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Geometric Convergence of Rational Approximations to e^{-z} in Infinite Sectors

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Summary. In this paper, we study the geometric convergence of rational approximations to e^{-z} in infinite sectors symmetric about the positive real axis.

1. Introduction

The study of the convergence of Padé and non-Padé rational approximations of e^{-z} on finite and infinite sets has received much recent interest, both for its applications in numerical analysis, as in the solution of systems of ordinary or partial differential equations (cf. [1–4, 16]), as well as for its pure approximation-theoretic interest (cf. [6, 8, 9, 12, 13, 15, 17]).

In particular, we have previously shown (cf. [9, 10]) that certain Padé approximants of e^{-z} can exhibit geometric convergence to e^{-z} in infinite *parabolic* regions of the complex plane, symmetric about the positive real axis. Because of this, it is natural to ask if certain Padé approximants of e^{-z} actually converge geometrically to e^{-z} on infinite *sectors*, symmetric about the positive real axis. Armed with facts (cf. [12]) about the location of the poles of particular Padé approximants of e^{-z}, we shall show here that such geometric convergence in infinite sectors is indeed possible. In establishing this, we also give precise convergence rates for Padé approximation to e^{-z} on $[0, +\infty)$. We also show that a sequence of rational functions can be found which both converges geometrically to e^{-z} on $[0, +\infty)$, and has all its poles in the left half-plane.

The outline of the paper is as follows. The remainder of this section gives the necessary background and notation needed. Then, in § 2, we state and discuss the new results of this paper. The proofs of the stated results in § 2 are then collected in § 3.

Let π_m denote the collection of all polynomials in the variable z having degree at most m, and let $\pi_{\nu,n}$ be the collection of all complex rational functions $r_{\nu,n}(z)$ of the form

$$r_{\nu,n}(z)=\frac{q_{\nu,n}(z)}{p_{\nu,n}(z)},\quad \text{where } q_{\nu,n}(z)\in\pi_\nu,\ p_{\nu,n}(z)\in\pi_n,\ p_{\nu,n}(0)=1.$$

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Then, the (ν, n)-th Padé approximant of e^{-z} is defined as that element $R_{\nu,n}(z)$ in $\pi_{\nu,n}$ for which

$$e^{-z}-R_{\nu,n}(z)=\mathcal{O}(|z|^{n+\nu+1}) \quad \text{as} \, |z|\to 0.$$

In explicit form, it is known (cf. Perron [7, p. 433]) that

$$R_{\nu,n}(z)=Q_{\nu,n}(z)/P_{\nu,n}(z),$$

where

$$Q_{\nu,n}(z)=\sum_{j=0}^{\nu}\frac{(n+\nu-j)!\,\nu!\,(-z)^j}{(n+\nu)!\,j!\,(\nu-j)!}, \tag{1.1}$$

$$P_{\nu,n}(z)=\sum_{j=0}^{n}\frac{(n+\nu-j)!\,n!\,z^j}{(n+\nu)!\,j!\,(n-j)!}. \tag{1.2}$$

Note that in the case $\nu=0$, $P_{0,n}(z)$ reduces to $s_n(z)=\sum_{j=0}^{n} z^j/j!$, the familiar n-th partial sum of e^z, so that $R_{0,n}(z)=1/s_n(z)$. Also, the error in the (ν, n)-th Padé approximation of e^{-z} has the following useful representation (cf. Perron [7, p. 436]):

$$\varepsilon_{\nu,n}(z):=R_{\nu,n}(z)-e^{-z}=\frac{(-1)^{\nu}\, z^{n+\nu+1}\int_0^1 e^{zt}\, t^{\nu}\,(1-t)^n\,dt}{(n+\nu)!\,e^z\,P_{\nu,n}(z)}, \tag{1.3}$$

for any finite z. For additional notation, we set

$$\eta_{\nu,n}:=\sup\{|\varepsilon_{\nu,n}(x)|: \quad x\geqq 0\}=\|R_{\nu,n}(x)-e^{-x}\|_{[0,+\infty)}. \tag{1.4}$$

For the study of either the uniform or geometric convergence of a sequence $\{R_{\nu(n),n}(x)\}_{n=1}^{\infty}$ of Padé approximants of e^{-x} on the nonnegative real axis $[0,+\infty)$, sharp bounds for $\eta_{\nu,n}$ are required. In Saff and Varga [9], the following upper bounds for $\eta_{\nu,n}$ were obtained:

$$\eta_{n,n}=1 \text{ for all } n\geqq 0; \tag{1.5}$$

$$\eta_{\nu,n}\leqq\prod_{j=1}^{n-\nu}\left(\frac{\nu+j}{3\nu+2j}\right)\leqq\frac{1}{2^{n-\nu}} \quad \text{for all } 0\leqq\nu<n; \tag{1.6}$$

$$\frac{A_1}{n}\leqq\eta_{n-1,n}\leqq\frac{A_2 \ln n}{n}, \quad \text{for all } n>1, \tag{1.7}$$

where A_1 and A_2 are positive constants, independent of n. From the above inequalities, it easily follows [9, Thm. 3.1] that a necessary and sufficient condition for the *uniform* convergence of a sequence of Padé approximants $\{R_{\nu(n),n}(x)\}_{n=1}^{\infty}$ of e^{-x} on $[0,+\infty)$, i.e.,

$$\lim_{n\to\infty}\eta_{\nu(n),n}=0, \tag{1.8}$$

is that $\nu(n)<n$ for all n sufficiently large. The analogous situation for finding necessary and sufficient conditions for the *geometric* convergence of a sequence of Padé approximants $\{R_{\nu(n),n}(x)\}_{n=1}^{\infty}$ of e^{-x} on $[0,+\infty)$, i.e.,

$$\overline{\lim_{n\to\infty}}\,(\eta_{\nu(n),n})^{1/n}<1, \tag{1.9}$$

was also studied in [9], and it was shown there in [9, Thm. 3.2] that

$$\varlimsup_{n\to\infty} \frac{\nu(n)}{n} < 1 \tag{1.10}$$

is a *sufficient* condition for geometric convergence. Based on new sharper estimates of $\eta_{\nu,n}$, to be described in § 2, one of our results here, Theorem 2.5, is that (1.10) is *both* necessary and sufficient for the geometric convergence of $\{R_{\nu(n),n}(x)\}_{n=1}^{\infty}$ to e^{-x} on $[0, +\infty)$.

As the title suggests, we are primarily interested here in the geometric convergence of rational approximations to e^{-z} in infinite sectors, symmetric about the positive real axis. To deduce such geometric convergence in infinite sectors, we need the following results and notation. First, for any θ with $0<\theta\leqq\pi$,

$$S(\theta) := \{z : |\arg z| < \theta\} \tag{1.11}$$

denotes an open infinite sector in the complex plane, symmetric about the positive real axis. Next, for an arbitrary set A in the complex plane, we denote by $\|\cdot\|_A$ the supremum norm on A, i.e., for f defined on A,

$$\|f\|_A := \sup\{|f(z)| : z\in A\}. \tag{1.12}$$

We now quote

Theorem 1.1. (Saff and Varga [12, Thm. 2.1]). For every $n\geqq 2$ and $\nu\geqq 0$, the Padé approximant $R_{\nu,n}(z)$ of e^{-z} has no poles in the infinite sector

$$S_{\nu,n} := \left\{z : |\arg z| \leqq \cos^{-1}\left(\frac{n-\nu-2}{n+\nu}\right)\right\}. \tag{1.13}$$

Consequently, if the sequence of Padé approximants $\{R_{\nu(n),n}(z)\}_{n=1}^{\infty}$ satisfies $\lim_{n\to\infty} \nu(n)/n=\sigma$ where $0<\sigma<1$, then for each $\varepsilon>0$ sufficiently small, no poles of $R_{\nu(n),n}(z)$ lie in the infinite sector $S\left(\cos^{-1}\left(\frac{1-\sigma}{1+\sigma}\right)-\varepsilon\right)$ (defined in (1.11)), for all n large.

The next result is a consequence of Saff and Varga [10, Thm. 2.4].

Theorem 1.2. Assume that the sequence of Padé approximants $\{R_{\nu(n),n}(z)\}_{n=1}^{\infty}$ of e^{-z} satisfies

$$\varlimsup_{n\to\infty} \{\|e^{-x} - R_{\nu(n),n}(x)\|_{[0,+\infty)}\}^{1/n} \leqq \frac{1}{q} < 1. \tag{1.14}$$

Assume further that for some θ_0 with $0<\theta_0\leqq\pi$, no poles of $R_{\nu(n),n}(z)$ lie in $S(\theta_0)$ for all n sufficiently large. Then, for every θ satisfying the inequality

$$0<\theta<4\tan^{-1}\left\{\left(\frac{\sqrt{q}-1}{\sqrt{q}+1}\right)\cdot\tan\left(\frac{\theta_0}{4}\right)\right\}, \tag{1.15}$$

there holds on the closure $\bar{S}(\theta)$,

$$\varlimsup_{n\to\infty} \{\|e^{-z} - R_{\nu(n),n}(z)\|_{\bar{S}(\theta)}\}^{1/n} \leqq \frac{1}{q}\left\{\frac{\sin[\frac{1}{4}(\theta_0+\theta)]}{\sin[\frac{1}{4}(\theta_0-\theta)]}\right\}^2 < 1. \tag{1.16}$$

We remark that the above result is a specialized form of Theorem 2.4 of [10], which makes use of the added fact that any sequence $\{R_{\nu(n),n}(z)\}_{n=1}^{\infty}$ of Padé approximants of e^{-z} converges uniformly and faster than geometrically to e^{-z} on any compact subset T of the complex plane, i.e.,

$$\lim_{n\to\infty} \|R_{\nu(n),n}(z)-e^{-z}\|_T^{1/n}=0.$$

This follows easily from the integral representation of (1.3), and the fact (cf. [7, p. 434]) that $\{R_{\nu(n),n}(z)\}_{n=1}^{\infty}$ converges for every finite z to e^{-z}.

Combining Theorems 1.1 and 1.2, a result, Theorem 2.7, concerning the geometric convergence of a sequence of Padé approximants $\{R_{\nu(n),n}(z)\}_{n=1}^{\infty}$ of e^{-z} in infinite sectors will then be deduced.

Finally, we pose the following natural question. Is it possible to find a sequence $\{\tilde{R}_{\nu(n),n}(z)\}_{n=1}^{\infty}$ of (not necessarily Padé) rational functions with $\tilde{R}_{\nu(n),n}\in\pi_{\nu(n),n}$ and $0\leqq\nu(n)\leqq n$ for all $n\geqq 1$, such that all the poles of $\{\tilde{R}_{\nu(n),n}(z)\}_{n=1}^{\infty}$ lie in the left-half plane $Re\, z<0$, and such that

$$\overline{\lim_{n\to\infty}}\, \{\|e^{-x}-\tilde{R}_{\nu(n),n}(x)\|_{[0,+\infty)}\}^{1/n}<1, \tag{1.17}$$

i.e., $\{\tilde{R}_{\nu(n),n}(z)\}_{n=1}^{\infty}$ converges geometrically to e^{-z} on $[0, +\infty)$. In view of the sharpness of the numerical results of [12], which in turn tend to indicate the sharpness of Theorem 1.1, this appears not to be possible for *Padé* rational approximations of e^{-z}. However, we will show that certain non-Padé rational functions do have this property. To prove this, we need the following result of

Theorem 1.3. (Saff and Varga [12, Thm. 2.4]). If $1<n<3\nu+4$, then all the poles of the Padé approximant $R_{\nu,n}(z)$ of e^{-z} lie in the half-plane

$$\mathrm{Re}\, z<n-\nu-2. \tag{1.18}$$

2. Statements of New Results

We now list and discuss our main results, deferring their proofs to the next section. For notation, $\binom{n}{\nu}=\frac{n!}{\nu!\,(n-\nu)!}$ denotes the familiar binomial coefficient.

Theorem 2.1. For any nonnegative integers ν and n with $0\leqq\nu\leqq n$,

$$\frac{Q(\nu,n)}{2^{n-\nu}\binom{n}{\nu}}\leqq\eta_{\nu,n}\leqq\frac{1}{2^{n-\nu}\binom{n}{\nu}}, \tag{2.1}$$

where there exists a positive constant γ, independent of ν and n, such that

$$Q(\nu,n)\geqq\frac{\gamma}{(n+1)^2}\ \text{for all}\ \nu\ \text{and}\ n. \tag{2.2}$$

It is interesting to compare the upper bound for $\eta_{\nu,n}$ in (2.1) with those of (1.5)–(1.7). First, as is directly verified,

$$\frac{1}{2^{n-\nu}\binom{n}{\nu}}\leqq\prod_{j=1}^{n-\nu}\left(\frac{\nu+j}{3\nu+2j}\right)\qquad\text{for all}\ \nu\ \text{and}\ n\ \text{with}\ 0\leqq\nu<n,$$

so that the upper bound of (2.1) for $\eta_{\nu,n}$ in these cases *improves* the corresponding upper bounds of (1.6). Moreover, since the case $\nu=n$ gives in (2.1) the upper bound $\eta_{n,n} \leqq 1$, we see from (1.5) that the upper bound of (2.1) is *sharp* in this case.

For the special case $\nu = n-1$, we also have from (2.1) that

$$\eta_{n-1,n} \leqq \frac{1}{2n} \qquad \text{for all } n \geqq 1,$$

which improves the corresponding upper bound of (1.7). More generally, we have as an immediate consequence of the upper bound of (2.1) the

Corollary 2.2. For any fixed positive integer μ,

$$\eta_{n-\mu,n} = \mathcal{O}\left(\frac{1}{n^{\mu}}\right) \qquad \text{as } n \to \infty. \tag{2.3}$$

We state now some further consequences of Theorem 2.1. First, define

$$g(\beta) := \beta^{\beta}(1-\beta)^{1-\beta}/2^{1-\beta} \qquad \text{for } 0 < \beta < 1, \tag{2.4}$$

and extend g by continuity to the closed interval $[0,1]$, so that $g(0) = 1/2$, and $g(1) = 1$. In what is to follow, $\{\nu(n)\}_{n=1}^{\infty}$ will denote a sequence of nonnegative integers with $0 \leqq \nu(n) \leqq n$ for all $n \geqq 1$, and $\{\nu(n_i)\}_{i=1}^{\infty}$ will denote a subsequence.

Theorem 2.3. If $\lim\limits_{i\to\infty} \dfrac{\nu(n_i)}{n_i} = \beta$, then

$$\lim_{i\to\infty} \eta_{\nu(n_i),n_i}^{1/n_i} = g(\beta). \tag{2.5}$$

Conversely, if $\lim\limits_{i\to\infty} \eta_{\nu(n_i),n_i}^{1/n_i} = \tau$, then every convergent subsequence of $\{\nu(n_i)/n_i\}_{i=1}^{\infty}$, i.e., $\lim\limits_{j\to\infty} \dfrac{\nu(n_j)}{n_j} = \beta$, has the property that $g(\beta) = \tau$. Moreover, if $\dfrac{1}{2} < \tau \leqq 1$, then the subsequence $\{\nu(n_i)/n_i\}_{i=1}^{\infty}$ itself converges:

$$\lim_{i\to\infty} \frac{\nu(n_i)}{n_i} = \beta, \quad \text{and} \quad g(\beta) = \tau.$$

Direct examination of the function g of (2.4) shows that

$$\min\{g(\beta) : 0 \leqq \beta \leqq 1\} = \tfrac{1}{3} = g(\tfrac{1}{3}). \tag{2.6}$$

This observation, coupled with Theorem 2.3, gives directly

Corollary 2.4. For any sequence $\{\nu(n)\}_{n=1}^{\infty}$,

$$\varliminf_{n\to\infty} \eta_{\nu(n),n}^{1/n} \geqq 1/3, \tag{2.7}$$

with equality being possible.

To couple the results of Theorem 2.3 and Corollary 2.4 with numerical experiments, we give in Table 1 the computed values for $\{\eta_{m,3m}\}_{m=1}^{16}$, along with the values $\{\eta_{m,3m}^{1/3m}\}_{m=1}^{16}$. Because $\eta_{\nu,n}$ is given from (1.4) by

$$\eta_{\nu,n} = \max\{|R_{\nu,n}(x) - e^{-x}| : x \geqq 0\}, \qquad 0 \leqq \nu < n,$$

a standard search procedure can be used to compute $\eta_{\nu,n}$. Note then from Theorem 2.3 that

$$\lim_{m\to\infty} \eta_{m,3m}^{1/3m} = 1/3. \tag{2.8}$$

Because the values $\{\eta_{m,3m}^{1/3m}\}_{m=1}^{16}$ in Table 1 appear to be converging, though rather slowly, to 1/3, we have added in columns 4 and 5 of Table 1 two applications of Shanks' extrapolation (cf. [14]) to speed convergence, i.e., if the original sequence is $\{\alpha_n^{(0)}\}$, then these extrapolations are defined recursively by

$$\alpha_n^{(j+1)} := \frac{\alpha_{n+1}^{(j)} \cdot \alpha_{n-1}^{(j)} - (\alpha_n^{(j)})^2}{\alpha_{n+1}^{(j)} + \alpha_{n-1}^{(j)} - 2\alpha_n^{(j)}}.$$

It is interesting to note here that this extrapolation technique is itself directly connected with Padé approximation (cf. [4, 14]). Similarly, in Table 2, we give the computed values of $\{\eta_{2m,4m}\}_{m=1}^{12}$, along with $\{\eta_{2m,4m}^{1/4m}\}_{m=1}^{12}$, and two Shanks'

Table 1. $\{\eta_{m,3m}\}_{m=1}^{16}$

m	$\eta_{m,3m}$	$\eta_{m,3m}^{1/3m}$	$\eta_{m,3m}^{1/3m(1)}$	$\eta_{m,3m}^{1/3m(2)}$
1	0.23920775 (−1)	0.28813217		
2	0.84062479 (−3)	0.30720883	0.32037478	
3	0.30533569 (−4)	0.31499860	0.32421092	0.32879485
4	0.11196202 (−5)	0.31921936	0.32629932	0.32983115
5	0.41217332 (−7)	0.32186369	0.32761173	0.33048299
6	0.15204101 (−8)	0.32367483	0.32851242	0.33092623
7	0.56149131 (−10)	0.32499260	0.32916836	0.33126026
8	0.20751025 (−11)	0.32599428	0.32966775	0.33148892
9	0.76726558 (−13)	0.32678134	0.33005966	0.33168144
10	0.28379098 (−14)	0.32741604	0.33037528	0.33193514
11	0.10499246 (−15)	0.32793870	0.33063778	0.33193044
12	0.38850582 (−17)	0.32837656	0.33085597	0.33213704
13	0.14377996 (−18)	0.32874871	0.33104242	0.33219469
14	0.53216633 (−20)	0.32906890	0.33120288	0.33229190
15	0.19698581 (−21)	0.32934730	0.33134275	
16	0.72921213 (−23)	0.32959159		

Table 2. $\{\eta_{2m,4m}\}_{m=1}^{12}$

m	$\eta_{2m,4m}$	$\eta_{2m,4m}^{1/4m}$	$\eta_{2m,4m}^{1/4m(1)}$	$\eta_{2m,4m}^{1/4m(2)}$
1	0.11766464 (−1)	0.32935290		
2	0.17865435 (−3)	0.34001784	0.34679534	
3	0.27615481 (−5)	0.34416185	0.34886191	0.35120926
4	0.42906830 (−7)	0.34636412	0.34996092	0.35175856
5	0.66810489 (−9)	0.34773006	0.35064296	0.35209889
6	0.10414761 (−10)	0.34865994	0.35110741	0.35233286
7	0.16245652 (−12)	0.34933380	0.35144420	0.35249899
8	0.25351536 (−14)	0.34984457	0.35169952	0.35262045
9	0.39572369 (−16)	0.35024505	0.35189945	0.35275971
10	0.61782415 (−18)	0.35056748	0.35206161	0.35285895
11	0.96471697 (−20)	0.35083265	0.35219631	
12	0.15065586 (−21)	0.35105463		

extrapolations. In this case, Theorem 2.3 gives that

$$\lim_{m\to\infty} \eta_{2m,4m}^{1/4m} = g\left(\frac{1}{2}\right) = \frac{1}{2\sqrt{2}} \doteq 0.3536,$$

which again is consistent with the numerical results of Table 2.

Next, as another consequence of Theorem 2.3 which settles the question of a necessary and sufficient condition for the geometric convergence in the uniform norm of a sequence of Padé approximants $\{R_{\nu(n),n}(x)\}_{n=1}^{\infty}$ to e^{-x} on $[0, +\infty)$, we have

Theorem 2.5. A necessary and sufficient condition that a sequence of Padé approximants $\{R_{\nu(n),n}(x)\}_{n=1}^{\infty}$ converges geometrically in the uniform norm to e^{-x} on $[0, +\infty)$ (cf. (1.9)) is that

$$\overline{\lim_{n\to\infty}} \frac{\nu(n)}{n} < 1. \tag{2.9}$$

As a consequence of Theorems 2.1 and 2.3, we also have

Theorem 2.6. For every $n \geqq 1$,

$$\left(\sum_{\nu=0}^{n} \frac{1}{\eta_{\nu,n}}\right)^{1/n} \geqq 3, \tag{2.10}$$

and

$$\lim_{n\to\infty} \left(\sum_{\nu=0}^{n} \frac{1}{\eta_{\nu,n}}\right)^{1/n} = 3. \tag{2.11}$$

It seems appropriate to comment on the many ways the particular constant 3 enters into the discussion of rational approximations of e^{-x} in the uniform norm on $[0, +\infty)$. It was first shown by Schönhage [13] for Chebyshev rational approximation to e^{-x} on $[0, +\infty)$ that if

$$\lambda_{0,n} := \inf\left\{\left\| e^{-x} - \frac{1}{p_n(x)} \right\|_{[0,+\infty)} : p_n \in \pi_n\right\},$$

then

$$\lim_{n\to\infty} \lambda_{0,n}^{1/n} = \tfrac{1}{3}. \tag{2.12}$$

Next, we see that this constant 3 appears explicitly in Corollary 2.4, and in Theorem 2.6. Furthermore, from (2.5) and (2.6), we have that

$$\lim_{n\to\infty} \{\| e^{-x} - R_{[n/3],n}(x) \|_{[0,+\infty)}\}^{1/n} = \tfrac{1}{3}, \tag{2.13}$$

where $[\alpha]$ denotes as usual the integer part of α. This incidentally shows that the *degree of convergence* of best Chebyshev rational approximation to e^{-x} on $[0, +\infty)$ by reciprocals of polynomials is *identical* with the best degree of convergence of Padé rational approximation to e^{-x} on $[0, +\infty)$ in the uniform norm. As a final example of the occurrence of the number 3, Theorem 1.1 shows that the associated Padé approximants $\{R_{[n/3],n}(z)\}_{n=1}^{\infty}$ of e^{-z} have no poles in the closed infinite sector $\overline{S}(\pi/3)$.

Having investigated the geometric convergence of $\{R_{\nu(n),n}(z)\}_{n=1}^{\infty}$ to e^{-z} on $[0, +\infty)$, this geometric convergence can be extended, by means of Theorems 1.1 and 1.2, to infinite sectors.

Theorem 2.7. Assume that the sequence of Padé approximants $\{R_{\nu(n),n}(z)\}_{n=1}^{\infty}$ of e^{-z} satisfies $\lim\limits_{n\to\infty} \nu(n)/n=\sigma$ where $0<\sigma<1$, and set $\theta_0=\cos^{-1}\left(\frac{1-\sigma}{1+\sigma}\right)$. If $g(\sigma)$ is defined by (2.4), then for every θ satisfying

$$0<\theta<4\tan^{-1}\left\{\left(\frac{1-\sqrt{g(\sigma)}}{1+\sqrt{g(\sigma)}}\right)\cdot\tan\left(\frac{\theta_0}{4}\right)\right\}, \tag{2.14}$$

there holds on the closure $\bar{S}(\theta)$,

$$\overline{\lim_{n\to\infty}}\ \{\|e^{-z}-R_{\nu(n),n}(z)\|_{\bar{S}(\theta)}\}^{1/n}\leqq g(\sigma)\left\{\frac{\sin\left[\frac{1}{4}(\theta_0+\theta)\right]}{\sin\left[\frac{1}{4}(\theta_0-\theta)\right]}\right\}^2<1. \tag{2.15}$$

Actually, a sufficient condition for the convergence of a sequence $\{R_{\nu(n),n}(z)\}_{n=1}^{\infty}$ of Padé approximants to e^{-z} in *some* infinite sector $S(\theta)$ with $\theta>0$, can also be deduced.

Theorem 2.8. A sufficient condition that the sequence of Padé approximants $\{R_{\nu(n),n}(z)\}_{n=1}^{\infty}$ of e^{-z} converges geometrically to e^{-z} in some infinite sector $S(\theta)=\{z: |\arg z|<\theta\}$ is that

$$0<\underline{\lim_{n\to\infty}}\ \nu(n)/n\leqq\overline{\lim_{n\to\infty}}\ \nu(n)/n<1. \tag{2.16}$$

To complete this section, we state a result which answers affirmatively the question posed in § 1.

Theorem 2.9. There exists a sequence $\{\tilde{R}_{\nu(n),n}(z)\}_{n=2}^{\infty}$ of rational functions with $\tilde{R}_{\nu(n),n}\in\pi_{\nu(n),n}$ and $0\leqq\nu(n)<n$ such that all the poles of $\{\tilde{R}_{\nu(n),n}(z)\}_{n=1}^{\infty}$ lie in $\operatorname{Re} z<0$, and such that $\{\tilde{R}_{\nu(n),n}(z)\}_{n=2}^{\infty}$ converges geometrically to e^{-z} on $[0, +\infty)$, i.e.,

$$\overline{\lim_{n\to\infty}}\ \{\|e^{-z}-\tilde{R}_{\nu(n),n}(z)\|_{[0,+\infty)}\}^{1/n}<1. \tag{2.17}$$

3. Proofs of New Results

In proving Theorem 2.1 of § 2, we find it convenient to first establish

Lemma 3.1. For any nonnegative integers m and n with $0\leqq m\leqq n+1$,

$$\int_0^x t^n e^{-t}\,dt\leqq\frac{(n+1-m)!\,x^m}{(n+1)}\qquad\text{for any } x\geqq 0. \tag{3.1}$$

Proof. As previously noted in § 1, $R_{0,n}(x)=1/s_n(x)$ where $s_n(x)=\sum\limits_{k=0}^{n} x^k/k!$, and in this case, the expression (1.3) directly implies that

$$e^x\int_0^x t^n e^{-t}\,dt=n!\,(e^x-s_n(x))=n!\sum_{j=n+1}^{\infty} x^j/j! \tag{3.2}$$

for all $x\geqq 0$, all $n\geqq 0$. Then, because

$$\frac{1}{j!}\leqq\frac{(n+1-m)!}{(n+1)!\,(j-m)!}\qquad\text{for any } j\geqq n+1\geqq m\geqq 0,$$

it follows from (3.2) that

$$e^x \int_0^x t^n e^{-t}\,dt \leqq \frac{(n+1-m)!\,x^m}{(n+1)} \sum_{k=n+1-m}^{\infty} x^k/k! \leqq \frac{(n+1-m)!\,x^m e^x}{(n+1)}$$

for all $x \geqq 0$ and all $n+1 \geqq m \geqq 0$, which gives the desired inequality of (3.1). ■

With Lemma 3.1, we now give the

Proof of Theorem 2.1. With the change of variables $t=1-u$ in the integral of (1.3), we have from (1.3) that

$$|\varepsilon_{\nu,n}(x)| = \frac{x^{n+\nu+1} \int_0^1 e^{-xu} u^n (1-u)^\nu\,du}{(n+\nu)!\,P_{\nu,n}(x)} \qquad \text{for all } x \geqq 0.$$

Now, the numerator of the above fraction is positive for all $0<x<\infty$, so that we can write with the definition of $P_{\nu,n}(x)$ in (1.2),

$$|\varepsilon_{\nu,n}(x)| = \frac{1}{\sum_{k=0}^{n} \{1/f_k(x)\}} \qquad \text{for all } x>0, \tag{3.3}$$

where

$$f_k(x) := \frac{k!\,(n-k)!\,x^{n+\nu+1-k} \int_0^1 e^{-xu} u^n (1-u)^\nu\,du}{n!\,(n+\nu-k)!}, \qquad 0 \leqq k \leqq n,\ x \geqq 0. \tag{3.4}$$

Expressing e^{-xu} in the above integrand in Maclaurin series form and integrating termwise (because of the uniform convergence), gives ,after multiplication by x^k,

$$x^k f_k(x) = \frac{k!\,(n-k)!\,\nu!}{n!\,(n+\nu-k)!} \sum_{j=0}^{\infty} (-1)^j \frac{x^{n+\nu+1+j}\,(n+j)!}{j!\,(n+\nu+1+j)!}.$$

Then, differentiating termwise $\nu+1$ times yields

$$\frac{d^{\nu+1}\,(x^k f_k(x))}{dx^{\nu+1}} = \frac{k!\,(n-k)!\,\nu!}{n!\,(n+\nu-k)!} \sum_{j=0}^{\infty} (-1)^j \frac{x^{n+j}}{j!} = \frac{k!\,(n-k)!\,\nu!\,x^n e^{-x}}{n!\,(n+\nu-k)!}.$$

Thus, on integrating,

$$\int_0^x \frac{d^{\nu+1}}{dt^{\nu+1}} (t^k f_k(t))\,dt = \frac{d^\nu}{dx^\nu} (x^k f_k(x)) = \frac{k!\,(n-k)!\,\nu!}{n!\,(n+\nu-k)!} \int_0^x t^n e^{-t}\,dt,$$

and, coupling this with the inequality (3.1) of Lemma 3.1 for the choice $m=k-\nu$, gives

$$\frac{d^\nu}{dx^\nu} (x^k f_k(x)) \leqq \frac{k!\,(n-k)!\,\nu!\,(n+\nu+1-k)\,x^{k-\nu}}{(n+1)!} \qquad \text{for all } x \geqq 0,$$

provided that k satisfies $\nu \leqq k \leqq n$. Integrating this inequality ν times and dividing through by x^k then gives the upper bound

$$f_k(x) \leqq \frac{(n-k)!\,\nu!\,(k-\nu)!\,(n+\nu+1-k)}{(n+1)!} \qquad \text{for all } x \geqq 0, \text{ all } \nu \leqq k \leqq n.$$

For the remaining cases, i.e., when $0 \leqq k < \nu$, we use the trivial upper bound

$$f_k(x) \leqq +\infty \qquad \text{for all } x \geqq 0, \text{ if } 0 \leqq k < \nu.$$

These upper bounds, when inserted in (3.3), give

$$|\varepsilon_{\nu,n}(x)| \leqq \frac{1}{\sum_{k=\nu}^{n}\left\{\frac{(n+1)!}{(n-k)!\,\nu!\,(k-\nu)!\,(n+\nu+1-k)}\right\}} \qquad \text{for all } x>0,$$

which implies, from the definition of $\eta_{\nu,n}$ in (1.4), that

$$\eta_{\nu,n} \leqq \frac{1}{\sum_{k=\nu}^{n}\left\{\frac{(n+1)!}{(n-k)!\,\nu!\,(k-\nu)!\,(n+\nu+1-k)}\right\}}. \tag{3.5}$$

However, since $(n+\nu+1-k) \leqq (n+1)$ for all $\nu \leqq k \leqq n$, then

$$\eta_{\nu,n} \leqq \frac{1}{\sum_{k=\nu}^{n}\left\{\frac{n!}{(n-k)!\,\nu!\,(k-\nu)!}\right\}} = \frac{1}{\binom{n}{\nu}\sum_{j=0}^{n-\nu}\binom{n-\nu}{j}} = \frac{1}{2^{n-\nu}\binom{n}{\nu}}, \tag{3.6}$$

which thus establishes the upper bound of (2.1) of Theorem 2.1.

We remark that the upper bound for $\eta_{\nu,n}$ of (3.5) is always *sharper* (i.e., not bigger) than the upper bound of (3.6), but the upper bound of (3.6) is, for our purposes, easier to work with.

We now establish the lower bound of (2.1) of Theorem 2.1. First, we know from (1.5) that $\eta_{n,n}=1$ for all $n \geqq 0$, so that the lower bound of (2.1) is valid for $Q(n,n) := \gamma/(n+1)^2$ with *any* $1>\gamma>0$. Thus, it suffices to establish the lower bound of (2.1) only for the cases $0 \leqq \nu < n$, $n \geqq 1$.

From the definition of $P_{\nu,n}(x)$ in (1.2), we have that

$$(n+\nu)!\,P_{\nu,n}(x) \leqq (n+1)! \max_{0\leqq k\leqq n}\left\{\frac{(n+\nu-k)!\,x^k}{k!\,(n-k)!}\right\} \qquad \text{for all } x \geqq 0.$$

Thus, there is a $\hat{k}=\hat{k}(x,\nu,n)$ satisfying $0 \leqq \hat{k} \leqq n$ for which

$$(n+\nu)!\,P_{\nu,n}(x) \leqq \frac{(n+1)!\,(n+\nu-\hat{k})!\,x^{\hat{k}}}{\hat{k}!\,(n-\hat{k})!} \qquad \text{for all } x \geqq 0. \tag{3.7}$$

Next, consider the integral of (1.3). For any $\sigma \in [0,1)$, this integral can be expressed as the sum

$$\int_0^1 e^{xt}\,t^\nu(1-t)^n\,dt = \int_0^\sigma e^{xt}\,t^\nu(1-t)^n\,dt + \int_\sigma^1 e^{xt}\,t^\nu(1-t)^n\,dt.$$

Hence, as the integrand is positive on $0<t<1$, then for $x>0$ and $\sigma \in [0,1)$,

$$\int_0^1 e^{xt}\,t^\nu(1-t)^n\,dt \geqq \int_\sigma^1 e^{xt}\,t^\nu(1-t)^n\,dt > e^{\sigma x}\sigma^\nu \int_\sigma^1 (1-t)^n\,dt,$$

whence

$$\int_0^1 e^{xt}\,t^\nu(1-t)^n\,dt > e^{\sigma x}\sigma^\nu(1-\sigma)^{n+1}/(n+1), \tag{3.8}$$

where the factor σ^ν is understood to be unity when $\nu=0$. Then, combining the inequalities of (3.7) and (3.8) with the expression for $\varepsilon_{\nu,n}(x)$ in (1.3) yields

$$|\varepsilon_{\nu,n}(x)| > \frac{\hat{k}!\,(n-\hat{k})!\,\sigma^\nu\,(1-\sigma)^{n+1}\,x^{n+\nu+1-\hat{k}}}{(n+1)^2\,n!\,(n+\nu-\hat{k})!\,e^{(1-\sigma)x}} \qquad \text{for any } \sigma \in [0,1),\ \text{any } x>0,$$

so that from (1.4),

$$\eta_{\nu,n} > \frac{\hat{k}!\,(n-\hat{k})!\,\sigma^{\nu}\,(1-\sigma)^{n+1}\,x^{n+\nu+1-\hat{k}}}{(n+1)^2\,n!\,(n+\nu-\hat{k})!\,e^{(1-\sigma)x}} \quad \text{for any } \sigma\in[0,1), \text{ any } x>0. \tag{3.9}$$

It remains to select appropriate x and σ for use in (3.9). Suggested by computer computations of $\eta_{\nu,n}$, the value of x for which $|\varepsilon_{\nu,n}(x)|=\eta_{\nu,n}$ is approximately given by

$$x=\frac{(n+\nu)^2}{2(n-\nu)}. \tag{3.10}$$

Our choice for σ is then the corresponding value of t which maximizes the integrand of the integral in (1.3) on $[0, 1]$, with x chosen as in (3.10):

$$\sigma=\frac{2\nu}{(n+\nu)}. \tag{3.11}$$

With the choice of x of (3.10), it can be shown that the value of $\hat{k}$ for which (3.7) is valid is given by

$$\hat{k}=\begin{cases} m \text{ if } n+\nu=2m; \\ m \text{ if } n+\nu=2m+1 \text{ and } \nu\leqq(2m^2+3m+1)/(4m+3); \\ m+1 \text{ if } n+\nu=2m+1 \text{ and } \nu\geqq(2m^2+3m+1)/(4m+3). \end{cases} \tag{3.12}$$

Then, inserting in (3.9) the values of x, σ, and $\hat{k}$, as given respectively by (3.10)–(3.12), determines a general lower bound for $\eta_{\nu,n}$. As a specific example, if $\nu=0$ and if $n=2m$ with $m\geqq 1$, this lower bound from (3.9)–(3.12) is just

$$\eta_{0,2m} > \frac{m!\,m^{m+1}\,e^{-m}}{(2m)!\,(2m+1)^2},$$

and, as

$$m^m\,e^{-m}\sqrt{2\pi m}\left(1+\frac{1}{4m}\right) > m! > m^m\,e^{-m}\sqrt{2\pi m} \qquad \text{for all } m\geqq 1,$$

this lower bound for $\eta_{0,2m}$ becomes

$$\eta_{0,2m} > \frac{Q(0,2m)}{2^{2m}}, \text{ with } Q(0,2m):=\frac{m}{\sqrt{2}\,(2m+1)^2\left(1+\frac{1}{8m}\right)} \geqq \frac{\gamma}{(2m+1)^2}$$

for all $m\geqq 1$, where $\gamma>0$ is independent of m. But this then establishes the lower bound (2.1), as well as (2.2), for the special case $\nu=0$, $n=2m$. The remaining cases which establish the lower bound (2.1), as well as (2.2) of Theorem 2.1, are proved in a similar (but more tedious) manner, thus completing the proof of Theorem 2.1. ■

Before proving Theorem 2.3, we list some properties of the function g, as defined in (2.4). First, g maps $[0, 1]$ into $[0, 1]$, and, as is readily verified, g is strictly decreasing on $(0, \frac{1}{3})$, strictly increasing on $(\frac{1}{3}, 1)$, and g^{-1} is moreover single-valued on $(\frac{1}{2}, 1]$. This then brings us to the

Proof of Theorem 2.3. To establish the first part, assume that $\lim_{i\to\infty}\frac{\nu(n_i)}{n_i}=\beta$. Clearly, since by convention here, $0\leqq\nu(n_i)\leqq n_i$, then $0\leqq\beta\leqq 1$. Assuming further that $0<\beta<1$, then all the numbers n_i, $\nu(n_i)$, and $n_i-\nu(n_i)$, tend to infinity. But with the upper bounds for $\eta_{\nu(n_i),n_i}$ from (2.1) of Theorem 2.1 and with Stirling's approximation of $m!$, we directly determine that

$$\overline{\lim_{i\to\infty}}\ \eta_{\nu(n_i),n_i}^{1/n_i} \leqq g(\beta),$$

and use of the lower bounds of (2.1) and (2.2), of Theorem 2.1 similarly gives us that

$$\underline{\lim_{i\to\infty}}\ \eta_{\nu(n_i),n_i}^{1/n_i} \geqq g(\beta),$$

whence

$$\lim_{i\to\infty} \eta_{\nu(n_i),n_i}^{1/n_i} = g(\beta), \tag{3.14}$$

the desired result of (2.5) of Theorem 2.3 when $0<\beta<1$. When $\beta=0$ or $\beta=1$, the proof is similar.

Conversely, assume that $\lim\limits_{i\to\infty} \eta_{\nu(n_i),n_i}^{1/n_i}=\tau$. Since the sequence $\{\nu(n_i)/n_i\}_{i=1}^{\infty}$ lies in $[0, 1]$, consider any convergent subsequence, i.e.,

$$\lim_{j\to\infty} \frac{\nu(n_j)}{n_j} = \beta.$$

But, the result above of (3.13) is then directly applicable ,whence $g(\beta)=\tau$. Finally, assume that $\frac{1}{2}<\tau\leqq 1$, and consider the two convergent subsequences $\{\nu(n_j)/n_j\}_{j=1}^{\infty}$ and $\{\nu(m_i)/m_i\}_{i=1}^{\infty}$ for which

$$\underline{\lim_{i\to\infty}} \frac{\nu(n_i)}{n_i} = \lim_{j\to\infty} \frac{\nu(n_j)}{n_j} = \beta_1 \leqq \beta_2 = \lim_{i\to\infty} \frac{\nu(m_i)}{m_i} = \overline{\lim_{i\to\infty}} \frac{\nu(n_i)}{n_i}.$$

From what has been established, it follows that $\frac{1}{2}<\tau=g(\beta_1)=g(\beta_2)\leqq 1$. But, as previously remarked, g^{-1} is single valued on $(\frac{1}{2}, 1]$, whence $\beta_1=\beta_2$ and $\{\nu(n_i)/n_i\}_{i=1}^{\infty}$ is itself convergent. ■

Proof of Theorem 2.5. As previously remarked in § 1, it is known [9] that $\overline{\lim\limits_{n\to\infty}} \dfrac{\nu(n)}{n} < 1$ is sufficient for the geometric convergence of the sequence $\{R_{\nu(n),n}(x)\}_{n=1}^{\infty}$ in the uniform norm to e^{-x} on $[0, +\infty)$. Conversely, if

$$\overline{\lim_{n\to\infty}}\ \eta_{\nu(n),n}^{1/n} < 1, \tag{3.14}$$

assume on the contrary that this sufficient condition fails, i.e., $\overline{\lim\limits_{n\to\infty}} \dfrac{\nu(n)}{n} = 1$. Then, there is a convergent subsequence with $\lim\limits_{i\to\infty} \dfrac{\nu(n_i)}{n_i} = 1$. For this subsequence, (2.5) of Theorem 2.3 gives us that $\lim\limits_{i\to\infty} \eta_{\nu(n_i),n_i}^{1/n_i} = g(1) = 1$, whence $\overline{\lim\limits_{n\to\infty}}\ \eta_{\nu(n),n} \geqq 1$, contradicting (3.14). ■

Proof of Theorem 2.6. Using the upper bound of (2.1) of Theorem 2.1, we see that $1/\eta_{\nu,n} \geqq 2^{n-\nu}\binom{n}{\nu}$, while use of the lower bounds of (2.1) and (2.2) similarly give that $1/\eta_{\nu,n} \leqq 2^{n-\nu}\binom{n}{\nu}\cdot(n+1)^2/\gamma$, for all $n\geqq 1$. Summing on ν,

$$3^n \leqq \sum_{\nu=0}^{n} \frac{1}{\eta_{\nu,n}} \leqq (n+1)^2\, 3^n/\gamma \qquad \text{for all } n \geqq 1.$$

Thus, on taking n-th roots, it follows that

$$3=\lim_{n\to\infty}\left(\sum_{\nu=0}^{n}\frac{1}{\eta_{\nu,n}}\right)^{1/n},$$

which establishes (2.10) and (2.11). ∎

Proof of Theorem 2.7. If the sequence of Padé approximants $\{R_{\nu(n),n}(z)\}_{n=1}^{\infty}$ of e^{-z} satisfies $\lim_{n\to\infty}\nu(n)/n=\sigma$ with $0<\sigma<1$, then from Theorem 2.5, this sequence converges geometrically to e^{-z} on the set $[0,+\infty)$. Moreover, from (2.5) of Theorem 2.3, we also have that

$$\lim_{n\to\infty}\{\|e^{-x}-R_{\nu(n),n}(x)\|_{[0,+\infty)}\}^{1/n}=g(\sigma)$$

where the function g is defined in (2.4), so that (1.14) of Theorem 1.2 is satisfied with $q=1/g(\sigma)$. Next, from Theorem 1.1, for each $\varepsilon>0$ sufficiently small, no poles of $R_{\nu(n),n}(z)$ lie in the open infinite sector $S(\theta_0-\varepsilon)$ with $\theta_0:=\cos^{-1}\left(\frac{1-\sigma}{1+\sigma}\right)$, for all n large. Thus, Theorem 1.2 can be applied, for each $\varepsilon>0$, and letting $\varepsilon\to 0$ then establishes Theorem 2.7. ∎

Proof of Theorem 2.8. Suppose that the sequence of Padé approximants $\{R_{\nu(n),n}(z)\}_{n=1}^{\infty}$ of e^{-z} satisfies $0<\sigma_1=\varliminf_{n\to\infty}\nu(n)/n\leqq\varlimsup_{n\to\infty}\nu(n)/n=\sigma_2<1$. Since $\sigma_2<1$, then from Theorem 2.5, this sequence converges geometrically to e^{-z} on $[0,+\infty)$. Next, because $\sigma_1>0$, then Theorem 1.1 gives us that the poles of $\{R_{\nu(n),n}(z)\}_{n=1}^{\infty}$ omit some infinite sector $S(\theta_0)$, $\theta_0>0$. Then, applying Theorem 1.2, we see that there is an infinite sector $S(\theta)$, $\theta>0$, on which $\{R_{\nu(n),n}(z)\}_{n=1}^{\infty}$ converges geometrically to e^{-z}. ∎

It remains an open question if (2.16) is also a *necessary* condition that $\{R_{\nu(n),n}(z)\}_{n=1}^{\infty}$ converges geometrically to e^{-z} in some infinite sector.

In answer to the question posed in § 1, we now give the

Proof of Theorem 2.9. If $[m]$ denotes the integer part of m, consider the particular sequence $\{R_{[n/3],n}(z)\}_{n=1}^{\infty}$ of Padé approximants of e^{-z}. Because $\nu(n):=[n/3]$ satisfies $\lim_{n\to\infty}\nu(n)/n=1/3$, it follows from Theorem 2.3 that

$$\lim_{n\to\infty}\{\|e^{-x}-R_{[n/3],n}(x)\|_{[0,+\infty)}\}^{1/n}=\tfrac{1}{3}. \tag{3.15}$$

Since $\|e^{-x}-R_{[n/3],n}(x)\|_{[n,+\infty)}\leqq\|e^{-x}-R_{[n/3],n}(x)\|_{[0,+\infty)}$, it is evident from (3.15) that

$$\varlimsup_{n\to\infty}\{\|e^{-x}-R_{[n/3],n}(x)\|_{[n,+\infty)}\}^{1/n}\leqq\tfrac{1}{3}.$$

Now, writing $x=n+t$ with $0\leqq t<\infty$, the above becomes

$$\varlimsup_{n\to\infty}\{\|e^{-t}-e^{n}R_{[n/3],n}(n+t)\|_{[0,+\infty)}\}^{1/n}\leqq\frac{e}{3}<1.$$

Thus, the sequence $\{\tilde{R}_{[n/3],n}(z):=e^{n}R_{[n/3],n}(n+z)\}_{n=2}^{\infty}$ converges geometrically to e^{-z} on $[0,+\infty)$. Next, because $\nu(n):=[n/3]$ and because $n\geqq 2$, then $1<n<3\nu(n)+4$, and Theorem 1.3 can be applied, i.e., all the poles of the Padé approximants $R_{[n/3],n}(z)$ lie in the half-plane

$$\operatorname{Re} z<n-\nu(n)-2,\qquad \text{for every } n\geqq 2.$$

But this implies that $e^n R_{[n/3],n}(n+z)$ has all its poles in

$$\operatorname{Re} z < -\nu(n) - 2 < 0 \qquad \text{for every } n \geqq 2,$$

whence $\{\tilde{R}_{[n/3],n}(z)\}_{n=2}^{\infty}$ has all its poles in Re $z<0$. ∎

We remark that the proof of Theorem 2.9 can be used to generate *other* sequences $\{\tilde{R}_{\nu(n),n}(z)\}_{n=2}^{\infty}$ of rational functions, with $\tilde{R}_{\nu(n),n} \in \pi_{\nu(n),n}$ and $0 \leqq \nu(n) < n$, converging geometrically to e^{-z} on $[0, +\infty)$ and having all poles in Re $z<0$. Indeed, choose any sequence $\{\hat{\nu}(n)\}_{n=2}^{\infty}$ of positive integers such that

$$3\hat{\nu}(n)+4>n \qquad \text{for } n \geqq 2;\ \lim_{n\to\infty} \hat{\nu}(n)/n = \beta \qquad \text{where } g(\beta) < \frac{1}{e}.$$

Then, the sequence $\{e^n R_{\hat{\nu}(n),n}(n+z)\}_{n=2}^{\infty}$ of rational functions also satisfies the requirements in Theorem 2.9. Note, moreover, that any sequence of rational functions having these properties will necessarily converge geometrically to e^{-z} in some infinite sector $S(\theta)$ with $\theta>0$.

Added in Proof. With regard to Theorem 2.9, it is proven in "Geometric convergence to e^{-z} by rational functions with real poles" by E. B. Saff, A. Schönhage, and R. S. Varga, in Numer. Math. **25**, 307-322 (1975), that there is a sequence of rational functions satisfying (2.17) for which the poles of the rational functions are all on the negative real axis.

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