A MINIMUM PRINCIPLE FOR POTENTIALS WITH APPLICATION TO CHEBYSHEV CONSTANTS

A. REZNIKOV, E. B. SAFF, AND O. V. VLASIUK

ABSTRACT. For "Riesz-like" kernels $K(x,y) = f(|x-y|)$ on $A\times A$, where A is a compact d-regular set $A\subset\mathbb{R}^p$, we prove a minimum principle for potentials $U_K^\mu = \int K(x,y)\mathrm{d}\mu(x)$, where μ is a Borel measure supported on A. Setting $P_K(\mu) = \inf_{y\in A} U^\mu(y)$, the K-polarization of μ, the principle is used to show that if $\{\nu_N\}$ is a sequence of measures on A that converges in the weak-star sense to the measure ν, then $P_K(\nu_N)\to P_K(\nu)$ as $N\to\infty$. The continuous Chebyshev (polarization) problem concerns maximizing $P_K(\mu)$ over all probability measures μ supported on A, while the N-point discrete Chebyshev problem maximizes $P_K(\mu)$ only over normalized counting measures for N-point multisets on A. We prove for such kernels and sets A, that if $\{\nu_N\}$ is a sequence of N-point measures solving the discrete problem, then every weak-star limit measure of ν_N as $N\to\infty$ is a solution to the continuous problem.

Keywords: Maximal Riesz polarization, Chebyshev constant, Hausdorff measure, Riesz potential, Minimum principle

Mathematics Subject Classification: Primary: 31C15, 31C20 ; Secondary: 30C80.

1. INTRODUCTION

For a nonempty compact set $A\subset\mathbb{R}^p$, a kernel $K\colon A\times A\to\mathbb{R}\cup\{\infty\}$ and a measure μ supported on A, the K-potential of μ is defined by

$$U_K^\mu(y) := \int_A K(x,y)\mathrm{d}\mu(x),\ \ y\in\mathbb{R}^p.$$

Assuming that K is lower semi-continuous, the Fatou lemma implies that if $y_n\to y$ as $n\to\infty$, we have

$$\liminf_n U_K^\mu(y_n) \geqslant U_K^\mu(y);$$

thus U_K^μ is a lower semi-continuous function on $\mathbb{R}^p$. We define the weak* topology on the space of positive Borel measures as follows.

Definition 1.1. Let $(\mu_n)_{n=1}^\infty$ be a sequence of positive Borel measures supported on a compact set A. We say that the measures μ_n converge to the measure μ in the weak* sense, $\mu_n\overset{*}{\to}\mu$, if for any function φ continuous on A we have

$$\int\varphi(x)\mathrm{d}\mu_n(x)\to\int\varphi(x)\mathrm{d}\mu(x),\quad n\to\infty.$$

For a measure μ supported on A its K-*polarization* is defined by

$$P_K(\mu) := \inf_{y\in A} U_K^\mu(y).$$

Date: July 2016.

The research of the authors was supported, in part, by the National Science Foundation grants DMS-1516400 and DMS-1412428.

In the following definition we introduce two special constants which denote the maximum value of $P_K(\mu)$ when μ ranges over all probability measures and when μ ranges over all probability measures supported on finite sets.

Definition 1.2. For a positive integer N the *discrete N-th K-polarization (or Chebyshev) constant of A* is defined by

$$\mathscr{P}_K(A,N) := \sup_{\omega_N \subset A} \inf_{y\in A} U_K^{\nu_{\omega_N}}(y), \tag{1}$$

where the supremum is taken over *N-point multisets* ω_N; i.e., N-point sets counting multiplicities, and where ν_{ω_N} is the normalized counting measure of ω_N:

$$\nu_{\omega_N} := \frac{1}{N}\sum_{x\in\omega_N}\delta_x.$$

Moreover, we say that the probability measure ν supported on A *solves the continuous K-polarization problem* if

$$\inf_{y\in A} U_K^{\nu}(y) = \sup_{\mu}\inf_{y\in A} U_K^{\mu}(y) =: T_K(A),$$

where the supremum is taken over all probability measures μ supported on A.

The following result has been known since 1960's; it relates the asymptotic behavior of $\mathscr{P}_K(A,N)$ as $N\to\infty$ with $T_K(A)$.

Theorem 1.3 (Ohtsuka, [8])**.** *Assume $A\subset\mathbb{R}^p$ is a compact set and $K\colon A\times A\to(-\infty,\infty]$ is a lower semi-continuous symmetric kernel bounded from below. Then*

$$\mathscr{P}_K(A,N)\to T_K(A),\ \ N\to\infty. \tag{2}$$

What has been as yet unresolved for integrable kernels on sets A of positive K-capacity is whether, under the mild assumptions of symmetry and lower semi-continuity of K, every limit measure (in the weak* sense) of a sequence of normalized counting measures ν_{ω_N} associated with optimal N-th K-polarization constants attains $T_K(A)$. We remark that such a result does not necessarily hold for non-integrable kernels. Consider a two-point set $A=\{0,1\}$ and any kernel K with $K(0,0)=K(1,1)=\infty$ and $K(0,1)=K(1,0)<\infty$. Then, for any $N\geqslant 2$, the measure $\nu_N := (1/N)\delta_0 + ((N-1)/N)\delta_1$ attains $\mathscr{P}_K(A,N)=\infty$. However, $\nu_N \xrightarrow{*} \delta_1$, which does not attain $T_K(A)=\infty$.

One case when such a result holds is for $K\in C(A\times A)$. Namely, the following is true, see [1], [4], [5] and [6].

Theorem 1.4. *Let $A\subset\mathbb{R}^p$ be a compact set and $K\in C(A\times A)$ be a symmetric function. A sequence $(\omega_N)_{N=1}^\infty$ of N-point multisets on A satisfies*

$$\lim_{N\to\infty} P_K(\nu_{\omega_N}) = T_K(A)$$

if and only if every weak*-limit measure ν^* of the sequence $(\nu_{\omega_N})_{N=1}^\infty$ attains $T_K(A)$.

Notice that this theorem does not cover cases when K is unbounded along the diagonal of $A\times A$; in particular, Riesz kernels $K(x,y)=|x-y|^{-s}$ when $s>0$. The following theorem by B. Simanek applies to Riesz kernels (as well as more general kernels) but under rather special conditions on the set A and the Riesz parameter.

Theorem 1.5 (Simanek, [9]). *Assume $A \subset \mathbb{R}^p$ is a compact set and $K\colon A\times A \to (-\infty,\infty]$ is a lower semi-continuous kernel bounded from below. Assume further there exists a unique probability measure μ_{eq} with $\operatorname{supp}(\mu_{eq}) = A$ and $U^{\mu_{eq}}(y) \equiv C$ for every $y \in A$. Then μ_{eq} is the unique measure that attains $T_K(A)$. Furthermore, if ν_N is an N-point normalized counting measure that attains $\mathscr{P}_K(A,N)$, then $\nu_N \xrightarrow{*} \mu_{eq}$ as $N \to \infty$.*

We remark that if $A = \mathbb{B}^d$, the d-dimensional unit ball and $f(t) = t^{-s}$ with $d-2 \le s < d$, then Theorem 1.5 applies, while if $0 < s < d-2$ or $f(t) = \log(2/t)$, then the assumptions of this theorem are not satisfied. However, it was shown by Erdélyi and Saff [3] that for this case the only N-point normalized counting measure ν_N that attains $\mathscr{P}_f(\mathbb{B}^d,N)$ is $\nu_N = \delta_0$.

In this paper we obtain a convergence theorem that holds for all integrable Riesz kernels provided the set A is d-regular.

Definition 1.6. A compact set $A \subset \mathbb{R}^p$ is called *d-regular*, $0 < d \leqslant p$, if there exist two positive constants c and C such that for any point $y \in A$ and any r with $0 < r < \operatorname{diam}(A)$, we have $cr^d \leqslant \mathscr{H}_d(B(y,r)\cap A) \leqslant Cr^d$, where $\mathscr{H}_d$ is the d-dimensional Hausdorff measure on $\mathbb{R}^p$ normalized by $\mathscr{H}_d([0,1]^d) = 1$.

Further, we introduce a special family of kernels.

Definition 1.7. A function $f\colon (0,\infty) \to (0,\infty)$ is called *d-Riesz-like* if it is continuous, strictly decreasing, and for some ε with $0 < \varepsilon < d$ and $t_\varepsilon > 0$ the function $t \mapsto t^{d-\varepsilon} f(t)$ is increasing on $[0,t_\varepsilon]$; the value at zero is formally defined by

$$\lim_{t\to 0^+} t^{d-\varepsilon} f(t).$$

The kernel K is called d-Riesz-like if $K(x,y) = f(|x-y|)$.

Remark. Examples of such functions f include s-Riesz potentials $f(t) = t^{-s}$ for $0 < s < d$, as well as $f(t) = \log(c/t)$, where the constant c is chosen so that $\log(c/|x-y|) > 0$ for any $x,y \in A$. Further, we can consider $f(t) := t^{-s}\cdot(\log(c/t))^\alpha$ for any $\alpha > 0$ and $0 < s < d$. We also do not exclude the case when f is bounded; e.g., $f(t) = e^{-ct^2}$, $c > 0$.

Under above assumptions on A and f, we first study the behavior of $P_K(\mu_N)$ as $\mu_N \xrightarrow{*} \mu$. In what follows, when $K(x,y) = f(|x-y|)$ we write U_f, P_f and $T_f(A)$ instead of U_K, P_K and $T_K(A)$. We prove the following.

Theorem 1.8. *Let A be a d-regular compact set, and f be a d-Riesz-like function. If $(\nu_N)_{N=1}^\infty$ is a sequence of measures on A with $\nu_N \xrightarrow{*} \nu$, then $P_f(\nu_N) \to P_f(\nu)$ as $N \to \infty$.*

This theorem is a direct consequence of a minimum principle for potentials, introduced below in Theorem 2.5. From Theorem 1.8 we derive the following result.

Theorem 1.9. *Let A and f satisfy the conditions of Theorem 1.8. For each N let ν_N be an N-point normalized counting measure that attains $\mathscr{P}_f(A,N)$. If ν^* is any weak*-limit measure of the sequence (ν_N), then ν^* solves the continuous f-polarization problem.*

Notice that whenever there is a unique measure ν that solves the continuous polarization problem on A, then Theorem 1.9 implies that the whole sequence $\{\nu_N\}$ converges to ν in the weak* sense.

2. A MINIMUM PRINCIPLE FOR RIESZ-LIKE POTENTIALS

We begin this section with some known results from potential theory. In what follows, all measures will have support on A. We proceed with the following definition, important in potential theory.

Definition 2.1. A set $E \subset A$ is called *K-negligible* if for any compact set $E_1 \subset E$ and any measure μ such that U_K^μ is bounded on E_1, we have $\mu(E_1) = 0$.

The following definition describes a useful class of kernels.

Definition 2.2. The kernel K is said to be *regular* if for any positive Borel measure μ the following is satisfied: if the potential U_K^μ is finite and continuous on $\operatorname{supp}\mu$, then it is finite and continuous in the whole space $\mathbb{R}^p$.

It is known, see [7], that a kernel of the form $K(x,y) = f(|x-y|)$, where f is a continuous non-negative strictly decreasing function, is regular. Regularity of a kernel implies the following two results.

Theorem 2.3 (Principle of descent, Lemma 2.2.1 in [7])**.** *Assume K is regular. If $\mu_n \xrightarrow{*} \mu$ and $y_n \to y_\infty$ as $n \to \infty$, then*

$$\liminf_n U_K^{\mu_n}(y_n) \geqslant U_K^\mu(y_\infty).$$

Theorem 2.4 (Lower envelope, Theorem 3 in [2])**.** *Assume K is regular. If $\mu_n \xrightarrow{*} \mu$, then the set*

$$E := \{y \in A \colon \liminf_n U_K^{\mu_n}(y) > U_K^\mu(y)\},$$

is K-negligible.

The new minimum principle mentioned in the title is the following.

Theorem 2.5. *Let A be a d-regular set, and f be a d-Riesz-like function on $(0,\infty)$. If for a measure μ on A and a constant M,*

$$U_f^\mu(y) \geqslant M, \quad y \in A \setminus E, \tag{3}$$

where E is f-negligible, then $U_f^\mu(y) \geqslant M$ for every $y \in A$.

We proceed with a proposition that is an analog of the Lebesgue differentiation theorem for potentials.

Definition 2.6. For a function $\varphi \colon A \to \mathbb{R}$ and a point $y \in A$, we call y a *weak d-Lebesgue point* of φ if

$$\varphi(y) = \lim_{r\to 0^+} \frac{1}{\mathscr{H}_d(A \cap B(y,r))} \int\limits_{A\cap B(y,r)} \varphi(z)\,\mathrm{d}\mathscr{H}_d(z),$$

where $B(y,r)$ denotes the open ball in $\mathbb{R}^p$ with center at y and radius r.

Proposition 2.7. *Suppose A and f satisfy conditions of Theorem 2.5, and μ is a measure supported on A. Then every point $y \in A$ is a weak d-Lebesgue point of U_f^μ.*

We start by proving the following technical lemma.

Lemma 2.8. *There exist positive numbers C_0 and r_0 such that for any $x \in A$ and any $r < r_0$:*

$$\frac{1}{\mathscr{H}_d(A\cap B(y,r))}\cdot\frac{1}{f(|x-y|)}\cdot\int_{A\cap B(y,r)} f(|x-z|)\mathrm{d}\mathscr{H}_d(z) \leqslant C_0. \tag{4}$$

Proof. Notice that the left-hand side of (4) is equal to

$$\frac{1}{\mathscr{H}_d(A\cap B(y,r))}\cdot\frac{1}{f(|x-y|)}\cdot\int_0^\infty \mathscr{H}_d(z\in A\cap B(y,r):\ f(|x-z|)>u)\mathrm{d}u.$$

Since f is decreasing, we see that

$$\{z\in A\cap B(y,r):\ f(|x-z|)>u\} = \{z\in A\cap B(y,r)\cap B(x,f^{-1}(u))\}.$$

This set is empty when $f^{-1}(u) < |x-y|-r$ or $u > f(|x-y|-r)$.

We consider two cases.

Case 1: $|x-y| > 2r$. Then we obtain the estimate

$$\begin{aligned}
&\frac{1}{\mathscr{H}_d(A\cap B(y,r))}\cdot\frac{1}{f(|x-y|)}\cdot\int_0^\infty \mathscr{H}_d(z\in A\cap B(y,r):\ f(|x-z|)>u)\mathrm{d}u\\
&= \frac{1}{\mathscr{H}_d(A\cap B(y,r))}\cdot\frac{1}{f(|x-y|)}\cdot\int_0^{f(|x-y|-r)} \mathscr{H}_d(z\in A\cap B(y,r):\ f(|x-z|)>u)\mathrm{d}u\\
&\leqslant \frac{f(|x-y|-r)}{f(|x-y|)}.
\end{aligned} \tag{5}$$

Since $|x-y|>2r$, we have $|x-y|-r \geqslant |x-y|/2$; thus,

$$\frac{1}{\mathscr{H}_d(A\cap B(y,r))}\cdot\frac{1}{f(|x-y|)}\cdot\int_0^\infty \mathscr{H}_d(z\in A\cap B(y,r):\ f(|x-z|)>u)\mathrm{d}u \leqslant \frac{f(|x-y|/2)}{f(|x-y|)}.$$

Finally, if recall that the function $t\mapsto f(t)\cdot t^{d-\varepsilon}$ is increasing for $t\in[0,t_\varepsilon]$. If $|x-y|>t_\varepsilon$, we get

$$\frac{f(|x-y|/2)}{f(|x-y|)} \leqslant \frac{f(t_\varepsilon/2)}{f(\mathrm{diam}(A))}.$$

If $|x-y|\leqslant t_\varepsilon$, we use that

$$f(|x-y|/2)\cdot(|x-y|/2)^{d-\varepsilon} \leqslant f(|x-y|)\cdot(|x-y|)^{d-\varepsilon};$$

thus

$$\frac{f(|x-y|/2)}{f(|x-y|)} \leqslant 2^{d-\varepsilon}.$$

Case 2: $|x-y|\leqslant 2r$. Again, we need only integrate for $u\leqslant f(|x-y|-r)$. Setting f equal to $f(0)$ for any negative argument, we write

$$\begin{aligned}
&\frac{1}{\mathscr{H}_d(A\cap B(y,r))}\cdot\frac{1}{f(|x-y|)}\cdot\int_0^{f(|x-y|-r)} \mathscr{H}_d(z\in A\cap B(y,r):\ f(|x-z|)>u)\mathrm{d}u =\\
&\frac{1}{\mathscr{H}_d(A\cap B(y,r))}\cdot\frac{1}{f(|x-y|)}\cdot\left(\int_0^{f(|x-y|+r)} + \int_{f(|x-y|+r)}^{f(|x-y|-r)} \mathscr{H}_d(z\in A\cap B(y,r):\ f(|x-z|)>u)\mathrm{d}u\right)
\end{aligned} \tag{6}$$

Trivially,

$$\begin{aligned}(7)\qquad &\frac{1}{\mathscr{H}_d(A\cap B(y,r))}\cdot\frac{1}{f(|x-y|)}\cdot\int_0^{f(|x-y|+r)}\mathscr{H}_d(z\in A\cap B(y,r)\colon f(|x-z|)>u)\mathrm{d}u\\ &\qquad\leqslant\frac{f(|x-y|+r)}{f(|x-y|)}\leqslant 1.\end{aligned}$$

Furthermore, since $|x-y|\leqslant 2r$, we have $f(|x-y|)\geqslant f(2r)$; thus,

$$\begin{aligned}(8)\qquad &\frac{1}{\mathscr{H}_d(A\cap B(y,r))}\cdot\frac{1}{f(|x-y|)}\int_{f(|x-y|+r)}^{f(|x-y|-r)}\mathscr{H}_d(z\in A\cap B(y,r)\colon f(|x-z|)>u)\mathrm{d}u\\ &\qquad\leqslant\frac{1}{c\cdot r^d}\cdot\frac{1}{f(2r)}\int_{f(|x-y|+r)}^{\infty}\mathscr{H}_d(z\in A\cap B(x,f^{-1}(u))\mathrm{d}u\\ &\qquad\leqslant\frac{C/c}{r^d f(2r)}\int_{f(|x-y|+r)}^{\infty}(f^{-1}(u))^d\mathrm{d}u.\end{aligned}$$

Note that the assumption $|x-y|\leqslant 2r$ implies $f(|x-y|+r)\geqslant f(3r)$, and thus

$$\begin{aligned}(9)\qquad &\frac{1}{\mathscr{H}_d(A\cap B(y,r))}\cdot\frac{1}{f(|x-y|)}\int_{f(|x-y|+r)}^{f(|x-y|-r)}\mathscr{H}_d(z\in A\cap B(y,r)\colon f(|x-z|)>u)\mathrm{d}u\\ &\qquad\leqslant\frac{C/c}{r^d f(2r)}\int_{f(3r)}^{\infty}(f^{-1}(u))^d\mathrm{d}u.\end{aligned}$$

We now observe that our assumption that $f(t)t^{d-\varepsilon}$ is increasing on $[0,t_\varepsilon]$ implies that the function $u^{1/(d-\varepsilon)}f^{-1}(u)$ is decreasing on $[f(t_\varepsilon),\infty)$. Therefore, for $3r<t_\varepsilon$ we have

$$\begin{aligned}(10)\qquad &\frac{C/c}{r^d f(2r)}\int_{f(3r)}^{\infty}(f^{-1}(u))^d\mathrm{d}u=\frac{C/c}{r^d f(2r)}\int_{f(3r)}^{\infty}(u^{1/(d-\varepsilon)}f^{-1}(u))^d\cdot u^{-d/(d-\varepsilon)}\mathrm{d}u\\ &\qquad\leqslant\frac{C_1}{r^d f(2r)}\cdot f(3r)^{d/(d-\varepsilon)}\cdot(3r)^d\cdot f(3r)^{1-d/(d-\varepsilon)}\leqslant C_2.\end{aligned}$$

□

Proof of Proposition 2.7. We formally define $f(0):=\lim_{t\to 0^+}f(t)\in(0,\infty]$. Without loss of generality, we consider the case $f(0)=\infty$; otherwise the potential U_f^μ is continuous on $\mathbb{R}^p$ and the proposition holds trivially.

Define

$$\Phi_r^\mu(y):=\frac{1}{\mathscr{H}_d(A\cap B(y,r))}\int_{A\cap B(y,r)}U_f^\mu(z)\mathrm{d}\mathscr{H}_d(z).$$

Tonneli's theorem and Lemma 2.8 imply

$$\begin{aligned}&\Phi_r^\mu(y)=\frac{1}{\mathscr{H}_d(A\cap B(y,r))}\int_{A\cap B(y,r)}\int_A f(|x-z|)\mathrm{d}\mu(x)\mathrm{d}\mathscr{H}_d(z)\\ (11)\qquad &=\int_A f(|x-y|)\cdot\frac{1}{\mathscr{H}_d(A\cap B(y,r))}\cdot\frac{1}{f(|x-y|)}\int_{A\cap B(y,r)}f(|x-z|)\mathrm{d}\mathscr{H}_d(z)\mathrm{d}\mu(x)\\ &\leqslant C_0\int_A f(|x-y|)\mathrm{d}\mu(x)=C_0U_f^\mu(y).\end{aligned}$$

We first suppose $U^\mu(y)=\infty$. Since U^μ is lower semi-continuous, we obtain that for any large number N there is a positive number r_N, such that $U^\mu(x)>N$ in $B(y,r_N)$. Then for any $r < r_N$ we get

$$\Phi_r^{\mu}(y) = \frac{1}{\mathscr{H}_d(A\cap B(y,r))}\int_{A\cap B(y,r)} U_f^{\mu}(z)\mathrm{d}\mathscr{H}_d(z) > N.$$

This implies that $\Phi_r^{\mu}(y) \to \infty = U_f^{\mu}(y)$ as $r \to 0^+$.

Now assume $U_f^{\mu}(y) = \int_A f(|x-y|)\mathrm{d}\mu(x) < \infty$. Notice that since $f(0) = \infty$, the measure μ cannot have a mass point at y. Consequently, for any $\eta > 0$ there exists a ball $B(y,\delta)$, such that

$$\int_{B(y,\delta)} f(|x-y|)\mathrm{d}\mu(x) < \eta.$$

Consider measures

$$\mathrm{d}\mu' := \mathbb{1}_{B(y,\delta)}\mathrm{d}\mu, \text{ and } \mu_c := \mu - \mu'.$$

Since $y \notin \mathrm{supp}(\mu_c)$, the potential $U_f^{\mu_c}$ is continuous at y. This implies

$$\Phi_r^{\mu_c}(y) = \frac{1}{\mathscr{H}_d(A\cap B(y,r))}\int_{A\cap B(y,r)} U_f^{\mu_c}(z)\mathrm{d}\mathscr{H}_d(z) \to U_f^{\mu_c}(y), \ \ r\to 0^+.$$

Also, on applying (11) to μ', it follows that

$$\Phi_r^{\mu}(y) = \Phi_r^{\mu_c}(y) + \Phi_r^{\mu'}(y) \leqslant \Phi_r^{\mu_c}(y) + C_0 U_f^{\mu'}(y) \leqslant \Phi_r^{\mu_c}(y) + C_0\eta.$$

Taking the lim sup, we obtain

$$\limsup_{r\to 0^+} \Phi_r^{\mu}(y) \leqslant U_f^{\mu_c}(y) + C_0\eta \leqslant U_f^{\mu}(y) + C_0\eta. \tag{12}$$

On the other hand, we know that

$$\Phi_r^{\mu}(y) = \Phi_r^{\mu_c}(y) + \Phi_r^{\mu'}(y) \geqslant \Phi_r^{\mu_c}(y),$$

and thus

$$\liminf_{r\to 0^+} \Phi_r^{\mu}(y) \geqslant U_f^{\mu_c}(y) = U_f^{\mu}(y) - U_f^{\mu'}(y) \geqslant U_f^{\mu}(y) - \eta.$$

This, together with (12) and the arbitrariness of η implies the assertion of Proposition 2.7. □

We are ready to deduce Theorem 2.5.

Proof of Theorem 2.5. We first claim that any f-negligible subset E of A has $\mathscr{H}_d$-measure zero. Indeed, take any compact set E_1 inside our f-negligible set E. Then for any $y \in A$

$$U_f^{\mathscr{H}_d}(y) = \int_A f(|x-y|)\mathrm{d}\mathscr{H}_d(x) = \int_0^{\infty} \mathscr{H}_d(x\in A\cap B(y, f^{-1}(u)))\mathrm{d}u.$$

We notice that if $u < f(\mathrm{diam}(A)) =: u_0$, then $f^{-1}(u) > \mathrm{diam}(A)$, and so $A\cap B(y, f^{-1}(u)) = A$. Thus,

$$\begin{aligned} U_f^{\mathscr{H}_d}(y) &= \left(\int_0^{u_0} + \int_{u_0}^{\infty}\right) \mathscr{H}_d(x\in A\cap B(y, f^{-1}(u)))\mathrm{d}u \\ &\leqslant u_0\mathscr{H}_d(A) + C\int_{u_0}^{\infty} (f^{-1}(u))^d \mathrm{d}u, \end{aligned} \tag{13}$$

which is bounded by a constant that does not depend on y, as proved in the Case 2 of Lemma 2.8; see inequality (10). Since the set E is negligible, we conclude that $\mathscr{H}_d(E_1) = 0$. Thus, for any compact subset E_1 of E we have $\mathscr{H}_d(E_1) = 0$ and so $\mathscr{H}_d(E) = 0$ as claimed. Now, let the measure μ satisfy (3). Then for any $y \in A$, we deduce from Proposition 2.7 and the fact that $U_f^\mu(z) \geqslant M$ holds $\mathscr{H}_d$-a.e. on A that $U_f^\mu(y) \geqslant M$. □

3. Proof of Theorem 1.8 and Theorem 1.9

Proof of Theorem 1.8. For any increasing infinite subsequence $\mathscr{N} \subset \mathbb{N}$, choose a subsequence $\mathscr{N}_1$, such that

$$\liminf_{N\in\mathscr{N}} P_f(\nu_N) = \lim_{N\in\mathscr{N}_1} P_f(\nu_N).$$

For each $N \in \mathscr{N}_1$ take a point y_N, such that $P_f(\nu_N) = U^{\nu_N}(y_N)$. Passing to a further subsequence $\mathscr{N}_0 \subset \mathscr{N}_1$ we can assume $y_N \to y_\infty$ as $N \to \infty$, $N \in \mathscr{N}_0$. Then the principle of descent, Theorem 2.3, implies

$$\liminf_{N\in\mathscr{N}} P_f(\nu_N) = \lim_{N\in\mathscr{N}_0} U_f^{\nu_N}(y_N) \geqslant U_f^\nu(y_\infty) \geqslant P_f(\nu). \tag{14}$$

Furthermore, for any $y \in A$ we have

$$\liminf_{N\in\mathscr{N}} U_f^{\nu_N}(y) \geqslant \liminf_{N\in\mathscr{N}} P_f(\nu_N) =: M_f(\mathscr{N}),\ \ y \in A.$$

By Theorem 2.4, $\liminf_{N\in\mathscr{N}} U_f^{\nu_N}(y) = U_f^\nu(y)$ for every $y \in A \setminus E$, where E is an f-negligible set that can depend on $\mathscr{N}$. Therefore,

$$U_f^\nu(y) \geqslant M_f(\mathscr{N}),\ \ y \in A \setminus E.$$

From the minimum principle, Theorem 2.5, we deduce that

$$U_f^\nu(y) \geqslant M_f(\mathscr{N}) = \liminf_{N\in\mathscr{N}} P_f(\nu_N),\ \ \forall y \in A,$$

and therefore

$$P_f(\nu) \geqslant \liminf_{N\in\mathscr{N}} P_f(\nu_N). \tag{15}$$

Combining estimates (14) and (15), we deduce that for any subsequence $\mathscr{N}$ we have

$$\liminf_{N\in\mathscr{N}} P_f(\nu_N) = P_f(\nu).$$

This immediately implies

$$\lim_N P_f(\nu_N) = P_f(\nu),$$

which completes the proof. □

Proof of Theorem 1.9. Assume for a subsequence $\mathscr{N}$ we have $\nu_N \xrightarrow{*} \nu^*$ as $N \to \infty$, $N \in \mathscr{N}$. From [8] we know that

$$P_f(\nu_N) \to T_f(A),\ \ N \to \infty.$$

On the other hand, from Theorem 1.8 we know that

$$P_f(\nu_N) \to P_f(\nu^*),\ \ N \to \infty,\ \ N \in \mathscr{N}.$$

Therefore,

$$P_f(\nu^*) = T_f(A),$$

which proves the theorem. □

Remark. Suppose $A = \cup_{k=1}^{m} A_k$, where A_k is a d_k-regular compact set, and that, for some positive number δ, we have $\operatorname{dist}(A_i, A_j) \geqslant \delta$ for $i \neq j$. Further assume that f is a d_k-Riesz-like kernel for every $k = 1, \ldots, m$, and μ is a measure supported on A. Then the result of Theorem 2.5, and thus of Theorems 1.8 and 1.9 hold. To see this, we first show that every $y \in A_k$ is a weak d_k-Lebesgue point of U_f^{μ}. Indeed, setting $\mathrm{d}\mu_k := \mathbb{1}_{A_k} \mathrm{d}\mu$ yields

$$U_f^{\mu}(y) = \sum_{k=1}^{m} U_f^{\mu_k}(y).$$

Proposition 2.7 implies that if $y \in A_k$, then y is a weak d_k-Lebesgue point of $U_f^{\mu_k}$. Moreover, for any $j \neq k$ we have $y \notin \operatorname{supp}(\mu_j)$; thus $U_f^{\mu_j}$ is continuous at y, and our assertion about weak Lebesgue points of U_f^{μ} follows. Similar to (13), we then deduce that if a set $E \subset A$ is f-negligible, then $\mathscr{H}_{d_k}(E \cap A_k) = 0$ for every $k = 1, \ldots, m$; therefore, the assertion of Theorem 2.5 remains true and the proofs of Theorems 1.8 and 1.9 go exactly as before.

REFERENCES

[1] S. V. Borodachov, D. P. Hardin, and E. B. Saff. *Minimal Discrete Energy on Rectifiable Sets*. Springer, 2017 (to appear).

[2] M. Brelot. Lectures on Potential Theory. Available at `www.math.tifr.res.in/~publ/ln/tifr19.pdf/`, 1960.

[3] T. Erdélyi and E. B. Saff. Riesz polarization inequalities in higher dimensions. *J. Approx. Theory*, 171:128–147, 2013.

[4] B. Farkas and B. Nagy. Transfinite diameter, Chebyshev constant and energy on locally compact spaces. *Potential Anal.*, 28:241–260, 2008.

[5] B. Farkas and S. G. Révész. Potential theoretic approach to rendezvous numbers. *Monatsh. Math.*, 148(4):309–331, 2006.

[6] B. Farkas and S. G. Révész. Rendezvous numbers of metric spaces—a potential theoretic approach. *Arch. Math. (Basel)*, 86(3):268–281, 2006.

[7] B. Fuglede. On the theory of potentials in locally compact spaces. *Acta Math.*, 103:139–215, 1960.

[8] M. Ohtsuka. On various definitions of capacity and related notions. *Nagoya Math. J.*, 30:121–127, 1967.

[9] B. Simanek. Extremal polarization configurations for integrable kernels. *arXiv preprint arXiv:1507.04813*, 2015.

CENTER FOR CONSTRUCTIVE APPROXIMATION, DEPARTMENT OF MATHEMATICS, VANDERBILT UNIVERSITY

E-mail address: `aleksandr.b.reznikov@vanderbilt.edu`

E-mail address: `edward.b.saff@vanderbilt.edu`

E-mail address: `oleksandr.v.vlasiuk@vanderbilt.edu`