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On the Sharpness of Theorems Concerning Zero-Free Regions for Certain Sequences of Polynomials

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Summary. In this paper, we establish the sharpness of a theorem concerning zero-free parabolic regions for certain sequences of polynomials satisfying a three-term recurrence relation. Similarly, we establish the sharpness of a zero-free sectorial region for certain sequences of Padé approximants to e^z.

1. Introduction

We begin by stating, in a slightly modified form, the result of [6, Thm. 2.1].

Theorem 1.1. Let $\{p_k(z)\}_{k=0}^n$ be a sequence of polynomials in the variable z of respective degrees k which satisfy the three-term recurrence relation

$$p_k(z)=\left(\frac{z}{b_k}+1\right)p_{k-1}(z)-\frac{z}{c_k}p_{k-2}(z), \qquad k=1,2,\ldots,n, \tag{1.1}$$

where the b_k's and c_k's are positive real numbers for all $1\leqq k\leqq n$, and where $p_{-1}(z):=0$, $p_0(z)=p_0\neq 0$. Set

$$\alpha:=\min\{b_k(1-b_{k-1}c_k^{-1})\colon k=1,2,\ldots,n\}, \qquad b_0:=0. \tag{1.2}$$

Then, if $\alpha>0$, the polynomials $\{p_k(\alpha z)\}_{k=1}^n$ have no zeros in the parabolic region

$$\mathscr{P}_1:=\{z=x+iy\in\mathbb{C}\colon y^2\leqq 4(x+1),\ x>-1\}. \tag{1.3}$$

One of our objects here is to show that the result of the above *parabola theorem* is *sharp*, in the sense that each boundary point of the parabola $\mathscr{P}_1$ is the limit point of zeros of an appropriate sequence of polynomials satisfying (1.1) and (1.2). Indeed, to show this, we shall use specific sequences of Padé numerators of e^z. In the same manner, we shall also show that the *sector theorem*, stated below as Theorem 1.3, for Padé numerators for e^z is sharp. These new results are explicitly stated in § 2, with their proofs being given in § 3. For the remainder of this section, we introduce necessary notation and cite relevant existing results.

Let π_m denote the collection of all polynomials in the variable z having degree at most m, and let $\pi_{n,\nu}$ be the set of all complex rational functions $r_{n,\nu}(z)$ of the

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form

$$r_{n,\nu}(z)=\frac{p_{n,\nu}(z)}{q_{n,\nu}(z)}, \quad \text{where} \quad p_{n,\nu}\in\pi_n,\, q_{n,\nu}\in\pi_\nu,\, q_{n,\nu}(0)=1.$$

Then, the (n,ν)-th Padé approximant to e^z is defined as that element $R_{n,\nu}(z)\in\pi_{n,\nu}$ for which

$$e^z-R_{n,\nu}(z)=\mathcal{O}(|z|^{n+\nu+1}), \qquad \text{as } |z|\to 0.$$

In explicit form, it is known [5, p. 433] that

$$R_{n,\nu}(z)=P_{n,\nu}(z)/Q_{n,\nu}(z),$$

where

$$P_{n,\nu}(z)=\sum_{j=0}^{n}\frac{(n+\nu-j)!\,n!\,z^j}{(n+\nu)!\,j!\,(n-j)!}, \tag{1.4}$$

and

$$Q_{n,\nu}(z)=\sum_{j=0}^{\nu}\frac{(n+\nu-j)!\,\nu!\,(-z)^j}{(n+\nu)!\,j!\,(\nu-j)!}. \tag{1.5}$$

We shall refer to the polynomials $P_{n,\nu}(z)$ and $P_{n,\nu}((\nu+1)\,z)$, respectively, as the *Padé numerator* and *normalized Padé numerator* of type (n,ν) for e^z.

Generally, one is interested in both the zeros and the poles of the Padé approximants $R_{n,\nu}(z)$. However, since the polynomials of (1.4) and (1.5) are related by the obvious identity

$$Q_{n,\nu}(z)=P_{\nu,n}(-z), \tag{1.6}$$

it suffices then to investigate only the zeros of $P_{n,\nu}(z)$, which are the zeros of $R_{n,\nu}(z)$.

For any fixed $\nu\geqq 0$, it readily follows from (1.4) that the elements of the sequence of Padé numerators $\{P_{k,\nu}(z)\}_{k=1}^{\infty}$ satisfy the following three-term recurrence relation, discovered by Frobenius [1]:

$$(k+\nu)\,P_{k,\nu}(z)=(z+k+\nu)\,P_{k-1,\nu}(z)-\frac{(k-1)\,z\,P_{k-2,\nu}(z)}{(k+\nu-1)}, \qquad k=1,2,\ldots, \tag{1.7}$$

where $P_{0,\nu}(z)=1$, and $P_{-1,\nu}(z):=0$, i.e., $\{P_{k,\nu}(z)\}_{k=1}^{\infty}$ satisfies (1.1) for any $n\geqq 1$ with $b_k=(k+\nu)$, $k\geqq 1$, and with $c_k=(k+\nu-1)\,(k+\nu)/(k-1)$, $k\geqq 2$. Moreover, it is easy to see that for these values of b_k and c_k, the constant α of (1.2) equals $\nu+1$ for any $n\geqq 1$. Hence, on applying Theorem 1.1, it follows that, for each fixed $\nu\geqq 0$ and any $k\geqq 1$, the normalized Padé numerator $P_{k,\nu}((\nu+1)\,z)$ has no zeros in the parabolic region $\mathscr{P}_1$ of (1.3). As this holds for each $\nu\geqq 0$, we then have the following consequence of Theorem 1.1.

Corollary 1.2. For every $n\geqq 0$ and every $\nu\geqq 0$, the normalized Padé numerator $P_{n,\nu}((\nu+1)\,z)$ for e^z has no zeros in the parabolic region $\mathscr{P}_1$ of (1.3).

We next state a slightly modified form of [7, Thm. 2.1].

Theorem 1.3. For every $n\geqq 2$ and every $\nu\geqq 0$, the Padé numerator $P_{n,\nu}(z)$ for e^z has no zeros in the infinite sector

$$\mathscr{S}_{n,\nu}:=\left\{z:\,|\arg z|\leqq\cos^{-1}\left(\frac{n-\nu-2}{n+\nu}\right)\right\}. \tag{1.8}$$

Consequently, for any fixed σ with $0<\sigma<\infty$, each element in the sequence of Padé numerators $\{P_{n_j,\nu_j}(z)\}_{j=1}^{\infty}$ satisfying

$$\lim_{j\to\infty} n_j=+\infty, \quad \lim_{j\to\infty}\frac{\nu_j}{n_j}=\sigma, \quad \text{and} \quad \left(\frac{\nu_j+1}{n_j-1}\right)\geqq\sigma \quad \text{for all} \quad j\geqq 1, \tag{1.9}$$

has no zeros in the infinite sector

$$\mathscr{S}_\sigma := \left\{z\colon |\arg z| \leqq \cos^{-1}\left(\frac{1-\sigma}{1+\sigma}\right)\right\}. \tag{1.10}$$

One of the objects here is to show that the above *sector theorem* is also *sharp*, i.e., for any $\varepsilon>0$, we can find a sequence of Padé numerators $\{P_{n_j,\nu_j}(z)\}_{j=1}^{\infty}$ for e^z, satisfying (1.9), which *do* have (infinitely many) zeros in $\mathscr{S}_{\sigma+\varepsilon}$, defined in (1.10). Note that since $\cos^{-1}\left(\frac{1-\sigma}{1+\sigma}\right)$ is strictly increasing on $[0,+\infty)$, then $\mathscr{S}_\sigma\subset\mathscr{S}_{\sigma+\varepsilon}$ for every $\varepsilon>0$.

2. Statement of New Results

We now list and discuss our main results, deferring their proofs to the next section.

Theorem 2.1. For any σ with $0<\sigma<\infty$, consider a sequence of Padé numerators $\{P_{n_j,\nu_j}(z)\}_{j=1}^{\infty}$ for e^z for which

$$\lim_{j\to\infty} n_j=+\infty, \quad \text{and} \quad \lim_{j\to\infty}\frac{\nu_j}{n_j}=\sigma. \tag{2.1}$$

Then, $P_{n_j,\nu_j}(z)$ has zeros of the form

$$(n_j+\nu_j+1)\exp\left[\pm i\cos^{-1}\left(\frac{n_j-\nu_j}{n_j+\nu_j+1}\right)\right]+\mathcal{O}((n_j+\nu_j+1)^{1/3}), \quad \text{as} \quad j\to\infty. \tag{2.2}$$

With the result of Theorem 2.1, we shall then establish the following sharpened form of Corollary 1.2.

Theorem 2.2. For every $n\geqq 0$ and every $\nu\geqq 0$, the normalized Padé numerator $P_{n,\nu}((\nu+1)z)$ for e^z has no zero in the parabolic region

$$\mathscr{P}_1 := \{z=x+iy\in\mathbb{C}\colon y^2\leqq 4(x+1),\ x>-1\},$$

and, moreover, each boundary point of $\mathscr{P}_1$ is the limit point of zeros of

$$\{P_{n,\nu}((\nu+1)z)\}_{n\geqq 0,\,\nu\geqq 0}.$$

Thus, the parabolic region $\mathscr{P}_1$ is the largest (connected) region in the complex plane containing the ray $[0,+\infty)$ which is devoid of all zeros of $\{P_{n,\nu}((\nu+1)z)\}_{n\geqq 0,\,\nu\geqq 0}$.

To indicate graphically the result of Theorem 2.2, we have plotted in Figure 1 all of those 8450 zeros $\tilde{z}$ of the normalized Padé numerators $\{P_{n,\nu}((\nu+1)z)\}_{n\geqq 0,\,\nu\geqq 0}^{25,\,25}$ which satisfy $-2\leqq\operatorname{Re}\tilde{z}\leqq+1$ and $0\leqq\operatorname{Im}\tilde{z}\leqq 3$, along with the boundary of the parabolic region $\mathscr{P}_1$. The limiting nature of the boundary of $\mathscr{P}_1$, as asserted in Theorem 2.2, is particularly evident in the neighborhood of $\operatorname{Re} z=-1$. One cannot help but notice the interesting troughs for these zeros $\tilde{z}$ appearing in Figure 1, but these may disappear for large values of n and ν. It also seems from Figure 1 that much more than the boundary of $\mathscr{P}_1$ is in the set of limit points of the zeros of $\{P_{n,\nu}((\nu+1)z)\}_{n\geqq 0,\,\nu\geqq 0}$.

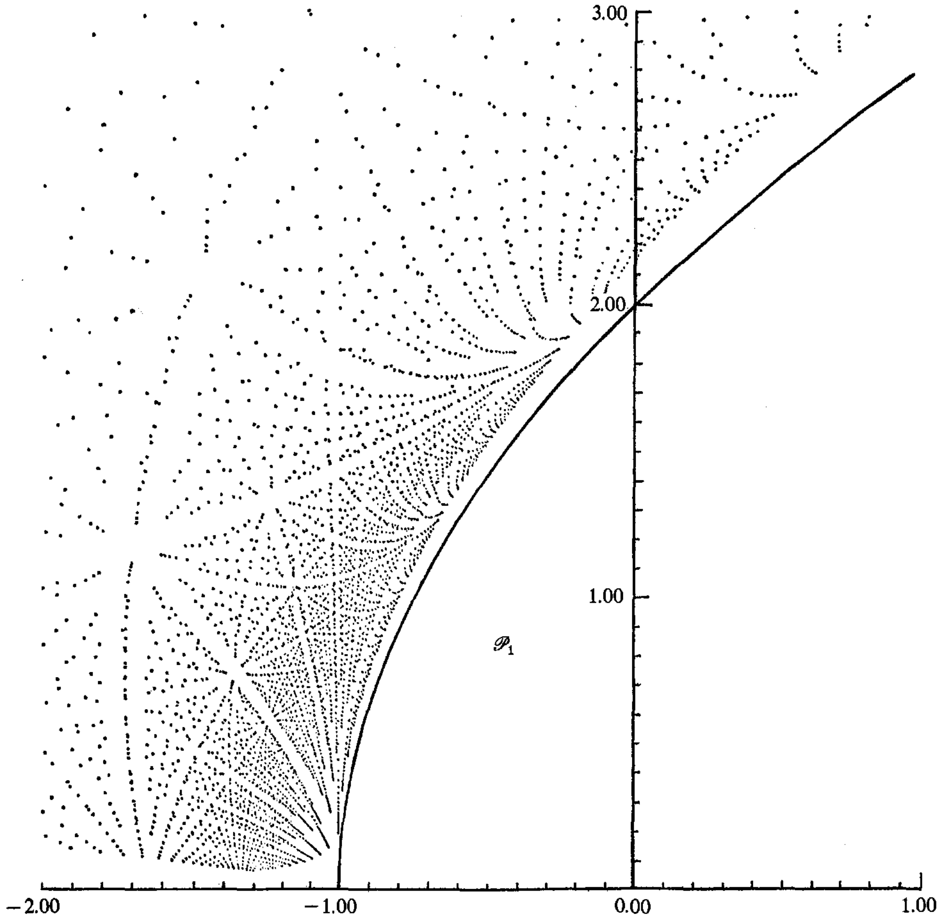

Fig. 1. Zeros of normalized Padé numerators $\{P_{n,\nu}((\nu+1)\,z)\}_{n=0,\,\nu=0}^{25,\;25}$ in $-2 \leqq \operatorname{Re} z \leqq 1$, $0 \leqq \operatorname{Im} z \leqq 3$, and zero-free parabolic region

Making use again of the result of Theorem 2.1, we shall establish the sharpness of the sector theorem, Theorem 1.3, as was originally conjectured in [7].

Theorem 2.3. For any σ with $0<\sigma<\infty$, any sequence of Padé numerators $\{P_{n_j,\nu_j}(z)\}_{j=0}^{\infty}$ for e^z satisfying (1.9) has no zeros in the infinite sector $\mathscr{S}_\sigma$ of (1.10), and, moreover, for any $\varepsilon>0$, this sequence $\{P_{n_j,\nu_j}(z)\}_{j=0}^{\infty}$ has infinitely many zeros in the sector $\mathscr{S}_{\sigma+\varepsilon}$. Thus, the sector $\mathscr{S}_\sigma$ is the largest region in the complex plane of the form $\{z: |\arg z| \leqq \mu\}$, which is devoid of all zeros of any sequence of Padé numerators $\{P_{n_j,\nu_j}(z)\}_{j=0}^{\infty}$ satisfying (1.9).

We remark that a graphical illustration of the result of Theorem 2.3, similar to the graphical illustration in Figure 1 of Theorem 2.2, can be found in Figures 1 and 2 of [7].

3. Proofs of New Results

Proof of Theorem 2.1. For any $n \geqq 0$ and $\nu \geqq 0$, set

$$w_{n,\nu}(z) := e^{-z/2} z^{-\left(\frac{n+\nu}{2}\right)} P_{n,\nu}(z), \tag{3.1}$$

where the Padé numerator $P_{n,\nu}(z)$ is defined in (1.4). In the case that $n+\nu$ is odd, $z^{-\left(\frac{n+\nu}{2}\right)}$ denotes the principal branch of $z^{-\left(\frac{n+\nu}{2}\right)}$. Then, as is known (cf. Olver [4, p. 260]), $w_{n,\nu}(z)$ satisfies Whittaker's equation

$$\frac{d^2 w(z)}{dz^2} = \left[\frac{1}{4} - \frac{k}{z} + \frac{m^2 - 1/4}{z^2}\right] w(z), \tag{3.2}$$

with

$$k := \frac{n-\nu}{2}, \qquad m := \frac{n+\nu+1}{2}. \tag{3.3}$$

On defining

$$W_{n,\nu}(z) := w_{n,\nu}(mz), \tag{3.4}$$

it then follows from (3.2) that $W_{n,\nu}(z)$ similarly satisfies

$$\frac{d^2 W(z)}{dz^2} = \left[\frac{m^2\left(z^2 - \frac{4k}{m} z + 4\right)}{4z^2} - \frac{1}{4z^2}\right] \cdot W(z). \tag{3.5}$$

Now, the roots of the quadratic $z^2 - \frac{4k}{m} z + 4$ are given by

$$\frac{2k}{m} \pm 2\sqrt{\frac{k^2}{m^2} - 1}, \tag{3.6}$$

and, with (3.3), we see that

$$0 \leqq \frac{k^2}{m^2} < 1.$$

Thus, the roots (3.6) can be expressed as

$$2e^{\pm i\theta_{n,\nu}}, \tag{3.7}$$

where

$$\cos\theta_{n,\nu} = \frac{k}{m} = \frac{n-\nu}{n+\nu+1}, \qquad 0 < \theta_{n,\nu} < \pi. \tag{3.8}$$

Hence, (3.5) can be written in the form

$$\frac{d^2 W(z)}{dz^2} = \left[\frac{m^2 (z - 2e^{i\theta_{n,\nu}})(z - 2e^{-i\theta_{n,\nu}})}{4z^2} - \frac{1}{4z^2}\right] \cdot W(z). \tag{3.9}$$

Next, set

$$Y_{n,\nu}(z) := W_{n,\nu}\left\{2e^{i\theta_{n,\nu}} + z\left(\frac{2}{m}\right)^{2/3} \frac{e^{\frac{i}{6}(4\theta_{n,\nu} - \pi)}}{\sin^{1/3}\theta_{n,\nu}}\right\}. \tag{3.10}$$

Then, as can be verified from (3.9), $Y_{n,\nu}(z)$ satisfies

$$\frac{d^2 Y(z)}{dz^2} = [z\, h_{n,\nu}(z) - g_{n,\nu}(z)] \cdot Y(z), \tag{3.11}$$

where

$$h_{n,\nu}(z) := \frac{1+\left(\dfrac{z e^{\frac{2}{3} i(\theta_{n,\nu}-\pi)}}{m^{2/3}\, 4^{2/3} \sin^{4/3}\theta_{n,\nu}}\right)}{\left[1+\dfrac{z e^{-\frac{i}{6}(2\theta_{n,\nu}+\pi)}}{m^{2/3}\, 2^{1/3} \sin^{1/3}\theta_{n,\nu}}\right]^2}, \tag{3.12}$$

and

$$g_{n,\nu}(z) := \frac{1}{4}\left\{\frac{\left(\dfrac{2}{m}\right)^{2/3} \dfrac{e^{\frac{i}{6}(4\theta_{n,\nu}-\pi)}}{\sin^{1/3}\theta_{n,\nu}}}{2e^{i\theta_{n,\nu}}+z\left(\dfrac{2}{m}\right)^{2/3} \dfrac{e^{\frac{i}{6}(4\theta_{n,\nu}-\pi)}}{\sin^{1/3}\theta_{n,\nu}}}\right\}^2. \tag{3.13}$$

Note from (3.1), (3.4), and (3.10), that

$$Y_{n,\nu}(z) = w_{n,\nu}\left\{2m e^{i\theta_{n,\nu}} + \frac{z m^{1/3}\, 2^{2/3}\, e^{\frac{i}{6}(4\theta_{n,\nu}-\pi)}}{\sin^{1/3}\theta_{n,\nu}}\right\} \tag{3.14}$$

is single-valued and analytic, except along the ray parameterized for $0 \leqq t < +\infty$ by

$$z(t) = m^{2/3}\, 2^{1/3} \sin^{1/3}\theta_{n,\nu} \cdot e^{i\left(\frac{\theta_{n,\nu}}{3}-\frac{5\pi}{6}\right)}\left[t\, e^{-i\theta_{n,\nu}}+1\right]. \tag{3.15}$$

Now, for any fixed σ with $0<\sigma<+\infty$, suppose that $\{(n_j, \nu_j)\}_{j=1}^{\infty}$ is an infinite sequence of pairs of nonnegative integers, such that

$$\lim_{j\to\infty} n_j = +\infty \quad \text{and} \quad \lim_{j\to\infty} \frac{\nu_j}{n_j} = \sigma. \tag{3.16}$$

For notational convenience, we drop subscripts and simply write (n, ν) for (n_j, ν_j). Then, from (3.8), it follows that

$$\lim_{j\to\infty} \cos\theta_{n,\nu} = \frac{1-\sigma}{1+\sigma}; \qquad \lim_{j\to\infty} \sin\theta_{n,\nu} = \frac{2\sqrt{\sigma}}{1+\sigma} > 0. \tag{3.17}$$

Next, from (3.12) and (3.13), it is clear that as $j\to\infty$, the common pole of $h_{n,\nu}(z)$ and $g_{n,\nu}(z)$ approaches infinity. Hence, we have

$$\lim_{j\to\infty} h_{n,\nu}(z) = 1, \qquad \lim_{j\to\infty} g_{n,\nu}(z) = 0,$$

uniformly on every compact subset of the complex plane. Consequently, the differential equation (3.11) "approaches" the *Airy differential equation* (cf. [4, p. 55]),

$$\frac{d^2 Y(z)}{dz^2} = z\, Y(z), \qquad \text{as } j\to\infty. \tag{3.18}$$

Now, consider the solutions $Y_{n,\nu}(z)$ of (3.11), as given in (3.14). With (3.1), we have

$$Y_{n,\nu}(0) = w_{n,\nu}(2m e^{i\theta_{n,\nu}}) = e^{-m e^{i\theta_{n,\nu}}}\{2m e^{i\theta_{n,\nu}}\}^{-\left(\frac{n+\nu}{2}\right)} P_{n,\nu}(2m e^{i\theta_{n,\nu}}),$$

and it follows from applying the first part of Theorem 1.3 to $P_{n,\nu}(2me^{i\theta_{n,\nu}})$ that, for $n \geqq 2$, $Y_{n,\nu}(0) \neq 0$. Thus, we finally set

$$y_{n,\nu}(z) := \frac{Y_{n,\nu}(z)}{Y_{n,\nu}(0)}, \qquad n \geqq 2. \tag{3.19}$$

Then, $y_{n,\nu}(z)$ satisfies the differential equation (3.11), and satisfies $y_{n,\nu}(0)=1$ for all $n \geqq 2$, i.e., for all j sufficiently large. Consider then the sequence $\{y'_{n,\nu}(0)\}_{j=1}^{\infty}$. If this sequence is bounded, there is a convergent subsequence $\{y'_{n_\ell,\nu_\ell}(0)\}_{\ell=1}^{\infty}$ such that

$$\lim_{\ell\to\infty} y'_{n_\ell,\nu_\ell}(0) = c, \qquad c \text{ a finite complex constant.} \tag{3.20}$$

If, on the other hand, the sequence $\{y'_{n,\nu}(0)\}_{j=1}^{\infty}$ is unbounded, there is similarly a subsequence $\{y'_{n_r,\nu_r}(0)\}_{r=1}^{\infty}$ such that

$$\lim_{r\to\infty} |y'_{n_r,\nu_r}(0)| = +\infty. \tag{3.21}$$

For simplicity again, we continue to denote these subsequences (n_ℓ, ν_ℓ) or (n_r, ν_r) by (n, ν).

If (3.20) is valid, then it follows from the fact (cf. [4, Ch. 5, § 3]) that solutions of linear equations depend *continuously* on the differential equation and on the initial data, that the subsequence $y_{n,\nu}$ converges, uniformly on every compact subset of the complex plane, to the solution $Y(z)$ of the Airy equation (3.18) which satisfies $Y(0)=1$, $Y'(0)=c$. From the asymptotic expansions of the particular Airy functions $\mathrm{Ai}(z)$, $\mathrm{Ai}\left(e^{\pm\left(\frac{2\pi i}{3}\right)}z\right)$ (cf. [3, p. 364]), it is easy to verify that each complex solution of the Airy differential equation (3.18) is an entire function with infinitely many zeros in the complex plane. Thus, the subsequence $y_{n,\nu}(z)$ converges to a not identically zero (since $Y(0)=1$) entire function $Y(z)$ which certainly has a finite zero z_0. Thus, by Hurwitz's Theorem, $y_{n,\nu}(z)$ has a zero at some finite point $z_{n,\nu}$, where $z_{n,\nu}\to z_0$ as $\ell\to\infty$. Tracing this all back through (3.14) and (3.1), then $P_{n,\nu}(z)$ has a zero of the form

$$2m\, e^{i\theta_{n,\nu}} + z_{n,\nu}\frac{m^{1/3}\, 2^{2/3} \exp\left[\frac{i}{6}(4\theta_{n,\nu} - \pi)\right]}{\sin^{1/3}\theta_{n,\nu}}, \tag{3.22}$$

or equivalently, from (3.3) and (3.8),

$$\begin{aligned}(n+\nu+1) &\exp\left[i\cos^{-1}\left(\frac{n-\nu}{n+\nu+1}\right)\right] \\ &+ z_{n,\nu}\frac{[2(n+\nu+1)]^{1/3}\exp\left[\frac{i}{6}\left(4\cos^{-1}\left(\frac{n-\nu}{n+\nu+1}\right)-\pi\right)\right]}{\sin^{1/3}\left(\cos^{-1}\left(\frac{n-\nu}{n+\nu+1}\right)\right)},\end{aligned} \tag{3.23}$$

where $z_{n,\nu}\to z_0$ as $\ell\to\infty$. Hence, with (3.17), $P_{n,\nu}(z)$ has a zero of the form

$$(n+\nu+1)\exp\left[i\cos^{-1}\left(\frac{n-\nu}{n+\nu+1}\right)\right] + \mathcal{O}((n+\nu+1)^{1/3}), \qquad \text{as } \ell\to\infty. \tag{3.24}$$

Of course, since $P_{n,\nu}(z)$ has real coefficients from (1.4), its zeros occur in conjugate complex pairs, so that $P_{n,\nu}(z)$ has zeros of the form

$$(n+\nu+1)\exp\left[\pm i\cos^{-1}\left(\frac{n-\nu}{n+\nu+1}\right)\right]+\mathcal{O}((n+\nu+1)^{1/3}),\qquad \text{as } \ell\to\infty. \quad (3.25)$$

If, on the other hand, (3.21) holds, then define

$$\mathcal{Y}_{n,\nu}(z)=\frac{y_{n,\nu}(z)}{y'_{n,\nu}(0)},\qquad \text{for all } r \text{ sufficiently large.} \quad (3.26)$$

Then, $\mathcal{Y}_{n,\nu}(0)\to 0$ as $r\to\infty$, and $\mathcal{Y}'_{n,\nu}(0)=1$. Thus, by the above argument, the subsequence $\{\mathcal{Y}_{n,\nu}(z)\}_{r=1}^{\infty}$ also converges, uniformly on every compact subset of the complex plane, to the not identically zero solution of the Airy equation (3.18) which satisfies $Y(0)=0$, $Y'(0)=1$, and this solution $Y(z)$ again is an entire function with a finite zero (at $z_0=0$). As in the previous case, then $P_{n,\nu}(z)$ has zeros of the form (3.25).

Having just established that for every sequence $\{(n_j, \nu_j)\}_{j=1}^{\infty}$ satisfying (3.16), there exists a subsequence such that the corresponding polynomials $P_{n,\nu}(z)$ have zeros of the form (3.25), it follows that (3.25) is valid for the original sequence $\{(n_j, \nu_j)\}_{j=1}^{\infty}$, which is the desired result of (2.2) of Theorem 2.1. ■

We note that Theorem 2.1 covers the case when $0<\sigma<\infty$, where

$$\lim_{j\to\infty} n_j=\infty, \quad \text{and} \quad \lim_{j\to\infty}\frac{\nu_j}{n_j}=\sigma.$$

Actually, the proof of Theorem 2.1 allows for extensions to certain limiting cases of $\sigma=0$ and $\sigma=\infty$, cases which will be of subsequent use to us. We now give these extensions as

Corollary 3.1. Consider a sequence of Padé numerators $\{P_{n_j,\nu_j}(z)\}_{j=1}^{\infty}$ for e^z for which (cf. (2.1))

$$\lim_{j\to\infty} n_j=\infty,\qquad \lim_{j\to\infty}\nu_j=\infty,\quad \text{and} \quad \lim_{j\to\infty}\frac{\nu_j}{n_j}=0. \quad (3.27)$$

Then, $P_{n_j,\nu_j}(z)$ has zeros of the form

$$(n_j+\nu_j+1)\exp\left[\pm i\cos^{-1}\left(\frac{n_j-\nu_j}{n_j+\nu_j+1}\right)\right]+\mathcal{O}(n_j^{1/2}\nu_j^{-1/6}),\qquad \text{as } j\to\infty. \quad (3.28)$$

Similarly, if

$$\lim_{j\to\infty} n_j=\infty,\qquad \lim_{j\to\infty}\nu_j=\infty,\quad \text{and} \quad \lim_{j\to\infty}\frac{n_j}{\nu_j}=0, \quad (3.29)$$

then $P_{n_j,\nu_j}(z)$ has zeros of the form

$$(\nu_j+n_j+1)\exp\left[\pm i\cos^{-1}\left(\frac{-\nu_j+n_j}{\nu_j+n_j+1}\right)\right]+\mathcal{O}(\nu_j^{1/2}n_j^{-1/6}),\qquad \text{as } j\to\infty. \quad (3.30)$$

Proof. Because the proof of (3.30) is completely similar, it suffices to establish (3.28) under the assumption of (3.27). For notational convenience again, we drop subscripts and write (n,ν) for (n_j,ν_j). Then, from the definition of $\theta_{n,\nu}$ in (3.8), it follows from (3.27) that $\theta_{n,\nu}$ satisfies

$$\theta_{n,\nu}=\cos^{-1}\left(\frac{n-\nu}{n+\nu+1}\right)=2\left(\frac{\nu}{n}\right)^{1/2}+\mathcal{O}\left(\left(\frac{\nu}{n}\right)^{3/2}\right),\qquad \text{as } j\to\infty. \quad (3.31)$$

Next, consider the functions $h_{n,\nu}(z)$ and $g_{n,\nu}(z)$ of (3.12) and (3.13). With (3.31) and the hypotheses of (3.27), a short calculation shows that

$$\begin{aligned} m^{2/3}\, 4^{2/3} \sin^{4/3}\theta_{n,\nu} &= 4\nu^{2/3}\left\{1+\mathcal{O}\left(\frac{\nu}{n}\right)\right\} \to \infty, \quad \text{as } j\to\infty,\\ m^{2/3}\, 2^{1/3} \sin^{1/8}\theta_{n,\nu} &= n^{1/2}\nu^{1/6}\left\{1+\mathcal{O}\left(\frac{\nu}{n}\right)\right\} \to \infty, \quad \text{as } j\to\infty, \end{aligned} \tag{3.32}$$

whence we deduce that

$$\lim_{j\to\infty} h_{n,\nu}(z)=1, \quad \text{and} \quad \lim_{j\to\infty} g_{n,\nu}(z)=0,$$

uniformly on any compact subset of the complex plane, just as in the proof of Theorem 2.1. The remainder of the proof of Theorem 2.1 can then be applied which gives that $P_{n,\nu}(z)$ has a zero of the form (3.22), with $z_{n,\nu}$ bounded. Coupling the form (3.22) with (3.31) and the assumptions of (3.27), then $P_{n,\nu}(z)$ has zeros of the form

$$(n+\nu+1)\exp\left[\pm i\cos^{-1}\left(\frac{n-\nu}{n+\nu+1}\right)\right]+\mathcal{O}(n^{1/2}\nu^{-1/6}), \quad \text{as } j\to\infty,$$

which is the desired result of (3.28). ∎

One useful consequence of Corollary 3.1 is the observation that the arguments of the zeros of (3.28), because of (3.31), tend to zero as $j\to\infty$. Thus, given *any* infinite sector $\mathcal{S}_\sigma$ (cf. (1.10)) with $\sigma>0$, and given any sequence $\{R_{n_j,\nu_j}(z)\}_{j=1}^{\infty}$ of Padé approximants to e^z satisfying (3.27), this sequence has infinitely many zeros in $\mathcal{S}_\sigma$. Alternatively, given any infinite sector $\mathcal{S}_\sigma$ with $\sigma>0$, and given any sequence $\{\mathcal{R}_{\nu_j,n_j}(z)\}_{j=1}^{\infty}$ of Padé approximants to e^{-z} satisfying (3.27), this sequence has infinitely many poles in $\mathcal{S}_\sigma$.

Proof of Theorem 2.2. To begin, it follows from (1.4) that $P_{1,\nu}((\nu+1)z)=1+z$ for every $\nu\geqq 0$, which implies that $z=-1$ is a zero of every $\{P_{1,\nu}((\nu+1)z)\}_{\nu=1}^{\infty}$. Thus, $z=-1$, a boundary point of $\mathcal{P}_1$ of (1.3), is trivially a limit point of zeros of $\{P_{n,\nu}((\nu+1)z)\}_{n\geqq 0,\,\nu\geqq 0}$.

Next, consider any $(\hat{x},\hat{y})$ on the boundary of $\mathcal{P}_1$ with $\hat{x}>-1$. We can write $\hat{x}=\frac{1-\sigma}{\sigma}$ where $0<\sigma<+\infty$, and, as $\hat{y}^2=4(\hat{x}+1)$, then $\hat{y}=\pm\frac{2}{\sqrt{\sigma}}$ Fixing σ, consider any sequence $\{(n_j,\nu_j)\}_{j=1}^{\infty}$ which satisfies (cf. (2.1))

$$\lim_{j\to\infty} n_j=+\infty \quad \text{and} \quad \lim_{j\to\infty}\frac{\nu_j}{n_j}=\sigma. \tag{3.33}$$

As a consequence of Theorem 2.1, $P_{n_j,\nu_j}(z)$ has zeros of the form

$$(n_j+\nu_j+1)\exp\left[\pm i\cos^{-1}\left(\frac{n_j-\nu_j}{n_j+\nu_j+1}\right)\right]+\mathcal{O}((n_j+\nu_j+1)^{1/3}), \quad \text{as } j\to\infty,$$

which implies that $P_{n_j,\nu_j}((\nu_j+1)z)$ has zeros of the form

$$\hat{z}_{n_j,\nu_j}=\left(\frac{n_j+\nu_j+1}{\nu_j+1}\right)\exp\left[\pm i\cos^{-1}\left(\frac{n_j-\nu_j}{n_j+\nu_j+1}\right)\right]+\mathcal{O}\left(\frac{(n_j+\nu_j+1)^{1/3}}{(\nu_j+1)}\right), \quad \text{as } j\to\infty.$$

But, it follows from (3.33) that

$$\hat{z}_{n_j, \nu_j} = \left(\frac{1+\sigma}{\sigma}\right) \exp\left[\pm i \cos^{-1}\left(\frac{1-\sigma}{1+\sigma}\right)\right] + o(1)$$
$$= \left(\frac{1-\sigma}{\sigma} \pm i \frac{2\sqrt{\sigma}}{\sigma}\right) + o(1) = \hat{x} \pm i\hat{y} + o(1), \quad \text{as } j \to \infty.$$

Thus, each boundary point of the parabolic region $\mathscr{P}_1$ is the limit point of zeros of $\{P_{n,\nu}((\nu+1)z)\}_{n \geqq 0, \nu \geqq 0}$. ■

Proof of Theorem 2.3. To establish Theorem 2.3, it suffices, from the sector theorem, Theorem 1.3, to show that, for any $\varepsilon > 0$, and any sequence $\{P_{n_j, \nu_j}(z)\}_{j=1}^{\infty}$ satisfying (1.9), there are infinitely many zeros of $\{P_{n_j, \nu_j}(z)\}_{j=1}^{\infty}$ in the sector $\mathscr{S}_{\sigma+\varepsilon}$, defined in (1.10). But, this is a direct consequence of (2.2) of Theorem 2.1. ■

We take this opportunity to say that the basic purpose of this paper is to give a brief and fairly direct proof of the sharpness of the parabola and sector theorems, Theorems 1.1 and 1.3. We do point out that sharper asymptotic results, concerning the zeros of Padé numerators, are known in special cases. For example, in the diagonal case $\nu = n$ (for which $\sigma = 1$ in (2.1)), Olver [2, 3] has obtained very sharp estimates on the limiting distribution of the zeros of $P_{n,n}(z)$. Similarly, in the case $\nu = 0$ (for which $\sigma = 0$ in (2.1)), Szegö [8] has obtained very sharp estimates on the limiting distributions of the zeros of $P_{n,0}(z) = s_n(z)$, the n-th partial sums of e^z. We hope in the future to extend the results of this paper to such sharper asymptotic results on the zeros of the Padé numerators $P_{n,\nu}(z)$, in the spirit of Olver and Szegö, to the general case, $0 \leqq \sigma < \infty$.

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Erratum

"On the sharpness of theorems concerning zero-free regions for certain sequences of polynomials", E. B. Saff and R. S. Varga, Numer. Math. 26(1976), 245-354.

p. 245, equation (1.3). Read: "$y^2 \leq 4(x+1)$" for "$y^2 \leq 4(\ +1)$".