bigr\}.$$

Assume, on the contrary, that (7.17) with $c = L$ does not hold, i.e. that $\mu\bigl(A_1^+(L)\bigr) > 0$. Since $W_{g,f}^\mu$ is μ-measurable one can choose $w' \in (L, \infty)$ so that $\mu\bigl(A_1^+(w')\bigr) > 0$. At the same time, as $w' > L$, relation (7.16) with $c = L$ yields

$$(\xi - \mu)\bigl(A_1^-(w')\bigr) > 0.$$

Therefore, there exist compact sets $K_1 \subset A_1^+(w')$ and $K_2 \subset A_1^-(w')$ such that

$$0 < \mu(K_1) < (\xi - \mu)(K_2). \tag{7.18}$$

Write $\tau := (\xi - \mu)|_{K_2}$; then $E_g(\tau) < E_\alpha(\tau) < \infty$. Since $\langle W_{g,f}^{\mu}, \tau \rangle \leqslant w'\tau(K_2) < \infty$, we thus get $\langle f, \tau \rangle < \infty$. Define $\theta := \mu - \mu|_{K_1} + b\tau$, where $b := \mu(K_1)/\tau(K_2) \in (0,1)$ by (7.18). Then, by straightforward verification, $\theta(A_1) = 1$ and $\theta \leqslant \xi$, hence $\theta \in \mathcal{E}_{g,f}^{\xi}(A_1, 1; D)$. On the other hand,

$$\begin{aligned}\langle W_{g,f}^{\mu}, \theta - \mu \rangle &= \langle W_{g,f}^{\mu} - w', \theta - \mu \rangle \\ &= -\langle W_{g,f}^{\mu} - w', \mu|_{K_1} \rangle + b\langle W_{g,f}^{\mu} - w', \tau \rangle < 0,\end{aligned}$$

which is impossible in view of Lemma 5.3 applied to $\lambda = \mu$ and $\nu = \theta$. This contradiction establishes (7.17), thus completing the proof that assertion (i′) implies assertion (ii′).

Conversely, let assertion (ii′) hold. Then $\mu\big(A_1^+(c)\big) = 0$ and $(\xi - \mu)\big(A_1^-(c)\big) = 0$. For any $\nu \in \mathcal{E}_{g,f}^{\xi}(A_1, 1; D)$ we therefore obtain

$$\begin{aligned}\langle W_{g,f}^{\mu}, \nu - \mu \rangle &= \langle W_{g,f}^{\mu} - c, \nu - \mu \rangle \\ &= \langle W_{g,f}^{\mu} - c, \nu|_{A_1^+(c)} \rangle + \langle W_{g,f}^{\mu} - c, (\nu - \xi)|_{A_1^-(c)} \rangle \geqslant 0.\end{aligned}$$

Application of Lemma 5.3 shows that, indeed, μ is the solution to Problem 5.1. □

7.4. **Proof of Theorem 6.4.** For any $x \in D$ let K_x be the inverse of $C\ell_{\overline{\mathbb{R}^n}} A_2$ relative to $S(x,1)$. Since K_x is compact, there is the (unique) κ_α-equilibrium measure $\gamma_x \in \mathcal{E}_\alpha^+(K_x; \mathbb{R}^n)$ on K_x with the properties $\|\gamma_x\|_\alpha^2 = \gamma_x(K_x) = c_\alpha(K_x)$,

$$U_\alpha^{\gamma_x} = 1 \ \text{ n.e. on } \ K_x, \tag{7.19}$$

and $U_\alpha^{\gamma_x} \leqslant 1$ on $\mathbb{R}^n$. Note that $\gamma_x \neq 0$, for $c_\alpha(K_x) > 0$ in consequence of $c_\alpha(A_2) > 0$, see [19, Chapter IV, Section 5, n° 19]. We assert that, under the stated requirements,

$$S_{\mathbb{R}^n}^{\gamma_x} = \begin{cases} \breve{K}_x & \text{if} \quad \alpha < 2, \\ \partial_{\mathbb{R}^n} K_x & \text{if} \quad \alpha = 2. \end{cases} \tag{7.20}$$

The latter equality in (7.20) follows from [19, Chapter II, Section 3, n° 13]. To establish the former equality,[12] we first note that $S_{\mathbb{R}^n}^{\gamma_x} \subset \breve{K}_x$ by the c_α-absolute continuity of γ_x. As for the converse inclusion, assume on the contrary that there is $x_0 \in \breve{K}_x$ such that $x_0 \notin S_{\mathbb{R}^n}^{\gamma_x}$, and let $r > 0$ be such that $\overline{B}(x_0, r) \cap S_{\mathbb{R}^n}^{\gamma_x} = \varnothing$. But $c_\alpha\big(B(x_0, r) \cap \breve{K}_x\big) > 0$, hence there is $y \in B(x_0, r)$ such that $U_\alpha^{\gamma_x}(y) = 1$. The function $U_\alpha^{\gamma_x}$ is α-harmonic on $B(x_0, r)$ [19, Chapter I, Section 5, n° 20], continuous on $\overline{B}(x_0, r)$, and takes at $y \in B(x_0, r)$ its maximum value 1. Applying [19, Theorem 1.28] we obtain $U_\alpha^{\gamma_x} = 1$ m_n-a.e. on $\mathbb{R}^n$, hence everywhere on $(\breve{K}_x)^c$ by the continuity of $U_\alpha^{\gamma_x}$ on $\big(S_{\mathbb{R}^n}^{\gamma_x}\big)^c$ $\big[\supset (\breve{K}_x)^c\big]$, and altogether n.e. on $\mathbb{R}^n$ by (7.19). This means that γ_x serves as the α-Riesz equilibrium measure on the whole of $\mathbb{R}^n$, which is impossible.

[12] We have brought here this proof, since we did not find a reference for this possibly known assertion.

Based on (5.5) and on the integral representation (3.3), we then arrive at the claimed relation (6.7) in view of the fact that, for every $x \in D$, ε'_x is the Kelvin transform of the equilibrium measure γ_x, see [17, Section 3.3].

7.5. **Proof of Theorem 6.5.** Since $\lambda^- = (\lambda^+)'$ by (5.5) and since $f = 0$ by assumption, the function

$$W_{\alpha,f}^{\lambda^\xi_{\mathbf{A}}} = U_\alpha^{\lambda^\xi_{\mathbf{A}}} = U_\alpha^{\lambda^+} - U_\alpha^{(\lambda^+)'}$$

is well defined and finite n.e. on $\mathbb{R}^n$. In particular, it is well defined on all of D and it equals there the strictly positive function $U_g^{\lambda^+}$, see Lemma 3.4. This together with (6.6) proves (6.8). Combining (6.8) with (6.4) shows that under the stated assumptions the number c from Theorem 6.3 is >0, while (6.3) now takes the (equivalent) form

$$U_\alpha^{\lambda^\xi_{\mathbf{A}}} \geqslant c > 0 \quad (\xi - \lambda^+)\text{-a.e.} \tag{7.21}$$

Having rewritten (6.4) as

$$U_\alpha^{\lambda^+} \leqslant U_\alpha^{\lambda^-} + c \quad \lambda^+\text{-a.e.},$$

we infer from [19, Theorems 1.27, 1.29, 1.30] that the same inequality holds on all of $\mathbb{R}^n$, which amounts to (6.10). In turn, (6.10) yields (6.9) when combined with (7.21). It follows directly from Theorem 6.3 that relations (6.9) and (6.10) together with $U_\alpha^{\lambda^\xi_{\mathbf{A}}} = 0$ n.e. on D^c determine uniquely the solution $\lambda^\xi_{\mathbf{A}}$ within the class of admissible measures.

Assume now that U_α^ξ is continuous on D. Then so is $U_\alpha^{\lambda^+}$. Indeed, since $U_\alpha^{\lambda^+}$ is l.s.c. and since $U_\alpha^{\lambda^+} = U_\alpha^\xi - U_\alpha^{\xi-\lambda^+}$ with U_α^ξ continuous on D and $U_\alpha^{\xi-\lambda^+}$ l.s.c., it follows that $U_\alpha^{\lambda^+}$ is also upper semicontinuous, hence continuous. Therefore, by the continuity of $U_\alpha^{\lambda^+}$ on D, (6.9) implies (6.11). Thus, by (6.8) and (6.11),

$$U_g^{\lambda^+} = c \;\text{ on }\; S_D^{\xi-\lambda^+},$$

which implies (6.12) in view of [14, Lemma 3.2.2] with $\kappa = g$.

Omitting now the requirement of the continuity of U_α^ξ, assume further that $\alpha < 2$ and that $m_n(D^c) > 0$. If on the contrary (6.13) is not fulfilled, then there is $x_0 \in S_D^\xi$ such that $x_0 \notin S_D^{\lambda^+}$. Thus one can choose $r > 0$ so that

$$\overline{B}(x_0, r) \subset D \quad\text{and}\quad \overline{B}(x_0, r) \cap S_D^{\lambda^+} = \varnothing. \tag{7.22}$$

Then $(\xi - \lambda^+)\bigl(\overline{B}(x_0, r)\bigr) > 0$, and hence by (6.9) there exists $y \in \overline{B}(x_0, r)$ with the property that $U_\alpha^{\lambda^\xi_{\mathbf{A}}}(y) = c$, or equivalently that

$$U_\alpha^{\lambda^+}(y) = U_\alpha^{\lambda^-}(y) + c. \tag{7.23}$$

As $U_\alpha^{\lambda^+}$ is α-harmonic on $B(x_0, r)$ and continuous on $\overline{B}(x_0, r)$, while $U_\alpha^{\lambda^-} + c$ is α-superharmonic on $\mathbb{R}^n$, we conclude from (6.10) and (7.23) with the aid of [19, Theorem 1.28] that

$$U_\alpha^{\lambda^+} = U_\alpha^{\lambda^-} + c \quad m_n\text{-a.e. on }\; \mathbb{R}^n. \tag{7.24}$$

This implies $c = 0$, for $U_\alpha^{\lambda^+} = U_\alpha^{(\lambda^+)'} = U_\alpha^{\lambda^-}$ holds n.e. on D^c, hence m_n-a.e. on D^c. A contradiction.

Similar arguments enable us to establish (6.14). Indeed, if (6.14) does not hold at some $x_1 \in D \setminus S_D^{\lambda^+}$, then relation (7.23) would be valid with x_1 in place of y, see (6.10); and moreover one could choose $r > 0$ so that (7.22) would be fulfilled with x_1 in place of x_0. Therefore, using the α-harmonicity of $U_\alpha^{\lambda^+}$ on $B(x_1, r)$, as well as the α-superharmonicity of $U_\alpha^{\lambda^-} + c$ on $\mathbb{R}^n$, we would arrive again at (7.24) and hence at the equality $c = 0$. The contradiction thus obtained completes the proof of the theorem.

8. Duality relation between non-weighted constrained and weighted unconstrained minimum α-Green energy problems

As above, fix a (not necessarily proper) subset A_1 of D which is relatively closed in D and fix a constraint $\xi \in \mathfrak{C}(A_1; \mathbb{R}^n)$, see (4.10), with $1 < \xi(A_1) < \infty$; such ξ exists because of the (permanent) assumption $c_\alpha(A_1) > 0$. According to Theorem 7.1, the non-weighted ($f = 0$) constrained minimum α-Green energy problem over the class $\mathcal{E}_g^\xi(A_1, 1; D)$ is (uniquely) solvable, i.e. there exists $\lambda = \lambda_{A_1}^\xi \in \mathcal{E}_g^\xi(A_1, 1; D)$ with

$$\|\lambda\|_g^2 = \min_{\nu \in \mathcal{E}_g^\xi(A_1,1;D)} \|\nu\|_g^2. \tag{8.1}$$

Write $q := [\xi(A_1) - 1]^{-1}$ and

$$\theta := q(\xi - \lambda), \quad f_0 := -qU_g^\xi.$$

Theorem 8.1. *Assume moreover that U_g^ξ is (finitely) continuous on D. Then the measure θ is a (unique) solution to the f_0-weighted unconstrained minimum α-Green energy problem over $\mathcal{E}_g^+(A_1, 1; D)$, i.e. $\theta \in \mathcal{E}_g^+(A_1, 1; D)$ and*

$$G_{g,f_0}(\theta) = \inf_{\nu \in \mathcal{E}_g^+(A_1,1;D)} G_{g,f_0}(\nu). \tag{8.2}$$

Moreover, there exists $\eta \in (0, \infty)$ such that

$$W_{g,f_0}^\theta = -\eta \quad \text{on} \quad S_D^\theta, \tag{8.3}$$

$$W_{g,f_0}^\theta \geqslant -\eta \quad \text{on} \quad D, \tag{8.4}$$

and these two relations (8.3) and (8.4) determine uniquely a solution to the problem (8.2) within the measures of the class $\mathcal{E}_g^+(A_1, 1; D)$.

Proof. Under the stated assumptions, relations (7.16) and (7.17) for the solution λ to the (non-weighted constrained) problem (8.1) take the (equivalent) form

$$U_g^\lambda \geqslant c \quad (\xi - \lambda)\text{-a.e.}, \tag{8.5}$$

$$U_g^\lambda \leqslant c \quad \lambda\text{-a.e.} \tag{8.6}$$

Thus $c > 0$, see (8.6). Applying Theorem 3.3 with $v = c$, from (8.6) we therefore obtain

$$U_g^\lambda \leqslant c \text{ on } D.$$

Combined with (8.5), this gives $U_g^\lambda = c$ $(\xi - \lambda)$-a.e., hence

$$U_g^\lambda = c \text{ on } S_D^{\xi-\lambda},$$

for U_g^λ is (finitely) continuous on D along with U_g^ξ. (Indeed, the continuity of U_g^λ follows in the same manner as in Section 7.5, see the second paragraph, with g in place of κ_α.)

With the chosen notations the two preceding displays can alternatively be rewritten as (8.3) and (8.4) with $\eta := qc$. In turn, (8.3) and (8.4) imply that θ, f_0 and $-\eta$ satisfy [25, Eqs. 7.9, 7.10], which according to [25, Theorem 7.3] establishes (8.2). □

9. Examples

The purpose of the examples below is to illustrate the assertions from Section 6.1. Observe that both in Example 9.1 and Example 9.2 the set $A_2 = D^c$ is not α-thin at infinity.

Example 9.1. Let $n \geqslant 3$, $0 < \alpha < 2$, $A_1 = D = B(0,r)$, where $r \in (0,\infty)$, and let $A_2 = D^c$, $f = 0$. Define $\xi := q\lambda_r$, where $q > 1$ and where λ_r is the κ_α-capacitary measure on $\overline{B}(0,r)$, see Remark 2.6. As follows from [19, Chapter II, Section 3, n° 13], $\xi \in \mathcal{E}_\alpha^+(A_1,q;\mathbb{R}^n)$, $S_D^\xi = D$ and U_α^ξ is continuous on $\mathbb{R}^n$. Since $f = 0$, Problem 4.4 reduces to the problem of minimizing $E_\alpha(\mu)$ over the class of all signed Radon measures $\mu \in \mathcal{E}_\alpha(\mathbf{A},\mathbf{1};\mathbb{R}^n)$ with $\mu^+ \leqslant \xi$, which by Theorem 5.2 is equivalent to the problem of minimizing $E_{g_D^\alpha}(\nu)$ where ν ranges over $\mathcal{E}_{g_D^\alpha}^\xi(A_1,1;D)$. According to Theorems 5.2, 6.1 and Lemma 4.6, these two constrained minimum energy problems are uniquely solvable (no short-circuit occurs) and their solutions, denoted respectively by $\lambda_{\mathbf{A}}^\xi = \lambda^+ - \lambda^-$ and $\lambda_{A_1}^\xi$, are related to each other as in (5.5). Furthermore, by (6.7), (6.12) and (6.13) we obtain

$$S_D^{\lambda^+} = S_D^{\lambda_{A_1}^\xi} = D, \quad S_{\mathbb{R}^n}^{\lambda^-} = D^c,$$

$$c_{g_D^\alpha}\left(S_D^{\xi-\lambda^+}\right) < \infty, \tag{9.1}$$

and finally by (6.6) and (6.11) we have

$$U_\alpha^{\lambda_{\mathbf{A}}^\xi} = \begin{cases} c & \text{on} \quad S_D^{\xi-\lambda^+}, \\ 0 & \text{on} \quad D^c, \end{cases} \tag{9.2}$$

where $c > 0$, while by (6.10)

$$U_\alpha^{\lambda_{\mathbf{A}}^\xi} \leqslant c \text{ on } D \setminus S_D^{\xi-\lambda^+}. \tag{9.3}$$

Moreover, according to Theorem 6.3 relations (9.2) and (9.3) determine uniquely the solution $\lambda_{\mathbf{A}}^\xi$ among the class of admissible measures.

Example 9.2. Let $n = 3$, $\alpha = 2$, $f = 0$ and let $D := \bigl\{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : \ x_1 > 0\bigr\}$. Define A_1 as the union of K_k over $k \in \mathbb{N}$, where

$$K_k := \Bigl\{(x_1, x_2, x_3) \in \mathbb{R}^3 : \ x_1 = \frac{1}{k}, \ x_2^2 + x_3^2 \leqslant k^2\Bigr\}, \quad k \in \mathbb{N}.$$

Let λ_k be the κ_2-capacitary measure on K_k, see Remark 2.6; hence $\lambda_k(K_k) = 1$ and $\|\lambda_k\|_2^2 = \pi^2/(2k)$ by [19, Chapter II, Section 3, n° 14]. Define

$$\xi := \sum_{k\in\mathbb{N}} \frac{\lambda_k}{k^2}.$$

In the same manner as in the proof of Theorem 6.2 one can see that ξ is a bounded positive Radon measure concentrated on A_1 with $E_2(\xi) < \infty$. Therefore it follows from Theorem 6.1 that Problem 4.4 for the constraint ξ and for the generalized condenser $\mathbf{A} = (A_1, D^c)$ has a solution (no short-circuit occurs), although $D^c \cap C\ell_{\mathbb{R}^3} A_1 = \partial D$ and hence

$$c_2\bigl(D^c \cap C\ell_{\mathbb{R}^3} A_1\bigr) = \infty.$$

Furthermore, since each $U_2^{\lambda_k}$, $k \in \mathbb{N}$, is continuous on $\mathbb{R}^n$ and bounded from above by $\pi^2/2k$, the potential U_2^{ξ} is continuous on $\mathbb{R}^n$ by uniform convergence of the sequence $\sum_{k\in\mathbb{N}} k^{-2} U_2^{\lambda_k}$. Hence (9.1), (9.2) and (9.3) also hold in the present case with $\alpha = 2$, again with $c > 0$, and relations (9.2) and (9.3) determine uniquely the solution $\lambda_{\mathbf{A}}^{\xi}$ within the class of admissible measures. Also note that $S_{\mathbb{R}^n}^{\lambda^-} = \partial D$ according to (6.7).

10. Appendix

The following example shows that even for positive bounded (hence extendible) measures on an open ball in $\mathbb{R}^3$ the finiteness of the α-Green energy does not necessarily imply the finiteness of the α-Riesz energy, contrary to what was stated in [11, Lemma 2.4].

Example 10.1. Let $\alpha = 2$. For technical simplicity we first construct the analogous example with the ball replaced by the halfspace $D = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \ x_1 > 0\}$ (next we apply a Kelvin transformation). The boundary ∂D (replacing the unit sphere) is then the hyperplane $\{x_1 = 0\}$. For $r > 0$ let μ_r denote the κ_2-capacitary measure on the closed 2-dimensional disc $K_r \subset \partial D$ of radius r centered at $(0, 0)$, see Remark 2.6. Such μ_r exists since $0 < c_2(K_r) < +\infty$ (in fact $c_2(K_r) = 2r/\pi^2$, see [19, Chapter II, Section 3, n° 14]). The Newtonian energy $E_2(\mu_r)$ equals $E_2(\mu_1)/r$, where $0 < E_2(\mu_1) = 1/c_2(K_1) < +\infty$. For $(z_1, z_2) \in \mathbb{R}^2$ and a measure ν on $\mathbb{R}^3$ denote by ν^{z_1,z_2} the translation of ν in $\mathbb{R}^3$ by the vector $(z_1, z_2, 0)$. Then $\mu_r^{z_1,z_2}$ is the κ_2-capacitary measure on the translation of the disk K_r by the vector $(z_1, z_2, 0)$, denoted by $K_r^{z_1,z_2}$.

For fixed $r > 0$ the potential $U_2^{\mu_r}$ on $\mathbb{R}^3$ equals 1 on the disc K_r by the Wiener criterion. By the continuity principle [19, Theorem 1.7], $U_2^{\mu_r}$ is continuous on $\mathbb{R}^3$, and even uniformly since $U_2^{\mu_r}(x) \to 0$ uniformly as $|x| \to +\infty$, $S_{\mathbb{R}^3}^{\mu_r}$ being compact (actually, $S_{\mathbb{R}^3}^{\mu_r} = K_r$).

For any positive measure ν on $\mathbb{R}^3$ we denote by $(\nu)\check{}$ the image of ν under the reflection $(x_1, x_2, x_3) \to (-x_1, x_2, x_3)$ with respect to ∂D. The 2-Green kernel on the halfspace D is given by

$$g(x, y) = U_2^{\varepsilon_y}(x) - U_2^{(\varepsilon_y)\check{}}(x),$$

see e.g. [1, Theorem 4.1.6], and therefore we obtain

$$\begin{aligned} E_g(\mu_r^{\varepsilon,0}) &= \int U_g^{\mu_r^{\varepsilon,0}}\, d\mu_r^{\varepsilon,0} = \int \Bigl(U_2^{\mu_r^{\varepsilon,0}} - U_2^{(\mu_r^{\varepsilon,0})\check{}} \Bigr)\, d\mu_r^{\varepsilon,0} \\ &= \int U_2^{\mu_r}\, d\mu_r - \int U_2^{\mu_r}(-2\varepsilon, x_2, x_3)\, d\mu_r(x_1, x_2, x_3) \to 0 \end{aligned} \tag{10.1}$$

as $\varepsilon \to 0$, noting that $U_2^{\mu_r}(-2\varepsilon, x_2, x_3) \to U_2^{\mu_r}(0, x_2, x_3)$ uniformly with respect to $(x_2, x_3) \in K_r$ as $\varepsilon \to 0$.

Consider decreasing sequences $\{c_k\}_{k\in\mathbb{N}}$ and $\{r_k\}_{k\in\mathbb{N}}$ of the numbers $c_k = 2^{-k}$ and $r_k = 2^{-2k}$. Then $c_k^2/r_k = 1$, hence

$$\sum_{k\in\mathbb{N}} c_k = 1 \quad \text{and} \quad \sum_{k\in\mathbb{N}} c_k^2/r_k = +\infty. \tag{10.2}$$

For $k \in \mathbb{N}$ choose $0 < \varepsilon_k < 1$ small enough so that

$$\bigl\|\mu_{r_k}^{\varepsilon_k,k}\bigr\|_g = \bigl\|\mu_{r_k}^{\varepsilon_k,0}\bigr\|_g < 1, \tag{10.3}$$

which is possible in view of (10.1). Now define the functional

$$\mu(\varphi) := \sum_{k\in\mathbb{N}} c_k \mu_{r_k}^{\varepsilon_k,k}(\varphi) \quad \text{for all} \quad \varphi \in C_0(D).$$

Since any compact subset of D has points in common with only finitely many (disjoint) disks $K_{r_k}^{\varepsilon_k,k}$, μ thus defined is a positive Radon measure on D with $\mu(D) = 1$, see the former equality in (10.2). Furthermore, the partial sums

$$\eta_\ell := \sum_{k=1}^{\ell} c_k \mu_{r_k}^{\varepsilon_k,k}, \quad \text{where} \quad \ell \in \mathbb{N},$$

belong to $\mathcal{E}_g^+(D)$ with $\|\eta_\ell\|_g < 1$, the latter being clear from (10.3) and the former equality in (10.2) in view of the triangle inequality in $\mathcal{E}_g(D)$. Since $\eta_\ell \to \mu$ vaguely in $\mathfrak{M}^+(D)$, hence $\eta_\ell \otimes \eta_\ell \to \mu \otimes \mu$ vaguely in $\mathfrak{M}^+(D \times D)$ [5, Chapitre III, Section 5, Exercice 5], from Lemma 2.1 with $X = D \times D$ and $\psi = g$ we obtain $\|\mu\|_g \leqslant 1$.

On the other hand, being bounded, μ is extendible to a positive Radon measure on $\mathbb{R}^3$ and

$$E_2(\mu) \geqslant \sum_{k\in\mathbb{N}} E_2\bigl(c_k \mu_{r_k}^{\varepsilon_k,k}\bigr) = \sum_{k\in\mathbb{N}} c_k^2 E_2(\mu_{r_k}) = \sum_{k\in\mathbb{N}} c_k^2 r_k^{-1} E_2(\mu_1) = +\infty,$$

where the last equality follows from the latter equality in (10.2). This verifies Example 10.1 for a halfspace.

For treating the ball, apply the inversion relative to the sphere with center $(2, 0, 0)$ and radius 2. It maps the above halfspace D on the ball D^* centered at $(1, 0, 0)$ and with radius 1. The above measure μ has bounded Newtonian potential U_2^μ at the point $(2, 0, 0)$ because μ is bounded and supported by the closed strip $\{0 \leqslant x_1 \leqslant 1\}$ not containing $(2, 0, 0)$. Therefore, the Kelvin transform μ^* of μ is a bounded measure, see [19, Eq. 4.5.3], and can be written in the form

$$\mu^* = \sum_{k\in\mathbb{N}} c_k \bigl(\mu_{r_k}^{\varepsilon_k,k}\bigr)^*,$$

the Kelvin transformation of positive measures being clearly countably additive. Since κ_2-energy is preserved by Kelvin transformation, so is g_D^2-energy of a measure $\mu_{r_k}^{\varepsilon_k,k} \in \mathcal{E}_2^+(D)$, as seen by combining [19, Eqs. 4.5.2, 4.5.4] and relation (3.9) above. Denoting by g^* the Green kernel for the above ball D^* we therefore obtain

$$\|\mu^*\|_{g^*} \leqslant \sum_{k\in\mathbb{N}} c_k \bigl\|\bigl(\mu_{r_k}^{\varepsilon_k,k}\bigr)^*\bigr\|_{g^*} = \sum_{k\in\mathbb{N}} c_k \bigl\|\mu_{r_k}^{\varepsilon_k,k}\bigr\|_g \leqslant 1.$$

And clearly $E_2(\mu^*) = E_2(\mu) = +\infty$. This verifies Example 10.1 also for a ball.

References

[1] D.H. Armitage, S.J. Gardiner, *Classical Potential Theory*, Springer, Berlin, 2001.

[2] B. Beckermann, A. Gryson, *Extremal rational functions on symmetric discrete sets and superlinear convergence of the ADI method*, Constr. Approx. **32** (2010), 393–428.

[3] J. Bliednter, W. Hansen, *Potential Theory: An Analytic and Probabilistic Approach to Balayage*, Springer, Berlin, 1986.

[4] N. Bourbaki, *Elements of Mathematics. General Topology. Chapters* I–IV, Springer, Berlin, 1989.

[5] N. Bourbaki, *Intégration. Chapitres* I–IV, Actualités Sci. Ind. 1175, Paris, 1952.

[6] N. Bourbaki, *Intégration. Intégration des Mesures. Chapitre* V, Hermann, Paris, 1956.

[7] M. Brelot, *On Topologies and Boundaries in Potential Theory*, Lecture Notes in Math., vol. 175. Springer, Berlin, 1971.

[8] H. Cartan, *Théorie du potentiel Newtonien: énergie, capacité, suites de potentiels*, Bull. Soc. Math. France **73** (1945), 74–106.

[9] J. Deny, *Les potentiels d'énergie finie*, Acta Math. **82** (1950), 107–183.

[10] J. Deny, *Sur la définition de l'énergie en théorie du potentiel*, Ann. Inst. Fourier Grenoble **2** (1950), 83–99.

[11] P.D. Dragnev, B. Fuglede, D.P. Hardin, E.B. Saff, N. Zorii, *Minimum Riesz energy problems for a condenser with touching plates*, Potential Anal. **44** (2016), 543–577.

[12] P.D. Dragnev, B. Fuglede, D.P. Hardin, E.B. Saff, N. Zorii, *Constrained minimum Riesz energy problems for a condenser with intersecting plates*, ArXiv:1710.01950 (2017).

[13] R. Edwards, *Functional analysis. Theory and applications*, Holt, Rinehart and Winston, New York, 1965.

[14] B. Fuglede, *On the theory of potentials in locally compact spaces*, Acta Math. **103** (1960), 139–215.

[15] B. Fuglede, *Capacity as a sublinear functional generalizing an integral*, Mat. Fys. Medd. Dan. Vid. Selsk. **38**, no. 7 (1971).

[16] B. Fuglede, *Symmetric function kernels and sweeping of measures*, Analysis Math. **42** (2016), 225–259.

[17] B. Fuglede, N. Zorii, *Green kernels associated with Riesz kernels*, ArXiv:1610.00268 (2016); Ann. Acad. Sci. Fenn. Math. **43** (2018). DOI:10.5186/aasfm.2018.4305

[18] H. Harbrecht, W.L. Wendland, N. Zorii, *Riesz minimal energy problems on $C^{k-1,k}$-manifolds*, Math. Nachr. **287** (2014), 48–69.
[19] N.S. Landkof, *Foundations of Modern Potential Theory*, Springer, Berlin, 1972.
[20] G. Of, W.L. Wendland, N. Zorii, *On the numerical solution of minimal energy problems*, Complex Var. Elliptic Equ. **55** (2010), 991–1012.
[21] M. Ohtsuka, *On potentials in locally compact spaces*, J. Sci. Hiroshima Univ. Ser. A-1 **25** (1961), 135–352.
[22] N. Zorii, *A noncompact variational problem in Riesz potential theory*. I; II, Ukrain. Math. J. **47** (1995), 1541–1553; **48** (1996), 671–682.
[23] N. Zorii, *Interior capacities of condensers in locally compact spaces*, Potential Anal. **35** (2011), 103–143.
[24] N. Zorii, *Constrained energy problems with external fields for vector measures*, Math. Nachr. **285** (2012), 1144–1165.
[25] N. Zorii, *Equilibrium problems for infinite dimensional vector potentials with external fields*, Potential Anal. **38** (2013), 397–432.
[26] N. Zorii, *Necessary and sufficient conditions for the solvability of the Gauss variational problem for infinite dimensional vector measures*, Potential Anal. **41** (2014), 81–115.

Department of Mathematical Sciences, Indiana University–Purdue University Fort Wayne, Fort Wayne, IN 46805, USA

E-mail address: dragnevp@ipfw.edu

Department of Mathematical Sciences, University of Copenhagen, Universitetsparken 5, 2100 Copenhagen, Denmark

E-mail address: fuglede@math.ku.dk

Center for Constructive Approximation, Department of Mathematics, Vanderbilt University, Nashville, TN 37240, USA

E-mail address: doug.hardin@vanderbilt.edu

Center for Constructive Approximation, Department of Mathematics, Vanderbilt University, Nashville, TN 37240, USA

E-mail address: edward.b.saff@vanderbilt.edu

Institute of Mathematics of National Academy of Sciences of Ukraine, Tereshchenkivska 3, 01601, Kyiv-4, Ukraine

E-mail address: natalia.zorii@gmail.com