COVERING AND SEPARATION OF CHEBYSHEV POINTS FOR NON-INTEGRABLE RIESZ POTENTIALS

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ABSTRACT. For Riesz s-potentials $K(x,y) = |x-y|^{-s}$, $s > 0$, we investigate separation and covering properties of N-point configurations $\omega_N^* = \{x_1, \ldots, x_N\}$ on a d-dimensional compact set $A \subset \mathbb{R}^\ell$ for which the minimum of $\sum_{j=1}^N K(x, x_j)$ is maximal. Such configurations are called N-point optimal Riesz s-polarization (or Chebyshev) configurations. For a large class of d-dimensional sets A we show that for $s > d$ the configurations ω_N^* have the optimal order of covering. Furthermore, for these sets we investigate the asymptotics as $N \to \infty$ of the best covering constant. For these purposes we compare best-covering configurations with optimal Riesz s-polarization configurations and determine the s-th root asymptotic behavior (as $s \to \infty$) of the maximal s-polarization constants. In addition, we introduce the notion of "weak separation" for point configurations and prove this property for optimal Riesz s-polarization configurations on A for $s > \dim(A)$, and for $d-1 \leqslant s < d$ on the sphere $\mathbb{S}^d$.

1. INTRODUCTION

Suppose A is a compact subset of a Euclidean space $\mathbb{R}^\ell$ and $\omega_N = \{x_1, \ldots, x_N\} \subset A$ is a *multiset* (or an *N-point configuration*); i.e., a set of points with possible repetitions and cardinality $\#\omega_N = N$, counting multiplicities. For a positive number s we put

$$P_s(A; \omega_N) := \inf_{y \in A} \sum_{j=1}^N \frac{1}{|y - x_j|^s}.$$

Then the *N-th s-polarization (or Chebyshev) constant of* A is defined by

$$\mathscr{P}_s(A; N) := \sup_{\omega_N \subset A} P_s(A; \omega_N).$$

We note that since A is compact, there exists for each $N \in \mathbb{N}$ a configuration $\omega_N^* = \{x_1^*, \ldots, x_N^*\}$ and a point y^* such that

$$\mathscr{P}_s(A; N) = P_s(A; \omega_N^*) = \sum_{j=1}^N \frac{1}{|y^* - x_j^*|^s}. \tag{1}$$

We call ω_N^* an *optimal (or extremal) Riesz s-polarization configuration* or simply an *optimal configuration*.

From an applications prospective, the maximal polarization problem, say on a compact surface (or body), can be viewed as the problem of determining the smallest number of sources (injectors) of a substance together with their optimal locations that can provide a required saturation of the substance at every point of the surface (body).

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The general notion of polarization (or Chebyshev constants) for potentials was likely first introduced by Ohtsuka [17]. Further investigations of the asymptotic behavior as $N \to \infty$ of polarization constants as well as the asymptotic behavior of optimal configurations appear, for example, in [1], [8], [10], [9], [3], [19], [2], [4], [18].

The following result is a special case of a theorem due to Borodachov, Hardin, Reznikov and Saff [4] (see also [2]). It describes the asymptotic behavior of optimal configurations for the case of non-integrable Riesz kernels on A. Here and throughout we denote by $\mathscr{H}_d$ the Hausdorff measure on $\mathbb{R}^\ell$, $d \leqslant \ell$, normalized by $\mathscr{H}_d([0,1]^d) = 1$.

Theorem 1.1. *Suppose A is a compact C^1-smooth d-dimensional manifold, embedded in $\mathbb{R}^\ell$ with $d \leqslant \ell$, and $\mathscr{H}_d(\partial A) = 0$, where ∂A denotes the boundary of A. If $s > d$, then there exists a positive finite constant $\sigma_{s,d}$ that does not depend on A such that*

$$\lim_{N\to\infty} \frac{\mathscr{P}_s(A;N)}{N^{s/d}} = \frac{\sigma_{s,d}}{\mathscr{H}_d(A)^{s/d}}. \tag{2}$$

Moreover, if $\{\omega_N^*\}_{N=1}^\infty$ is any sequence of optimal configurations satisfying (1)*, then the normalized counting measures μ_N^* for the multisets ω_N^* satisfy*

$$\mu_N^* := \frac{1}{N} \sum_{x\in\omega_N^*} \delta_x \xrightarrow{*} \mu,$$

where $\xrightarrow{*}$ denotes convergence in the weak* topology, and μ is the uniform measure on A; i.e., for any Borel set $B \subset \mathbb{R}^\ell$

$$\mu(B) = \frac{\mathscr{H}_d(B\cap A)}{\mathscr{H}_d(A)}.$$

In other words, in the limit, optimal polarization configurations ω_N^* for non-integrable Riesz potentials are uniformly distributed in the weak* sense. In this paper we study more distributional properties of optimal configurations ω_N^*. In particular, we investigate their separation, their covering (or mesh) radius, and their connection to the "best covering problem" for the set A.

Definition 1.2. Let A be a compact subset of a Euclidean space. For any N-point configuration $\omega_N \subset A$, the *separation constant* of ω_N is defined by

$$\delta(\omega_N) := \min_{i\neq j} |x_i - x_j|$$

and the *covering radius* of ω_N is defined by

$$\rho_A(\omega_N) := \max_{y\in A} \min_{x\in\omega_N} |y-x|. \tag{3}$$

The *best N-point covering radius for A* $\rho_A(N)$ is given by

$$\rho_A(N) := \min_{\omega_N\subset A} \rho_A(\omega_N), \tag{4}$$

where the minimum is taken over all N-point configurations $\omega_N \subset A$.

In approximation theory (for example, in interpolation by splines), the separation constant $\delta(\omega_N)$ often measures "stability" of approximation, while the covering radius $\rho_A(\omega_N)$ is involved in bounds for the error of the approximation (see, e.g., [5]). Quasi-uniform sequences; i.e., sequences $\{\omega_N\}_{N=2}^\infty$ for which the ratios $\rho_A(\omega_N)/\delta(\omega_N)$ are bounded from above, appear, for example, in a number of applications involving approximation by radial basis functions, see, e.g., [16]. Thus they play an important role in the complexity analysis for such applications.

Regarding the asymptotic behavior of polarization constants as s grows large, it is known, see [2], that for a fixed N we have

$$\lim_{s\to\infty}\left(\frac{\mathscr{P}_s(A;N)}{N^{s/d}}\right)^{1/s}=\frac{1}{N^{1/d}\rho_A(N)}.$$

However, the proof in [2] does not guarantee that this limit is uniform in N; thus it does not imply any asymptotic behavior of the constants $\sigma_{s,d}$ in (2) as $s\to\infty$. One of our main results, Theorem 2.8, shows that for a large class of d-dimensional sets A,

$$\lim_{s\to\infty}\left(\frac{\sigma_{s,d}}{\mathscr{H}_d(A)^{s/d}}\right)^{1/s}=\lim_{s\to\infty}\lim_{N\to\infty}\left(\frac{\mathscr{P}_s(A;N)}{N^{s/d}}\right)^{1/s}=\frac{1}{\lim_{N\to\infty}N^{1/d}\rho_A(N)}. \tag{5}$$

In the case when $A\subset\mathbb{R}^2$ is a compact set with $\mathscr{H}_2(A)>0$, it is known [13] that

$$\lim_{N\to\infty}N^{1/2}\rho_A(N)=\frac{\sqrt{2}}{\sqrt[4]{27}}\mathscr{H}_2(A)^{1/2};$$

thus from (5),

$$\lim_{s\to\infty}\sigma_{s,2}^{1/s}=\frac{\sqrt[4]{27}}{\sqrt{2}}.$$

For higher dimensions we prove that all limits in (5) exist.

We shall work primarily with the class of d-regular sets.

Definition 1.3. A compact set $A\subset\mathbb{R}^\ell$ is called *d-regular* if there exist a measure μ supported on A and two positive constants c_1 and c_2 such that for any $x\in A$ and any positive $r<\operatorname{diam}(A)$, we have

$$c_1r^d\leqslant\mu(A\cap B(x,r))\leqslant c_2r^d, \tag{6}$$

where $B(x,r)$ is the open ball in $\mathbb{R}^\ell$ with center x and radius r.

The following estimate from above for $\mathscr{P}_s(A;N)$, which follows from [8, Theorem 2.4] and its proof, will be useful for our investigation.

Theorem 1.4. *If $A\subset\mathbb{R}^\ell$, $\ell\geqslant d$, $\mathscr{H}_d(A)>0$ and $s>d$, then there exists a constant $C_s>0$, that depends on d, A and s such that, for any positive integer N,*

$$\mathscr{P}_s(A;N)\leqslant C_sN^{s/d}. \tag{7}$$

Moreover, C_s can be chosen so that there exists a constant C_0 with the property that for large values of s we have $1\leqslant(C_s)^{1/s}\leqslant C_0$.

The following immediate consequence of this theorem will be proved in Section 8.

Proposition 1.5. *With the hypotheses of Theorem 1.4, let $\omega_N=\{x_j\}_{j=1}^N$ be a fixed N-point configuration on A. There exists a positive constant c_s, independent of N and ω_N, with the following property: if $y^*=y_s^*\in A$ is a point such that*

$$\sum_{j=1}^N\frac{1}{|y^*-x_j|^s}=\inf_{y\in A}\sum_{j=1}^N\frac{1}{|y-x_j|^s},$$

then $|y^* - x_j| \geqslant c_s N^{-1/d}$ *for each* $j = 1, \ldots, N$. *Moreover,* c_s *can be chosen so that* $\lim_{s\to\infty} c_s^{1/s} = 1$.

Furthermore, the same is true for $s \in [d-1, d)$ *when* $A = \mathbb{S}^d$*, the d-dimensional unit sphere in* $\mathbb{R}^{d+1}$.

We next introduce the main class of sets A that we will consider.

Definition 1.6. A compact set $A \subset \mathbb{R}^d$ is called a *body* if $A \neq \emptyset$ and $A = \mathrm{Clos}(\mathrm{Int}(A))$. We say that a body $A \subset \mathbb{R}^d$ is *strongly convex* if it is convex and its boundary ∂A is a $(d-1)$-dimensional C^2-smooth manifold with non-degenerate Gaussian curvature *.

This class includes the closed unit ball

$$\mathbb{B}^d := \{x \in \mathbb{R}^d \colon |x| \leqslant 1\}$$

and ellipsoids

$$\{(x_1, \ldots, x_d) \colon x_1^2/a_1^2 + \cdots + x_d^2/a_d^2 \leqslant 1\};$$

however, it does not include cubes and polyhedra.

The paper is organized as follows. In Section 2 we state and discuss our main results. In Section 3 we prove a 'weak separation' result for strongly convex bodies. In Section 4 we prove the 'weak separation' for the unit cube $[0,1]^d$, and in Section 5 we prove it for the unit sphere $\mathbb{S}^d$ and spherical caps in $\mathbb{S}^d$. Further, in Section 6, we derive a criterion for a sequence of configurations to have an optimal order of covering radius $\rho_A(\omega_N)$. We also show that configurations ω_N^* that are optimal for $\mathscr{P}_s(A;N)$ satisfy this criterion if A is strongly convex, a cube, a sphere, or a spherical cap. And, in Section 7, we connect the asymptotic behavior of the constant $\sigma_{s,d}$ as $s \to \infty$ with the asymptotic behavior of the best covering radius $\rho_N(A)$, where A is any of the sets just mentioned. We prove Proposition 1.5 in Section 8 and in the Appendix (Section 9) we present equivalent definitions of best covering for the space $\mathbb{R}^d$.

2. Main results

For strongly convex bodies $A \subset \mathbb{R}^d$ the separation and covering properties of extremal configurations ω_N^* for $\mathscr{P}_s(A;N)$, in general, depend on the parameter s. Here we shall prove 'weak separation' and covering properties for $s > d$. In contrast, it is known [8] that for the closed d-dimensional unit ball $\mathbb{B}^d \subset \mathbb{R}^d$ and for $0 < s \leqslant d-2$, the unique optimal N-point s-polarization configuration ω_N^* is $\omega_N^* = \{0, \ldots, 0\}$; thus,

$$\delta(\omega_N^*) = 0, \quad \rho_A(\omega_N^*) = 1, \quad \forall N.$$

The main reason behind this is that the function

$$x \mapsto |x-y|^{-s}$$

is superharmonic when $s \leqslant d-2$.

Our first goal is to establish for the non-integrable case $s > d$ a weak-separation property in the following sense.

Definition 2.1. A family Ω of multisets ω from A, where $A \subset \mathbb{R}^\ell$ has Hausdorff dimension d, is called *weakly well-separated with parameter* $\eta > 0$ if there exists an $M \in \mathbb{N}$ such that for every $\omega \in \Omega$ and every point $z \in \mathbb{R}^\ell$, we have

$$\#\Big(\omega \cap B(z, \eta \cdot (\#\omega)^{-1/d})\Big) \leqslant M. \tag{8}$$

*Such conditions appear in many problems in harmonic analysis, see, e.g., [12].

It is easy to see that for a d-regular set A there exists a positive constant C such that for any configuration $\omega \subset A$ we have

$$\delta(\omega) \leqslant C \cdot (\#\omega)^{-1/d}. \tag{9}$$

If for some $\eta > 0$ inequality (8) holds with $M = 1$ for every $\omega \in \Omega$, then

$$\delta(\omega) \geqslant \eta \cdot (\#\omega)^{-1/d};$$

therefore, we get the optimal order of separation with respect to the cardinality of ω.

Definition 2.2. A set A is called *d-admissible* if $A \subset \mathbb{R}^d$ is strongly convex, or $A = \mathbb{S}^d \subset \mathbb{R}^{d+1}$, or $A \subset \mathbb{S}^d$ is a spherical cap.

We prove the following theorems.

Theorem 2.3. *If $d \in \mathbb{N}$, $s > d$, and the set A is d-admissible, then there exists an $\eta > 0$ such that the family $\Omega = \Omega_s := \{\omega\colon P_s(A;\omega) = \mathscr{P}_s(A;\#\omega)\}$ is weakly well-separated with parameter η and $M = 2d-1$. Moreover, $\eta = \eta_s$ can be chosen so that $\lim_{s\to\infty} \eta_s^{1/s} = 1$.*

The same is true for $s \in [d-1, d)$ when $A = \mathbb{S}^d$.

The result for strongly convex bodies is proved in Section 3, while the results for the sphere and spherical caps are proved in Section 5.

Remark. If $d = 1$ and $A = [0,1]$, then for every $s > 1$, the family $\Omega = \Omega_s$ is weakly well-separated with some $\eta > 0$ and $M = 1$.

As a consequence of the proof of Theorem 2.3, we obtain the following.

Corollary 2.4. *Assume $A \subset \mathbb{R}^d$ is a compact set and $s > d$. For every $r > 0$, there exists an $\eta > 0$ that depends on r with the following property: if for some $z \in A$ we have $B(z,r) \subset A$, then $\#(\omega_N^* \cap B(z, \eta N^{-1/d})) \leqslant 2d-1$, where ω_N^* is optimal for $\mathscr{P}_s(A;N)$.*

Remark. As we shall show in Lemma 3.1, if A is strongly convex then no points from ω_N^* can lie on the boundary ∂A; moreover, the distance from any point in ω_N^* to ∂A is at least of the order $N^{-2/d}$.

The next theorem deals with the unit cube. For this case, our methods impose a stronger condition on the Riesz parameter s.

Theorem 2.5. *If $[0,1]^d \subset \mathbb{R}^d$, $d \geqslant 2$, denotes the unit cube and $s > 3d-4$, then there exists a $\eta > 0$ such that the family $\Omega = \Omega_s = \{\omega\colon P_s(A;\omega) = \mathscr{P}_s(A;\#\omega)\}$ is weakly well-separated with parameter η and $M = 2d-1$. Moreover, $\eta = \eta_s$ can be chosen so that $\lim_{s\to\infty} \eta_s^{1/s} = 1$.*

Regarding the covering radius of N-point configurations having a weak separation property we prove the following.

Theorem 2.6. *Let ℓ, d and s be positive integers with $\ell \geqslant d$ and $s > d$. Suppose the compact set $A \subset \mathbb{R}^\ell$ with $\mathscr{H}_d(A) > 0$ is contained in some d-regular compact set $\tilde{A}$. If the N-point configuration $\omega_N \subset A$ is such that for some $\eta > 0$ and $M \in \mathbb{N}$ we have $\#(B(z, \eta N^{-1/d}) \cap \omega_N) \leqslant M$ for all $z \in A$, then*

$$\rho_A(\omega_N) = \max_{y \in A} \min_{x \in \omega_N} |y - x| \leqslant R_s N^{-1/d}, \tag{10}$$

where

$$R_s := \left(\frac{7^s \cdot C_d \cdot M \cdot s}{5^s \cdot p_s \cdot (s-d) \cdot \eta^d} \right)^{\frac{1}{s-d}}, \tag{11}$$

C_d is a positive constant that depends only on d and A, and p_s is any positive constant such that

$$\inf_{y\in A} \sum_{x\in\omega_N} \frac{1}{|y-x|^s} \geqslant p_s N^{s/d}. \tag{12}$$

From this theorem and Theorem 2.3 we deduce the following.

Corollary 2.7. *If the set A is d-admissible and $s > d$, then there exists a positive constant R_s such that for any N-point configuration ω_N^* that is optimal for $\mathscr{P}_s(A;N)$, we have $\rho_A(\omega_N^*) \leqslant R_s N^{-1/d}$. Moreover, there exists a positive constant R_0 such that for large values of s we have $R_s \leqslant R_0$.*

The same is true if $A = [0,1]^d$ and $s > 3d-4$.

Corollary 2.7 implies that if A is an d-admissible set or a unit cube, then $\rho_A(N) \leqslant R_s N^{-1/d}$ for some positive constant R_s. On the other hand, it is easy to see that in this case, for some positive constant b, we have $\rho_A(N) \geqslant bN^{-1/d}$. Fine estimates on the constant R_s for large values of s result in the following theorem dealing with the asymptotic behavior of $\mathscr{P}_s(A;N)^{1/s}$ as $s \to \infty$.

Theorem 2.8. *Suppose the set A is d-admissible or $A = [0,1]^d$. Then with $\sigma_{s,d}$ as defined in Theorem 1.1, the following limits exist as positive real numbers and satisfy*

$$\lim_{s\to\infty} \left(\frac{\sigma_{s,d}}{\mathscr{H}_d(A)^{s/d}} \right)^{1/s} = \lim_{s\to\infty} \left(\lim_{N\to\infty} \frac{\mathscr{P}_s(A;N)}{N^{s/d}} \right)^{1/s} = \frac{1}{\lim_{N\to\infty} N^{1/d} \rho_A(N)}. \tag{13}$$

In particular, taking $A = [0,1]^d$ we obtain

$$\lim_{s\to\infty} \sigma_{s,d}^{1/s} = \frac{1}{\lim_{N\to\infty} N^{1/d} \rho_{[0,1]^d}(N)} = \left(\frac{V_d}{\Gamma_d} \right)^{1/d}, \tag{14}$$

where the constant Γ_d is the optimal covering density † *of the space $\mathbb{R}^d$ (see [7, Chapter 2] and Section 9) and $V_d := \mathscr{H}_d(\mathbb{B}^d) = \pi^{d/2}/\Gamma(d/2+1)$.*

We remark that $\Gamma_1 = 1$ and $\Gamma_2 = 2\pi/\sqrt{27}$.

A consequence Theorem 2.8 is that, in the limit as $s \to \infty$, the covering radius of optimal Riesz s-polarization configurations become asymptotically best possible.

Corollary 2.9. *Suppose the set A is d-admissible or $A = [0,1]^d$. For every $s > 3d-4$, let ω_N^s be an N-point configuration such that $\mathscr{P}_s(A;N) = P_s(A;\omega_N^s)$. Then*

$$\lim_{s\to\infty} \lim_{N\to\infty} N^{1/d} \rho_A(\omega_N^s) = \lim_{N\to\infty} N^{1/d} \rho_A(N).$$

† The problem of finding Γ_d is known in [7] as “finding the thinnest covering of $\mathbb{R}^d$.”

3. Weak separation for strongly convex bodies

In what follows, we always assume $s > d$ and $A \subset \mathbb{R}^d$ is a strongly convex body. By $\overline{B(x,r)}$ we denote the closure of $B(x,r)$ and I_{d-1} denotes the $(d-1)\times(d-1)$ identity matrix. Furthermore, the j'th coordinate of a point $x \in \mathbb{R}^d$ will be denoted by $x(j)$; we also denote by x' the $(d-1)$-dimensional vector that consists of the first $d-1$ coordinates of x; thus, $x = (x', x(d))$. By $e_1, \dots, e_d$ we denote the canonical basis in $\mathbb{R}^d$. If we have a $d \times d$ matrix M, we put

$$(Mx,x) := (Mx)\cdot x, \quad x \in \mathbb{R}^d.$$

To establish Theorem 2.3 we begin with two lemmas about the behavior of extremal configurations for $\mathscr{P}_s(A;N)$ near the boundary ∂A.

Lemma 3.1. *There exists a constant $b_s > 0$ with the following property: for all $N \geqslant 1$, if ω_N^* is an extremal configuration for $\mathscr{P}_s(A;N)$ and $x \in \omega_N^*$, then* $\operatorname{dist}(x,\partial A) > b_s N^{-2/d}$. *Moreover, b_s can be chosen so that* $\lim_{s\to\infty} b_s^{1/s} = 1$.

Remark. Let $x_\partial \in \partial A$ and make a rotation so that in the neighborhood $B(x_\partial, r)$ the manifold ∂A is given by $\{(x', x(d)) \colon x(d) = f(x')\}$ with $\nabla f(x'_\partial) = 0$ and the matrix $\mathrm{d}^2 f(x)$ is non-positive for $x \in \partial A \cap \overline{B(x_\partial, r)}$ (this can be done since A is convex). Moreover, r can be chosen sufficiently small so that

$$\overline{B(x_\partial, r)} \cap A = \overline{B(x_\partial, r)} \cap \{x \colon x(d) \leqslant f(x')\}.$$

We notice that the Gaussian curvature of ∂A at x_∂ is equal to the product of eigenvalues of the matrix $\mathrm{d}^2 f(x'_\partial)$. Since in Theorem 2.3 we assume the Gaussian curvature is non-zero, the manifold ∂A is compact and C^2-smooth and $\mathrm{d}^2 f \leqslant 0$, we deduce that there exists a constant $C_A > 0$ such that $\mathrm{d}^2 f(x') \leqslant -C_A I_{d-1}$ for every $x \in B(x_\partial, r)$, where C_A does not depend on x_∂.

Proof of Lemma 3.1. Take a point $x_\partial \in \partial A$ for which $|x - x_\partial| = \operatorname{dist}(x, \partial A)$. We can make a rotation and assume $x = x_\partial - cN^{-2/d} \cdot e_d$. We show that this is impossible if c is sufficiently small.

Let f be the function from the above remark. For a small positive number ε consider a point

$$\tilde{x} := x - \varepsilon e_d \in A$$

and a configuration $\widetilde{\omega}_N := (\omega_N^* \setminus \{x\}) \cup \{\tilde{x}\}$. Consider a point $\tilde{y}$ such that

$$P(A;\widetilde{\omega}_N) = \sum_{\tilde{x}_j \in \widetilde{\omega}_N} \frac{1}{|\tilde{y} - \tilde{x}_j|^s}.$$

Since ω_N^* is an extremal configuration, we have

$$P_s(A;\omega_N^*) \geqslant P_s(A;\widetilde{\omega}_N),$$

which after utilizing the definition of $\widetilde{\omega}_N$ implies

$$|\tilde{y} - x| \leqslant |\tilde{y} - \tilde{x}|.$$

Using that $\tilde{x} = x - \varepsilon e_d$, we get

$$\tilde{y}(d) - x(d) \geqslant -\varepsilon/2,$$

or

$$\tilde{y}(d) \geqslant x(d) - \varepsilon/2 = x_\partial(d) - cN^{-2/d} - \varepsilon/2.$$

Since ε is an arbitrarily small number, we can assume $\varepsilon/2 \leqslant cN^{-2/d}$. Then we obtain

$$\tilde{y}(d) \geqslant x_\partial(d) - 2cN^{-2/d}.$$

On the other hand, since A is a convex set, and the plane $\{z \in \mathbb{R}^d : z(d) = x_\partial(d)\}$ is tangent to ∂A, we have $\tilde{y}(d) \leqslant x_\partial(d)$.

We now estimate the diameter of the set

$$S(N,c) := \{y \in A : x_\partial(d) - 2cN^{-2/d} \leqslant y(d) \leqslant x_\partial(d)\}.$$

Since A is strongly convex, we obviously have $A \cap \{z \in \mathbb{R}^d : z(d) = x_\partial(d)\} = \{x_\partial\}$. Thus, $\operatorname{diam}(S(N,c)) \to 0$ as $c \to 0$. If c is chosen small enough, then $S(N,c) \subset B(x_\partial, \eta) \cap A$ for some $\eta > 0$. Therefore, if y belongs to $S(N,c)$, then for some $\xi \in B(x_\partial, \eta)$ we have

$$\begin{aligned} x_\partial(d) - 2cN^{-2/d} \leqslant y(d) \leqslant f(y') = f(x'_\partial) + \frac{1}{2}(\mathrm{d}^2 f(\xi')(y' - x'_\partial), (y' - x'_\partial)) \\ \leqslant x_\partial(d) - \frac{C_A}{2} \cdot |y' - x'_\partial|^2, \end{aligned} \tag{15}$$

which implies

$$|y' - x'_\partial|^2 \leqslant \frac{4c}{C_A} \cdot N^{-2/d}; \tag{16}$$

thus, for a suitable constant C_B,

$$|y - x_\partial|^2 \leqslant \frac{4c}{C_A} \cdot N^{-2/d} + 4c^2 N^{-4/d} \leqslant C_B \cdot c \cdot N^{-2/d}.$$

Therefore, since $\varepsilon \leqslant 2cN^{-2/d}$,

$$|\tilde{y} - \tilde{x}| \leqslant |\tilde{y} - x_\partial| + 2cN^{-2/d} \leqslant \tau \cdot \sqrt{c} \cdot N^{-1/d}$$

for some constant τ that does not depend on s. For c sufficiently small, this inequality contradicts Proposition 1.5 and so the lemma follows. □

In the next lemma we show that if $x \in A$ is close to ∂A in one direction, then its distance in orthogonal directions can be estimated from below.

Lemma 3.2. *Let ω_N^* be an extremal configuration for $\mathscr{P}_s(A;N)$ and $x \in \omega_N^*$. Assume τ is a sufficiently small positive number that does not depend on N. If $\operatorname{dist}(x, \partial A) = |x - x_\partial|$ with $x - x_\partial$ parallel to e_d, then the estimate $|x - x_\partial| < \tau N^{-1/d}$ implies $x \pm \tau N^{-1/d} e_j \in A$ for every $j = 1, \dots, d-1$.*

Proof. Again let f be as in the above remark. Arguing as in the preceding lemma, we see that we need to show that $|x - x_\partial| < \tau N^{-1/d}$ implies $x(d) \leqslant f(x' \pm \tau N^{-1/d} e'_j)$. Notice that since $x \in \omega_N^*$, we know that $|x - x_\partial| > cN^{-2/d}$ for some constant c. We apply the Taylor formula again:

$$f(x' \pm \tau N^{-1/d} e'_j) = x_\partial(d) + \frac{\tau^2 N^{-2/d}}{2} (\mathrm{d}^2 f(\xi') e'_j, e'_j). \tag{17}$$

Since the boundary ∂A is compact and smooth, we can always assume $\mathrm{d}^2 f(\xi') > -C I_{d-1}$ for some positive constant C. Thus,

$$f(x' \pm \tau N^{-1/d} e'_j) \geqslant x_\partial(d) - C\tau^2 N^{-2/d} \geqslant x(d) + (c - C\tau^2) N^{-2/d} \geqslant x(d)$$

if τ is sufficiently small. □

We are ready to prove Theorem 2.3.

Proof of Theorem 2.3 for a strongly convex set A. We argue by contradiction. Suppose there exists small number $\eta > 0$ and an extremal configuration $\omega_N^* = \{x_1,\dots,x_N\}$ such that $\{x_1,\dots,x_{2d}\} \subset B(z,\eta N^{-1/d})$. Consider

$$\hat{x} := \frac{x_1+\cdots+x_{2d}}{2d} \in A.$$

Since $\hat{x} \in B(z,\eta N^{-1/d})$, we have $|x_j - \hat{x}| \leqslant 2\eta N^{-1/d}$ for every $j = 1,\dots,2d$.

Fix a small number $\tau > \eta$. We will choose it later to be a multiple of η. Set $\varepsilon := \tau N^{-1/d}$. We consider two cases.

Case 1: $\operatorname{dist}(\hat{x},\partial A) \geqslant \varepsilon$. Define $2d$ points as follows:

$$\begin{aligned} \tilde{x}_1 &:= \hat{x} - \varepsilon e_1, & \tilde{x}_2 &:= \hat{x} + \varepsilon e_1,\\ \tilde{x}_3 &:= \hat{x} - \varepsilon e_2, & \tilde{x}_4 &:= \hat{x} + \varepsilon e_2,\\ &\dots\\ \tilde{x}_{2d-1} &:= \hat{x} - \varepsilon e_d, & \tilde{x}_{2d} &:= \hat{x} + \varepsilon e_d. \end{aligned}$$

Since $\operatorname{dist}(\hat{x},\partial A) \geqslant \varepsilon$, these points belong to A. Define $\widetilde{\omega}_N := \{\tilde{x}_1,\dots,\tilde{x}_{2d},\tilde{x}_{2d+1},\dots,\tilde{x}_N\}$, where $\tilde{x}_j := x_j$ for $j \geqslant 2d+1$. Let $\tilde{y}$ be such that

$$P_s(A;\widetilde{\omega}_N) = \sum_{j=1}^N \frac{1}{|\tilde{y}-\tilde{x}_j|^s}. \tag{18}$$

We have

$$\sum_{j=1}^N \frac{1}{|\tilde{y}-\tilde{x}_j|^s} \leqslant \mathscr{P}_s(A;N) = P_s(A;\omega_N^*) \leqslant \sum_{j=1}^N \frac{1}{|\tilde{y}-x_j|^s},$$

and thus

$$\sum_{j=1}^{2d} \frac{1}{|\tilde{y}-\tilde{x}_j|^s} \leqslant \sum_{j=1}^{2d} \frac{1}{|\tilde{y}-x_j|^s}, \tag{19}$$

Set $f(x) := |\tilde{y}-x|^{-s}$. Then, from the Taylor formula about $\hat{x}$, we have for $x \in \{x_1,\dots,x_{2d}\}$

$$f(x) = f(\hat{x}) + s\frac{(\tilde{y}-\hat{x})\cdot(x-\hat{x})}{|\tilde{y}-\hat{x}|^{s+2}} + \frac{1}{2}\cdot\left(-s\cdot\frac{|x-\hat{x}|^2}{|\tilde{y}-\xi|^{s+2}} + s(s+2)\cdot\frac{((\tilde{y}-\xi)\cdot(x-\hat{x}))^2}{|\tilde{y}-\xi|^{s+4}}\right),$$

for some $\xi = \xi(x) \in B(\hat{x},|x-\hat{x}|)$. From Proposition 1.5 we know that $|\tilde{y}-\tilde{x}_1| \geqslant c_s N^{-1/d}$. Without loss of generality we assume $\tau < c_s/2$, and so

$$|\tilde{y}-\hat{x}| = |\tilde{y}-\tilde{x}_1+\varepsilon e_1| \geqslant (c_s-\tau)N^{-1/d} \geqslant (c_s/2)\cdot N^{-1/d}, \tag{20}$$

and

$$|\tilde{y}-\xi| \geqslant |\tilde{y}-\hat{x}| - |\hat{x}-\xi| \geqslant |\tilde{y}-\hat{x}| - |x-\hat{x}| \geqslant |\tilde{y}-\hat{x}| - 2\eta N^{-1/d} \geqslant (1-4\eta/c_s)|\tilde{y}-\hat{x}|.$$

Therefore, for every $j = 1,\dots,2d$ we have

$$f(x_j) \leqslant f(\hat{x}) + s\frac{(\tilde{y}-\hat{x})\cdot(x_j-\hat{x})}{|\tilde{y}-\hat{x}|^{s+2}} + \frac{2s(s+3)\eta^2 N^{-2/d}(1-4\eta/c_s)^{-s-2}}{|\tilde{y}-\hat{x}|^{s+2}}.$$

Summing these inequalities over j and recalling that $x_1+\cdots+x_{2d} = 2d\hat{x}$ yields

$$\sum_{j=1}^{2d} \frac{1}{|\tilde{y}-x_j|^s} \leqslant 2d\cdot f(\hat{x}) + \frac{4sd(s+3)\cdot\eta^2 N^{-2/d}\cdot(1-4\eta/c_s)^{-s-2}}{|\tilde{y}-\hat{x}|^{s+2}}. \tag{21}$$

Plugging this estimate into (19), we obtain

$$f(\hat{x}) \geqslant \frac{1}{2d}\sum_{j=1}^{2d} f(\tilde{x}_j) - \frac{\eta^2 N^{-2/d}\cdot 2s(s+3)(1-4\eta/c_s)^{-s-2}}{|\tilde{y}-\hat{x}|^{s+2}}. \tag{22}$$

We proceed with the Taylor formula for $f(\tilde{x}_j)$. We first write it for $j=1$. Recall that $\tilde{x}_1 = \hat{x} - \varepsilon e_1$. Since $|e_1| = 1$, we get for some $\xi \in B(\hat{x}, |\tilde{x}_1 - \hat{x}|) = B(\hat{x}, \varepsilon)$,

$$\begin{aligned}
f(\tilde{x}_1) &= f(\hat{x} - \varepsilon e_1) \\
&= f(\hat{x}) - s\varepsilon \frac{(\tilde{y}-\hat{x})\cdot e_1}{|\tilde{y}-\hat{x}|^{s+2}} + \frac{\varepsilon^2}{2}\cdot\left(-s\cdot\frac{e_1\cdot e_1}{|\tilde{y}-\hat{x}|^{s+2}} + s(s+2)\frac{((\tilde{y}-\hat{x})\cdot e_1)^2}{|\tilde{y}-\hat{x}|^{s+4}}\right) \\
&\quad + \frac{\varepsilon^3}{6}\cdot\left(-3s(s+2)\cdot\frac{((\tilde{y}-\xi)\cdot e_1)\cdot(e_1\cdot e_1)}{|\tilde{y}-\xi|^{s+4}} + s(s+2)(s+4)\cdot\frac{((\tilde{y}-\xi)\cdot e_1)^3}{|\tilde{y}-\xi|^{s+6}}\right) \\
&= f(\hat{x}) - s\varepsilon \frac{(\tilde{y}-\hat{x})\cdot e_1}{|\tilde{y}-\hat{x}|^{s+2}} + \frac{\varepsilon^2}{2}\cdot\left(-s\cdot\frac{1}{|\tilde{y}-\hat{x}|^{s+2}} + s(s+2)\frac{((\tilde{y}-\hat{x})\cdot e_1)^2}{|\tilde{y}-\hat{x}|^{s+4}}\right) \\
&\quad + \frac{\varepsilon^3}{6}\cdot\left(-3s(s+2)\cdot\frac{((\tilde{y}-\xi)\cdot e_1)}{|\tilde{y}-\xi|^{s+4}} + s(s+2)(s+4)\cdot\frac{((\tilde{y}-\xi)\cdot e_1)^3}{|\tilde{y}-\xi|^{s+6}}\right).
\end{aligned} \tag{23}$$

Next we estimate the remainder term involving ξ. As before,

$$|\tilde{y}-\xi| \geqslant |\tilde{y}-\hat{x}| - |\xi-\hat{x}| \geqslant |\tilde{y}-\hat{x}| - \tau N^{-1/d} \geqslant (1-2\tau/c_s)|\tilde{y}-\hat{x}|.$$

This implies

$$\begin{aligned}
&\left|-3s(s+2)\cdot\frac{(\tilde{y}-\xi)\cdot e_1}{|\tilde{y}-\xi|^{s+4}} + s(s+2)(s+4)\cdot\frac{((\tilde{y}-\xi)\cdot e_1)^3}{|\tilde{y}-\xi|^{s+6}}\right| \\
&\qquad \leqslant s(s+2)(s+7)\cdot(1-2\tau/c_s)^{-s-3}\cdot\frac{1}{|\tilde{y}-\hat{x}|^{s+3}}.
\end{aligned} \tag{24}$$

Using the formula (23) with $\tilde{x}_1$ replaced by $\tilde{x}_j$ we obtain an equation for $f(\tilde{x}_j)$ which, when substituted along with (24) into (22), yields

$$\begin{aligned}
&\frac{\varepsilon^2}{2}\left(-s\frac{1}{|\tilde{y}-\hat{x}|^{s+2}} + \frac{s(s+2)}{d}\cdot\frac{1}{|\tilde{y}-\hat{x}|^{s+2}}\right) \\
&\qquad - \frac{\varepsilon^3}{6}s(s+2)(s+7)\cdot(1-2\tau/c_s)^{-s-3}\cdot\frac{1}{|\tilde{y}-\hat{x}|^{s+3}} \\
&\qquad - \eta^2 N^{-2/d}\cdot 4s(s+3)(1-4\eta/c_s)^{-s-2}\cdot\frac{1}{|\tilde{y}-\hat{x}|^{s+2}} \leqslant 0.
\end{aligned} \tag{25}$$

We remark that the first term in (25) is, up to a constant factor, the Laplacian, in x, of the function $f(x)$. Although $f(x)$ is neither convex nor concave (for some choices of $\tilde{y}$, about which we have no information), the Laplacian $\Delta f(x)$ is always positive, which plays an essential role in our argument. Indeed, the need for at least $2d$ points $\{x_j\}_{j=1}^{2d}$ enables the definition of $\{\tilde{x}_j\}_{j=1}^{2d}$ so that the leading terms in the Taylor formula vanish leaving the positive second term. This will enable us to arrive at a contradiction to (25) as we now explain.

Recalling from (20) that $|\tilde{y}-\hat{x}| \geqslant (c_s/2)\cdot N^{-1/d}$, we multiply (25) by $2|\tilde{y}-\hat{x}|^{s+2}$ and divide by $sN^{-2/d}$ to obtain

$$\begin{aligned}(26)\quad \frac{s+2-d}{d}\tau^2 - 2/3\tau^3 N^{-1/d}\cdot(s+2)(s+7)(1-2\tau/c_s)^{-s-3}c_s^{-1}\\ -8\eta^2(s+3)(1-4\eta/c_s)^{-s-2}\leqslant 0.\end{aligned}$$

Since $s>d$, this is impossible if τ is a suitable large multiple (depending on s) of η and η is small, and so the first assertion of Theorem 2.3 holds in this case. Observe that (26) fails if $\eta=\eta_s=c_s/s$ and s is sufficiently large. Hence from Proposition 1.5 the family Ω_s is weakly well-separated with $M=2d-1$ and parameter η_s with $\lim_{s\to\infty}\eta_s^{1/s}=1$.

Case 2: $\operatorname{dist}(\hat{x},\partial A)<\varepsilon$. Without loss of generality, we assume $\bar{x}+\varepsilon e_d\notin A$. We again take the point $x_\partial\in\partial A$ that achieves this distance and argue as in Lemma 3.2. We see that for any $j\leqslant 2d-2$ the points $\tilde{x}_j$, defined as above, lie in the set A. We redefine

$$\tilde{x}_{2d-1}:=\tilde{x}_{2d}:=\hat{x}-\varepsilon e_d,$$

and let $\tilde{y}$ be as in (18). The Taylor expansions of the terms on the left in (19) yield the following analog of (25):

$$\begin{aligned}(27)\quad -s\frac{\varepsilon}{d}\cdot\frac{\tilde{y}(d)-\hat{x}(d)}{|\tilde{y}-\hat{x}|^{s+2}}+\frac{\varepsilon^2}{2}\left(-s\frac{1}{|\tilde{y}-\hat{x}|^{s+2}}+\frac{s(s+2)}{d}\cdot\frac{1}{|\tilde{y}-\hat{x}|^{s+2}}\right)\\ -\frac{\varepsilon^3}{6}s(s+2)(s+7)\cdot(1-2\tau/c_s)^{-s-3}\cdot\frac{1}{|\tilde{y}-\hat{x}|^{s+3}}\\ -\eta^2N^{-2/d}\cdot 4s(s+3)(1-4\eta/c_s)^{-s-2}\cdot\frac{1}{|\tilde{y}-\hat{x}|^{s+2}}\leqslant 0,\end{aligned}$$

and, consequently, we have the following analog of (26),

$$\begin{aligned}(28)\quad -2\tau\frac{N^{1/d}}{d}(\tilde{y}(d)-\hat{x}(d))+\frac{s+2-d}{d}\tau^2-\\ -2/3\tau^3N^{-1/d}\cdot(s+2)(s+7)(1-2\tau/c_s)^{-s-3}c_s^{-1}-\\ 8\eta^2(s+3)(1-4\eta/c_s)^{-s-2}\leqslant 0.\end{aligned}$$

Since $\tilde{y}(d)\leqslant x_\partial(d)$ and $\hat{x}(d)>x_\partial(d)-\tau N^{-1/d}$, we obtain

$$-2\tau\frac{N^{1/d}}{d}(\tilde{y}(d)-\hat{x}(d))+\frac{s+2-d}{d}\tau^2\geqslant\frac{s-d}{d}\tau^2;$$

therefore, (28) is impossible for suitably small choices of η and τ, which as in the Case 1 yields the assertion of Theorem 2.3. □

4. Weak separation for the cube

In this section we show how to modify the proof of Theorem 2.3 to a case when the boundary ∂A is not smooth. Namely, we prove the weak well-separation result for the unit cube, Theorem 2.5.

We begin with the following lemma.

Lemma 4.1. *If $s > d$, ω_N^* is optimal for $\mathscr{P}_s([0,1]^d;N)$, and $x \in \omega_N^*$, then there exists a constant b_s that does not depend on N such that*

$$\max_{j=1,\dots,d} x(j) \geqslant b_s N^{-1/d}.$$

Moreover, one can choose b_s so that $\lim_{s\to\infty} b_s^{1/s} = 1$.

Proof. We proceed as in Lemma 3.1. Denote $v := (1,\dots,1)$ and $\tilde{x} := x + \varepsilon v$. If for some small number c we have $\max_{j=1,\dots,d} x(j) \leqslant cN^{-1/d}$, then $\tilde{x} \in [0,1]^d$. Further, set $\widetilde{\omega}_N := (\omega_N^* \setminus \{x\}) \cup \{\tilde{x}\}$. If $\tilde{y}$ minimizes $P_s([0,1]^d, \widetilde{\omega}_N)$, then we have

$$|\tilde{y} - x| \leqslant |\tilde{y} - \tilde{x}|,$$

which implies

$$(\tilde{y} - x) \cdot v \leqslant d\varepsilon.$$

Utilizing the definition of v and taking $\varepsilon \leqslant cN^{-1/d}$, we obtain

$$\tilde{y}(j) \leqslant \sum_{j=1}^{d} \tilde{y}(j) \leqslant \sum_{j=1}^{d} x(j) + d\varepsilon \leqslant d(cN^{-1/d} + \varepsilon) \leqslant 2dcN^{-1/d}.$$

Therefore,

$$|\tilde{y} - \tilde{x}| \leqslant \sqrt{d}(\max_{j=1,\dots,d} \tilde{y}(j) + \max_{j=1,\dots,d} \tilde{x}(j)) \leqslant 4d\sqrt{d} \cdot cN^{-1/d}.$$

If c is small enough, this contradicts Proposition 1.5. □

We are ready to prove Theorem 2.5.

Weak separation for the cube. We again argue by contradiction. Suppose for $\eta > 0$ and an optimal Riesz s-polarization configuration $\omega_N^* = \{x_1, \dots, x_N\}$ we have $\{x_1, \dots, x_{2d}\} \subset B(z, \eta N^{-1/d})$. Define

$$\hat{x} := \frac{x_1 + \dots + x_{2d}}{2d} \in [0,1]^d.$$

Since $\hat{x} \in B(z, \eta N^{-1/d})$, we have $|x_j - \hat{x}| \leqslant 2\eta N^{-1/d}$ for every $j = 1, \dots, 2d$.

Consider a small number $\tau > \eta$. We will choose it later to be a multiple of η. Set $\varepsilon := \tau N^{-1/d}$. We consider two cases.

Case 1: $\operatorname{dist}(\hat{x}, \partial[0,1]^d) \geqslant \varepsilon$. In this case we proceed exactly as in the first case of Section 3 and get the same contradiction.

Case 2: $\operatorname{dist}(\hat{x}, \partial[0,1]^d) < \varepsilon$. We notice that since $|\hat{x} - x_j| < 2\eta N^{-1/d}$, Lemma 4.1 implies that $\hat{x}$ cannot be close to any vertex of the cube. Therefore, there exists at least one number j such that $\hat{x} \pm \varepsilon e_j \in [0,1]^d$. Without loss of generality, $j = 1$. We now assume that for some $j_0 = 1, \dots, N$ we have $\hat{x} \pm \varepsilon e_j \in [0,1]^d$ for $j \leqslant j_0$, and $\hat{x} - \varepsilon e_j \notin [0,1]^d$ for $j > j_0$. Cases when $\hat{x} + \varepsilon e_j \notin [0,1]^d$ are treated similarly. We define

$$\tilde{x}_1 := \hat{x} - \varepsilon e_1, \qquad \tilde{x}_2 := \hat{x} + \varepsilon e_1,$$
$$\dots$$
$$\tilde{x}_{2j_0-1} := \hat{x} - \varepsilon e_{j_0}, \qquad \tilde{x}_{2j_0} := \hat{x} + \varepsilon e_{j_0},$$

$\tilde{x}_k := \hat{x} + \varepsilon e_{\lfloor (k+1)/2 \rfloor}$ for $k = 2j_0+1, \ldots, 2d$, and $\widetilde{\omega}_N := \{\tilde{x}_1, \ldots, \tilde{x}_N\}$, where $\tilde{x}_j := x_j$ for $j > 2d$. Let $\tilde{y}$ such that

$$P_s(A; \widetilde{\omega}_N) = \sum_{j=1}^{N} \frac{1}{|\tilde{y} - \tilde{x}_j|^s}.$$

Similarly to (27), we get

$$\begin{aligned}(29)\quad & s\frac{\varepsilon}{d} \cdot \sum_{j>j_0} \frac{\tilde{y}(j) - \hat{x}(j)}{|\tilde{y} - \hat{x}|^{s+2}} + \frac{\varepsilon^2}{2}\left(-s\frac{1}{|\tilde{y} - \hat{x}|^{s+2}} + \frac{s(s+2)}{d} \cdot \frac{1}{|\tilde{y} - \hat{x}|^{s+2}}\right) \\ & \qquad - \frac{\varepsilon^3}{6} s(s+2)(s+7) \cdot (1 - 2\tau/c_s)^{-s-3} \cdot \frac{1}{|\tilde{y} - \hat{x}|^{s+3}} \\ & \qquad - \eta^2 N^{-2/d} \cdot 4s(s+3)(1 - 4\eta/c_s)^{-s-2} \cdot \frac{1}{|\tilde{y} - \hat{x}|^{s+2}} \leqslant 0,\end{aligned}$$

Notice that if $\tilde{y}(j) \geqslant \hat{x}(j)$, then

$$s\frac{\varepsilon}{d} \cdot \frac{\tilde{y}(j) - \hat{x}(j)}{|\tilde{y} - \hat{x}|^{s+2}} \geqslant 0.$$

If $\tilde{y}(j) < \hat{x}(j)$, then we estimate $\tilde{y}(j) - \hat{x}(j) \geqslant -\hat{x}(j) \geqslant -\varepsilon$. Since $j_0 > 1$, we have at most $d-1$ numbers j with $j > j_0$. Therefore, (29) implies

$$\begin{aligned}(30)\quad & \varepsilon^2 \cdot \frac{s + 2 - d - 2(d-1)}{2d} \cdot \frac{1}{|\tilde{y} - \hat{x}|^{s+2}} \\ & \qquad - \frac{\varepsilon^3}{6} s(s+2)(s+7) \cdot (1 - 2\tau/c_s)^{-s-3} \cdot \frac{1}{|\tilde{y} - \hat{x}|^{s+3}} \\ & \qquad - \eta^2 N^{-2/d} \cdot 4s(s+3)(1 - 4\eta/c_s)^{-s-2} \cdot \frac{1}{|\tilde{y} - \hat{x}|^{s+2}} \leqslant 0,\end{aligned}$$

which for suitably chosen η and τ gives a contradiction if $s > 3d - 4$. As with Theorem 2.3, it follows that $\eta = \eta_s$ can be taken so that $\lim_{s\to\infty} \eta_s^{1/s} = 1$.

□

5. Weak separation on the sphere and spherical caps

In this section we prove Theorem 1.6 when $A = \mathbb{S}^d$ or when $A \subset \mathbb{S}^d$ is a spherical cap. We proceed as in Section 3. However, computations will be different since the sphere $\mathbb{S}^d$ is not "flat". We start with the following result.

Theorem 5.1 (Weak separation on the sphere)**.** *Consider the unit sphere $\mathbb{S}^d \subset \mathbb{R}^{d+1}$, and $s > d$ of $s \in [d-1, d)$. Then there exists a number $\eta > 0$ such that for any N, any optimal configuration ω_N^* and any point $z \in \mathbb{R}^{d+1}$, we have*

$$\#(\omega_N \cap B(z, \eta N^{-1/d})) \leqslant 2d - 1.$$

Moreover, for large values of s we can choose $\eta = \eta_s$ with

$$\lim_{s\to\infty} \eta_s^{1/s} = 1.$$

Proof. Assume the theorem is false: there exists a ball $B(z, \eta N^{-1/d})$ and an optimal configuration $\omega_N^* = \{x_1, \ldots, x_N\}$ such that $\{x_1, \ldots, x_{2d}\} \subset B(z, \eta N^{-1/d})$.

Without loss of generality, we can assume $z'=0$ and $z(d+1)>0$. Denote

$$\hat{x}' := \frac{x_1'+\cdots+x_{2d}'}{2d},$$

and

$$\hat{x}(d+1) := \sqrt{1-|\hat{x}'|^2}.$$

Since $|x_j - z| < \eta N^{-1/d}$ for $j=1,\dots,2d$, then $|x_j'| = |x_j' - z'| < \eta N^{-1/d}$; thus

$$|\hat{x}'| < \eta N^{-1/d},$$

and

$$1-\eta^2 N^{-2/d} \leqslant \hat{x}(d+1) \leqslant 1, \quad 1-\eta^2 N^{-2/d} \leqslant x_j(d+1) \leqslant 1.$$

Therefore,

$$-\eta^2 N^{-2/d} \leqslant x_j(d+1) - \hat{x}(d+1) \leqslant \eta^2 N^{-2/d},$$

which implies for η sufficiently small

$$|x_j - \hat{x}|^2 = |x_j' - \hat{x}'|^2 + (x_j(d+1) - \hat{x}(d+1))^2 \leqslant 4\eta^2 N^{-2/d} + \eta^4 N^{-4/d} \leqslant 5\eta^2 N^{-2/d}.$$

We conclude that

$$\{x_1,\dots,x_{2d}\} \subset B(\hat{x}, \sqrt{5}\eta N^{-1/d}).$$

Since the problem is rotation-invariant, we can assume $\hat{x} = e_{d+1} = (0,0,\dots,0,1)$ — the North pole of the sphere.

Fix a small number τ, with $\eta < \tau < c_s/20$. We will choose τ at the end of the proof. Set

$$\varepsilon := \tau N^{-1/d}.$$

Note that $\{e_1',\dots,e_d'\}$ is the canonical orthonormal basis in $\mathbb{R}^d$; denote

$$\begin{aligned} v_1 &:= e_1, & v_2 &:= -e_1,\\ v_3 &:= e_2, & v_4 &:= -e_2,\\ &\dots\\ v_{2d-1} &:= e_d, & v_{2d} &:= -e_d. \end{aligned}$$

For $j=1,\dots,2d$ set

$$\tilde{x}_j' := \hat{x}' + \varepsilon v_j = \varepsilon v_j, \quad \tilde{x}_j(d+1) := \sqrt{1-|\tilde{x}_j'|^2},$$

and $\tilde{x}_j := x_j$ if $j > 2d$. For $\widetilde{\omega}_N := \{\tilde{x}_1,\dots,\tilde{x}_N\}$ let $\tilde{y}$ be such that

$$P_s(\mathbb{S}^d, \widetilde{\omega}_N) = \sum_{j=1}^N \frac{1}{|\tilde{y} - \tilde{x}_j|^s}.$$

As before, denote

$$f(x) := \frac{1}{|\tilde{y} - x|^s}.$$

Estimates

$$\sum_{j=1}^N \frac{1}{|\tilde{y} - x_j|^s} \geqslant \inf_{y\in\mathbb{S}^d} \sum_{j=1}^N \frac{1}{|y-x_j|^s} = \mathscr{P}_s(\mathbb{S}^d; N) \geqslant P_s(\mathbb{S}^d; \widetilde{\omega}_N) = \sum_{j=1}^N \frac{1}{|\tilde{y} - \tilde{x}_j|^s}, \tag{31}$$

imply, after utilizing that $x_j = \tilde{x}_j$ for $j \geqslant 2d+1$, that

$$\sum_{j=1}^{2d} \frac{1}{|\tilde{y}-x_j|^s} \geqslant \sum_{j=1}^{2d} \frac{1}{|\tilde{y}-\tilde{x}_j|^s}. \tag{32}$$

Then from Taylor formula about $\hat{x}$ we have for $x \in \{x_1, \dots, x_{2d}\}$ for some $\xi = \xi(x) \in B(\hat{x}, |x-\hat{x}|)$,

$$f(x) = f(\hat{x}) + s\frac{(y-\hat{x})\cdot(x-\hat{x})}{|y-\hat{x}|^{s+2}} + \left(-s\cdot\frac{|x-\hat{x}|^2}{|y-\xi|^{s+2}} + s(s+2)\cdot\frac{((y-\xi)\cdot(x-\hat{x}))^2}{|y-\xi|^{s+4}}\right).$$

Recall that if $x = x_j$, $1 \leqslant j \leqslant 2d$, then $|x-\hat{x}| \leqslant \sqrt{5}\eta N^{-1/d}$. Moreover, we know from Lemma 1.5 that $|\tilde{y}-\tilde{x}_j| \geqslant c_s N^{-1/d}$. This implies

$$|\tilde{y}-\hat{x}| = |\tilde{y}-\tilde{x}_1+\varepsilon e_1| \geqslant (c_s-\tau)N^{-1/d} \geqslant (c_s/2)\cdot N^{-1/d},$$

and

$$|\tilde{y}-\xi| \geqslant |\tilde{y}-\hat{x}| - |\hat{x}-\xi| \geqslant |\tilde{y}-\hat{x}| - |x-\hat{x}| \geqslant |\tilde{y}-\hat{x}| - \sqrt{5}\eta N^{-1/d} \geqslant (1-2\sqrt{5}\eta/c_s)|\tilde{y}-\hat{x}|.$$

Therefore, for every $j = 1, \dots, 2d$ we have

$$f(x_j) \leqslant f(\hat{x}) + s\frac{(\tilde{y}-\hat{x})\cdot(x_j-\hat{x})}{|\tilde{y}-\hat{x}|^{s+2}} + 5s(s+3)\eta^2 N^{-2/d}(1-2\sqrt{5}\eta/c_s)^{-s-2}\cdot\frac{1}{|\tilde{y}-\hat{x}|^{s+2}}.$$

Summing these inequalities over j and recalling that $x_1' + \cdots + x_{2d}' = (2d)\cdot e' = 0$, we obtain

$$\begin{aligned}\sum_{j=1}^{2d} \frac{1}{|\tilde{y}-x_j|^s} \leqslant{}& 2d\cdot f(\hat{x}) + s\frac{(\tilde{y}(d+1)-1)\cdot(x_1(d+1)+\cdots+x_{2d}(d+1)-2d)}{|\tilde{y}-\hat{x}|^{s+2}} \\ &+ 10sd(s+3)\cdot\eta^2 N^{-2/d}\cdot(1-2\sqrt{5}\eta/c_s)^{-s-2}\cdot\frac{1}{|\tilde{y}-\hat{x}|^{s+2}}.\end{aligned} \tag{33}$$

From $|\tilde{y}(d+1)-1| \leqslant 2$ and $|x_j(d+1)-1| = 1-x_j(d+1) \leqslant \eta^2 N^{-2/d}$, we get

$$\sum_{j=1}^{2d} \frac{1}{|\tilde{y}-x_j|^s} \leqslant 2d\cdot f(\hat{x}) + \eta^2 N^{-2/d}\cdot\left(4sd + 10sd(s+3)(1-2\sqrt{5}\eta)^{-s-2}\right)\cdot\frac{1}{|\tilde{y}-\hat{x}|^{s+2}}.$$

Plugging this estimate in (31), we obtain

$$f(\hat{x}) \geqslant \frac{1}{2d}\sum_{j=1}^{2d} f(\tilde{x}_j) - \eta^2 N^{-2/d}\cdot\left(2s + 5s(s+3)(1-2\sqrt{5}\eta)^{-s-2}\right)\cdot\frac{1}{|\tilde{y}-\hat{x}|^{s+2}}. \tag{34}$$

We proceed with the Taylor formula for $f(\tilde{x}_j)$ about $\hat{x}$. We first write it for $j=1$. Recall that $\tilde{x}_1 = (\varepsilon e_1', \sqrt{1-\varepsilon^2})$. Setting $v := \tilde{x}_1 - \hat{x} = (\varepsilon e_1', \sqrt{1-\varepsilon^2}-1)$, we obtain for some $\xi \in B(\hat{x}, |\tilde{x}_1-\hat{x}|) \subset B(\hat{x}, \sqrt{2}\varepsilon)$,

$$\begin{aligned} f(\tilde{x}_1) &= f(\hat{x}+v) \\ &= f(\hat{x}) + s\frac{(\tilde{y}-\hat{x})\cdot v}{|\tilde{y}-\hat{x}|^{s+2}} + \frac{1}{2}\cdot\left(-s\cdot\frac{v\cdot v}{|\tilde{y}-\hat{x}|^{s+2}} + s(s+2)\frac{((\tilde{y}-\hat{x})\cdot v)^2}{|\tilde{y}-\hat{x}|^{s+4}}\right) \\ &\quad + \frac{1}{6}\cdot\left(-3s(s+2)\cdot\frac{((\tilde{y}-\xi)\cdot v)\cdot(v\cdot v)}{|\tilde{y}-\xi|^{s+4}} + s(s+2)(s+4)\cdot\frac{((\tilde{y}-\xi)\cdot v)^3}{|\tilde{y}-\xi|^{s+6}}\right).\end{aligned} \tag{35}$$

We first estimate the remainder term involving ξ. As before,

$$|\tilde{y}-\xi| \geqslant |\tilde{y}-\hat{x}| - |\xi-\hat{x}| \geqslant |\tilde{y}-\hat{x}| - \sqrt{2}\tau N^{-1/d} \geqslant (1-2\sqrt{2}\tau/c_s)|\tilde{y}-\hat{x}|.$$

Thus,

$$\begin{aligned}(36)\quad &\left|-3s(s+2)\cdot\frac{((\tilde{y}-\xi)\cdot v)\cdot(v\cdot v)}{|\tilde{y}-\xi|^{s+4}} + s(s+2)(s+4)\cdot\frac{((\tilde{y}-\xi)\cdot v)^3}{|\tilde{y}-\xi|^{s+6}}\right| \\ &\leqslant s(s+2)(s+7)\cdot|v|^3\cdot(1-2\sqrt{2}\tau/c_s)^{-s-3}\cdot\frac{1}{|\tilde{y}-\hat{x}|^{s+3}} \\ &\leqslant 2\sqrt{2}s(s+2)(s+7)\varepsilon^3\cdot(1-2\sqrt{2}\tau/c_s)^{-s-3}\cdot\frac{1}{|\tilde{y}-\hat{x}|^{s+3}}.\end{aligned}$$

For every $j=1,\dots,2d$ write the Taylor formula similar to (35); in view of the estimate (36), we get from(34),

$$\begin{aligned}(37)\quad & s\cdot\frac{(\tilde{y}(d+1)-1)(\sqrt{1-\varepsilon^2}-1)}{|\tilde{y}-\hat{x}|^{s+2}} \\ &+\frac{1}{2}\left(-s\frac{2-2\sqrt{1-\varepsilon^2}}{|\tilde{y}-\hat{x}|^{s+2}} + \frac{s(s+2)}{2d}\cdot\frac{2\varepsilon^2|\tilde{y}'|^2 + 2d(\tilde{y}(d+1)-1)^2(\sqrt{1-\varepsilon^2}-1)^2}{|\tilde{y}-\hat{x}|^{s+4}}\right) \\ &-2\sqrt{2}s(s+2)(s+7)\varepsilon^3\cdot(1-2\sqrt{2}\tau/c_s)^{-s-3}\cdot\frac{1}{|\tilde{y}-\hat{x}|^{s+3}} \\ &-\eta^2 N^{-2/d}\cdot\left(2s+5s(s+3)(1-2\sqrt{5}\eta/c_s)^{-s-2}\right)\cdot\frac{1}{|\tilde{y}-\hat{x}|^{s+2}} \leqslant 0.\end{aligned}$$

Using

$$2d(\tilde{y}(d+1)-1)^2(\sqrt{1-\varepsilon^2}-1)^2 \geqslant 0,$$

dividing by s and multiplying by $|\tilde{y}-e|^{s+4}$, we obtain

$$\begin{aligned}(38)\quad & (\tilde{y}(d+1)-1)(\sqrt{1-\varepsilon^2}-1)|\tilde{y}-\hat{x}|^2 \\ &+\frac{1}{2}\left(-(2-2\sqrt{1-\varepsilon^2})|\tilde{y}-\hat{x}|^2 + \frac{s+2}{2d}\cdot 2\varepsilon^2|\tilde{y}'|^2\right) \\ &-2\sqrt{2}(s+2)(s+7)\varepsilon^3\cdot(1-2\sqrt{2}\tau/c_s)^{-s-3}\cdot|\tilde{y}-\hat{x}| \\ &-\eta^2 N^{-2/d}\cdot\left(2+5(s+3)(1-2\sqrt{5}\eta/c_s)^{-s-2}\right)\cdot|\tilde{y}-\hat{x}|^2 \leqslant 0\end{aligned}$$

Let us simplify first two terms. Notice that $|\tilde{y}-\hat{x}|^2 = 2-2\tilde{y}\cdot\hat{x} = 2-2\tilde{y}(d+1)$. We have:

$$\begin{aligned}(39)\quad & (\tilde{y}(d+1)-1)(\sqrt{1-\varepsilon^2}-1)|\tilde{y}-\hat{x}|^2 \\ &+\frac{1}{2}\left(-(2-2\sqrt{1-\varepsilon^2})|\tilde{y}-\hat{x}|^2 + \frac{s+2}{2d}\cdot 2\varepsilon^2|\tilde{y}'|^2\right) \\ &= \tilde{y}(d+1)(\sqrt{1-\varepsilon^2}-1)(2-2\tilde{y}(d+1)) + \frac{s+2}{2d}(1-\tilde{y}(d+1)^2)\varepsilon^2 \\ &= |\tilde{y}-\hat{x}|^2\cdot\left((\sqrt{1-\varepsilon^2}-1)\tilde{y}(d+1) + \varepsilon^2\frac{s+2}{4d}(1+\tilde{y}(d+1))\right).\end{aligned}$$

If $\tilde{y}(d+1)<0$, we use that $\sqrt{1-\varepsilon^2}-1\leqslant -\frac{\varepsilon^2}{2}$ to get

$$(\sqrt{1-\varepsilon^2}-1)\tilde{y}(d+1)+\varepsilon^2\frac{s+2}{4d}(1+\tilde{y}(d+1))$$
$$\geqslant \frac{\varepsilon^2}{2}\left(-\tilde{y}(d+1)+\frac{s+2}{2d}(1+\tilde{y}(d+1))\right)\geqslant \frac{\varepsilon^2}{2}\cdot\min\left(\frac{s+2}{2d},1\right). \tag{40}$$

If $\tilde{y}(d+1)\geqslant 0$, we use $\sqrt{1-\varepsilon^2}-1\geqslant -\frac{\varepsilon^2}{2}-\frac{\varepsilon^4}{8}$ to get

$$(\sqrt{1-\varepsilon^2}-1)\tilde{y}(d+1)+\varepsilon^2\frac{s+2}{4d}(1+\tilde{y}(d+1))$$
$$\geqslant \frac{\varepsilon^2}{2}\left(-\tilde{y}(d+1)+\frac{s+2}{2d}(1+\tilde{y}(d+1))\right)-\frac{\varepsilon^4}{8}\geqslant \frac{\varepsilon^2}{2}\min\left(\frac{s+2}{2d},\frac{s+2-d}{d}\right)-\frac{\varepsilon^4}{8}. \tag{41}$$

Combining estimates (40) and (41), we get

$$(\tilde{y}(d+1)-1)(\sqrt{1-\varepsilon^2}-1)|\tilde{y}-\hat{x}|^2$$
$$+\frac{1}{2}\left(-(2-2\sqrt{1-\varepsilon^2})|\tilde{y}-\hat{x}|^2+\frac{s+2}{2d}\cdot 2\varepsilon^2|\tilde{y}'|^2\right)$$
$$\geqslant |\tilde{y}-\hat{x}|^2\cdot\left(\varepsilon^2\min\left(\frac{1}{2},\frac{s+2}{4d},\frac{s+2-d}{2d}\right)-\frac{\varepsilon^4}{8}\right). \tag{42}$$

Plugging this estimate into (38) and dividing by $|\tilde{y}-\hat{x}|^2$, we obtain:

$$\varepsilon^2\min\left(\frac{1}{2},\frac{s+2}{4d},\frac{s+2-d}{2d}\right)-\frac{\varepsilon^4}{8}$$
$$-2\sqrt{2}(s+2)(s+7)\varepsilon^3\cdot(1-2\sqrt{2}\tau/c_s)^{-s-3}\cdot|\tilde{y}-\hat{x}|^{-1}$$
$$-\eta^2N^{-2/d}\cdot\left(2+5(s+3)(1-2\sqrt{5}\eta/c_s)^{-s-2}\right)\leqslant 0 \tag{43}$$

We now recall that $\varepsilon=\tau N^{-1/d}$. Denote

$$C(s,d):=\min\left(\frac{1}{2},\frac{s+2}{4d},\frac{s+2-d}{2d}\right).$$

Then

$$C(s,d)\tau^2-\frac{\tau^4N^{-2/d}}{8}$$
$$-4d\sqrt{2}(s+2)(s+7)(1-2\sqrt{2}\tau/c_s)^{-s-3}\tau^3\cdot(N^{-1/d}|\tilde{y}-\hat{x}|^{-1})$$
$$-\eta^2\cdot\left(4d+10d(s+3)(1-2\sqrt{5}\eta/c_s)^{-s-2}\right)\leqslant 0. \tag{44}$$

We should finally recall that $N^{-1/d}|\tilde{y}-\hat{x}|^{-1}\leqslant 2/c_s$. Thus, we can choose sufficiently small η and τ such that the left-hand side of (44) is strictly positive, which is a contradiction. Finally, as in Section 3, for large values of s we can choose $\eta=\eta_s$ with $\eta_s^{1/s}\to 1$ as $s\to\infty$. □

We proceed with the same statement for spherical caps $A\subset\mathbb{S}^d$. As in the case of bodies in $\mathbb{R}^d$, we will need to deal with the case when point $\hat{x}$ is near the boundary.

Corollary 5.2 (Weak separation on the caps)**.** *Consider the unit sphere $\mathbb{S}^d \subset \mathbb{R}^{d+1}$, and $s > d$. Let $A \subset \mathbb{S}^d$ be a spherical cap, $A = \{x \in \mathbb{S}^d \colon x(1) \geqslant t_0\}$. Then there exists a number $\eta > 0$ such that for any N, any optimal configuration ω_N^* for $\mathscr{P}_s(A;N)$, and any point $z \in \mathbb{R}^{d+1}$ we have*

$$\#(\omega_N \cap B(z, \eta N^{-1/d})) \leqslant 2d - 1.$$

Moreover, for large values of s we can choose $\eta = \eta_s$ so that

$$\lim_{s\to\infty} \eta_s^{1/s} = 1.$$

Proof. For the sake of simplicity, we prove this corollary for $d = 2$. The case of general d can be treated similarly. We also assume $t_0 \geqslant 0$. The case $t_0 < 0$ is done through the same estimates.

We again argue by contradiction. Assume for some small $\eta > 0$ there exists a ball $B(z, \eta N^{-1/2})$ and an extremal configuration $\omega_N^* = \{x_1, \dots, x_N\}$ such that $\{x_1, \dots, x_4\} \subset B(z, N^{-1/2})$. Set

$$\hat{x}' := \frac{x_1' + \dots + x_4'}{4},$$

and

$$\hat{x}(3) := \sqrt{1 - |\hat{x}'|^2}.$$

Recall that $x \in A$ if and only if $x(1) \geqslant t_0$. Thus, we see that $\hat{x}' \in A$, and, as before,

$$\{x_1, \dots, x_4\} \subset B(\hat{x}, \sqrt{5}\eta N^{-1/2}).$$

Since the problem is rotation invariant, we can assume $\hat{x} = (\hat{t}, 0, \sqrt{1-\hat{t}^2})$ for some $\hat{t} \geqslant t_0$.

We denote

$$v_1 := (-\sqrt{1-\hat{t}^2}, 0, \hat{t}), \quad v_2 := (0, 1, 0).$$

Set $\varepsilon := \tau N^{-1/2}$ and consider

$$\begin{aligned}
\tilde{x}_1 &:= \Big(\varepsilon\sqrt{1-\hat{t}^2} + \hat{t}\sqrt{1-\varepsilon^2(1-\hat{t}^2)}, 0, -\varepsilon\hat{t} + \sqrt{1-\hat{t}^2}\cdot\sqrt{1-\varepsilon^2(1-\hat{t}^2)}\Big),\\
\tilde{x}_2 &:= \Big(-\varepsilon\sqrt{1-\hat{t}^2} + \hat{t}\sqrt{1-\varepsilon^2(1-\hat{t}^2)}, 0, \varepsilon\hat{t} + \sqrt{1-\hat{t}^2}\cdot\sqrt{1-\varepsilon^2(1-\hat{t}^2)}\Big),\\
\tilde{x}_3 &:= \Big(\sqrt{1-\varepsilon^2}\hat{t}, \varepsilon, \sqrt{1-\hat{t}^2}\cdot\sqrt{1-\varepsilon^2}\Big),\\
\tilde{x}_4 &:= \Big(\sqrt{1-\varepsilon^2}\hat{t}, -\varepsilon, \sqrt{1-\hat{t}^2}\cdot\sqrt{1-\varepsilon^2}\Big).
\end{aligned}$$

If $\tilde{x}_1, \dots, \tilde{x}_4 \in A$, then we get the same contradiction as for the sphere $\mathbb{S}^d$. Thus, the only case we need to consider is when one of these points is not in A.

A direct computation shows that

$$\tilde{x}_1 - \hat{x} = \left(\varepsilon\sqrt{1-\hat{t}^2} - \frac{\hat{t}(1-\hat{t}^2)}{2}\varepsilon^2, 0, -\varepsilon t - \frac{(1-\hat{t}^2)^{3/2}}{2}\varepsilon^2\right) + O(\varepsilon^3),$$

$$\tilde{x}_2 - \hat{x} = \left(-\varepsilon\sqrt{1-\hat{t}^2} - \frac{\hat{t}(1-\hat{t}^2)}{2}\varepsilon^2, 0, \varepsilon t - \frac{(1-\hat{t}^2)^{3/2}}{2}\varepsilon^2\right) + O(\varepsilon^3),$$

$$\tilde{x}_3 - \hat{x} = \left(-\frac{\hat{t}}{2}\varepsilon^2, \varepsilon, -\frac{\sqrt{1-\hat{t}^2}}{2}\varepsilon^2\right) + O(\varepsilon^3),$$

$$\tilde{x}_4 - \hat{x} = \left(-\frac{\hat{t}}{2}\varepsilon^2, -\varepsilon, -\frac{\sqrt{1-\hat{t}^2}}{2}\varepsilon^2\right) + O(\varepsilon^3).$$

Thus, $\tilde{x}_1(1)$ and $\tilde{x}_3(1)$ are greater or equal than t_0, and if $\tilde{x}_2(1) < t_0$ or $\tilde{x}_4(1) < t_0$, then

$$\hat{t} - \varepsilon\sqrt{1-\hat{t}^2} - \frac{\hat{t}(1-\hat{t}^2)}{2}\varepsilon^2 \leqslant t_0. \tag{45}$$

If this is the case, we define the points $\tilde{x}_1, \ldots, \tilde{x}_4$ differently; namely,

$$\begin{aligned}
\tilde{x}_1 &:= \left(\varepsilon\sqrt{1-\hat{t}^2} + \hat{t}\sqrt{1-\varepsilon^2(1-\hat{t}^2)}, 0, -\varepsilon\hat{t} + \sqrt{1-\hat{t}^2}\cdot\sqrt{1-\varepsilon^2(1-\hat{t}^2)}\right),\\
\tilde{x}_2 &:= \left(\varepsilon\sqrt{1-\hat{t}^2} + \hat{t}\sqrt{1-\varepsilon^2(1-\hat{t}^2)}, 0, -\varepsilon\hat{t} + \sqrt{1-\hat{t}^2}\cdot\sqrt{1-\varepsilon^2(1-\hat{t}^2)}\right),\\
\tilde{x}_3 &:= \left(\hat{t}, \varepsilon, \sqrt{1-\hat{t}^2-\varepsilon^2}\right),\\
\tilde{x}_4 &:= \left(\hat{t}, -\varepsilon, \sqrt{1-\hat{t}^2-\varepsilon^2}\right).
\end{aligned}$$

We set $\tilde{x}_j := x_j$ for $j > 4$, $\widetilde{\omega}_N := \{\tilde{x}_1, \ldots, \tilde{x}_N\}$ and write the same Taylor formulas as before. We get

$$f(\hat{x}) \geqslant \frac{1}{2d}\sum_{j=1}^{2d} f_{\tilde{y}}(\tilde{x}_j) - \eta^2 N^{-2/d}\cdot\left(2s5s(s+3)(1-2\sqrt{5}\eta/c_s)^{-s-2}\right)\cdot\frac{1}{|\tilde{y}-\hat{x}|^{s+2}}. \tag{46}$$

Expanding $f(\tilde{x}_j)$ about $\hat{x}$ as before, we get

$$\begin{aligned}
&\varepsilon^2\cdot\left(\frac{|\tilde{y}-\hat{x}|^2}{2}(2-\hat{t}-(s+2)/2)+s\right)\\
&\qquad\qquad + 2\varepsilon\left((\tilde{y}(1)-\hat{t})\sqrt{1-\hat{t}^2} - (\tilde{y}(3)-\sqrt{1-\hat{t}^2})\hat{t}\right)\\
&\qquad + \varepsilon^2\cdot\left((\tilde{y}(1)-\hat{t})\hat{t} + (\tilde{y}(3)-\sqrt{1-\hat{t}^2})\sqrt{1-\hat{t}^2} - \frac{\tilde{y}(3)-\sqrt{1-\hat{t}^2}}{\sqrt{1-\hat{t}^2}}\right)\\
&-4\eta^2 N^{-2/d}\cdot\left(2s+5s(s+3)(1-2\sqrt{5}\eta/c_s)^{-s-2}\right) - \text{remainder terms involving } \xi \leqslant 0,
\end{aligned} \tag{47}$$

where the remainder terms are handled exactly as in (36).

We proceed with showing that the third term can not be a large negative number. In fact,

$$(\tilde{y}(1)-\hat{t})\hat{t} + (\tilde{y}(3)-\sqrt{1-\hat{t}^2})\sqrt{1-\hat{t}^2} - \frac{\tilde{y}(3)-\sqrt{1-\hat{t}^2}}{\sqrt{1-\hat{t}^2}} = \tilde{y}(1)\hat{t} - \frac{\hat{t}^2}{\sqrt{1-\hat{t}^2}}\tilde{y}(3). \tag{48}$$

If $\tilde{y}(3) < 0$, we see that this expression is non-neagtive. Otherwise, plugging

$$\tilde{y}(1) \geqslant t_0 \geqslant \hat{t} - \varepsilon\sqrt{1-\hat{t}^2} - \frac{\hat{t}(1-\hat{t}^2)}{2}\varepsilon^2,$$

and $\tilde{y}(3) \leqslant \sqrt{1-t_0^2}$ into (48), we obtain

$$(\tilde{y}(1)-\hat{t})\hat{t} + (\tilde{y}(3)-\sqrt{1-\hat{t}^2})\sqrt{1-\hat{t}^2} - \frac{\tilde{y}(3)-\sqrt{1-\hat{t}^2}}{\sqrt{1-\hat{t}^2}} \geqslant -c\varepsilon$$

for some non-negative constant c, which depends only on t_0. We finally show how to estimate the second term of (47). Without loss of generality, we can assume this term is negative, in particular, $\hat{t} \neq 0$. The equality

$$(\tilde{y}(1)-\hat{t})\sqrt{1-\hat{t}^2}-(\tilde{y}(3)-\sqrt{1-\hat{t}^2})\hat{t}=\tilde{y}(1)\sqrt{1-\hat{t}^2}-\tilde{y}(3)\hat{t}.$$

yields

$$|\tilde{y}-\hat{x}|^2=2-2\tilde{y}(1)\hat{t}-2\tilde{y}(3)\sqrt{1-\hat{t}^2}\leqslant 2-2\tilde{y}(1)/\hat{t}\leqslant 2-2t_0/\hat{t}\leqslant \varepsilon\sqrt{1-\hat{t}^2}+\frac{\hat{t}(1-\hat{t}^2)}{2}\varepsilon^2\leqslant c\varepsilon,$$

where again c is a positive constant which depends only on t_0. On the other hand,

$$\tilde{y}(1)\sqrt{1-\hat{t}^2}-\tilde{y}(3)\hat{t}\geqslant -\varepsilon-c\varepsilon^2.$$

Thus, inequality (47) implies

$$\varepsilon^2(c\varepsilon(2-\hat{t}-(s+2)/2)+s)-2\varepsilon^2-c\varepsilon^3-\text{ remainder terms }\leqslant 0,$$

which is impossible since $s>2$. □

6. Proofs of covering results

Proof of Theorem 2.6. Fix an integer N. Since $\tilde{A}$ is a d-regular compact set, there exists a finite family of sets $\{Q_\alpha\}_\alpha$ with the following properties:

(i) $\tilde{A}=\cup_\alpha Q_\alpha$ and the interiors of the sets Q_α are disjoint; furthermore, $\mu(Q_\alpha)=0$ for every α, where μ is the measure from Definition 1.3;

(ii) There exists a positive constant a_1 that does not depend on N, and points $z_\alpha\in Q_\alpha$ such that $B(z_\alpha,a_1\eta N^{-1/d})\cap\tilde{A}\subset Q_\alpha\subset B(z_\alpha,\eta N^{-1/d})$.

For the construction of such sets see, e.g., [6]. Notice that since $Q_\alpha\subset B(z_\alpha,\eta N^{-1/d})$, we have $\#(Q_\alpha\cap\omega_N)\leqslant M$.

Let $\mathscr{A}$ denote the set of indices α such that $Q_\alpha\cap\omega_N\neq\emptyset$. Since every Q_α can contain no more than M points from ω_N, we deduce that number of such indices is at least as large as N/M.

Hereafter we follow an argument in [11].

Without loss of generality, we assume $\rho_A(\omega_N)\geqslant 5\eta N^{-1/d}$. Let $y\in A$ be such that $\min_{x_k\in\omega_N}|y-x_k|=\rho_A(\omega_N)$. For every $x_j\in\omega_N$ let $\alpha_j=\alpha$ denote the index such that $x_j\in Q_\alpha$ for some α. If $x\in Q_\alpha$, then

$$|y-x|\leqslant|y-x_j|+|x_j-x|\leqslant|y-x_j|+2\eta N^{-1/d}\leqslant|y-x_j|+\frac{2}{5}\rho_A(\omega_N)\leqslant\frac{7}{5}|y-x_j|.$$

Consequently,

$$|y-x_j|^{-s}\leqslant\left(\frac{7}{5}\right)^s\cdot\min_{x\in Q_\alpha}|y-x|^{-s}. \tag{49}$$

Furthermore,

$$|y-x|\geqslant|y-x_j|-|x_j-x|\geqslant|y-x_j|-2\eta N^{-1/d}\geqslant|y-x_j|-\frac{2}{5}\rho_A(\omega_N)\geqslant\frac{3}{5}\rho_A(\omega_N),$$

which implies

$$A\cap B(y,(3/5)\rho_A(\omega_N))\subset A\setminus\bigcup_{\alpha\in\mathscr{A}}Q_\alpha.$$

For each $x_j \in Q_\alpha$ we see from (49) that

$$\frac{1}{|y-x_j|^s} \leqslant \left(\frac{7}{5}\right)^s \frac{1}{\mu(Q_\alpha)} \int_{Q_\alpha} \frac{\mathrm{d}\mu(x)}{|y-x|^s}.$$

Since $B(z_\alpha, a_1\eta N^{-1/d}) \cap \tilde{A} \subset Q_\alpha$, we have by the d-regularity condition that $\mu(Q_\alpha) \geqslant c_1 \cdot \eta^d/N$, where the positive constant c_1 does not depend on s. This implies from assumption (12) that

$$\begin{aligned}
(50) \quad p_s N^{s/d} &\leqslant \sum_{x_j\in\omega_N} \frac{1}{|y-x_j|^s} \leqslant M\cdot\left(\frac{7}{5}\right)^s \sum_{\alpha\in\mathscr{A}} \frac{1}{\mu(Q_\alpha)} \int\limits_{Q_\alpha} \frac{\mathrm{d}\mu(x)}{|y-x|^s} \\
&\leqslant c_1^{-1} M \cdot \left(\frac{7}{5}\right)^s \cdot \eta^{-d} \cdot N \int\limits_{A\setminus B(y,(3/5)\rho_A(\omega_N))} \frac{\mathrm{d}\mu(x)}{|y-x|^s} \\
&\leqslant c_1^{-1} \cdot c_2 \cdot \frac{s}{s-d} \cdot M \cdot \left(\frac{7}{5}\right)^s \cdot \eta^{-d} \cdot N \cdot ((3/5)\rho_A(\omega_N))^{d-s},
\end{aligned}$$

where c_2 does not depend on s. This yields, for $C_d := c_1^{-1} \cdot c_2$,

$$\rho_A(\omega_N)^{s-d} \leqslant C_d \cdot \frac{s}{s-d} \cdot \left(\frac{7}{5}\right)^s \cdot \frac{1}{p_s} \cdot \eta^{-d} \cdot M \cdot N^{-\frac{s-d}{d}},$$

which implies

$$\rho_A(\omega_N) \leqslant \left(C_d \cdot \frac{s}{s-d}\right)^{\frac{1}{s-d}} \cdot \left(\frac{7}{3}\right)^{\frac{s}{s-d}} \cdot p_s^{-\frac{1}{s-d}} \cdot \eta^{-\frac{d}{s-d}} \cdot M^{\frac{1}{s-d}} \cdot N^{-1/d},$$

as claimed. □

Proof of Corollary 2.7. First, we prove that for any ω_N that is extremal for $\mathscr{P}_s(A;N)$, there exists a positive constant p_s with

$$\inf_{y\in A} \sum_{x_j\in\omega_N} \frac{1}{|y-x_j|^s} \geqslant p_s N^{s/d}.$$

We prove it for strongly convex $A \subset \mathbb{R}^d$ or $A = [0,1]^d$. The case $A = \mathbb{S}^d$ is similar. First, notice that for any $z \in A$ we have $A \subset z + [-\operatorname{diam}(A), \operatorname{diam}(A)]^d =: Q$. For a fixed N and a fixed constant a, consider a maximal set $\mathscr{E}$ such that for any $x, y \in \mathscr{E}$ we have $|x-y| \geqslant aN^{-1/d}$. The maximality of $\mathscr{E}$ implies that

$$A \subset \bigcup_{x\in\mathscr{E}} B(x, aN^{-1/d});$$

thus $\rho_A(\mathscr{E}) \leqslant aN^{-1/d}$.

On the other hand, we see that the sets $B(x,(a/3)N^{-1/d}) \cap Q$ are disjoint. Thus,

$$\mathscr{H}_d(Q) \geqslant c_1 \cdot a^d \cdot N^{-1} \cdot \#(\mathscr{E}),$$

which implies

$$\#(\mathscr{E}) \leqslant c_2 a^{-d} N,$$

where c_1 and c_2 are positive constants that depend on d. We now choose a such that $c_2 a^{-d} = 1$. This implies that there exists an N-point set $\widetilde{\omega}_N$ such that

$$A \subset \bigcup_{\tilde{x}_j \in \widetilde{\omega}_N} B(\tilde{x}_j, aN^{-1/d}),$$

where the number a depends only on A and d. In particular, $\rho_A(\widetilde{\omega}_N) \leqslant aN^{-1/d}$.

Observe that

$$\inf_{y\in A} \sum_{x_j \in \omega_N} \frac{1}{|y-x_j|^s} = \mathscr{P}_s(A;N) \geqslant P_s(A;\widetilde{\omega}_N) = \inf_{y\in A} \max_{\tilde{x}_j \in \widetilde{\omega}_N} \frac{1}{|y-\tilde{x}_j|^s} \tag{51}$$

$$= \frac{1}{\max_{y\in A} \min_{\tilde{x}_j \in \widetilde{\omega}_N} |y-\tilde{x}_j|^s} = \rho_A(\widetilde{\omega}_N)^{-s} \geqslant a^{-s} N^{s/d}.$$

Thus, we can apply Theorem 2.6 with $p_s = a^{-s}$ to obtain

$$\rho_A(\omega_N) \leqslant R_s N^{-1/d}$$

for

$$R_s = \left(\frac{C_d \cdot M \cdot s \cdot 7^s \cdot a^s}{(s-d) \cdot 5^s \cdot \eta_s^d} \right)^{\frac{1}{s-d}},$$

where η_s is the constant from Theorem 2.3 or Theorem 2.5.

To complete the proof, recall that we have $\lim_{s\to\infty} \eta_s^{1/s} = 1$, therefore for large values of s we have $R_s \leqslant R_0$ for some positive R_0. □

7. Proof of Best Covering Results

We begin by remarking that in Section 6 we have seen that if A is d-regular, then for some positive constants a and b we have $aN^{-1/d} \leqslant \rho_A(N) \leqslant bN^{-1/d}$, where $\rho_A(N)$ is defined in (4).

Proof of Theorem 2.8. Using the same argument as in (51), we see that

$$\mathscr{P}_s(A;N) \geqslant \frac{1}{\rho_A(N)^s}.$$

Therefore,

$$\left(\lim_{N\to\infty} \frac{\mathscr{P}_s(A;N)}{N^{s/d}} \right)^{1/s} \geqslant \frac{1}{\liminf_{N\to\infty}(N^{1/d}\rho_A(N))},$$

which implies

$$\liminf_{s\to\infty} \left(\lim_{N\to\infty} \frac{\mathscr{P}_s(A;N)}{N^{s/d}} \right)^{1/s} \geqslant \frac{1}{\liminf_{N\to\infty}(N^{1/d}\rho_A(N))}. \tag{52}$$

On the other hand, for a fixed positive integer N and large s consider an N-point configuration $\omega_N^* = \{x_1, \ldots, x_N\}$ such that $\mathscr{P}_s(A;N) = P_s(A;\omega_N^*)$. Corollary 2.7 implies that if s is large enough, then $\rho_A(\omega_N^*) \leqslant R_0 N^{-1/d}$, where R_0 depends neither on N, nor on s. We also recall that the Theorems 2.3 and 2.5 imply that for any large value of s there exists a number $\eta_s > 0$ such that for any $z \in \mathbb{R}^d$ we have $\#(\omega_N^* \cap B(z, \eta_s N^{-1/d})) \leqslant 2d-1$ and $\lim_{s\to\infty} \eta_s^{1/s} = 1$.

We now take a point $y \in A$ such that

$$\min_{j=1,\ldots,N} |y-x_j| = \rho_A(\omega_N^*), \tag{53}$$

and set

$$B_n := B(y, n\rho_A(\omega_N^*)) \setminus B(y, (n-1)\rho_A(\omega_N^*)),$$

where n is an integer with $n \geqslant 2$. Since the open ball $B(y, \rho_A(\omega_N^*))$ does not intersect ω_N^*, we have

$$\omega_N^* \subset \bigcup_{n=2}^{\infty} B_n.$$

Notice that for any $n \geqslant 2$ we have $B_n \subset B(y, nR_0 N^{-1/d})$; thus, there exists a constant $\tilde{C}_1$ that does not depend on s such that the annulus B_n can be covered by $\tilde{C}_1 R_0^d n^d \eta_s^{-d} =: C_2 n^d \eta_s^{-d}$ balls of radius $\eta N^{-1/d}$. Thus, for any $n \geqslant 2$ we have

$$\#(B_n \cap \omega_N^*) \leqslant C_2(2d-1)n^d \eta_s^{-d} =: C_3 n^d \eta_s^{-d}.$$

For y defined in (53) we have

$$\mathscr{P}_s(A;N) \leqslant \sum_{x \in \omega_N^*} \frac{1}{|y-x|^s} \leqslant \sum_{n=2}^{\infty} \left(\sum_{x \in \omega_N^* \cap B_n} \frac{1}{|y-x|^s} \right).$$

By the definition of B_n, for any $x \in B_n$ we have $|y-x| \geqslant (n-1)\rho_A(\omega_N^*)$, which implies

$$\mathscr{P}_s(A;N) \leqslant \sum_{n=2}^{\infty} C_3 n^d \eta_s^{-d} (n-1)^{-s} \rho_A(\omega_N^*)^{-s} = C_3 \eta_s^{-d} \rho_A(\omega_N^*)^{-s} \sum_{n=2}^{\infty} n^d (n-1)^{-s}. \tag{54}$$

Dividing by $N^{s/d}$ and using that $\rho_A(\omega_N^*) \geqslant \rho_A(N)$, we obtain

$$\frac{\mathscr{P}_s(A;N)}{N^{s/d}} \leqslant C_3 \eta_s^{-d} \sum_{n=1}^{\infty} n^{d-s} \cdot \left(\frac{1}{N^{1/d} \rho_A(N)} \right)^s, \tag{55}$$

which implies

$$\left(\lim_{N \to \infty} \frac{\mathscr{P}_s(A;N)}{N^{s/d}} \right)^{1/s} \leqslant C_3^{1/s} \eta_s^{-d/s} \left(\sum_{n=2}^{\infty} n^{d-s} \right)^{1/s} \cdot \frac{1}{\limsup_{N \to \infty} (N^{1/d} \rho_A(N))}. \tag{56}$$

Taking $\limsup_{s \to \infty}$, we obtain

$$\limsup_{s \to \infty} \left(\lim_{N \to \infty} \frac{\mathscr{P}_s(A;N)}{N^{s/d}} \right)^{1/s} \leqslant \frac{1}{\limsup_{N \to \infty} (N^{1/d} \rho_A(N))}. \tag{57}$$

Estimates (52) and (57) imply that $\lim_{N \to \infty} N^{1/d} \rho_A(N)$ and $\lim_{s \to \infty} \left(\lim_{N \to \infty} \mathscr{P}_s(A;N) N^{-s/d} \right)^{1/s}$ exist and satisfy

$$\lim_{s \to \infty} \left(\lim_{N \to \infty} \frac{\mathscr{P}_s(A;N)}{N^{s/d}} \right)^{1/s} = \frac{1}{\lim_{N \to \infty} (N^{1/d} \rho_A(N))}.$$

□

As an immediate consequence of Theorem 2.8 we state the following corollary about behavior of covering radii of optimal s-Riesz polarization configurations as $s \to \infty$.

Corollary 7.1. *Suppose A is a d-admissible set or $A = [0,1]^d$. For every $N \geqslant 1$ and every $s > d$ fix an N-point configuration ω_N^s such that $\mathscr{P}_s(A;N) = P_s(A;\omega_N^s)$. Then the following limits exist and satisfy*

$$\lim_{s \to \infty} \lim_{N \to \infty} N^{1/d} \rho_A(\omega_N^s) = \lim_{N \to \infty} N^{1/d} \rho_A(N). \tag{58}$$

Proof. Arguing as in (51), we get that

$$\mathscr{P}_s(A;N) \geqslant \frac{1}{\rho_A(\omega_N^s)^s},$$

which implies from (13) that

$$\lim_{N\to\infty} N^{1/d}\rho_A(N) = \lim_{s\to\infty}\left(\lim_{N\to\infty}\frac{\mathscr{P}_s(A;N)}{N^{s/d}}\right)^{-1/s} \leqslant \liminf_{s\to\infty}\left[\liminf_{N\to\infty} N^{1/d}\rho_A(\omega_N^s)\right].$$

On the other hand, arguing as in (54), (55) and (56) we get

$$\lim_{N\to\infty} N^{1/d}\rho_A(N) = \lim_{s\to\infty}\left(\lim_{N\to\infty}\frac{\mathscr{P}_s(A;N)}{N^{s/d}}\right)^{-1/s} \geqslant \limsup_{s\to\infty}\left[\limsup_{N\to\infty}(N^{1/d}\rho_A(\omega_N^s))\right],$$

and (58) follows. □

8. Proof of Proposition 1.5

Proof of Proposition 1.5 for $s > d$. Take a positive integer N, an N-point configuration ω_N and the point y^*. Theorem 1.4 implies, for any $j = 1,\dots,N$,

$$\begin{aligned} &C_s \cdot N^{s/d} \geqslant \mathscr{P}_s(A;N) \\ &\geqslant P_s(A;\omega_N) = \sum_{x\in\omega_N}\frac{1}{|y^*-x|^s} \geqslant \frac{1}{|y^*-x_j|^s} = N^{s/d}\cdot(N^{1/d}\cdot|y^*-x_j|)^{-s}; \end{aligned} \tag{59}$$

therefore, $|y^*-x_j| \geqslant C_s^{-1/s}\cdot N^{-1/d} =: c_s N^{-1/d}$. □

To prove Proposition 1.5 for the case $A = \mathbb{S}^d$ and $s \in [d-1,d)$ we set

$$U(y) = U_s(y) := \frac{1}{\mathscr{H}_d(\mathbb{S}^d)}\int_{\mathbb{S}^d}\frac{\mathrm{d}\mathscr{H}_d(x)}{|x-y|^s}.$$

Then it is well known (see, e.g., [15]) that if $s \in (0,d)$ then $U(y)$ is constant of $\mathbb{S}^d$, and we denote this constant by $\gamma_{s,d}$ [‡].

We need the following lemma, which can be found in [14].

Lemma 8.1. *For each $s \in [d-1,d)$ there exists a constant $C = C(s,d)$ such that for every y with $|y| = 1+N^{-1/d}$ we have*

$$U(y) \geqslant \gamma_{s,d} - CN^{-1+s/d}. \tag{60}$$

Furthermore, if for a constant c and an N-point configuration $\omega_N \subset \mathbb{S}^d$ we have $U(y) \leqslant c\cdot U^{\omega_N}(y)$, where

$$U^{\omega_N}(y) = U_s^{\omega_N}(y) := \frac{1}{N}\sum_{x\in\omega_N}\frac{1}{|x-y|^s},$$

then the same inequality holds for every $y \in \mathbb{R}^{d+1}$.

Proof of Proposition 1.5 for $A = \mathbb{S}^d$ and $s \in [d-1,d)$. Fix an N-point configuration $\omega_N = \{x_1,\dots,x_N\}$ and set $\gamma := P_s(\mathbb{S}^d;\omega_N)$. For every $y \in \mathbb{S}^d$ we have

$$U^{\omega_N}(y) \geqslant \frac{\gamma}{N} = \frac{\gamma}{\gamma_{s,d}\cdot N}\cdot U(y);$$

[‡] $\gamma_{s,d}$ is the Wiener constant (maximal s-energy constant) on $\mathbb{S}^d$.

thus, for every y with $|y| = 1 + N^{-1/d}$ we have

$$U^{\omega_N}(y) \geqslant \frac{\gamma}{\gamma_{s,d} \cdot N} \cdot (\gamma_{s,d} - CN^{-1+s/d}) = \frac{\gamma - C_1 \cdot \gamma \cdot N^{-1+s/d}}{N}.$$

Notice that

$$\gamma = \inf_{y \in \mathbb{S}^d} \sum_{j=1}^{N} \frac{1}{|x_j - y|^s} \leqslant \frac{1}{\mathscr{H}_d(\mathbb{S}^d)} \sum_{j=1}^{N} \int_{\mathbb{S}^d} \frac{\mathrm{d}\mathscr{H}_d(y)}{|x_j - y|^s} = \gamma_{s,d} \cdot N,$$

which implies that for every y with $|y| = 1 + N^{-1/d}$, we have

$$\sum_{j=1}^{N} \frac{1}{|x_j - y|^s} = NU^{\omega_N}(y) \geqslant \gamma - C_2 N^{s/d}. \tag{61}$$

With y^* as in the statement of Proposition 1.5, set $y := (1 + N^{-1/d}) \cdot y^*$. Then for every $j = 1, \ldots, N$ we have $|x_j - y| \geqslant |x_j - y^*|$. Therefore, for every $i = 1, \ldots, N$, if follows from (61) that

$$\gamma - C_2 N^{s/d} - \frac{1}{|y - x_i|^s} \leqslant \sum_{j \neq i} \frac{1}{|y - x_j|^s} \leqslant \sum_{j \neq i} \frac{1}{|y^* - x_j|^s} = \gamma - \frac{1}{|y^* - x_i|^s}.$$

We now use that $|x_i - y| \geqslant N^{-1/d}$ to get

$$\frac{1}{|y^* - x_i|^s} \leqslant (C_2 + 1) N^{s/d},$$

which completes the proof. □

9. Appendix: equivalent definition of best covering of the Euclidean space $\mathbb{R}^d$

Assume $\mathscr{B} \subset \mathbb{R}^d$ is a family of unit balls. The density of $\mathscr{B}$ is defined by

$$\Delta(\mathscr{B}) := \lim_{R \to \infty} \frac{\sum_{B \in \mathscr{B}} \mathscr{H}_d(B \cap [-R, R]^d)}{(2R)^d} \tag{62}$$

whenever the limit exists. The optimal covering density for $\mathbb{R}^d$ is defined by

$$\Gamma_d := \inf \Delta(\mathscr{B}),$$

where the infimum is taken over all families $\mathscr{B}$ that cover $\mathbb{R}^d$.

It is known, see [7, Chapter 2] and [2], that Γ_1 is attained for balls centered on the lattice $2\mathbb{Z}$ and Γ_2 is attained for balls centered on the properly rescaled equi-triangular lattice. For higher dimensions no explicit results are known; however, if we minimize only over lattices, then it is known that for $d \leqslant 5$ an optimal lattice is the properly rescaled $A_d := \{(x_1, \ldots, x_{d+1}) \in \mathbb{Z}^{d+1} \colon x_1 + \cdots + x_{d+1} = 0\}$, which is a lattice in a d-dimensional hyperplane.

We start by proving the following lemma.

Lemma 9.1. *If $V_d = \mathscr{H}_d(\mathbb{B}^d)$, $\mathscr{B}$ covers $\mathbb{R}^d$ and the limit* (62) *exists, then*

$$\frac{\Delta(\mathscr{B})}{V_d} = \lim_{R \to \infty} \frac{\#\{B \in \mathscr{B} \colon \text{center of } B \text{ is in } [-R, R]^d\}}{(2R)^d}.$$

Conversely, if the limit in the right-hand side exists, then $\Delta(\mathscr{B})$ exists as well and $\Delta(\mathscr{B})/V_d$ is equal to this limit.

Proof. Define $\mathscr{B}_R := \left\{B \in \mathscr{B} \colon \text{center of } B \text{ is in } [-R,R]^d\right\}$. We estimate

$$\sum_{B\in\mathscr{B}} \mathscr{H}_d(B\cap[-R,R]^d) \geqslant \sum_{B\in\mathscr{B}_{R-2}} \mathscr{H}_d(B\cap[-R,R]^d) = V_d\cdot \#\mathscr{B}_{R-2}. \tag{63}$$

On the other hand, if $B\cap[-R,R]^d \neq \emptyset$, then the center of B is in $[-R-2,R+2]^d$. Therefore,

$$\sum_{B\in\mathscr{B}} \mathscr{H}_d(B\cap[-R,R]^d) \leqslant \sum_{B\in\mathscr{B}_{R+2}} \mathscr{H}_d(B\cap[-R,R]^d) \leqslant V_d\cdot \#\mathscr{B}_{R+2}. \tag{64}$$

Estimates (63) and (64) obviously imply assertion of the lemma. □

We continue with more equivalent definitions of Γ_d. For a compact set $A\subset\mathbb{R}^d$ and a positive number r put

$$N_A(r) := \min\left\{N\in\mathbb{N}\colon \exists\omega_N=\{x_1,\dots,x_N\}\subset A \text{ such that } A\subset\cup_{j=1}^N B(x_j,r)\right\}.$$

A simple rescaling argument yields for every $R>0$

$$N_{[-R,R]^d}(1) = N_{[0,1]}(1/2R).$$

We show the following.

Theorem 9.2. *For every $d\in\mathbb{N}$ we have*

$$\frac{\Gamma_d}{V_d} = \lim_{R\to\infty}\frac{N_{[-R,R]^d}(1)}{(2R)^d} = \lim_{r\to 0} r^d N_{[0,1]^d}(r) = \lim_{N\to\infty} N\cdot\rho_{[0,1]^d}(N)^d = \lim_{s\to\infty}(\sigma_{s,d})^{-d/s}. \tag{65}$$

Proof. The existence of

$$\lim_{N\to\infty} N\cdot\rho_{[0,1]^d}(N)^d$$

as well as the last equality follows from Theorem 2.8. The equalities

$$\lim_{R\to\infty}\frac{N_{[-R,R]^d}(1)}{(2R)^d} = \lim_{r\to 0} r^d N_{[0,1]^d}(r) = \lim_{N\to\infty} N\cdot\rho_{[0,1]^d}(N)^d$$

are straightforward and left to the reader. We derive the first equality in (65). For a small $\varepsilon>0$ take a set $\mathscr{B}$ such that

$$\frac{\Gamma_d}{V_d} \geqslant \lim_{R\to\infty}\frac{\#\mathscr{B}_R}{(2R)^d} - \varepsilon$$

and

$$\mathbb{R}^d = \bigcup_{B\in\mathscr{B}} B,$$

where $\mathscr{B}_R$ is defined as in preceding proof. As in the proof of Lemma 9.1, we have

$$[-(R-2),R-2]^d \subset \bigcup_{B\in\mathscr{B}_R} B;$$

therefore

$$\frac{N_{[-(R-2),R-2]^d]}(1)}{(2(R-2))^d} \leqslant \frac{\#\mathscr{B}_R}{(2R)^d}\cdot\frac{(2R)^d}{(2(R-2))^d}.$$

Consequently,

$$\lim_{R\to\infty}\frac{N_{[-R,R]^d}(1)}{(2R)^d} \leqslant \frac{\Gamma_d}{V_d} + \varepsilon.$$

In view of the arbitrariness of ε, we get

$$\lim_{R\to\infty} \frac{N_{[-R,R]^d}(1)}{(2R)^d} \leqslant \frac{\Gamma_d}{V_d}. \tag{66}$$

To prove the opposite inequality, we fix a large number R_0 and choose a configuration ω with $\#\omega = N_{[-R_0,R_0]^d}(1)$ and

$$[-R_0,R_0]^d \subset \bigcup_{x\in\omega} B(x,1).$$

Define

$$\mathscr{B} := \{B(x,1) \colon x \in ((2R_0\mathbb{Z}^d)+\omega)\};$$

then obviously

$$\mathbb{R}^d = \bigcup_{B\in\mathscr{B}} B.$$

Fix a number $R > R_0$ and choose an integer n such that $(2n-1)R_0 \leqslant R \leqslant (2n+1)R_0$. Then

$$\#\mathscr{B}_{(2n-1)R_0} \leqslant \#\mathscr{B}_R \leqslant \#\mathscr{B}_{(2n+1)R_0}.$$

Since

$$\#\mathscr{B}_{(2n-1)R_0} = (2n-1)^d N_{[-R_0,R_0]^d}(1)$$

and

$$\#\mathscr{B}_{(2n+1)R_0} = (2n+1)^d N_{[-R_0,R_0]^d}(1),$$

we get

$$\left(\frac{2n-1}{2n+1}\right)^d \cdot \frac{N_{[-R_0,R_0]^d}(1)}{(2R_0)^d} \leqslant \frac{\#\mathscr{B}_R}{(2R)^d} \leqslant \left(\frac{2n+1}{2n-1}\right)^d \cdot \frac{N_{[-R_0,R_0]^d}(1)}{(2R_0)^d}.$$

Therefore,

$$\lim_{R\to\infty} \frac{\#\mathscr{B}_R}{(2R)^d} = \frac{N_{[-R_0,R_0]^d}(1)}{(2R_0)^d},$$

which implies, in view of Lemma 9.1, that

$$\frac{\Gamma_d}{V_d} \leqslant \frac{N_{[-R_0,R_0]^d}(1)}{(2R_0)^d}.$$

From of the arbitrariness of R_0 and the estimate (66), the lemma follows. □

REFERENCES

[1] G. Ambrus, K. M. Ball, and T. Erdélyi. Chebyshev constants for the unit circle. *Bull. Lond. Math. Soc.*, 45(2):236–248, 2013.

[2] S. V. Borodachov, D. P. Hardin, and E. B. Saff. *Minimal Discrete Energy on Rectifiable Sets*. Springer, 2017, to appear.

[3] S.V. Borodachov and N. Bosuwan. Asymptotics of discrete Riesz d-polarization on subsets of d-dimensional manifolds. *Potential Anal.*, 41(1):35–49, 2014.

[4] S.V. Borodachov, D.P. Hardin, A. Reznikov, and E.B. Saff. Optimal discrete measures for Riesz potentials. *To appear in Trans. Amer. Math. Soc., arXiv preprint arXiv:1606.04128*, 2016.

[5] A. Breger, M. Ehler, and M. Graef. Points on manifolds with asymptotically optimal covering radius. *arXiv preprint arXiv:1607.06899*, 2016.

[6] M. Christ. A $T(b)$ theorem with remarks on analytic capacity and the Cauchy integral. *Colloq. Math.*, 60/61(2):601–628, 1990.

[7] J. H. Conway and N. J. A. Sloane. *Sphere Packings, Lattices and Groups*, volume 290 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, third edition, 1999. With additional contributions by E. Bannai, R. E. Borcherds, J. Leech, S. P. Norton, A. M. Odlyzko, R. A. Parker, L. Queen and B. B. Venkov.
[8] T. Erdélyi and E. B. Saff. Riesz polarization inequalities in higher dimensions. *J. Approx. Theory*, 171:128–147, 2013.
[9] B. Farkas and B. Nagy. Transfinite diameter, Chebyshev constant and energy on locally compact spaces. *Potential Anal.*, 28:241–260, 2008.
[10] B. Farkas and S. G. Révész. Potential theoretic approach to rendezvous numbers. *Monatsh. Math.*, 148(4):309–331, 2006.
[11] D. P. Hardin, E. B. Saff, and J. T. Whitehouse. Quasi-uniformity of minimal weighted energy points on compact metric spaces. *J. Complexity*, 28(2):177–191, 2012.
[12] A. Iosevich, N. H. Katz, and T. Tao. Convex bodies with a point of curvature do not have Fourier bases. *Amer. J. Math.*, 123(1):115–120, 2001.
[13] R. Kershner. The number of circles covering a set. *Amer. J. Math.*, 61:665–671, 1939.
[14] A. B. J. Kuijlaars, E. B. Saff, and X. Sun. On separation of minimal Riesz energy points on spheres in Euclidean spaces. *J. Comput. Appl. Math.*, 199(1):172–180, 2007.
[15] N.S. Landkof. *Foundations of Modern Potential Theory*, volume 180 of *Die Grundlehren der mathematischen Wissenschaften, Band 180*. Springer-Verlag, New York-Heidelberg, 1972. Translated from the Russian by A. P. Doohovskoy.
[16] Q. T. Le Gia, I. H. Sloan, and H. Wendland. Multiscale analysis in Sobolev spaces on the sphere. *SIAM J. Numer. Anal.*, 48(6):2065–2090, 2010.
[17] M. Ohtsuka. On various definitions of capacity and related notions. *Nagoya Math. J.*, 30:121–127, 1967.
[18] A. Reznikov, E.B. Saff, and O.V. Vlasiuk. A minimum principle for potentials with application to Chebyshev constants. *To appear in Potential Anal., arXiv preprint arXiv:1607.07283*, 2016.
[19] B. Simanek. Asymptotically optimal configurations for Chebyshev constants with an integrable kernel. *New York J. Math*, 22:667–675, 2016.

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