OPTIMAL DISCRETE MEASURES FOR RIESZ POTENTIALS

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ABSTRACT. For $s \geqslant d$, we obtain the leading term as $N \to \infty$ of the maximal weighted N-point Riesz s-polarization (or Chebyshev constant) for a certain class of d-rectifiable compact subsets of $\mathbb{R}^p$. This class includes compact subsets of d-dimensional C^1 manifolds whose boundary relative to the manifold has $\mathscr{H}_d$-measure zero, as well as finite unions of such sets when their pairwise intersections have $\mathscr{H}_d$-measure zero. We also explicitly find the weak* limit distribution of asymptotically optimal N-point polarization configurations as $N \to \infty$.

Keywords: maximal Riesz polarization, Chebyshev constant, rectifiable set, Hausdorff measure, Riesz potential

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1. INTRODUCTION

Suppose $A \subset \mathbb{R}^p$ and s is a positive number. Fix a function $w\colon A \times A \to [0,\infty]$ and an N-point *multiset* $\omega_N = \{x_1, \ldots, x_N\} \subset A$; i.e., a set of points with possible repetitions and cardinality $\#\omega_N = N$, counting multiplicities. We denote

$$P_s^w(A;\omega_N) := \inf_{y\in A} \sum_{j=1}^{N} \frac{w(y,x_j)}{|y-x_j|^s}. \tag{1}$$

Further, define the *weighted N-th s-polarization (or Chebyshev) constant of A* by

$$\mathscr{P}_s^w(A;N) := \sup_{\omega_N \subset A} P_s^w(A;\omega_N). \tag{2}$$

We note that if A is a compact set and the weight w is lower semi-continuous and strictly positive on $A\times A$, then for any N there exists a configuration $\omega_N^* = \{x_1^*, \ldots, x_N^*\}$ and a point y^*, such that

$$\mathscr{P}_s^w(A;N) = P_s^w(A;\omega_N^*) = \sum_{j=1}^{N} \frac{w(y^*,x_j^*)}{|y^*-x_j^*|^s}.$$

Thus,

$$U_s^w(y) := \sum_{j=1}^{N} \frac{w(y,x_j^*)}{|y-x_j^*|^s}$$

is an optimal N-point weighted Riesz potential for A.

In the unweighted case; i.e., when $w(x,y) \equiv 1$, we will omit the superscript w in $\mathscr{P}_s^w(A,N)$, $P_s^w(A;\omega_N)$, $U_s^w(y)$, etc. In this case the problem of finding quantity (2) and N-point configurations ω_N^* that attain the supremum in (2) is the standard maximal discrete s-polarization problem. It was studied earlier in the works [1], [2], [9], [13], [5], [19], [23],

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and others (see the book [7] and the papers [10], [11], [12] and [8] for more references). In the case $s = \log$; i.e., when in (1) we use the kernel $\log(1/|y-x|)$ instead of $|y-x|^{-s}$, the Chebyshev constant of a set A, defined in (3), is related to the transfinite diameter of A, see [16], [20], [3], [10], [24] and [4] for more details.

The exact solution to the maximal discrete s-polarization problem is, in particular, known in the following cases. When $A = \mathbb{S}^1$ (the unit circle in $\mathbb{R}^2$) and $s > 0$, an N-point configuration is optimal if and only if it consists of N equally spaced points (partial cases of this result were obtained in [1], [17], [2], and [9] and a complete solution was found in [13]). When $A = \mathbb{S}^2$ (the unit sphere in $\mathbb{R}^3$) and $s > 0$ a configuration of four points is optimal if and only if it forms the set of vertices of a regular simplex inscribed in $\mathbb{S}^2$ (cf. [23]). When A is the unit sphere $\mathbb{S}^d$ in $\mathbb{R}^{d+1}$ it is not difficult to show that a configuration of $2 \leqslant N \leqslant d+1$ points with the center of mass at the origin is optimal for every $s > 0$. When A is the closed unit ball $\mathscr{B}^d \subset \mathbb{R}^d$, $0 < s \leqslant d-2$, $d \geqslant 3$, and $N \in \mathbb{N}$ is arbitrary, the N-point configuration with all its points lying at the center of $\mathscr{B}^d$ is optimal [9]. Several other cases when the exact solution is known are described, for example, in the book [7].

Since the exact solution is known only in a handful number of cases, it makes sense to search for asymptotic solution to the polarization problem. A number of applications (such as computer-aided geometric design) require placing a large number of points on a surface according to a prescribed non-uniform distribution. This can be achieved by solving the maximal polarization problem for a weighted potential as in (1). In this paper we will be interested in the asymptotic behavior of $\mathscr{P}_s^w(A;N)$ for large values of N and in the weak* limit distribution of asymptotically optimal configurations when $s \geqslant d$, where d is the Hausdorff dimension of A, and A is as in Theorem 2.3.

To discuss known results, by $\mathfrak{L}_p$ we denote the Lebesgue measure on $\mathbb{R}^p$ and by $\mathscr{H}_d$ we denote the d-dimensional Hausdorff measure on $\mathbb{R}^p$, scaled so that $\mathscr{H}_d(\mathscr{Q}_d) = 1$, where $\mathscr{Q}_d := [0,1]^d$ is a d-dimensional unit cube. If $x \in \mathbb{R}^p$ and $r > 0$, by $B(x,r)$ we denote the open ball $\{y \in \mathbb{R}^p \colon |y-x| < r\}$ and by $B[x,r]$ the closed ball $\{y \in \mathbb{R}^p : |y-x| \leqslant r\}$.

In the case $0 < s < d$ the set A has a positive s-capacity and one can use methods from Potential Theory. The result of Ohtsuka proved in [18] implies that for every infinite compact set A and $s > 0$,

$$\lim_{N\to\infty} \frac{\mathscr{P}_s(A,N)}{N} = T_s(A), \tag{3}$$

(here and in what follows we understand the limit as an extended real number), where

$$T_s(A) := \sup_{\mu\in\mathscr{M}(A)} \inf_{x\in A} U_s^\mu(x), \tag{4}$$

$\mathscr{M}(A)$ is the set of all Borel probability measures supported on A, and

$$U_s^\mu(x) := \int_A \frac{1}{|x-y|^s}\, d\mu(y).$$

If A is a compact set of Hausdorff dimension d, then $0 < T_s(A) < \infty$ for $0 < s < d$, see, for example, the chapter on polarization in [7]. Ohtsuka also showed that the value $T_s(A)$ is greater than or equal to the Wiener s-constant for A. Consequently, $T_s(A) = \infty$ when $s > d$ or $s = d$ and $\mathscr{H}_d(A) < \infty$ (see [15, Theorem 8.7]).

More precisely the order of growth of the quantity $\mathscr{P}_s(A,N)$ in the case $s \geqslant d$ found by Erdélyi and Saff [9, Theorems 2.3 and 2.4] is as follows. If $\mathscr{H}_d(A) > 0$, then

$$\mathscr{P}_s(A,N) = O(N^{s/d}),\ \ s > d,\ \text{ and }\ \mathscr{P}_d(A,N) = O(N\ln N),\ \ N \to \infty. \tag{5}$$

If, in addition, A is d-rectifiable, then in the case $s > d$ we have $\mathscr{P}_s(A,N) \asymp N^{s/d}$ (cf. [9]). If we further assume A to be a compact subset of a d-dimensional C^1-manifold in $\mathbb{R}^p$, $d \leqslant p$, then (cf. Borodachov and Bosuwan [5])

$$\lim_{N\to\infty} \frac{\mathscr{P}_d(A,N)}{N\ln N} = \frac{\beta_d}{\mathscr{H}_d(A)}, \tag{6}$$

where

$$\beta_d = \mathscr{H}_d(B(0,1)) = \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)}$$

(the cases $A = \mathscr{B}^d$ and $A = S^d$ of (6) were earlier established in [9]).

Thus, the unweighted maximal s-polarization problem behaves differently for different values of s and the following three cases can be distinguished: the potential theoretic case $(0 < s < d)$, the hypersingular case $(s > d)$, and the transition case $(s = d)$. This behavior suggests to define

$$\tau_{s,d}(N) := \begin{cases} N^{s/d}, & s > d, \\ N\ln N, & s = d, \\ N, & s < d. \end{cases} \tag{7}$$

and

$$\underline{h}^w_{s,d}(A) := \liminf_{N\to\infty} \frac{\mathscr{P}^w_s(A;N)}{\tau_{s,d}(N)},\ \ \overline{h}^w_{s,d}(A) := \limsup_{N\to\infty} \frac{\mathscr{P}^w_s(A;N)}{\tau_{s,d}(N)}, \tag{8}$$

If $\underline{h}^w_{s,d}(A) = \overline{h}^w_{s,d}(A)$, we denote

$$h^w_{s,d}(A) := \lim_{N\to\infty} \frac{\mathscr{P}^w_s(A;N)}{\tau_{s,d}(N)}. \tag{9}$$

If the function w is identically equal to 1, we drop the superscript and write $\underline{h}_{s,d}$, $\overline{h}_{s,d}$, and $h_{s,d}$. Define

$$\sigma_{s,p} := h_{s,p}(\mathscr{Q}_p).$$

When $0 < s < p$ from the finiteness of the limit (3) we can deduce that $\sigma_{s,p}$ is a positive finite number. When $s = p$ from (6) we have $\sigma_{p,p} = \beta_p$.

Another fundamental difference between the cases $s < d$ and $s \geqslant d$ is the weak* limit distribution of asymptotically extreme sequences of configurations.

Definition 1.1. Assume $A \subset \mathbb{R}^p$, $w\colon A\times A \to [0,\infty]$ and $s > 0$. For every positive integer N let a configuration $\omega_N \subset A$ be chosen. We call the family $\Omega := \{\omega_N\}_{N\geqslant 1}$ *asymptotically extremal* (or *asymptotically optimal*) if

$$\lim_{N\to\infty} \frac{P^w_s(A;\omega_N)}{\mathscr{P}^w_s(A,N)} = 1.$$

In the case $w \equiv 1$ and $0 < s < d$, where d is the Hausdorff dimension of A, if a measure μ^s exists that attains a supremum in (4), then every sequence $\{\omega_N\}_{N=1}^{\infty}$ of N-point configurations on A having weak* limit distribution according to the measure μ^s, is asymptotically optimal [7]. Simanek [22] found sufficient conditions on the set A and the kernel under which the weak* limit measure of optimal configurations coincides with the equilibrium measure for that kernel. In particular, he showed that for $0 < s < d$, optimal configurations for the maximal s-polarization problem on the sphere S^d are uniformly distributed in the weak* limit. His result also implies that optimal configurations on the ball $\mathscr{B}^d$ in $\mathbb{R}^d$ for $d-2 < s < d$ are distributed in the weak* limit according to the equilibrium measure on $\mathscr{B}^d$ for the Riesz s-potential. This implies that the weak* limit distribution of asymptotically optimal sequences on A is, in general, non-uniform for $0 < s < d$. In contrast to this, when $s = d$ and A is a compact subset of a d-dimensional C^1-manifold in $\mathbb{R}^p$, $d \leqslant p$, with $\mathscr{H}_d(A) > 0$, *every* asymptotically optimal sequence of configurations on A is uniformly distributed on A (in the weak* sense) with respect to $\mathscr{H}_d$ [5].

The main goal of this paper is to study the existence of the limit $h^w_{s,d}(A)$ and find its value as well as describe the weak* limit distribution of asymptotically optimal sequences of N-point configurations on A for $s \geqslant d$. This result, in particular, establishes the previously unknown value $h_{s,d}(A)$ in the case $s > d$. First, we will prove that the limit $\sigma_{s,p}$ exists for every $s > p \geqslant 1$ and that this limit is a finite and positive number. Further, we will find the limit $h_{s,p}(A)$ for $s > p \geqslant 1$ and Jordan measurable compact sets $A \subset \mathbb{R}^p$ (for $s = p$, the limit $h_{p,p}(A)$ is known from (6)). Finally, we will give conditions on the set A and weight w for which the value of $h^w_{s,d}(A)$, $s \geqslant d$, and the density of the weak* limit distribution of asymptotically optimal sequences of configurations can be explicitly written in terms of w and the constant $\sigma_{s,d}$.

The paper is structured as follows. Section 2 contains the main results of this paper: Theorems 2.1, 2.2, and 2.3 and Corollary 2.4, which immediately follows from Theorem 2.3. Section 3 compares our results with their known analogues for the minimal discrete Riesz energy. In Section 4 we establish a certain analogue of the Vitali's covering lemma which is used in the proof of Theorem 2.3. In Section 5 we prove Theorem 2.1. Section 6 establishes the subadditivity of the set function $[\underline{h}^w_{s,d}(\cdot)]^{-d/s}$, which is another key ingredient in the proof of Theorem 2.3. In Section 7 we prove sharp lower bounds in Theorem 2.2 and in equality (13) of Theorem 2.3. Section 8 contains the proof of the distribution part of Theorem 2.3. Finally, in Section 9 we establish sharp upper bounds in Theorem 2.2 and in equality (13) of Theorem 2.3.

2. Main results

2.1. **Results for compact bodies.** In this subsection we state our main theorems for compact sets $A \subset \mathbb{R}^p$ such that $\mathfrak{L}_p(A) > 0$ and $\mathfrak{L}_p(\partial A) = 0$ and the case $s \geqslant p$.

Theorem 2.1. *For $\mathscr{Q}_p = [0,1]^p$ and any $s \geqslant p$, the limit $\sigma_{s,p} = h_{s,p}(\mathscr{Q}_p)$ exists as a finite and positive number. Moreover,*

$$\sigma_{p,p} = \beta_p = \frac{\pi^{p/2}}{\Gamma(1+p/2)}.$$

The case $s = p$ of Theorem 2.1 was earlier obtained in [5]. In the case $p = 1$ and $s > 1$, Hardin, Kendall and Saff [13] proved that

$$\sigma_{s,1} = 2(2^s-1)\zeta(s), \quad s > 1,$$

where $\zeta(s)$ is the classical Riemann zeta-function. For $p=2$, we conjecture that the value of $\sigma_{s,2}$ for $s>2$ is

$$\sigma_{s,2}=\frac{3^{s/2}-1}{2}\zeta_\Lambda(s), \tag{10}$$

where

$$\zeta_\Lambda(s)=\sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}}\frac{1}{((n+m/2)^2+3m^2/4)^{s/2}}$$

is the Epstein zeta-function for the hexagonal lattice $\Lambda\subset\mathbb{R}^2$. These values for $\sigma_{s,p}$ come from the following "unbounded" analog of our problem. For an infinite configuration $\omega_\infty=\{x_1,x_2,\dots\}$ with

$$\limsup_{R\to\infty}\frac{\#(\omega_\infty\cap[-R/2,R/2]^p)}{R^p}\leqslant 1, \tag{11}$$

define

$$P_s(\omega_\infty):=\limsup_{R\to\infty}P_s([-R/2,R/2]^p;\omega_\infty\cap[-R/2,R/2]^p).$$

Further, define

$$\mathscr{P}_s(\mathbb{R}^p):=\sup P_s(\omega_\infty),$$

where the supremum is taken over all configurations ω_∞ with (11). We conjecture that for some special dimensions p, such as 2, 8 and 24 the supremum is attained when the points of ω_∞ form a lattice. For $p=1$ it is easy to see that an integer lattice $\mathbb{Z}$ gives the right constant $\sigma_{s,1}$. For $p=2$ it is conjectured that the hexagonal lattice $\Lambda=\{(n+m/2,m\sqrt{3}/2)\colon n,m\in\mathbb{Z}\}$ is optimal. Moreover, it can be shown, see [21], that for any p and any $s>p$ we have $\mathscr{P}_s(\mathbb{R}^p)=\sigma_{s,p}$. Thus, if the conjectured optimality of the hexagonal lattice Λ is true, then $\sigma_{s,2}$ for $s>2$ is given by (10).

Our next theorem extends the case $A=\mathscr{Q}_p$ to a larger family of sets.

Theorem 2.2. *Suppose $A\subset\mathbb{R}^p$ is a compact set with $\mathfrak{L}_p(\partial A)=0$. Then for any $s\geqslant p$,*

$$h_{s,p}(A)=\lim_{N\to\infty}\frac{\mathscr{P}_s(A;N)}{\tau_{s,p}(N)}=\frac{\sigma_{s,p}}{\mathfrak{L}_p(A)^{s/p}}.$$

In particular, if $\mathfrak{L}_p(A)=0$, then $h_{s,p}(A)=\infty$.

A particular case of this theorem for $A=B(0,1)$ and $s=p$ was proved by Erdélyi and Saff in [9] and later extended by Borodachov and Bosuwan in [5].

2.2. **Results for embedded compact sets.** In this section we state our most general theorem. For this purpose we need some definitions. First, we introduce some regularity properties.

Definition 2.1. A set $A\subset\mathbb{R}^p$ is called $(\mathscr{H}_d,d)$*-rectifiable*, $d\leqslant p$, if $\mathscr{H}_d(A)<\infty$ and A is the union of at most countably many images of bounded sets in $\mathbb{R}^d$ under Lipschitz maps, and a set of $\mathscr{H}_d$-measure zero.

Further, we say that A is *d-bi-Lipschitz at $x\in A$* if, for any $\varepsilon>0$, there exists a number $\delta>0$, an open set $U\subset\mathbb{R}^d$ and a function $\varphi_{x,\delta}\colon B(x,\delta)\cap A\to\mathbb{R}^d$ with $\varphi_{x,\delta}(B(x,\delta)\cap A)=U$ and

$$(1+\varepsilon)^{-1}|y-z|\leqslant|\varphi_{x,\delta}(y)-\varphi_{x,\delta}(z)|\leqslant(1+\varepsilon)|y-z|,\ \ \forall\, y,z\in B(x,\delta)\cap A.$$

By A_d we denote the set of all points $x \in A$ at which A is d-bi-Lipschitz. Further, denote $A'_d := A \setminus A_d$.

Notice that any set $A \subset \mathbb{R}^p$ is $(\mathscr{H}_p, p)$-rectifiable with $A'_p = \partial A$. We remark that any compact set A with $\mathscr{H}_d(A) < \infty$ and $\mathscr{H}_d(A'_d) = 0$ is $(\mathscr{H}_d, d)$-rectifiable. Thus, any embedded compact C^1-smooth d-dimensional manifold with $\mathscr{H}_d(\partial A) = 0$ is $(\mathscr{H}_d, d)$-rectifiable. In particular, if this manifold is closed, then $A'_d = \emptyset$. Further, a finite union of C^1-smooth arcs is a $(\mathscr{H}_1, 1)$-rectifiable set.

The following notion of Minkowski content comes from Geometric Measure Theory.

Definition 2.2. Let $A \subset \mathbb{R}^p$ be a bounded set, $A_\varepsilon := \{x \in \mathbb{R}^p \colon \operatorname{dist}(x, A) < \varepsilon\}$ and recall that β_{p-d} denotes the volume of the $(p-d)$-dimensional unit ball (if $p = d$ we formally set $\beta_0 = 1$). If the limit

$$\mathscr{M}_d(A) := \lim_{\varepsilon \to 0^+} \frac{\mathfrak{L}_p(A_\varepsilon)}{\beta_{p-d}\varepsilon^{p-d}}$$

exists, then it is called the *d-Minkowski content* of A.

The notion of Minkowski content is useful because the equality $\mathscr{H}_d(A) = \mathscr{M}_d(A)$ implies a nice behavior of discrete s-energy on A and on its compact subsets, see Theorem 3.1.

We equip the set $A \times A$, with the metric

$$\operatorname{dist}((x_1, y_1), (x_2, y_2)) = \sqrt{|x_1 - x_2|^2 + |y_1 - y_2|^2},$$

where $x_1, x_2, y_1, y_2 \in A$. Concerning the weight $w(x,y)$ we utilize the following definition from [6].

Definition 2.3. Suppose $A \subset \mathbb{R}^p$ is a compact set. We call a function $w \colon A \times A \to [0, \infty]$ a *CPD-weight on $A \times A$ with parameter d* if the following properties hold:

- (i) w is continuous (as a function on $A \times A$) at $\mathscr{H}_d$-almost every point of the diagonal $D(A) := \{(x,x) \colon x \in A\}$;
- (ii) there is a neighborhood G of $D(A)$ (relative to $A \times A$), such that $\inf_G w(x,y) > 0$;
- (iii) w is bounded on any closed subset $B \subset A \times A$ with $B \cap D(A) = \emptyset$.

Suppose A is a compact set with $\mathscr{H}_d(A) < \infty$. For a CPD-weight w on $A \times A$ with parameter d and an $\mathscr{H}_d$-measurable set $B \subset A$ define

$$\mathscr{H}_d^{s,w}(B) := \int_B w^{-d/s}(x,x) \mathrm{d}\mathscr{H}_d(x). \tag{12}$$

We are ready to state our main theorem.

Theorem 2.3. *Let d and p be positive integers with $d \leqslant p$. Suppose $A \subset \mathbb{R}^p$ is a compact set with $\mathscr{H}_d(A) = \mathscr{M}_d(A) < \infty$ and $\mathscr{H}_d(\operatorname{clos}(A'_d)) = 0$. Assume w is a CPD-weight on $A \times A$ with parameter d. Then for any $s \geqslant d$, we have*

$$h^w_{s,d}(A) = \lim_{N \to \infty} \frac{\mathscr{P}^w_s(A; N)}{\tau_{s,d}(N)} = \frac{\sigma_{s,d}}{[\mathscr{H}_d^{s,w}(A)]^{s/d}}. \tag{13}$$

Moreover, if $\mathscr{H}_d(A) > 0$, then for any asymptotically extremal family $\Omega = \{\omega_N\}_{N \geqslant 1}$ and every compact set $B \subset A$ with $\mathscr{H}_d(\partial_A B) = 0$, we have

$$\lim_{N \to \infty} \frac{\#(\omega_N \cap B)}{N} = \frac{\mathscr{H}_d^{s,w}(B)}{\mathscr{H}_d^{s,w}(A)}. \tag{14}$$

In the case $w=1$ and $s=d$, Borodachov and Bosuwan [5] proved the above theorem for sets $A=\cup_{j=1}^m A_j$, where each A_j is a compact subset of a C^1-smooth d-dimensional manifold in $\mathbb{R}^p$, with $\mathscr{H}_d(A_j\cap A_k)=0$ if $j\neq k$. Thus, for $s=d$, the novelty of Theorem 2.3 is in the presence of a *CPD* weight w.

We remark that the equality $\mathscr{H}_d(A)=\mathscr{M}_d(A)$ holds if A is a *d-rectifiable* compact set; that is, A is the image of a compact subset of $\mathbb{R}^d$ under a Lipschitz map (in particular, this equality holds if $d=p$). Moreover, if A is $(\mathscr{H}_d,d)$-rectifiable with $\mathscr{H}_d(A)=\mathscr{M}_d(A)$, then the same is true for every compact subset of A. For details, see [7, Chapter 7].

We further remark that any embedded d-dimensional compact C^1-smooth manifold A with $\mathscr{H}_d(\partial A)=0$ satisfies conditions of the theorem. Moreover, any finite union of C^1-smooth arcs also satisfies these conditions. On the other hand, a "fat" Cantor set $\mathscr{C}\subset[0,1]$ with $\mathscr{H}_1(\mathscr{C})>0$ (thus, of dimension 1) does not satisfy the condition $\mathscr{H}_1(\mathscr{C}_1')=0$.

As a corollary of Theorem 2.3 we state the result for the unit sphere $\mathbb{S}^{p-1}\subset\mathbb{R}^p$. In the case $s=p-1$ equality (15) below was originally proved by Erdérlyi and Saff, [9].

Corollary 2.4. *For a unit sphere $\mathbb{S}^{p-1}\subset\mathbb{R}^p$ and $s\geqslant p-1$, we have*

$$h_{s,p-1}(\mathbb{S}^{p-1})=\frac{\sigma_{s,p-1}}{\mathscr{H}_{p-1}(\mathbb{S}^{p-1})}=\sigma_{s,p-1}\cdot\left(\frac{2\pi^{p/2}}{\Gamma(p/2)}\right)^{-s/(p-1)}. \tag{15}$$

Furthermore, for any compact set $B\subset\mathbb{S}^{p-1}$ with $\mathscr{H}_{p-1}(\partial_{\mathbb{S}^{p-1}}B)=0$ and any asymptotically extremal family $\Omega=\{\omega_N\}_{N\geqslant1}$, we have

$$\lim_{N\to\infty}\frac{\#(\omega_N\cap B)}{N}=\frac{\mathscr{H}_{p-1}(B)}{\mathscr{H}_{p-1}(\mathbb{S}^{p-1})}=\frac{\Gamma(p/2)}{2\pi^{p/2}}\cdot\mathscr{H}_{p-1}(B). \tag{16}$$

For a unit ball $B[0,1]\subset\mathbb{R}^p$ and $s\geqslant p$, we have

$$h_{s,p}(B[0,1])=\sigma_{s,p}\cdot\left(\frac{\pi^{p/2}}{\Gamma(p/2+1)}\right)^{-s/p}. \tag{17}$$

Furthermore, for any asymptotically extremal family $\Omega=\{\omega_N\}_{N\geqslant1}$ and any compact set $B\subset B[0,1]$ with $\mathfrak{L}_p(\partial B)=0$, we have

$$\lim_{N\to\infty}\frac{\#(\omega_N\cap B)}{N}=\frac{\mathscr{H}_p(B)}{\mathscr{H}_p(B[0,1])}=\frac{\Gamma(p/2+1)}{\pi^{p/2}}\cdot\mathfrak{L}_p(B). \tag{18}$$

Finally, for any C^1-smooth curve Γ with $0<\mathscr{H}_1(\Gamma)<\infty$ and any $s\geqslant1$ we have

$$h_{s,1}(\Gamma)=2(2^s-1)\zeta(s)\cdot(\mathscr{H}_1(\Gamma))^{-s}. \tag{19}$$

3. Comparison with energy asymptotics

In this section we state for the purpose of comparison a result about the asymptotics of weighted discrete energy in the hyper-singular case $s\geqslant d$. For a compact set $A\subset\mathbb{R}^p$, weight $w\colon A\times A\to[0,\infty]$ and an integer $N\geqslant2$, define

$$\mathscr{E}_s^w(A;N):=\inf\left\{\sum_{\substack{x_i,x_j\in\omega_N\\ i\neq j}}\frac{w(x_i,x_j)}{|x_i-x_j|^s}\colon\ \omega_N\subset A,\#\omega_N=N\right\}.$$

If the weight w is identically equal to 1, we drop the superscript w. For an infinite set A, any $s>0$, and a non-negative weight w on $A\times A$ we, similarly to [9, Theorem 2.3]), obtain

$$\mathscr{P}_s^w(A;N)\geqslant\frac{\mathscr{E}_s^w(A;N)}{N-1}\quad N\geqslant 2. \tag{20}$$

The following theorem, proved by Borodachov, Hardin and Saff, [6, 7], describes the asymptotic behavior of $\mathscr{E}_s^w(A;N)$.

Theorem 3.1. *Let d and p be positive integers with $d\leqslant p$. Suppose $A\subset\mathbb{R}^p$ is a compact $(\mathscr{H}_d,d)$-rectifiable set with $\mathscr{M}_d(A)=\mathscr{H}_d(A)$ and w is a CPD-weight on $A\times A$ with parameter d. If $s>d$, then for any compact set $B\subset A$,*

$$\lim_{N\to\infty}\frac{\mathscr{E}_s^w(B;N)}{N\tau_{s,d}(N)}=\frac{C_{s,d}}{[\mathscr{H}_d^{s,w}(B)]^{s/d}},$$

where $C_{s,d}$ is a finite positive constant that depends only on s and d. If A is a compact subset of a d-dimensional C^1-smooth manifold, then for any compact set $B\subset A$,

$$\lim_{N\to\infty}\frac{\mathscr{E}_d^w(B;N)}{N\tau_{d,d}(N)}=\frac{\beta_d}{\mathscr{H}_d^{d,w}(B)},$$

where β_d is the Lebesgue measure of the d-dimensional unit ball.

In particular, if $d=p$ and $A\subset\mathbb{R}^p$ is a compact set with $\mathfrak{L}_p(A)=0$, then both limits above are equal to ∞.

The following corollary of Theorem 3.1 proves a particular case of Theorem 2.3.

Corollary 3.2. *Suppose $A\subset\mathbb{R}^p$ is a compact set with $\mathscr{H}_d(A)=\mathscr{M}_d(A)=0$. Assume w is a CPD-weight on A with parameter d. Then*

$$h_{s,d}^w(A)=\lim_{N\to\infty}\frac{\mathscr{P}_s^w(A;N)}{\tau_{s,d}(N)}=\infty.$$

Proof. If A is a finite set of cardinality N_0, then for any $N>N_0$ we have $\mathscr{P}_s^w(A;N)=\infty$. Thus, $h_{s,p}^w(A)=\infty$. For an infinite set A we divide (20) by $\tau_{s,p}(N)$ and pass to $\liminf$ as $N\to\infty$. From Theorem 3.1 we obtain

$$\underline{h}_{s,p}^w(A)\geqslant\lim_{N\to\infty}\frac{\mathscr{E}_s^w(A;N)}{(N-1)\tau_{s,p}(N)}=\infty,$$

which completes the proof of the corollary. □

4. A COVERING CONSTRUCTION

In this section we prove the existence of a sufficiently regular covering for subsets of A_d. Namely, we prove the following lemma.

Lemma 4.1. *Let ε and γ be positive numbers, $A\subset\mathbb{R}^p$ be a compact set with $\mathscr{H}_d(A)<\infty$, and $B\subset A\setminus\operatorname{clos}(A_d')$ be a nonempty set open relative to A. Then there exists a finite collection $\mathscr{Q}_{\varepsilon,\gamma}=\{Q_\alpha\}$ of sets with the following properties:*

(i) Sets Q_α are pairwise disjoint, for any α we have $Q_\alpha\subset B$ and

$$\mathscr{H}_d\left(B\setminus\bigcup_\alpha Q_\alpha\right)<\gamma;$$

(ii) For any α, there is a "center" $x_\alpha \in Q_\alpha$, and a number $\rho_\alpha \leqslant \min\left\{\frac{\delta_\alpha}{10\cdot(1+\varepsilon)}, \varepsilon\right\}$ such that

$$Q_\alpha = \varphi^{-1}_{x_\alpha,\delta_\alpha}(B[\varphi_{x_\alpha,\delta_\alpha}(x_\alpha), \rho_\alpha]),$$

where δ_α and $\varphi_{x_\alpha,\delta_\alpha}$ are from Definition 2.1.

Proof. First, we need to introduce some notation. Since $B \subset A_d$, we know that for any $x \in B$ there is a number $\delta = \delta(x,\varepsilon)$ and a bi-Lipschitz function $\varphi_{x,\delta} \colon B(x,\delta) \cap A \to \mathbb{R}^d$ with Lipschitz constants $(1+\varepsilon)^{-1}$ and $1+\varepsilon$, such that $U_x := \varphi_{x,\delta}(B(x,\delta)\cap A)$ is an open set. Thus, we can find a number $r = r(x) > 0$, such that $B[\varphi_{x,\delta}(x), r] \subset U_x$ and $\varphi^{-1}_{x,\delta}(B[\varphi_{x,\delta}(x), r]) \subset B$. Since the Lipschitz constant of $\varphi_{x,\delta}$ is less than $1+\varepsilon$, we deduce that $r < \delta\cdot(1+\varepsilon)$. For any positive number $\rho \leqslant r$, define

$$Q_{x,\rho} := \varphi^{-1}_{x,\delta}(B[\varphi_{x,\delta}(x), \rho]) \tag{21}$$

and

$$V_\varepsilon(B) := \left\{ Q_{x,\rho} \colon 0 < \rho \leqslant \min\left\{\frac{r(x)}{10(1+\varepsilon)^2}, \varepsilon\right\},\ \ x \in B \right\}.$$

Without loss of generality, we assume that B cannot be covered by a finite disjoint collection of sets from $V_\varepsilon(B)$. We now choose the sets Q_n inductively. First, take any set $Q_1 = Q_{x_1,\rho_1} \in V_\varepsilon(B)$. Suppose the sets $Q_1, \dots, Q_{n-1}$ are chosen. By our assumption, $B \neq \cup_{i=1}^{n-1} Q_i$. Since the sets Q_i are closed, there is a point $y \in B$ with $\operatorname{dist}(y, \cup_{i=1}^{n-1} Q_i) > 0$. Therefore, for some small number ρ, we should have $Q_{y,\rho} \cap Q_i = \emptyset$ for any $i = 1, \dots, n-1$. Define

$$a_n := \sup\{\rho \colon \text{ for some } y \in B,\ Q_{y,\rho} \in V_\varepsilon(B) \text{ and } Q_{y,\rho} \cap Q_i = \emptyset \text{ for any } i = 1, \dots, n-1\}.$$

Now take $Q_n := Q_{x_n,\rho_n} \subset V_\varepsilon(B)$ such that the sets $Q_1, \dots, Q_n$ are pairwise disjoint and $\rho_n > a_n/2$. We immediately deduce that

$$\sum_{i=1}^{\infty} \mathscr{H}_d(Q_i) = \mathscr{H}_d(\cup_{i=1}^{\infty} Q_i) \leqslant \mathscr{H}_d(B) \leqslant \mathscr{H}_d(A) < \infty.$$

Therefore, $\mathscr{H}_d(Q_n) \to 0$ as $n \to \infty$ and for some large number n, we have

$$\sum_{i=n+1}^{\infty} \mathscr{H}_d(Q_i) < \frac{\gamma}{10^d(1+\varepsilon)^{4d}}.$$

Moreover, since $\mathscr{H}_d(Q_n) \geqslant (1+\varepsilon)^{-d}\beta_d\rho_n^d$, we obtain that $\rho_n \to 0$ as $n \to \infty$. It remains to show that $\mathscr{H}_d\left(B \setminus (\bigcup_{i=1}^n Q_i)\right) < \gamma$. Suppose $y \in B \setminus (\bigcup_{i=1}^n Q_i)$. Then there exists a positive number ρ such that $Q := Q_{y,\rho}$ is disjoint from Q_i for any $i = 1, \dots, n$. Since $\rho_n \to 0$ as $n \to \infty$, the set Q cannot be disjoint from every Q_i, $i = 1, \dots, \infty$. Thus, we can choose the smallest number $m > n$, such that $Q \cap Q_m \neq \emptyset$. Since Q is disjoint from Q_i for any $i = 1, \dots, m-1$, we obtain $\rho \leqslant a_m < 2\rho_m$. Fix a point $z \in Q \cap Q_m$. We have

$$\begin{aligned} |y - x_m| &\leqslant |y - z| + |z - x_m| \\ &\leqslant (1+\varepsilon)\left(|\varphi_{y,\delta_y}(y) - \varphi_{y,\delta_y}(z)| + |\varphi_{x_m,\delta_{x_m}}(z) - \varphi_{x_m,\delta_{x_m}}(x_m)|\right) \\ &\leqslant (1+\varepsilon)(\rho + \rho_m) < 3(1+\varepsilon)\rho_m. \end{aligned} \tag{22}$$

Since $Q_m \in V_\varepsilon(B)$, we deduce that $\rho_m \leqslant r(x_m)/(10(1+\varepsilon)^2)$, thus $|y-x_m| < r(x_m)/(1+\varepsilon) < \delta_m$. Therefore, the point y is in the domain of φ_{x_m,δ_m}. Thus,

$$|\varphi_{x_m,\delta_m}(y) - \varphi_{x_m,\delta_m}(x_m)| \leqslant (1+\varepsilon)|y-x_m| \leqslant 3(1+\varepsilon)^2\rho_m,$$

so $y \in Q_{x_m,3(1+\varepsilon)^2\rho_m}$. Therefore,

$$B\setminus\bigcup_{i=1}^{n} Q_i \subset \bigcup_{m=n+1}^{\infty} Q_{x_m,3(1+\varepsilon)^2\rho_m}$$

and

$$\mathscr{H}_d\left(B\setminus\bigcup_{i=1}^{n} Q_i\right) \leqslant \sum_{m=n+1}^{\infty} \mathscr{H}_d(Q_{x_m,3(1+\varepsilon)^2\rho_m}) \tag{23}$$
$$\leqslant (1+\varepsilon)^d\beta_d \sum_{m=n+1}^{\infty} 3^d(1+\varepsilon)^{2d}\rho_m^d \leqslant 3^d(1+\varepsilon)^{4d} \sum_{m=n+1}^{\infty} \mathscr{H}_d(Q_{x_m,\rho_m}) < \gamma,$$

which completes our proof.

□

5. Proof of Theorem 2.1

In this section we prove that the limit $h_{s,p}(\mathscr{Q}_p)$ exists for any $s>p$ and that $\sigma_{s,p} = h_{s,p}(\mathscr{Q}_p)$ is a positive finite number. For the case $s=p$, the theorem was proved by Borodachov and Bosuwan [5] using a different method. Our proof for $s>p$ utilizes an argument similar to the one used in [14].

Fix a positive number ε. For an integer N take a configuration $\omega_N \subset \mathscr{Q}_p$ such that

$$P_s(\mathscr{Q}_p;\omega_N) \geqslant \mathscr{P}_s(\mathscr{Q}_p;N) - \varepsilon.$$

Fix a positive integer m. For a vector $\mathbf{j} = (j_1, j_2, \dots, j_p) \in \mathbb{Z}^p$ with $0 \leqslant j_k \leqslant m-1$, define

$$Q^{\mathbf{j}} := \left[\frac{j_1}{m}, \frac{j_1+1}{m}\right] \times \dots \times \left[\frac{j_p}{m}, \frac{j_p+1}{m}\right] = \frac{1}{m}(\mathscr{Q}_p + \mathbf{j}).$$

Further set

$$\omega_N^{\mathbf{j}} := \frac{1}{m}(\omega_N + \mathbf{j}) \subset Q^{\mathbf{j}}.$$

Finally, define $\overline{\omega}_{m^pN} := \bigcup_{\mathbf{j}} \omega_N^{\mathbf{j}} \subset \mathscr{Q}_p$. We obtain

$$\mathscr{P}_s(\mathscr{Q}_p;m^pN) \geqslant P_s(\mathscr{Q}_p;\overline{\omega}_{m^pN}) = \min_{\mathbf{j}} \inf_{y\in Q^{\mathbf{j}}} \sum_{x\in\overline{\omega}_{m^pN}} \frac{1}{|y-x|^s} \geqslant \tag{24}$$
$$\min_{\mathbf{j}} \inf_{y\in Q^{\mathbf{j}}} \sum_{x\in\omega_N^{\mathbf{j}}} \frac{1}{|y-x|^s} = \min_{\mathbf{j}} P_s(Q^{\mathbf{j}};\omega_N^{\mathbf{j}}).$$

Since all cubes $Q^{\mathbf{j}}$ and configurations $\omega_N^{\mathbf{j}}$ can be obtained from each other by a shift, the value of $P_s(Q^{\mathbf{j}};\omega_N^{\mathbf{j}})$ does not depend on $\mathbf{j}$. Thus, we can take $\mathbf{j} = \mathbf{0} = (0,\dots,0)$ and obtain

$$\mathscr{P}_s(\mathscr{Q}_p;m^pN) \geqslant \inf_{y\in Q^{\mathbf{0}}} \sum_{x\in\omega_N^{\mathbf{0}}} \frac{1}{|y-x|^s}.$$

Since $\mathscr{Q}_p = mQ^{\mathbf{0}}$, we have

$$\mathscr{P}_s(\mathscr{Q}_p; m^p N) \geqslant m^s \inf_{y\in\mathscr{Q}_p} \sum_{x\in\omega_N} \frac{1}{|y-x|^s} = m^s P_s(\mathscr{Q}_p;\omega_N) \geqslant m^s(\mathscr{P}_s(\mathscr{Q}_p;N)-\varepsilon).$$

In view of arbitrariness of ε, we obtain

$$\mathscr{P}_s(\mathscr{Q}_p; m^p N) \geqslant m^s \mathscr{P}_s(\mathscr{Q}_p;N).$$

From inequality (5) we have $\overline{h}_{s,p}(\mathscr{Q}_p) < \infty$. We now fix a positive number $\varepsilon > 0$ and a positive integer N_0 with

$$\frac{\mathscr{P}_s(\mathscr{Q}_p;N_0)}{N_0^{s/p}} > \overline{h}_{s,p}(\mathscr{Q}_p) - \varepsilon.$$

For any $N > N_0$ choose the non-negative integer m_N, such that $m_N^p N_0 \leqslant N < (m_N+1)^p N_0$. If N_0 is fixed and $N \to \infty$, then $m_N \to \infty$. Moreover,

$$\overline{h}_{s,p}(\mathscr{Q}_p) < \frac{\mathscr{P}_s(\mathscr{Q}_p;N_0)}{N_0^{s/p}} + \varepsilon = \frac{m_N^s \mathscr{P}_s(\mathscr{Q}_p;N_0)}{m_N^s N_0^{s/p}} + \varepsilon \leqslant \frac{\mathscr{P}_s(\mathscr{Q}_p; m_N^p N_0)}{m_N^s N_0^{s/p}} + \varepsilon.$$

Notice that inequality $m_N^p N_0 \leqslant N$ implies $\mathscr{P}_s(\mathscr{Q}_p; m_N^p N_0) \leqslant \mathscr{P}_s(\mathscr{Q}_p;N)$. Therefore,

$$\overline{h}_{s,p}(\mathscr{Q}_p) < \frac{\mathscr{P}_s(\mathscr{Q}_p;N)}{N^{s/p}} \cdot \left(\frac{m_N+1}{m_N}\right)^s + \varepsilon.$$

Since this inequality holds for any $N > N_0$, we can pass to lim inf as $N \to \infty$. We obtain

$$\overline{h}_{s,p}(\mathscr{Q}_p) \leqslant \underline{h}_{s,p}(\mathscr{Q}_p) + \varepsilon.$$

In view of arbitrariness of ε, we obtain that $\sigma_{s,p} = h_{s,p}(\mathscr{Q}_p)$ exists as a finite real number. Inequality (20) together with Theorem 3.1 (which deals with the energy) imply that $0 < \sigma_{s,p} < \infty$. □

We remark that to prove the positivity of $\sigma_{s,p}$ we do not need to use Theorem 3.1. If we break the cube $\mathscr{Q}_p$ into $N = n^p$ equal subcubes and let ω_N be the configuration consisting of the centers of these cubes, then $P_s(\mathscr{Q}_p;\omega_N)$ will have order $N^{s/d}$ as $N \to \infty$.

6. Sub-additivity of $[\underline{h}^w_{s,d}(\cdot)]^{-d/s}$

In this section we prove the following lemma, which will play an important role in the proof of the lower bound in (13), see Lemmas 7.1 and 7.2.

Lemma 6.1. *Suppose B and C are subsets of $\mathbb{R}^p$, and $w\colon (B\cup C)\times(B\cup C) \to [0,\infty]$. Then for any positive $d \leqslant p$ and any $s \geqslant d$, we have*

$$\underline{h}^w_{s,d}(B\cup C)^{-d/s} \leqslant \underline{h}^w_{s,d}(B)^{-d/s} + \underline{h}^w_{s,d}(C)^{-d/s}. \tag{25}$$

Proof. Fix three positive integer numbers N, N_1, N_2 with $N_1+N_2 = N$. For an arbitrary $\varepsilon > 0$, take configurations $\omega_{N_1} \subset B$ and $\omega_{N_2} \subset C$ with

$$P^w_s(B;\omega_{N_1}) > \mathscr{P}^w_s(B;N_1) - \varepsilon, \quad P^w_s(C;\omega_{N_2}) > \mathscr{P}^w_s(C;N_2) - \varepsilon.$$

Define $\omega_N := \omega_{N_1} \cup \omega_{N_2}$. Then

$$\mathscr{P}_s^w(B\cup C;N) \geqslant P_s^w(B\cup C;\omega_N) = \min\left\{\inf_{y\in B}\sum_{x\in\omega_N}\frac{w(y,x)}{|y-x|^s}, \inf_{y\in C}\sum_{x\in\omega_N}\frac{w(y,x)}{|y-x|^s}\right\} \tag{26}$$

$$\geqslant \min\left\{\inf_{y\in B}\sum_{x\in\omega_{N_1}}\frac{w(y,x)}{|y-x|^s}, \inf_{y\in C}\sum_{x\in\omega_{N_2}}\frac{w(y,x)}{|y-x|^s}\right\} \geqslant \min\{\mathscr{P}_s^w(B;N_1), \mathscr{P}_s^w(C;N_2)\} - \varepsilon.$$

In view of arbitrariness of ε we obtain that

$$\mathscr{P}_s^w(B\cup C;N) \geqslant \min\{\mathscr{P}_s^w(B;N_1), \mathscr{P}_s^w(C;N_2)\}. \tag{27}$$

Notice that if $\underline{h}_{s,d}^w(B) = 0$ or $\underline{h}_{s,d}^w(C) = 0$ then lemma holds trivially. Thus, we can assume that both $\underline{h}_{s,d}^w(B)$ and $\underline{h}_{s,d}^w(C)$ are positive. We now assign particular values to N_1 and N_2. For a fixed number α with $0 < \alpha < 1$, define $N_1 := \lfloor \alpha N \rfloor$ and $N_2 := N - N_1$. Since $\alpha \in (0,1)$, we have $N_1 \to \infty$ and $N_2 \to \infty$ as $N \to \infty$. Estimate (27) implies

$$\frac{\mathscr{P}_s^w(B\cup C;N)}{\tau_{s,d}(N)} \geqslant \min\left\{\frac{\mathscr{P}_s^w(B;N_1)}{\tau_{s,d}(N_1)}\cdot\frac{\tau_{s,d}(N_1)}{\tau_{s,d}(N)}, \frac{\mathscr{P}_s^w(C;N_2)}{\tau_{s,d}(N_2)}\cdot\frac{\tau_{s,d}(N_2)}{\tau_{s,d}(N)}\right\},$$

which, together with

$$\lim_{N\to\infty}\frac{\tau_{s,d}(N_1)}{\tau_{s,d}(N)} = \alpha^{s/d}, \quad \lim_{N\to\infty}\frac{\tau_{s,d}(N_2)}{\tau_{s,d}(N)} = (1-\alpha)^{s/d}, \quad s \geqslant d,$$

yields

$$\underline{h}_{s,d}^w(B\cup C) \geqslant \min\left\{\alpha^{s/d}\underline{h}_{s,d}^w(B), (1-\alpha)^{s/d}\underline{h}_{s,d}^w(C)\right\} \quad \text{for any } \alpha \in (0,1). \tag{28}$$

If $\underline{h}_{s,d}^w(B) = \underline{h}_{s,d}^w(C) = \infty$ then the right-hand side of (28) is equal to ∞, and the lemma holds trivially. If $\underline{h}_{s,d}^w(B) < \infty$ and $\underline{h}_{s,d}^w(C) = \infty$ then the right-hand side of (28) is equal to $\alpha^{s/d}\underline{h}_{s,d}^w(B)$. Letting α go to 1 we obtain the lemma. The case $\underline{h}_{s,d}^w(B) = \infty$ and $\underline{h}_{s,d}^w(C) < \infty$ is treated similarly.

If both $\underline{h}_{s,d}^w(B)$ and $\underline{h}_{s,d}^w(C)$ are positive and finite, then we set

$$\alpha := \frac{\underline{h}_{s,d}^w(C)^{d/s}}{\underline{h}_{s,d}^w(B)^{d/s} + \underline{h}_{s,d}^w(C)^{d/s}} \in (0,1).$$

This choice of α together with inequality (28) implies the estimate (25). □

7. An estimate of $\underline{h}_{s,d}^w(A)$ from below

In this section we prove important corollaries of Lemma 6.1. We start with the unweighted case (i.e., $w = 1$) for $d = p$.

Lemma 7.1. *Suppose $A \subset \mathbb{R}^p$ is a compact set with $\mathfrak{L}_p(\partial A) = 0$. Then for any $s \geqslant p$,*

$$\underline{h}_{s,p}(A) \geqslant \frac{\sigma_{s,p}}{\mathfrak{L}_p(A)^{s/p}}. \tag{29}$$

Proof. If $\mathfrak{L}_p(A) = 0$ then the lemma follows from Corollary 3.2. Thus, we assume $\mathfrak{L}_p(A) > 0$.

Fix a number $\varepsilon > 0$. Our assumptions on the set A imply that there exists a finite family $\mathscr{D} = \{Q_i\}$ of closed cubes with disjoint interiors, such that $Q_i \subset A$ and

$$\mathfrak{L}_p\left(A \setminus \cup_i Q_i\right) < \varepsilon.$$

Denote $D := A \setminus \cup_i Q_i$. Since $\mathfrak{L}_p(\partial A) = 0$, we also get $\mathfrak{L}_p(\partial D) = 0$. Thus, $\mathfrak{L}_p(\mathrm{clos}(D)) = \mathfrak{L}_p(D) < \varepsilon$. From inequality (20) and Theorem 3.1 we obtain

$$\underline{h}_{s,p}(\mathrm{clos}(D)) \geqslant \lim_{N\to\infty} \frac{\mathscr{E}_s(\mathrm{clos}(D);N)}{(N-1)\tau_{s,p}(N)} \geqslant C_{s,p}\varepsilon^{-s/p}.$$

Further, inequality (25) yields

$$\underline{h}_{s,p}(A)^{-p/s} \leqslant \sum_i \underline{h}_{s,p}(Q_i)^{-p/s} + \underline{h}_{s,p}(\mathrm{clos}(D))^{-p/s} \leqslant \sum_i \underline{h}_{s,p}(Q_i)^{-p/s} + C_{s,p}^{-p/s}\varepsilon.$$

Theorem 2.1 implies that $\underline{h}_{s,p}(Q_i) = \sigma_{s,p}\mathfrak{L}_p(Q_i)^{-s/p}$. Thus,

$$\begin{aligned}\underline{h}_{s,p}(A)^{-p/s} &\leqslant \sum_i \sigma_{s,p}^{-p/s}\mathfrak{L}_p(Q_i) + C_{s,p}^{-p/s}\varepsilon \\ &= \sigma_{s,p}^{-p/s}\mathfrak{L}_p(\cup_i Q_i) + C_{s,p}^{-p/s}\varepsilon \leqslant \sigma_{s,p}^{-p/s}\mathfrak{L}_p(A) + C_{s,p}^{-p/s}\varepsilon.\end{aligned}$$

Since ε can be arbitrarily small, estimate (29) follows. □

Next, we deduce a general estimate for $\underline{h}^w_{s,d}$. Namely, we prove the following lemma.

Lemma 7.2. *Suppose $d, p \in \mathbb{N}$, $d \leqslant p$, $A \subset \mathbb{R}^p$ is a compact set with $\mathscr{H}_d(A) = \mathscr{M}_d(A) < \infty$ and $\mathscr{H}_d(\mathrm{clos}(A'_d)) = 0$. Suppose w is a CPD weight on $A \times A$ with parameter d. Then for any $s \geqslant d$, we have*

$$\underline{h}^w_{s,d}(A) \geqslant \frac{\sigma_{s,d}}{\mathscr{H}_d^{s,w}(A)^{s/d}}. \tag{30}$$

Proof. Denote $B := A \setminus \mathrm{clos}(A'_d)$. Then B is a subset of A_d open relative to A. By assumption, $\mathrm{clos}(A'_d)$ is a compact subset of A of zero $\mathscr{H}_d$-measure. Then taking into account (20) we obtain

$$\underline{h}^w_{s,d}(\mathrm{clos}(A'_d)) \geqslant \lim_{N\to\infty} \frac{\mathscr{E}^w_s(\mathrm{clos}(A'_d);N)}{(N-1)\tau_{s,d}(N)} = C_{s,d}[\mathscr{H}_d^{s,w}(\mathrm{clos}(A'_d))]^{-s/d} = \infty. \tag{31}$$

Fix a number $\varepsilon > 0$, set $\gamma = \varepsilon$ and take the finite collection $\mathscr{Q}_{\varepsilon,\varepsilon}$ of disjoint sets $\{Q_\alpha\}$ from Lemma 4.1. Denote $D := B \setminus \cup_\alpha Q_\alpha$. Since $\mathrm{clos}(D)$ is a compact subset of A, we have

$$\underline{h}^w_{s,d}(\mathrm{clos}(D)) \geqslant \lim_{N\to\infty} \frac{\mathscr{E}^w_s(\mathrm{clos}(D);N)}{(N-1)\tau_{s,d}(N)} = C_{s,d}[\mathscr{H}_d^{s,w}(\mathrm{clos}(D))]^{-s/d}. \tag{32}$$

Next, we will estimate $\underline{h}^w_{s,d}(Q_\alpha)$ for each α. Denote

$$\tilde{Q}_\alpha := \varphi_{x_\alpha,\delta_\alpha}(Q_\alpha) = B[\varphi_{x_\alpha,\delta_\alpha}(x_\alpha), \rho_\alpha]$$

and take any N-point configuration $\tilde{\omega}_N = \{\tilde{x}_1, \ldots, \tilde{x}_N\} \subset \tilde{Q}_\alpha$. For any $j = 1, \ldots, N$, denote $x_j := \varphi^{-1}_{x_\alpha,\delta_\alpha}(\tilde{x}_j)$ and set $\omega_N := \{x_1, \ldots, x_N\} \subset Q_\alpha$. Further, denote

$$\underline{w}_{Q_\alpha} := \inf_{(x,y)\in Q_\alpha\times Q_\alpha} w(x,y).$$

Since the function $\varphi_{x_\alpha,\delta_\alpha}$ is Lipschitz on Q_α with Lipschitz constant $1+\varepsilon$, we have

$$\begin{aligned}(33)\quad \mathscr{P}_s^w(Q_\alpha;N) &\geqslant P_s^w(Q_\alpha;\omega_N) = \inf_{y\in Q_\alpha}\sum_{x\in\omega_N}\frac{w(y,x)}{|y-x|^s}\\ &\geqslant \underline{w}_{Q_\alpha}\inf_{\tilde{y}\in\tilde{Q}_\alpha}\sum_{\tilde{x}\in\tilde{\omega}_N}\frac{1}{|\varphi^{-1}_{x_\alpha,\delta_\alpha}(\tilde{y})-\varphi^{-1}_{x_\alpha,\delta_\alpha}(\tilde{x})|^s}\\ &\geqslant (1+\varepsilon)^{-s}\underline{w}_{Q_\alpha}\inf_{\tilde{y}\in\tilde{Q}_\alpha}\sum_{\tilde{x}\in\tilde{\omega}_N}\frac{1}{|\tilde{y}-\tilde{x}|^s} = (1+\varepsilon)^{-s}\underline{w}_{Q_\alpha}P_s(\tilde{Q}_\alpha;\tilde{\omega}_N).\end{aligned}$$

Therefore,

$$\mathscr{P}_s^w(Q_\alpha;N)\geqslant(1+\varepsilon)^{-s}\underline{w}_{Q_\alpha}\mathscr{P}_s(\tilde{Q}_\alpha;N).$$

Dividing by $\tau_{s,d}(N)$ and passing to $\liminf$ as $N\to\infty$, we obtain from Lemma 7.1

$$\begin{aligned}(34)\quad \underline{h}^w_{s,d}(Q_\alpha) &\geqslant (1+\varepsilon)^{-s}\underline{w}_{Q_\alpha}\underline{h}_{s,d}(\tilde{Q}_\alpha)\geqslant(1+\varepsilon)^{-s}\underline{w}_{Q_\alpha}\sigma_{s,d}\mathfrak{L}_d(\tilde{Q}_\alpha)^{-s/d}\\ &\geqslant(1+\varepsilon)^{-2s}\sigma_{s,d}\underline{w}_{Q_\alpha}\mathscr{H}_d(Q_\alpha)^{-s/d}.\end{aligned}$$

Finally, we apply Lemma 6.1 to $A=\operatorname{clos}(A'_d)\cup\operatorname{clos}(D)\cup\bigcup_\alpha Q_\alpha$. Combining (31), (32) and (34), we obtain

$$\begin{aligned}(35)\quad [\underline{h}^w_{s,d}(A)]^{-d/s} &\leqslant [\underline{h}^w_{s,d}(\operatorname{clos}(A'_d))]^{-d/s}+[\underline{h}^w_{s,d}(\operatorname{clos}(D))]^{-d/s}+\sum_\alpha[\underline{h}^w_{s,d}(Q_\alpha)]^{-d/s}\\ &\leqslant C_{s,d}^{-d/s}\mathscr{H}_d^{s,w}(\operatorname{clos}(D))+(1+\varepsilon)^{2d}\sigma_{s,d}^{-d/s}\sum_\alpha\underline{w}_{Q_\alpha}^{-d/s}\mathscr{H}_d(Q_\alpha).\end{aligned}$$

Define a function

$$\underline{w}_\varepsilon(x):=\begin{cases}\underline{w}_{Q_\alpha}^{-d/s}, & x\in Q_\alpha \text{ for some } \alpha,\\ 0, & x\notin\cup_\alpha Q_\alpha.\end{cases}$$

Then (35) implies

$$[\underline{h}^w_{s,d}(A)]^{-d/s}\leqslant C_{s,d}^{-d/s}\mathscr{H}_d^{s,w}(\operatorname{clos}(D))+(1+\varepsilon)^{2d}\sigma_{s,d}^{-d/s}\int_A\underline{w}_\varepsilon(x)d\mathscr{H}_d(x).$$

Observe that $\mathscr{H}_d(\partial_A Q_\alpha)=0$ for every α and that the set $J:=A\setminus(\cup_\alpha\operatorname{int}_A Q_\alpha)$ is closed, where $\operatorname{int}_A Q_\alpha$ is the interior of Q_α relative to A. Recall also that $Q_\alpha\subset B$ for all α and that the sets Q_α are pairwise disjoint. Then

$$D\subset\operatorname{clos}(D)\subset D\cup\operatorname{clos}(A'_d)\cup(\cup_\alpha\partial_A Q_\alpha)=J.$$

Consequently, $\mathscr{H}_d(\operatorname{clos}(D))=\mathscr{H}_d(D)<\gamma=\varepsilon$. Then $\mathscr{H}_d^{s,w}(\operatorname{clos}(D))\to0$ as $\varepsilon\to0$. Since $\operatorname{diam}(Q_\alpha)\leqslant2\varepsilon(1+\varepsilon)$ for all α, for every $\varepsilon>0$ sufficiently small, we have $Q_\alpha\times Q_\alpha\subset G$ for every α, where the set G a neighborhood of $D(A)$ relative to $A\times A$ such that $a:=\inf_G w>0$, see Definition 2.3.

Denote $\varepsilon_k:=\varepsilon2^{-k}$, $k\in\mathbb{N}$. Then letting $k\to\infty$, we have

$$(36)\qquad \left[\underline{h}^w_{s,d}(A)\right]^{-d/s}\leqslant\sigma_{s,d}^{-d/s}\limsup_{k\to\infty}\int_A\underline{w}_{\varepsilon_k}(x)\,d\mathscr{H}_d(x).$$

For every $k\in\mathbb{N}$, let $\{Q_\alpha^k\}$ be the family of sets Q_α constructed for $\varepsilon=\varepsilon_k$. Denote by M the set of all points $y\in A$ such that w is continuous at (y,y) (as a function on $A\times A$) and $y\in\cup_\alpha Q_\alpha^k$ for every $k\in\mathbb{N}$. Then $\mathscr{H}_d(A\setminus M)\leqslant\sum_{k=1}^\infty\varepsilon_k=\varepsilon$.

Let $x \in M$ be arbitrary. Since $w^{-d/s}$ is continuous at (x,x), there is $\tau > 0$ such that whenever $(y,z) \in A \times A$ and $\operatorname{dist}((x,x),(y,z)) < \tau$, we have $w(x,x)^{-d/s} > w(y,z)^{-d/s} - \varepsilon$. Let $k_\varepsilon \in \mathbb{N}$ be such that for all $k \geqslant k_\varepsilon$, $\operatorname{diam} Q_\alpha^k = 2\varepsilon_k(\varepsilon_k + 1) < \tau/\sqrt{2}$ and $Q_\alpha^k \times Q_\alpha^k \subset G$ for all α. If α_1 is such that $x \in Q_{\alpha_1}^k$, then for $k \geqslant k_\varepsilon$,

$$w(x,x)^{-d/s} \geqslant \underline{w}_{Q_{\alpha_1}^k}^{-d/s} - \varepsilon = \underline{w}_{\varepsilon_k}(x) - \varepsilon.$$

Consequently, $\limsup\limits_{k\to\infty} \underline{w}_{\varepsilon_k}(x) \leqslant w(x,x)^{-d/s} + \varepsilon$. Then in view of the Fatou's lemma, from (36) we have

$$\begin{aligned}
\left[\underline{h}_{s,d}^w(A)\right]^{-d/s} &\leqslant \sigma_{s,d}^{-d/s} \int_A \limsup_{k\to\infty} \underline{w}_{\varepsilon_k}(x)\, d\mathscr{H}_d(x) \\
&\leqslant \sigma_{s,d}^{-d/s} \left(\int_M \left(w(x,x)^{-d/s} + \varepsilon \right) d\mathscr{H}_d(x) + a^{-d/s} \mathscr{H}_d(A \setminus M) \right) \\
&\leqslant \sigma_{s,d}^{-d/s} \left(\mathscr{H}_d^{s,w}(A) + \varepsilon \mathscr{H}_d(A) + \varepsilon a^{-d/s} \right).
\end{aligned}$$

In view of arbitrariness of ε, we obtain (30). $\square$

8. Limit distribution of asymptotically extremal configurations

In this section we study how point configurations from asymptotically extremal sequences $\{\omega_N\}_{N\geqslant 1}$ for weighted polarization $\mathscr{P}_s^w(A;N)$ are distributed over the set A.

Throughout this section, the set A satisfies the assumptions of Theorem 2.3 (including the assumption that $\mathscr{H}_d(A) > 0$). Fix a relatively open subset $B \subset A$; fix numbers $\varepsilon > 0$ and $\gamma > 0$, and the family $\mathscr{D}_{\varepsilon,\gamma}$ from Lemma 4.1. Fix a sequence $\{\omega_N\}_{N\geqslant 1}$ of asymptotically extreme configurations (see Definition 1.1).

We start with the following lemma.

Lemma 8.1. *For a fixed $Q_\alpha \in \mathscr{D}_{\varepsilon,\gamma}$ let $\tilde{Q}_\alpha = \varphi_{x_\alpha,\delta_\alpha}(Q_\alpha) = B[\varphi_{x_\alpha,\delta_\alpha}(x_\alpha), \rho_\alpha] \subset \mathbb{R}^d$. Suppose $\tilde{\Gamma} \subset \tilde{Q}_\alpha$ is a d-dimensional open cube. Denote $\Gamma := \varphi_{x_\alpha,\delta_\alpha}^{-1}(\tilde{\Gamma})$ and $N_\Gamma := \#(\omega_N \cap \Gamma)$. Then $N_\Gamma \to \infty$ as $N \to \infty$.*

Proof. Suppose there is an unbounded set $\mathscr{N}$ of positive integer numbers such that N_Γ are uniformly bounded from above when $N \in \mathscr{N}$. Since $\varphi_{x_\alpha,\delta_\alpha}$ is a bi-Lipschitz function, there is a positive number a_0 (that does not depend on N) and a point $z \in A$ (that can depend on N) such that $B(z,a_0) \cap A \subset \Gamma$ and $B(z,a_0) \cap \omega_N = \emptyset$. Therefore, $|z - x| \geqslant a_0$ for any $x \in \omega_N$. Since the set $F := \operatorname{clos}(\cup_{N\in\mathscr{N}}\{(z,x) \colon x \in \omega_N\})$ is a closed subset of $A \times A$ with $F \cap D(A) = \emptyset$, we conclude that the weight w is bounded from above on F. Then for some constant C and any large enough $N \in \mathscr{N}$ we have

$$P_s^w(A;\omega_N) \leqslant \sum_{x\in\omega_N} \frac{w(z,x)}{|z-x|^s} \leqslant C \cdot N.$$

Since ω_N is asymptotically extreme, we have $\underline{h}_{s,d}^w(A) = 0$, which contradicts the fact that $\underline{h}_{s,d}^w(A) > 0$ established in Lemma 7.2. Then $N_\Gamma \to \infty$ as $N \to \infty$. $\square$

Next lemma makes the asymptotic behavior of N_Γ more precise.

Lemma 8.2. *Assume Γ, $\{\omega_N\}_{N\geqslant 1}$, and N_Γ are as above. Then*

$$\liminf_{N\to\infty} \frac{\tau_{s,d}(N_\Gamma)}{\tau_{s,d}(N)} \geqslant \frac{\mathscr{H}_d(\Gamma)^{s/d}\underline{h}_{s,d}^w(A)}{\sigma_{s,d}(1+\varepsilon)^{2s}\overline{w}_\Gamma} \tag{37}$$

and

$$\limsup_{N\to\infty}\frac{\tau_{s,d}(N_\Gamma)}{\tau_{s,d}(N)}\geqslant\frac{\mathscr{H}_d(\Gamma)^{s/d}\overline{h}^w_{s,d}(A)}{\sigma_{s,d}(1+\varepsilon)^{2s}\overline{w}_\Gamma},\tag{38}$$

where

$$\overline{w}_\Gamma:=\sup_{(y,x)\in\Gamma\times\Gamma}w(y,x).$$

Proof. Let $\varphi_{x_\alpha,\delta_\alpha}$ be chosen for the given ε from Definition 2.1. Denote the sidelength of $\tilde\Gamma$ by r. Fix a number $\upsilon\in(0,1)$. By $\tilde\Gamma_\upsilon$ we denote the closed d-dimensional cube with the same center as $\tilde\Gamma$ and sidelength $r-\upsilon$. Denote $\Gamma_\upsilon:=\varphi^{-1}_{x_\alpha,\delta_\alpha}(\tilde\Gamma_\upsilon)$.

For any $N\geqslant 1$, we have

$$P^w_s(A;\omega_N)\leqslant\inf_{y\in\Gamma_\upsilon}\sum_{x\in\omega_N}\frac{w(y,x)}{|y-x|^s}=\inf_{y\in\Gamma_\upsilon}\left(\sum_{x\in\omega_N\cap\Gamma}\frac{w(y,x)}{|y-x|^s}+\sum_{x\in\omega_N\setminus\Gamma}\frac{w(y,x)}{|y-x|^s}\right).$$

If $y\in\Gamma_\upsilon$ and $x\in(B(x_\alpha,\delta_\alpha)\cap A)\setminus\Gamma$ then $|\varphi_{x_\alpha,\delta_\alpha}(y)-\varphi_{x_\alpha,\delta_\alpha}(x)|\geqslant\upsilon/2$, thus $|y-x|\geqslant(1+\varepsilon)^{-1}\upsilon/2$. Furthermore, $h:=\operatorname{dist}(\Gamma_\upsilon,A\setminus B(x_\alpha,\delta_\alpha))>0$ because both sets are compact. Then for any $y\in\Gamma_\upsilon$ and $x\in A\setminus\Gamma=(A\setminus B(x_\alpha,\delta_\alpha))\cup((B(x_\alpha,\delta_\alpha)\cap A)\setminus\Gamma)$, we have $|y-x|\geqslant\min\{h,v/(2(1+\varepsilon))\}>0$. This means that the set $F_1:=\operatorname{clos}(\Gamma_\upsilon\times(A\setminus\Gamma))\subset A\times A$ does not intersect the diagonal $D(A)$. Thus, the weight w is bounded above on F_1 by a constant (which can depend on υ). Consequently,

$$P^w_s(A;\omega_N)\leqslant\inf_{y\in\Gamma_\upsilon}\sum_{x\in\omega_N\cap\Gamma}\frac{w(y,x)}{|y-x|^s}+C_{\upsilon,\varepsilon}\cdot N\leqslant\overline{w}_\Gamma\cdot\inf_{y\in\Gamma_\upsilon}\sum_{x\in\omega_N\cap\Gamma}\frac{1}{|y-x|^s}+C_{\upsilon,\varepsilon}\cdot N,$$

where $C_{\upsilon,\varepsilon}$ is a constant independent on N and ω_N. Denote $\tilde y:=\varphi_{x_\alpha,\delta_\alpha}(y)$ and $\tilde x:=\varphi_{x_\alpha,\delta_\alpha}(x)$, $x\in\omega_N\cap B(x_\alpha,\delta_\alpha)$. Then

$$P^w_s(A;\omega_N)\leqslant(1+\varepsilon)^s\overline{w}_\Gamma\inf_{\tilde y\in\tilde\Gamma_\upsilon}\sum_{\tilde x\in\tilde\omega^\Gamma_N}\frac{1}{|\tilde y-\tilde x|^s}+C_{\gamma,\varepsilon}\cdot N,$$

where $\tilde\omega^\Gamma_N:=\varphi_{x_\alpha,\delta_\alpha}(\omega_N\cap\Gamma)\subset\tilde\Gamma$.

For any $\tilde x\in\tilde\omega^\Gamma_N$, define $\tilde x'$ to be the point in $\tilde\Gamma_\upsilon$ closest to $\tilde x$ (in particular, $\tilde x'=\tilde x$ if $\tilde x\in\tilde\Gamma_\upsilon$). Denote $\tilde\omega'_N:=\{\tilde x':\tilde x\in\tilde\omega^\Gamma_N\}$. Notice that $\#\tilde\omega'_N=N_\Gamma$. Since $\tilde\Gamma_\upsilon$ is a convex set, for any $\tilde y\in\tilde\Gamma_\upsilon$ we have $|\tilde y-\tilde x|\geqslant|\tilde y-\tilde x'|$. Thus,

$$\begin{aligned}P^w_s(A;\omega_N)&\leqslant(1+\varepsilon)^s\overline{w}_\Gamma\inf_{\tilde y\in\tilde\Gamma_\upsilon}\sum_{\tilde x'\in\tilde\omega'_N}\frac{1}{|\tilde y-\tilde x'|^s}+C_{\upsilon,\varepsilon}\cdot N\leqslant\\&(1+\varepsilon)^s\overline{w}_\Gamma\mathscr{P}_s(\tilde\Gamma_\upsilon;N_\Gamma)+C_{\upsilon,\varepsilon}\cdot N=\\&(1+\varepsilon)^s\overline{w}_\Gamma\mathscr{H}_d(\tilde\Gamma_\upsilon)^{-s/d}\mathscr{P}_s(\mathscr{Q}_d,N_\Gamma)+C_{\upsilon,\varepsilon}\cdot N.\end{aligned}\tag{39}$$

We now divide by $\tau_{s,d}(N)$ and take the $\liminf_{N\to\infty}$. Using Lemma 8.1 and Theorem 2.1, we obtain

$$\underline{h}^w_{s,d}(A)\leqslant(1+\varepsilon)^s\overline{w}_\Gamma(r-\upsilon)^{-s}\sigma_{s,d}\cdot\liminf_{N\to\infty}\frac{\tau_{s,d}(N_\Gamma)}{\tau_{s,d}(N)}.$$

Since the number υ can be arbitrarily small, the function $\varphi_{x_\alpha,\delta_\alpha}$ is bi-Lipschitz, and $\mathscr{H}_d(\tilde\Gamma)=r^d$, we further obtain

$$\underline{h}^w_{s,d}(A)\leqslant(1+\varepsilon)^{2s}\overline{w}_\Gamma\mathscr{H}_d(\Gamma)^{-s/d}\sigma_{s,d}\cdot\liminf_{N\to\infty}\frac{\tau_{s,d}(N_\Gamma)}{\tau_{s,d}(N)},$$

which proves (37). Similarly, passing to $\limsup_{N\to\infty}$ in (39), we obtain

$$\overline{h}^w_{s,d}(A) \leqslant (1+\varepsilon)^{2s}\overline{w}_\Gamma \mathscr{H}_d(\Gamma)^{-s/d}\sigma_{s,d}\cdot \limsup_{N\to\infty}\frac{\tau_{s,d}(N_\Gamma)}{\tau_{s,d}(N)},$$

which proves (38). ☐

Finally, we state the main lemma of this section, which proves the limiting behavior (14).

Lemma 8.3. *Suppose $B\subset A$ is a set with $\mathscr{H}_d(\partial_A B)=0$. Suppose $\Omega=\{\omega_N\}_{N\geqslant 1}$ is an asymptotically extreme sequence of configurations for $\mathscr{P}^w_s(A;N)$. Then*

$$\lim_{N\to\infty}\frac{\#(\omega_N\cap B)}{N}=\frac{\mathscr{H}^{s,w}_d(B)}{\mathscr{H}^{s,w}_d(A)}.$$

Proof. If $\mathscr{H}_d(B)=0$ then clearly

$$\liminf_{N\to\infty}\frac{\#(\omega_N\cap B)}{N}\geqslant\frac{\mathscr{H}^{s,w}_d(B)}{\mathscr{H}^{s,w}_d(A)}.$$

Therefore, it remains to prove this inequality for B with $\mathscr{H}_d(B)>0$. Denote $B_d:=\mathrm{int}_A(B\setminus\mathrm{clos}(A'_d))$, where $\mathrm{int}_A X$ denotes the interior of a set $X\subset A$ relative to A. Consider the family $\mathscr{D}_{\varepsilon,\varepsilon}=\{Q_\alpha\}$ from Lemma 4.1 constructed for the set B_d. Then $B_d=(\cup_\alpha Q_\alpha)\cup D$ with $\mathscr{H}_d(D)<\varepsilon$. For each ball $\tilde{Q}_\alpha:=\varphi_{x_\alpha,\delta_\alpha}(Q_\alpha)=B[\varphi_{x_\alpha,\delta_\alpha}(x_\alpha),\rho_\alpha]$, consider a finite family $\tilde{\mathfrak{G}}_\alpha$ of disjoint open cubes $\tilde{\Gamma}\subset\tilde{Q}_\alpha$ (the families $\mathfrak{G}_\alpha$ will be specified later). Denote $\mathfrak{G}_\alpha:=\{\varphi^{-1}_{x_\alpha,\delta_\alpha}(\tilde{\Gamma})\colon\tilde{\Gamma}\in\tilde{\mathfrak{G}}_\alpha\}$ and let $\mathfrak{G}:=\bigcup_\alpha\mathfrak{G}_\alpha$. Recall that for any $\Gamma\in\mathfrak{G}$ we define $N_\Gamma:=\#(\omega_N\cap\Gamma)$.

Notice that if $s>d$ then Lemma 8.2 implies that for every $\Gamma=\varphi^{-1}_{x_\alpha,\delta_\alpha}(\tilde{\Gamma})\in\mathfrak{G}$, we have

$$\liminf_{N\to\infty}\frac{N_\Gamma}{N}\geqslant(1+\varepsilon)^{-2d}\overline{w}_\Gamma^{-d/s}\left(\frac{\underline{h}^w_{s,d}(A)}{\sigma_{s,d}}\right)^{d/s}\cdot\mathscr{H}_d(\Gamma). \tag{40}$$

In the case $s=d$, the inequality $\ln N_\Gamma\leqslant\ln N$ implies that $\tau_{d,d}(N_\Gamma)/\tau_{d,d}(N)\leqslant N_\Gamma/N$. Consequently,

$$\liminf_{N\to\infty}\frac{N_\Gamma}{N}\geqslant(1+\varepsilon)^{-2d}\overline{w}_\Gamma^{-1}\frac{\underline{h}^w_{d,d}(A)}{\sigma_{d,d}}\cdot\mathscr{H}_d(\Gamma). \tag{41}$$

Therefore, for any $s\geqslant d$ we obtain

$$\liminf_{N\to\infty}\frac{N_\Gamma}{N}\geqslant(1+\varepsilon)^{-2d}\overline{w}_\Gamma^{-d/s}\left(\frac{\underline{h}^w_{s,d}(A)}{\sigma_{s,d}}\right)^{d/s}\cdot\mathscr{H}_d(\Gamma). \tag{42}$$

Since all sets $\Gamma\in\mathfrak{G}$ are disjoint, from (42) we have

(43)
$$\liminf_{N\to\infty}\frac{\#(\omega_N\cap B)}{N}\geqslant\liminf_{N\to\infty}\frac{\#(\omega_N\cap(\cup_\alpha Q_\alpha))}{N}\geqslant\liminf_{N\to\infty}\frac{1}{N}\sum_{\Gamma\in\mathfrak{G}}N_\Gamma\geqslant\sum_{\Gamma\in\mathfrak{G}}\liminf_{N\to\infty}\frac{N_\Gamma}{N}\geqslant$$
$$(1+\varepsilon)^{-2d}\left(\frac{\underline{h}^w_{s,d}(A)}{\sigma_{s,d}}\right)^{d/s}\cdot\sum_{\Gamma\in\mathfrak{G}}\overline{w}_\Gamma^{-d/s}\mathscr{H}_d(\Gamma).$$

Fix a positive number υ. For every α, choose the family $\tilde{\mathfrak{G}}_\alpha$ such that

$$\mathfrak{L}_d\left(\tilde{Q}_\alpha \setminus \bigcup_{\tilde{\Gamma}\in\tilde{\mathfrak{G}}_\alpha} \tilde{\Gamma}\right) < \upsilon. \tag{44}$$

This is possible since $\mathfrak{L}_d(\partial\tilde{Q}_\alpha) = 0$. Denote

$$B_{\mathfrak{G}} := \bigcup_{\Gamma\in\mathfrak{G}} \Gamma.$$

Since the family $\{Q_\alpha\}$ is finite, for some constant C_ε, which does not depend on υ, we have

$$\mathscr{H}_d(B_d \setminus (B_{\mathfrak{G}} \cup D)) \leqslant C_\varepsilon \cdot \upsilon,$$

where $D = B_d \setminus (\cup_\alpha Q_\alpha)$.

From the family $\mathfrak{G}$ we can construct a new family $\mathfrak{G}'$ by splitting each cube $\tilde{\Gamma} \in \tilde{\mathfrak{G}}_\alpha$ into 2^d equal open cubes (we do this for every α). Observe that $\operatorname{clos}(B_{\mathfrak{G}}) = \operatorname{clos}(B_{\mathfrak{G}'})$ and that inequality (44) still holds. Denote $\mathfrak{G}_0 := \mathfrak{G}$ and let $\mathfrak{G}_n := (\mathfrak{G}_{n-1})'$, $n\in\mathbb{N}$. Define the function

$$u_n(x) := \begin{cases} \overline{w}_\Gamma^{-d/s}, & x\in\Gamma,\ \ \Gamma\in\mathfrak{G}_n, \\ 0, & x\in \operatorname{clos}(B_{\mathfrak{G}})\setminus B_{\mathfrak{G}_n}, \quad n = 0,1,2,\ldots. \end{cases}$$

Denote $M := \cap_{n=0}^{\infty} B_{\mathfrak{G}_n}$ and let K be the set of points $x\in M$ such that w is continuous at the point (x,x) (as a function on $A\times A$). Then $\mathscr{H}_d(K) = \mathscr{H}_d(M) = \mathscr{H}_d(B_{\mathfrak{G}}) = \mathscr{H}_d(\operatorname{clos}(B_{\mathfrak{G}}))$. It is not difficult to see that $\lim\limits_{n\to\infty} u_n(x) = w(x,x)^{-d/s}$ for every $x\in K$; i.e., $\mathscr{H}_d$-almost everywhere on $\operatorname{clos}(B_{\mathfrak{G}})$. Furthermore, with $y\in\Gamma$ we have $\overline{w}_\Gamma \geqslant w(y,y) \geqslant a = \inf_G w > 0$ (see Definition 2.3). Consequently, $|u_n(x)| \leqslant a^{-d/s}$, $x\in \operatorname{clos}(B_{\mathfrak{G}})$. Then by the Lebesgue Dominated Convergence Theorem, we have

$$\sum_{\Gamma\in\mathfrak{G}_n} \overline{w}_\Gamma^{-d/s}\mathscr{H}_d(\Gamma) = \int\limits_{B_{\mathfrak{G}_n}} u_n(x)\,d\mathscr{H}_d(x) = \int\limits_{\operatorname{clos}(B_{\mathfrak{G}})} u_n(x)\,d\mathscr{H}_d(x) \to \int\limits_{\operatorname{clos}(B_{\mathfrak{G}})} w^{-d/s}(x,x)\,d\mathscr{H}_d(x) \tag{45}$$

as $n\to\infty$. In view of the arbitrariness of the family $\mathfrak{G}$ in (43), we have

$$\liminf_{N\to\infty}\frac{\#(\omega_N\cap B)}{N} \geqslant (1+\varepsilon)^{-2d}\left(\frac{\underline{h}^w_{s,d}(A)}{\sigma_{s,d}}\right)^{d/s}\cdot \int\limits_{\operatorname{clos}(B_{\mathfrak{G}})} w^{-d/s}(x,x)d\mathscr{H}_d(x). \tag{46}$$

Since $w^{-d/s}(x,x) \leqslant a^{-d/s}$, $x\in A$, taking $\upsilon\to 0$, we deduce that

$$\liminf_{N\to\infty}\frac{\#(\omega_N\cap B)}{N} \geqslant (1+\varepsilon)^{-2d}\left(\frac{\underline{h}^w_{s,d}(A)}{\sigma_{s,d}}\right)^{d/s}\cdot \int\limits_{\cup_\alpha Q_\alpha} w^{-d/s}(x,x)d\mathscr{H}_d(x). \tag{47}$$

Finally, since ε can be made arbitrarily small, we have

$$\liminf_{N\to\infty}\frac{\#(\omega_N\cap B)}{N} \geqslant \left(\frac{\underline{h}^w_{s,d}(A)}{\sigma_{s,d}}\right)^{d/s}\cdot \mathscr{H}_d^{s,w}(B_d). \tag{48}$$

Using the assertion of Lemma 7.2, we obtain

$$\liminf_{N\to\infty}\frac{\#(\omega_N\cap B)}{N} \geqslant \frac{\mathscr{H}_d^{s,w}(B_d)}{\mathscr{H}_d^{s,w}(A)} = \frac{\mathscr{H}_d^{s,w}(B)}{\mathscr{H}_d^{s,w}(A)}. \tag{49}$$

Notice that a similar estimate is true for the set $A\setminus B$. Thus,

$$\limsup_{N\to\infty}\frac{\#(\omega_N\cap B)}{N}=1-\liminf_{N\to\infty}\frac{\#(\omega_N\cap(A\setminus B))}{N}\leqslant 1-\frac{\mathscr{H}_d^{s,w}(A\setminus B)}{\mathscr{H}_d^{s,w}(A)}=\frac{\mathscr{H}_d^{s,w}(B)}{\mathscr{H}_d^{s,w}(A)}. \tag{50}$$

Combining estimates (49) and (50), we obtain

$$\lim_{N\to\infty}\frac{\#(\omega_N\cap B)}{N}=\frac{\mathscr{H}_d^{s,w}(B)}{\mathscr{H}_d^{s,w}(A)}.$$

□

9. An estimate for $\overline{h}^w_{s,d}$ from above

In view of Lemmas 7.2 and 8.3, it remains to prove an estimate for $\overline{h}^w_{s,d}$ from above, which is given by the following lemma.

Lemma 9.1. *Suppose $A\subset\mathbb{R}^p$ is a compact set with $\mathscr{H}_d(A)=\mathscr{M}_d(A)<\infty$ and that $\mathscr{H}_d(\operatorname{clos}(A'_d))=0$. Suppose w is a CPD-weight on $A\times A$ with parameter d. Then for any $s\geqslant d$, we have*

$$\overline{h}^w_{s,d}(A)\leqslant\frac{\sigma_{s,d}}{\mathscr{H}_d^{s,w}(A)^{s/d}}. \tag{51}$$

Proof. If $\mathscr{H}_d(A)=0$, then inequality (51) holds trivially. Assume that $\mathscr{H}_d(A)>0$. Set $B_d:=A\setminus\operatorname{clos}(A'_d)$. Then B_d is a relatively open subset of A. For a positive number $\varepsilon>0$, fix the family $\mathscr{D}_{\varepsilon,\varepsilon}$ from Lemma 4.1. Let $\{\omega_N\}_{N\geqslant 1}$ be an asymptotically optimal sequence of configurations for $\mathscr{P}^w_s(A;N)$. Let $\Gamma\subset B_d$ be a set as in Lemma 8.1. Recall the estimate in (38):

$$\limsup_{N\to\infty}\left(\frac{N_\Gamma}{N}\right)^{s/d}\geqslant\limsup_{N\to\infty}\frac{\tau_{s,d}(N_\Gamma)}{\tau_{s,d}(N)}\geqslant(1+\varepsilon)^{-2s}\overline{w}_\Gamma^{-1}\frac{\overline{h}^w_{s,d}(A)}{\sigma_{s,d}}\mathscr{H}_d(\Gamma)^{s/d}.$$

Since $\mathscr{H}_d(\partial_A\Gamma)=0$, Lemma 8.3 implies that the limit $\lim\limits_{N\to\infty}\frac{N_\Gamma}{N}$ exists. Then

$$(1+\varepsilon)^{-2d}\left(\frac{\overline{h}^w_{s,d}(A)}{\sigma_{s,d}}\right)^{d/s}\overline{w}_\Gamma^{-d/s}\mathscr{H}_d(\Gamma)\leqslant\lim_{N\to\infty}\frac{N_\Gamma}{N}.$$

We now argue exactly as in Lemma 8.3. That is, we take the sequence of families $\{\mathfrak{G}_n\}_{n=0}^\infty$ from the proof of Lemma 8.3 and obtain

$$(1+\varepsilon)^{-2d}\left(\frac{\overline{h}^w_{s,d}(A)}{\sigma_{s,d}}\right)^{d/s}\sum_{\Gamma\in\mathfrak{G}_n}\overline{w}_\Gamma^{-d/s}\mathscr{H}_d(\Gamma)\leqslant\sum_{\Gamma\in\mathfrak{G}_n}\lim_{N\to\infty}\frac{N_\Gamma}{N}=\lim_{N\to\infty}\frac{1}{N}\sum_{\Gamma\in\mathfrak{G}_n}N_\Gamma\leqslant 1.$$

Using (45) and passing to the limit as $n\to\infty$ we obtain

$$(1+\varepsilon)^{-2d}\left(\frac{\overline{h}^w_{s,d}(A)}{\sigma_{s,d}}\right)^{d/s}\mathscr{H}_d^{s,w}(\operatorname{clos}(B_{\mathfrak{G}}))\leqslant 1.$$

Letting $\upsilon\to 0$ (see (44)) and then letting $\varepsilon\to 0$, we obtain

$$\left(\frac{\overline{h}^w_{s,d}(A)}{\sigma_{s,d}}\right)^{d/s}\mathscr{H}_d^{s,w}(B_d)\leqslant 1.$$

Consequently,

$$\overline{h}^{w}_{s,d}(A) \leqslant \frac{\sigma_{s,d}}{\mathscr{H}_d^{s,w}(B_d)^{s/d}} = \frac{\sigma_{s,d}}{\mathscr{H}_d^{s,w}(A)^{s/d}},$$

which completes the proof. □

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