ENERGY BOUNDS FOR CODES AND DESIGNS IN HAMMING SPACES

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Abstract. We obtain universal bounds on the energy of codes and for designs in Hamming spaces. Our bounds hold for a large class of potential functions, allow unified treatment, and can be viewed as a generalization of the Levenshtein bounds for maximal codes.

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1. Introduction

Let $Q = \{0, 1, \dots, q-1\}$ be the alphabet of q symbols and $\mathbb{H}(n,q)$ be the set of all q-ary vectors $x = (x_1, x_2, \dots, x_n)$ over Q. The Hamming distance $d(x,y)$ between points $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ from $\mathbb{H}(n,q)$ is equal to the number of coordinates in which they differ. The use of q suggests that the alphabet is a finite field and most coding theory applications assume this but we will not make use of a field structure. In particular, q is not necessarily a power of a prime.

It is convenient (cf. [24]) to use the "inner product" $\langle x,y\rangle := 1 - \frac{2d(x,y)}{n}$ instead of distances. Let $T_n = \{t_0, t_1, \dots, t_n\}$, where $t_i := -1 + \frac{2i}{n}$, $i = 0, 1, \dots, n$, are all possible values of inner products in $\mathbb{H}(n,q)$, written in increasing order.

The above setting allows consideration of $\mathbb{H}(n,q)$ as a *polynomial metric space* [14, 22, 24]. This gives us major advantages, in particular in the binary case $q = 2$, where $\mathbb{H}(n,2)$ is antipodal with great similarities to the Euclidean spheres $\mathbb{S}^{n-1}$ as a polynomial metric space.

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We refer to any nonempty set $C \subset \mathbb{H}(n,q)$ as a *code*. For a given function $h(t) : [-1,1) \to (0,+\infty)$, we define the *h-energy* (or potential energy) of C by

$$E(n,C;h) := \frac{1}{|C|} \sum_{x,y\in C, x\neq y} h(\langle x,y\rangle).$$

A commonly arising problem (cf. [2, 3, 4, 12]) is to minimize the potential energy provided the cardinality $|C|$ of C is fixed; that is, to determine

$$\mathcal{E}(n,M;h) := \min\{E(n,C;h) : |C| = M\},$$

the minimum possible h-energy of a code $C \subset \mathbb{H}(n,q)$ of cardinality M.

As our bounds utilize information for the potential between the discrete nodes T_n, we assume h to be *absolutely monotone* on $[-1,1)$; i.e., $h^{(k)}(t) \geq 0$ for all $k \geq 0$ and all $t \in [-1,1)$. Thus our setting, while more restrictive than the setting in Cohn-Zhou's paper [12] where the authors use the discrete version[1] of absolute monotonicity, allows for unified definition, proof and investigation of universal (in sense of Levenshtein) bounds (Theorems 4.1, 4.2 and 5.1). In particular, we obtain bounds which are valid for all absolute monotone potentials. Furthermore, for the investigation of the asymptotic consequences (as $n \to \infty$) of our bounds, the assumption on the whole interval $[-1,1)$ seems better suited.

Energy minimizing codes $C \subset \mathbb{H}(n,q)$ for the potential function $h(t) = [2/n(1-t)]^\alpha$, where $\alpha \to \infty$, maximize the minimum distance $d(C) := \min\{d(x,y) : x,y \in C, x \neq y\}$ for fixed cardinality $M = |C|$. Another interesting potential function is $f(t) = \gamma^{2/n(1-t)}$, where γ is the Bhattacharyya parameter (cf. [25]). Further motivations are given in [2, 12].

In this paper we obtain universal lower bounds for $\mathcal{E}(n,M;h)$ where the universality is meant in Levenshtein's sense (bounds hold for all dimensions and cardinalities, cf. [24]), as well as in expressions which are common for a large class of potential functions. Our bounds are attained for many well known good codes (e.g. Hamming, Golay, MDS, or Nordstrom-Robinson codes) which are universally optimal in the sense of [12] (see also [24, Section 6.2, Table 6.4], [22, 23], [2]). Furthermore, the method of deriving our bounds is based on Hermite interpolation at nodes that are universally determined by Levenshtein in [22] (see also [23, 24]) and are independent of the particular potential.

In Section 2 we collect the main notions and results that are necessary for the derivation and explanation of our bounds. Section 3 is devoted to general bounds on the energy of codes and designs in $\mathbb{H}(n,q)$ are derived in a unified way from identity (8). These results are more or less folklore. In Section 4 we state and prove our main universal lower bounds for codes and designs in Hamming spaces. Although our bounds are optimal in the sense that they cannot be improved in a certain wide framework, it is still possible to find better bounds by linear programming. Section 5 describes three ways for finding such improvements – using the discrete structure of the inner products (the set T_n),

[1] Indeed, the discrete version of the absolute monotonicity from [12] follows from the continuous one.

using higher degrees polynomials, or using preliminary information on the structure of codes and designs under consideration. Examples of upper bounds are given in Section 6 where we investigate in detail the case of 2-designs in the binary case $\mathbb{H}(n,2)$. Section 7 is devoted to asymptotic consequences of our lower bounds in a natural process when the length n and the cardinality M tend to infinity in certain relation. In Section 8 we provide some examples.

2. Preliminaries

2.1. **Krawtchouk polynomials and the linear programming framework.** For fixed n and q, the (normalized) Krawtchouk polynomials are defined by

$$Q_i^{(n,q)}(t) := \frac{1}{r_i} K_i^{(n,q)}(d), \tag{1}$$

where $d = \frac{n(1-t)}{2}$, $r_i = (q-1)^i \binom{n}{i}$, and

$$K_i^{(n,q)}(d) = \sum_{j=0}^{i} (-1)^j (q-1)^{i-j} \binom{d}{j} \binom{n-d}{i-j}, \quad i = 0, 1, \ldots, n,$$

are the (usual) Krawtchouk polynomials corresponding to $\mathbb{H}(n,q)$. The polynomials $K_i^{(n,q)}(d)$ can be also defined by the three-term recurrence relation

$$K_0^{(n,q)}(d) = 1, \quad K_1^{(n,q)}(d) = n(q-1) - qd,$$

$$(i+1)K_{i+1}^{(n,q)}(d) = [i + (q-1)(n-i) - qd]K_i^{(n,q)}(d) - (q-1)(n-i+1)K_{i-1}^{(n,q)}(d),$$

for $1 \leq k \leq n-1$. The measure of orthogonality for the system $\{Q_i^{(n,q)}(t)\}_{i=0}^n$ is a discrete measure given by

$$d\mu_n(t) := q^{-n} \sum_{i=0}^{n} \binom{n}{i} (q-1)^i \delta_{t_i}, \tag{2}$$

where δ_{t_i} is the Dirac-delta measure at $t_i \in T_n$. Note that the form

$$\langle f, g \rangle = \int f(t) g(t) d\mu_n(t) \tag{3}$$

defines an inner product over the class $\mathcal{P}_n$ of polynomials of degree less than or equal to n.

We also need the so-called adjacent polynomials as introduced by Levenshtein (cf. [24, Section 6.2], see also [22, 23])

$$Q_i^{(1,0,n,q)}(t) = \frac{K_i^{(n-1,q)}(d-1)}{\sum_{j=0}^{i}\binom{n}{j}(q-1)^j}, \tag{4}$$

$$Q_i^{(1,1,n,q)}(t) = \frac{K_i^{(n-2,q)}(d-1)}{\sum_{j=0}^{i}\binom{n-1}{j}(q-1)^j}, \tag{5}$$

$$Q_i^{(0,1,n,q)}(t) = \frac{K_i^{(n-1,q)}(d)}{\binom{n-1}{i}(q-1)^i}, \tag{6}$$

where $d = n(1-t)/2$. The corresponding measures of orthogonality are, respectively,

$$(1-t)d\mu_n(t), \quad (1-t)(1+t)d\mu_n(t), \quad (1+t)d\mu_n(t). \tag{7}$$

If $f(t) \in \mathbb{R}[t]$ is a real polynomial of degree $m \le n$, then $f(t)$ can be uniquely expanded in terms of the Krawtchouk polynomials as $f(t) = \sum_{i=0}^{m} f_i Q_i^{(n,q)}(t)$. The identity (see, for example, [21, Equation (1.7)], [22, Equation (1.20)], [23, Equation (26)])

$$|C|f(1) + \sum_{x,y\in C, x\neq y} f(\langle x,y\rangle) = |C|^2 f_0 + \sum_{i=1}^{m} \frac{f_i}{r_i} \sum_{j=1}^{r_i} \left(\sum_{x\in C} Y_{ij}(x) \right)^2 \tag{8}$$

serves as a major source of estimations by linear programming. Here $C \subset \mathbb{H}(n,q)$ is an arbitrary code, $f(t)$ is as above, $Y_{ij}(x)$, $j = 1, 2, \ldots, r_i$, are the functions[2] of the space $V_i = \{u(x) : \mathbb{H}(n,q) \to \mathbb{C} \mid u(x) = \xi^{\alpha_1 x_{j_1} + \cdots + \alpha_i x_{j_i}}, 1 \le j_1 < \cdots < j_i \le n, \alpha_1, \ldots, \alpha_i \in \{1, \ldots, q-1\}\}$, ξ is a complex primitive q-th root of unity and $r_i = |V_i| = (q-1)^i \binom{n}{i}$ as above.

The Rao bound (subsection 2.3) and the Levenshtein bound (subsection 2.4) can be obtained after the sums on both sides of (8) are neglected and suitable polynomials (optimal in some sense) are applied.

2.2. **Designs in $\mathbb{H}(n,q)$ and their energy.** We also need the notion of designs (see [20, 24]) in $\mathbb{H}(n,q)$, which play an important role in the understanding of energy problems in Hamming spaces. The designs in Hamming spaces are well studied since they are, in a certain sense, an approximation of the whole space $\mathbb{H}(n,q)$.

We first give a combinatorial definition.

Definition 2.1. Let τ and λ be positive integers. A τ-design $C \subset \mathbb{H}(n,q)$ of strength τ and index λ is a code $C \subset \mathbb{H}(n,q)$ of cardinality $|C| = M = \lambda q^\tau$ such that the $M \times n$ matrix obtained from the codewords of C as rows has the following property: every $M \times \tau$ submatrix contains all ordered τ-tuples of $\mathbb{H}(\tau,q)$, each one exactly $\lambda = \frac{M}{q^\tau}$ times as rows.

[2] In fact, these functions are orthonormal w.r.t. the inner product $\langle u,v\rangle = q^{-n}\sum_{x\in\mathbb{H}(n,q)} u(x)\overline{v(x)}$, and hence are linear independent (see [23, Theorem 2.1]).

The characterization of codes by their strength as designs started with Delsarte [13], where $\tau+1=d'$ is the dual distance of the code C (see also [14, 23, 24]).

An equivalent definition asserts that $C \subset \mathbb{H}(n,q)$ is a τ-design if $\sum_{x\in C} Y_{ij}(x)=0$ for every $1\leq i\leq \tau$ and every $1 \leq j \leq r_i$. This reduces the right-hand side of (8) to $f_0|C|$ for polynomials of degree at most τ and thus suggests that such polynomials could be very useful in derivation and investigation of linear programming bounds.

In this paper we obtain bounds for the minimum and maximum possible potential energies of designs in $\mathbb{H}(n,q)$. Denote by

$$\mathcal{L}(n,M,\tau;h) := \min\{E(n,C;h) : |C|=M, C\subset \mathbb{H}(n,q) \text{ is a } \tau\text{-design}\}$$

the minimum possible h-energy of τ-designs in $\mathbb{H}(n,q)$ of M points, and by

$$\mathcal{U}(n,M,\tau;h) := \max\{E(n,C;h) : |C|=M, C\subset \mathbb{H}(n,q) \text{ is a } \tau\text{-design}\}$$

the maximum possible h-energy of τ-designs in $\mathbb{H}(n,q)$ of M points.

We also consider important to have energy estimates for codes with prescribed minimum distance instead of cardinality. The minimum distance requirement yields interesting upper bounds on energies as well. So we denote by

$$\mathcal{F}(n,d;h) := \min\{E(n,C;h) : d(C)=d\}$$

the minimum possible h-energy of a code $C\subset \mathbb{H}(n,q)$ of fixed minimum distance d, and by

$$\mathcal{G}(n,d;h) := \max\{E(n,C;h) : d(C)=d\}$$

the maximum possible h-energy of a code $C\subset \mathbb{H}(n,q)$ of fixed minimum distance d. One can make transitions from bounds on $\mathcal{E}(n,M;h)$ to bounds for $\mathcal{F}(n,d;h)$ and $\mathcal{G}(n,d;h)$ by using suitable cardinalities.

All the quantities $\mathcal{E}(n,M;h)$, $\mathcal{L}(n,M,\tau;h)$, $\mathcal{U}(n,M,\tau;h)$, $\mathcal{F}(n,d;h)$ and $\mathcal{G}(n,d;h)$ can be estimated by polynomials techniques (linear programming method) by using suitable polynomials in (8).

2.3. **Rao bound.** For fixed strength τ and dimension n denote

$$B(n,\tau)=\min\{|C| : \exists\ \tau\text{-design } C\subset \mathbb{H}(n,q)\}.$$

The classical universal lower bound on $B(n,\tau)$ is due to Rao [27] (see also [20, 14, 24])

$$B(n,\tau)\geq R(n,\tau)=\begin{cases} q\displaystyle\sum_{i=0}^{k-1}\binom{n-1}{i}(q-1)^i, & \text{if } \tau=2k-1,\\ \displaystyle\sum_{i=0}^{k}\binom{n}{i}(q-1)^i, & \text{if } \tau=2k.\end{cases} \tag{9}$$

The bound (9) can be obtained by linear programming using in (8) the following polynomials of degree τ:

$$f^{(\tau)}(t) = (t+1)^{\varepsilon} \left(\sum_{i=0}^{k} r_i^{(0,1)} Q_i^{(0,1,n,q)}(t) \right)^2 ,$$

where $\tau = 2k-1+\varepsilon$, $\varepsilon \in \{0,1\}$, $r_i^{(0,1)} = (q-1)^i \binom{n-1}{i}$ for $i = 0,1,\dots,n-1$.

We use the Rao bound to indicate which parameters must be chosen in order to obtain universal lower bounds on $\mathcal{E}(n,M;h)$, $\mathcal{L}(n,M,\tau;h)$ and $\mathcal{F}(n,d;h)$ and upper bounds for $\mathcal{U}(n,M,\tau;h)$ and $\mathcal{G}(n,d;h)$. More precisely, for given length n and cardinality M, we find the unique

$$\tau := \tau(n,M) \quad \text{such that} \quad M \in (R(n,\tau), R(n,\tau+1)] .$$

Then all other necessary parameters come with n, M and τ as shown in Subsection 2.5.

2.4. **Levenshtein bound.** Let

$$A_q(n,s) := \max\{|C| \colon C \subset \mathbb{H}(n,q), \langle x,y \rangle \le s,\, x \neq y \in C\}$$

denote the maximal possible cardinality of a code in $\mathbb{H}(n,q)$ of prescribed maximal inner product s. We remark that in Coding Theory this quantity is usually denoted by $A_q(n,d)$, where $s = 1 - \frac{2d}{n}$, so we have replaced the condition $d(x,y) \ge d$ by $\langle x,y \rangle \le s$.

For $a,b \in \{0,1\}$ and $i \in \{1,2,\dots,n-a-b\}$, denote by $t_i^{a,b}$ the greatest zero of the adjacent polynomial $Q_i^{(a,b,n,q)}(t)$ (see (4)) and also define $t_0^{1,1} = -1$. We have the interlacing properties $t_{k-1}^{1,1} < t_k^{1,0} < t_k^{1,1}$, see [24, Lemmas 5.29, 5.30]. For positive integer τ, let $\mathcal{I}_\tau$ denote the interval

$$\mathcal{I}_\tau := \begin{cases} \left[t_{k-1}^{1,1}, t_k^{1,0} \right], & \text{if } \tau = 2k-1, \\ \left[t_k^{1,0}, t_k^{1,1} \right], & \text{if } \tau = 2k. \end{cases}$$

Then the intervals $\mathcal{I}_\tau$ are well defined and partition $\mathcal{I} = [-1,1)$ into subintervals with non-overlapping interiors.

For every $s \in \mathcal{I}_\tau$, Levenshtein used linear programming by (8) with the following polynomials of degree τ:

$$f_\tau^{(n,s)}(t) = (t-s)(t+1)^{\varepsilon} \left(\sum_{i=0}^{k-1} r_i^{(1,\varepsilon)} Q_i^{(1,\varepsilon,n,q)}(t) Q_i^{(1,\varepsilon,n,q)}(s) \right)^2 \tag{10}$$

(see [24, Equations (5.81) and (5.82)]), where $\tau = 2k-1+\varepsilon$, $\varepsilon \in \{0,1\}$, and

$$r_i^{(1,\varepsilon)} = \left(\sum_{j=0}^{i} \binom{n-\varepsilon}{j} (q-1)^j \right)^2 / (q-1)^i \binom{n-1-\varepsilon}{i}.$$

This yields the bound (see [24, Equation (6.45) and (6.46)])

$$A_q(n,s) \leq \begin{cases} L_{2k-1}(n,s) = \left(1 - \frac{Q_{k-1}^{(1,0,n,q)}(s)}{Q_k^{(n,q)}(s)}\right) \sum_{j=0}^{k-1} \binom{n}{j}(q-1)^j, & \text{if } s \in \mathcal{I}_{2k-1}, \\ L_{2k}(n,s) = q\left(1 - \frac{Q_{k-1}^{(1,1,n,q)}(s)}{Q_k^{(0,1,n,q)}(s)}\right) \sum_{j=0}^{k-1} \binom{n-1}{j}(q-1)^j, & \text{if } s \in \mathcal{I}_{2k}. \end{cases} \tag{11}$$

An important connection between the Rao (9) and the Levenshtein (11) bounds is given by the equalities

$$\begin{gathered} L_{2k-2}(n, t_{k-1}^{1,1}) = L_{2k-1}(n, t_{k-1}^{1,1}) = R(n, 2k-1), \\ L_{2k-1}(n, t_k^{1,0}) = L_{2k}(n, t_k^{1,0}) = R(n, 2k) \end{gathered} \tag{12}$$

at the ends of the intervals $\mathcal{I}_\tau$. The relations (12) explain and justify our connection between the cardinality M and the strength (degree) $\tau = \tau(n, M)$.

2.5. **Useful quadrature.** Levenshtein [22] proves (see also [24, Section 5, Theorem 5.39] and [23]) that for every fixed (cardinality) $M > R(n, 2k-1)$ there exist uniquely determined real numbers $-1 < \alpha_0 < \alpha_1 < \cdots < \alpha_{k-1} < 1$ and positive numbers $\rho_0, \rho_1, \ldots, \rho_{k-1}$, such that the equality

$$f_0 = \frac{f(1)}{M} + \sum_{i=0}^{k-1} \rho_i f(\alpha_i) \tag{13}$$

holds for every real polynomial $f(t)$ of degree at most $2k-1$.

The numbers α_i, $i = 0, 1, \ldots, k-1$, are the roots of the equation

$$P_k(t)P_{k-1}(\alpha_{k-1}) - P_k(\alpha_{k-1})P_{k-1}(t) = 0, \tag{14}$$

where $P_i(t) = Q_i^{(1,0,n,q)}(t)$. In fact, α_i, $i = 0, 1, \ldots, k-1$, are also the roots of the polynomial $f_{2k-1}^{(n,s)}(t)$ from (10) (see [22, 24]). In our approach, it is convenient to find $\alpha_{k-1} = s$ from the equation $M = L_{2k-1}(n, s)$ and to solve then (14).

Similarly, for every fixed (cardinality) $M > R(n, 2k)$ there exist uniquely determined real numbers $-1 = \beta_0 < \beta_1 < \cdots < \beta_k < 1$ and positive numbers $\gamma_0, \gamma_1, \ldots, \gamma_k$, such that the equality

$$f_0 = \frac{f(1)}{M} + \sum_{i=0}^{k} \gamma_i f(\beta_i) \tag{15}$$

holds for every real polynomial $f(t)$ of degree at most $2k$. The numbers β_i, $i = 1, \ldots, k$, are the roots of the equation

$$P_k(t)P_{k-1}(\beta_k) - P_k(\beta_k)P_{k-1}(t) = 0, \tag{16}$$

where $P_i(t) = Q_i^{(1,1,n,q)}(t)$ and also roots of the polynomial $f_{2k}^{(n,s)}(t)$ from (10) (see [22, 24]). Similarly to the odd case, $\beta_k = s$ can be found from the equation $M = L_{2k}(n, s)$ and then (16) can be solved.

As mentioned in the end of subsection 2.3 we always take care where the cardinality M is located with respect to the Rao bound. We actually associate M with the corresponding numbers:

$$\alpha_0, \alpha_1, \ldots, \alpha_{k-1}, \rho_0, \rho_1, \ldots, \rho_{k-1} \text{ when } M = L_{2k-1}(n,s) \in (R(n,2k-1), R(n,2k)]$$

or, analogously, with the corresponding

$$\beta_0, \beta_1, \ldots, \beta_k, \gamma_0, \gamma_1, \ldots, \gamma_k \text{ when } M = L_{2k}(n,s) \in (R(n,2k), R(n,2k+1)].$$

Then the quadratures (13) and (15) can be applied for deriving and calculation of our bounds.

We also use the kernels (see (2.69) in [24, Section 2]; also Section 5 in [24])

$$T_k(u,v) = \sum_{i=0}^{k} r_i Q_i^{(n,q)}(u) Q_i^{(n,q)}(v) = c \cdot \frac{Q_{k+1}^{(n,q)}(u) Q_k^{(n,q)}(v) - Q_{k+1}^{(n,q)}(v) Q_k^{(n,q)}(u)}{u-v} \tag{17}$$

(c is a positive constant, $u \neq v$, this is in fact the Christoffel-Darboux formula). Note that the $(1,0)$ and $(1,1)$ analogs of $T_k(u,v)$ define the Levenshtein polynomials – see (10).

2.6. **Bounds for the extreme inner products of designs in $\mathbb{H}(n,q)$.** Denote by $s(C) = \max\{\langle x,y\rangle : x,y \in C, x \neq y\}$ and $\ell(C) = \min\{\langle x,y\rangle : x,y \in C, x \neq y\}$ the extreme inner products of a code C. The quantities

$$s(n,M,\tau) = \max\{s(C) : C \subset \mathbb{H}(n,q) \text{ is a } \tau\text{-design}, |C| = M\},$$

and

$$\ell(n,M,\tau) = \min\{\ell(C) : C \subset \mathbb{H}(n,q) \text{ is a } \tau\text{-design}, |C| = M\},$$

can be estimated (see [17, 6]) by using the following equivalent algebraic definition of designs in $\mathbb{H}(n,q)$.

Definition 2.2. A τ-design $C \subset \mathbb{H}(n,q)$ is a code such that the equality

$$\sum_{y \in C} f(\langle x,y\rangle) = f_0 |C| \tag{18}$$

holds for any point $x \in \mathbb{H}(n,q)$ and any real polynomial $f(t) = \sum_{i=0}^{r} f_i Q_i^{(n,q)}(t)$ of degree $r \leq \tau$.

As shown in [17, 6], suitable polynomials in Definition 2.2 give non-trivial upper and lower bounds on $s(n,N,\tau)$ for every n, τ and cardinality $N \in [D(n,\tau), D(n,\tau+1)]$ and on $\ell(n,N,2k)$ for every n, even $\tau = 2k$ and cardinality $N \in [D(n,2k), D(n,2k+1))$. We will derive and use such bounds in Subsection 5.3.

3. General linear programming bounds for $\mathcal{E}(n,M;h)$, $\mathcal{F}(n,d;h)$, $\mathcal{L}(n,M,\tau;h)$, $\mathcal{U}(n,M,\tau;h)$ and $\mathcal{G}(n,d;h)$

The next assertion follows from (8) and is well known (see [12, Proposition 5]).

Theorem 3.1. *Let n and h be fixed and $f(t)$ be a real polynomial such that:*

(A1) *$f(t) \leq h(t)$ for every $t \in T_n$;*

(A2) *the coefficients in the Krawtchouk expansion $f(t) = \sum_{i=0}^{n} f_i Q_i^{(n,q)}(t)$ satisfy $f_i \geq 0$ for every $i \geq 1$.*

Then $\mathcal{E}(n,M;h) \geq f_0 M - f(1)$ for every M.

The next four assertions are more or less immediate corollaries of (8) but are not explicit in the literature. The design's property allows relaxation of the conditions on the coefficients in the Krawtchouk expansion (Theorems 3.2 and 3.4), and the use of the minimum distance d allows formulations (Theorems 3.3 and 3.5) which involve the function $A_q(n,s)$.

Theorem 3.2. *Let n, h and τ be fixed and $f(t)$ be a real polynomial that satisfies* (A1) *and*

(A2′) *the coefficients in the Krawtchouk expansion $f(t) = \sum_{i=0}^{n} f_i Q_i^{(n,q)}(t)$ satisfy $f_i \geq 0$ for every $i \geq \tau + 1$.*

Then $\mathcal{L}(n,M,\tau;h) \geq f_0 M - f(1)$ for every $M \geq R(n,\tau)$.

The quantity $\mathcal{F}(n,d;h)$ can be bounded from below in a similar way.

Theorem 3.3. *Let n, $d = n(1-s)/2$ and h be fixed and $f(t)$ be a real polynomial that satisfies* (A2) *and*

(A1′) *$f(t) \leq h(t)$ for every $t \in T_n \cap [-1, 1-2d/n] = T_n \cap [-1, s]$.*

Then $\mathcal{F}(n,d;h) \geq f_0 M - f(1)$, where M is a feasible size of a code of minimum distance d. In particular, $\mathcal{F}(n,d;h) \geq f_0 A_q(n,s) - f(1)$.

Denote by $A_{n,M;h}$ (respectively $A_{n,M,\tau;h}$ or $B_{n,d;h}$) the set of polynomials that satisfy the conditions (A1) and (A2) (respectively (A1) and (A2′) or (A1′) and (A2)).

One similarly obtains general upper bounds for $\mathcal{U}(n,M,\tau;h)$ and $\mathcal{G}(n,d;h)$.

Theorem 3.4. *Let n, τ and h be fixed and $g(t)$ be a real polynomial such that:*

(B1) *$g(t) \geq h(t)$ for every $t \in T_n \cap [\ell(n,M,\tau), s(n,M,\tau)]$;*

(B2) *the coefficients in the Krawtchouk expansion $g(t) = \sum_{i=0}^{n} g_i Q_i^{(n,q)}(t)$ satisfy $g_i \leq 0$ for $i \geq \tau + 1$.*

Then $\mathcal{U}(n,M,\tau;h) \leq g_0 M - g(1)$ for every $M \geq R(n,\tau)$.

Theorem 3.5. *Let n, $d = n(1-s)/2$ and h be fixed and $g(t)$ be a real polynomial such that:*

(B1′) *$g(t) \geq h(t)$ for every $t \in T_n \cap [-1, 1-2d/n] = T_n \cap [-1, s]$;*

(B2′) *the coefficients in the Krawtchouk expansion* $g(t) = \sum_{i=0}^{n} g_i Q_i^{(n,q)}(t)$ *satisfy* $g_i \leq 0$ *for every* $i \geq 1$.

Then $\mathcal{G}(n,d;h) \leq g_0 M - g(1)$, *where* M *is a feasible size of a code of minimum distance* d. *In particular,* $\mathcal{G}(n,d;h) \leq g_0 A_q(n,s) - g(1)$.

Denote by $B_{n,M,\tau;h}$ (respectively $C_{n,d;h}$) the set of polynomials satisfying the conditions (B1) and (B2) (respectively (B1′) and (B2′)).

Theorems 3.1–3.5 impose corresponding optimization problems.

Problem 3.6. Find polynomial(s) $f \in A_{n,M;h}$ ($f \in A_{n,M,\tau;h}$, $f \in B_{n,d;h}$, respectively) which give maximum value of $f_0 M - f(1) = f_0(M-1) - (f_1 + \cdots + f_n)$.

Problem 3.7. Find polynomial(s) $g \in B_{n,M,\tau;h}$ ($g \in C_{n,d;h}$, respectively) which give minimum value of $g_0 M - g(1) = g_0(M-1) - (g_1 + \cdots + g_n)$.

Another general problem asks for finding universally optimal codes [3, 12]. Such codes attain the minimum possible energy with respect to all absolute monotone potential functions simultaneously.

4. Universal lower bounds for $\mathcal{L}(n,M,\tau;h)$ and $\mathcal{E}(n,M;h)$

We begin with the bound for $\mathcal{L}(n,M,\tau;h)$ which is easier to prove.

Theorem 4.1. *Let* n *and* τ *be fixed and* h *be absolutely monotone on* $[-1,1)$. *Then*

$$\mathcal{L}(n,M,\tau;h) \geq \begin{cases} M \displaystyle\sum_{i=0}^{k-1} \rho_i h(\alpha_i), & \tau = 2k-1,\ \forall M \in (R(n,2k-1), R(n,2k)]\,, \\ M \displaystyle\sum_{i=0}^{k} \gamma_i h(\beta_i), & \tau = 2k,\ \forall M \in (R(n,2k), R(n,2k+1)]\,. \end{cases} \tag{19}$$

Moreover, the bounds (19) *cannot be improved by utilizing polynomials* $f(t)$ *of degree at most* τ *satisfying* $f(t) \leq h(t)$ *for every* $t \in [-1,1]$.

Proof. Let $\tau = 2k-1$ and the polynomial $f(t)$ be the Hermite interpolant of $h(t)$ at the points $\alpha_0, \alpha_1, \ldots, \alpha_{k-1}$, i.e. $f(\alpha_i) = h(\alpha_i)$ and $f'(\alpha_i) = h'(\alpha_i)$ for every $i = 0, 1, \ldots, k-1$. Then $\deg(f) \leq 2k-1$ and the condition (A2)′ is trivially satisfied. Furthermore, it follows from the Role theorem that $f(t) \leq h(t)$ for every $t \in [-1,1)$. Therefore (A1) is also satisfied and $f(t) \in A_{n,M,2k-1;h}$. We calculate the bound by using the quadrature formula (13):

$$f_0 = \frac{f(1)}{M} + \sum_{i=0}^{k-1} \rho_i f(\alpha_i) \iff f_0 M - f(1) = M \sum_{i=0}^{k-1} \rho_i f(\alpha_i)$$

and the last equality implies $f_0 M - f(1) = M \sum_{i=0}^{k-1} \rho_i h(\alpha_i)$ since $f(\alpha_i) = h(\alpha_i)$ from the interpolation.

Furthermore, for any polynomial $F(t)$ of degree at most $2k-1$ satisfying $F(t) \le h(t)$ for every $t \in [-1,1]$, we have from the quadrature formula (13) for $f(t)$

$$F_0 M - F(1) = M \sum_{i=0}^{k-1} \rho_i h(\alpha_i) \ge M \sum_{i=0}^{k-1} \rho_i f(\alpha_i) = f_0 M - f(1),$$

which proves the optimality property of $f(t)$.

The case $\tau = 2k$ is similar, with single intersection $f(\beta_0) = h(\beta_0)$ of the graphs of $f(t)$ and $h(t)$ at the point $\beta_0 = -1$ and $f(\beta_i) = h(\beta_i)$ and $f'(\beta_i) = h'(\beta_i)$ for every $i = 1, \ldots, k$. Now the degree of $f(t)$ is at most $2k$ and again (A2)$'$ is trivially satisfied and (A1) follows from the Role theorem. $\square$

The Hermite interpolant of $h(t)$ at the $\{\alpha_i\}$ nodes can be also utilized for obtaining the same bounds for $\mathcal{E}(n, M; h)$. The proof of the positive-definiteness of these Hermite interpolants (i.e. the condition (A2)) follows the framework of [11, Section 3], applied to discrete orthogonal polynomials.

Theorem 4.2. *Let n be fixed and h be absolutely monotone on $[-1,1)$. Then*

$$\mathcal{E}(n, M; h) \ge \begin{cases} M \sum_{i=0}^{k-1} \rho_i h(\alpha_i), & \forall M \in (R(n,2k-1), R(n,2k)]\,, \\ M \sum_{i=0}^{k} \gamma_i h(\beta_i), & \forall M \in (R(n,2k), R(n,2k+1)]\,. \end{cases} \tag{20}$$

Proof. The polynomial from Theorem 4.1 serves as the solution of the linear program given in Theorem 3.1. We already have established (A1). Since we do not have the design property, we need to verify that (A2) is satisfied. We shall do this by adapting the approach in [11, Sections 3 and 5], where for the two cases in the right-hand side of (20) we shall consider the discrete measures $(1-t)d\mu_n(t)$ and $(1-t^2)d\mu_n(t)$ and the associated orthogonal polynomials $\{Q_i^{(1,0,n,q)}(t)\}$ and $\{Q_i^{(1,1,n,q)}(t)\}$ respectively.

The results in [11, Section 3] (and in particular Theorem 3.1 there) are proved for a Borel measure μ such that

$$\int p(t)^2 \, d\mu(t) > 0$$

for all polynomials p that are not identically zero and for the associated orthogonal polynomials. However, a careful inspection of the proofs in that section reveals that the results remain true for the discrete measures defined in (2) and (7) and the associated orthogonal polynomials of degree up to n. This is consequence from the fact that for such polynomials the bilinear form (3) indeed defines an inner product.

Since the conductivity property for Hermite interpolants, as discussed in [11, Section 5], is independent of the measure of orthogonality, what remains to prove is the positivity of the constant $-Q_k^{(1,0,n,q)}(\alpha_{k-1})/Q_{k-1}^{(1,0,n,q)}(\alpha_{k-1})$, respectively $-Q_k^{(1,0,n,q)}(\beta_k)/Q_{k-1}^{(1,0,n,q)}(\beta_k)$.

This follows from the normalization $Q_i^{(1,0,n,q)}(1) = 1$, respectively $Q_i^{(1,1,n,q)}(1) = 1$, and the interlacing property of the zeros of $Q_k^{(1,0,n,q)}(t)$ and $Q_{k-1}^{(1,0,n,q)}(t)$, respectively $Q_k^{(1,1,n,q)}(t)$ and $Q_{k-1}^{(1,1,n,q)}(t)$.

The optimality of the polynomial from Theorem 4.1 was already derived in the proof of the previous theorem. This completes the proof. □

Remark 4.3. We note that the roots of (14) and (16) interlace with the zeros of $\{Q_k^{(1,0,n,q)}(t)\}$ and $\{Q_k^{(1,1,n,q)}(t)\}$ respectively (see [28, Theorem 3.3.4]), which in turn interlace the Krawtchouk polynomials $\{Q_k^{(n,q)}(t)\}$ defined in (1) (see [24, Lemmas 5.29, 5.30]). Therefore, the asymptotic distribution of the quadrature nodes $\{\alpha_i\}$, respectively $\{\beta_i\}$, as $k/n \to const$ when $k, n \to \infty$, is governed by a constrained energy problem as studied in [15] and [16]. We shall investigate this behavior in details in a future work.

It is clear that all maximal codes which attain the Levenshtein bounds $L_\tau(n, s)$ have the necessary strength and the suitable inner products and therefore achieve our bounds (19) and (20) as well. In [12], a table of universally optimal codes in $\mathbb{H}(n, q)$ is presented. Some of these codes attain (11) and our bounds. Clearly, all codes on (19) and (20) are universally optimal.

For fixed n and $d \geq 3$, assume that bounds $M_1 \leq A_q(n, s) \leq M_2$, $s = 1 - 2d/n$, are known (see [1, 5, 18, 26, 29, 30]). Then every putative code $C \subset \mathbb{H}(n, q)$ of minimum distance $d(C) = d$ and cardinality $M \in [M_1, M_2]$ has energy $E(n, C; h)$ bounded from below by (20). This results in bounds for $\mathcal{F}(n, d; h)$ that will be discussed elsewhere.

In the end of this section we remark that the bounds (19) and (20) are discrete analogs of bounds on the potential energy of spherical codes and designs recently obtained by the authors [9, 10].

5. On optimality of the universal lower bounds

In this section we consider three different approaches for finding better than the universal bounds.

5.1. **Using the discrete nature of the inner products.** Utilizing Cohn-Zhao's [12] approach to finding good polynomials we could sometimes improve the universal bounds by using Lagrange instead of Hermite interpolation in a set of inner products which correspond to what is called pair covering in [12]. In fact, the nodes (α_i) for the odd case and (β_i) for the even case show which pairs must be covered.

Indeed, if $\alpha_i \in [t_j, t_{j+1})$, then we can replace the Hermite's touching of the graphs of f and h at the point α_i by intersection in the points t_j and t_{j+1}. This means that the graph of f goes above the graph of h in the interval (t_j, t_{j+1}) and, in particular, $f(\alpha_i) \geq h(\alpha_i)$. In other words, the idea is to replace the conditions $f(\alpha_i) = h(\alpha_i)$ and $f'(\alpha_i) = h'(\alpha_i)$ by $f(t_j) = h(t_j)$ and $f(t_{j+1}) = h(t_{j+1})$. Of course, this sometimes needs adjustments as explained below.

We consider in more detail the odd case $\tau = 2k-1$. Observe that the nodes α_i, $i = 0, 1, \dots, k-2$, as zeros of orthogonal polynomials w.r.t. the discrete measure $(t - \alpha_{k-1}(1-t)d\mu_n(t)$ are separated by the mass points of μ_n, that is the inner products T_n.

For every $i = 0, 1, \dots, k-1$, define the pair $(t_{j(i)}, t_{j(i)+1})$ of neighbour elements of T_n by $t_{j(i)} \le \alpha_i < t_{j(i)+1}$; i.e., $\alpha_i \in [t_{j(i)}, t_{j(i)+1})$. If $t_{j(i)+1} < t_{j(i+1)}$ for every $i \in \{0, 1 \dots, k-2\}$, then the pairs $(t_{j(i)}, t_{j(i)+1})$, $i = 0, 1, \dots, k-1$, are disjoint. Then the Lagrange interpolant $f(t)$ of $h(t)$ in the points

$$t_{j(0)}, t_{j(0)+1}, t_{j(1)}, t_{j(1)+1}, \dots, t_{j(k-1)}, t_{j(k-1)+1}$$

has degree at most $2k-1$ and satisfies the property (A1).

If $t_{j(i)+1} = t_{j(i+1)}$ for some i (that is the right end of the interval of α_i coincides with the left end of the interval of α_{i+1}) then we apply the Hermite requirements $f(t_{j(i+1)}) = h(t_{j(i+1)})$ and $f'(t_{j(i+1)}) = h'(t_{j(i+1)})$. Now the graph of $f(t)$ touches from above the graph of $h(t)$ at the coincidence point $t_{j(i+1)}$ and (A1) is again satisfied.

Summarizing, we see that this construction always implies that the condition (A1) is satisfied. The condition (A2$'$) is trivially satisfied and therefore $f(t) \in A_{n,M,2k-1;h}$. Furthermore, our construction implies, as mentioned above, that $f(\alpha_i) \ge h(\alpha_i)$ for every $i = 0, 1, \dots, k-1$. Hence we obtain

$$\mathcal{L}(n, M, 2k-1; h) \ge f_0 M - f(1) \ge M \sum_{i=0}^{k-1} \rho_i h(\alpha_i)$$

and the bound of $f(t)$ is at least as good as the universal bounds (19). Clearly, the improvement is strict if $\alpha_i \notin T_n$ for at least one i. We conjecture (see Proposition 21 in [12]) that the condition (A2) is also satisfied and (20) can be correspondingly improved.

The even case is dealt analogously with the corresponding attention to $\beta_0 = -1$.

5.2. **Test functions and improvements by higher degree polynomials.** Let n and M be fixed and $\tau = \tau(n, M)$ be as explained in the end of subsection 2.3. As above, the equation $L_\tau(n, s) = M$, $s = \alpha_{k-1}$ or β_k, defines all necessary parameters as in subsection 2.5. Let $j \ge \tau + 1$ be a positive integer. We consider the following *test-functions* in n and s:

$$P_j(n, s) := \begin{cases} \dfrac{1}{M} + \displaystyle\sum_{i=0}^{k-1} \rho_i Q_j^{(n,q)}(\alpha_i) & \text{for} \quad s \in \mathcal{I}_{2k-1} \\ \dfrac{1}{M} + \displaystyle\sum_{i=0}^{k} \gamma_i Q_j^{(n,q)}(\beta_i) & \text{for} \quad s \in \mathcal{I}_{2k}. \end{cases} \tag{21}$$

The test functions (21) were introduced in 1998 for polynomial metric spaces (which include $\mathbb{H}(n, q)$) by Boyvalenkov and Danev [7], where the binary case $\mathbb{H}(n, 2)$ was considered in detail.

The next theorem shows that the functions $P_j(n,s)$ give necessary and sufficient conditions for existence of improving polynomials of higher degrees satisfying the condition $f(t) \leq h(t)$ in $[-1,1]$.

Theorem 5.1. *Let h be strictly absolutely monotone function. The bounds (19) or (20) can be improved by a polynomial of degree at least $\tau+1$ from $A_{n,M,\tau;h}$ or $A_{n,M,h}$, satisfying $f(t) \leq h(t)$ in $[-1,1]$ if and only if $P_j(n,s) < 0$ for some $j \geq \tau+1$. Furthermore, if $P_j(n,s) < 0$ for some $j \geq \tau+1$, then (19) or (20) can be improved by a polynomial from $A_{n,M,\tau;h}$ or $A_{n,M,h}$, satisfying $f(t) \leq h(t)$ in $[-1,1]$ of degree exactly j.*

Proof. We give a proof for $\tau = 2k-1$.

(Necessity) Suppose $f(t) \in A_{n,M,\tau;h}$ or $A_{n,M,h}$ and $f(t) \leq h(t)$ in $[-1,1]$. Then

$$f(t) = g(t) + \sum_{j \geq \tau+1} f_j Q_j^{(n,q)}(t),$$

for some g of degree at most τ and $f_j \geq 0$, for $j \geq \tau+1$. Note that $f_0 = g_0$. Furthermore, using (13) for $g(t)$, we obtain

$$\begin{aligned} Mf_0 - f(1) &= Mg_0 - f(1) = g(1) + M\sum_{i=0}^{k-1} \rho_i g(\alpha_i) - \left(g(1) + \sum_{j\geq\tau+1} f_j \right) \\ &= M\sum_{i=0}^{k-1} \rho_i \left(f(\alpha_i) - \sum_{j\geq\tau+1} f_j Q_j^{(n,q)}(\alpha_i) \right) - \left(\sum_{j\geq\tau+1} f_j \right) \\ &= M\sum_{i=0}^{k-1} \rho_i f(\alpha_i) - M\sum_{j\geq\tau+1} f_j \left(\frac{1}{M} + \sum_{i=0}^{k-1} \rho_i Q_j^{(n,q)}(\alpha_i) \right) \\ &= M\sum_{i=0}^{k-1} \rho_i f(\alpha_i) - M\sum_{j\geq\tau+1} f_j P_j(n,s) \leq M\sum_{i=0}^{k-1} \rho_i h(\alpha_i), \end{aligned}$$

where, for the last inequality, we used $f(t) \in A_{n,M,\tau;h}$ or $A_{n,M,h}$ and $P_j(n,s) \geq 0$.

(Sufficiency) Conversely, assume that h is strictly absolutely monotone and suppose that $P_j(n,s) < 0$ for some $j \geq 2k$.

We shall improve the bound (19) or (20) by using the polynomial

$$f(t) = \epsilon Q_j^{(n,q)}(t) + g(t),$$

where $\epsilon > 0$ and $g(t)$ of degree at most $2k-1$ will be properly chosen.

Denote $\tilde{h}(t) := h(t) - \epsilon Q_j^{(n,q)}(t)$ and select ϵ such that $\tilde{h}(t)^{(i)}(t) \geq 0$ on $[-1,1]$ for all $i = 0,1,\ldots,j$. For this choice of ϵ the function $\tilde{h}(t)$ is absolutely monotone. The polynomial $g(t)$ is chosen then to be the Hermite interpolant of $\tilde{h}$ at the nodes $\{\alpha_i\}$, i.e.

$$g(\alpha_i) = \tilde{h}(\alpha_i), \quad g'(\alpha_i) = \tilde{h}'(\alpha_i), \ i = 0,1,\ldots,k-1.$$

Then $g \in A_{n,M,\tau;\tilde{h}}$ implying that $f \in A_{n,M,\tau;\tilde{h}}$ and, since $\tilde{h}(t)$ is an absolutely monotone function, we can infer as in Theorem 4.2 that $g \in A_{n,M;\tilde{h}}$, implying that $f \in A_{n,M;h}$.

Let $g(t) = \sum_{\ell=0}^{2k-1} g_\ell Q_\ell^{(n,q)}(t)$. Note that $f_0 = g_0$ and $f(1) = g(1) + \epsilon$. We next prove that the bound given by $f(t)$ is better than the odd branch of (19) or (20). To this end, we multiply by ρ_i and sum up the first interpolation equalities:

$$\sum_{i=0}^{k-1} \rho_i g(\alpha_i) = \sum_{i=0}^{k-1} \rho_i h(\alpha_i) - \epsilon \sum_{i=0}^{k-1} \rho_i Q_j^{(n,q)}(\alpha_i).$$

Since

$$M \sum_{i=0}^{k-1} \rho_i g(\alpha_i) = M g_0 - g(1)$$

by (13) and

$$M \sum_{i=1}^{k} \rho_i Q_j^{(n,q)}(\alpha_i) = M P_j(n,s) - 1$$

by the definition of the test functions (21), we obtain

$$M g_0 - g(1) = M \sum_{i=0}^{k-1} \rho_i h(\alpha_i) + \epsilon - M\epsilon P_j(n,s)$$

which is equivalent to

$$M f_0 - f(1) = M \sum_{i=0}^{k-1} \rho_i h(\alpha_i) - M\epsilon P_j(n,s),$$

i.e. the polynomial $f(t)$ gives better bound indeed. □

The investigation of the test functions $P_{2k+3}(n,s)$ for $\tau \in \{2k-1, 2k\}$ in the binary case from [7, Section 4.2] gives the following.

Theorem 5.2. *Let $q = 2$.*

a) If $2k+3 \le n \le k^2 + 4k + 2$, then the even branch of the bounds (19) *and* (20) *can be improved for every $s \in \left(t_k^{1,0}, t_k^{1,1}\right)$.*

b) If $k \ge 5$ and $2k+3 \le n \le \left(k^2 + 8k + 1 + \sqrt{(k^2+4k+5)(k^2-4k-3)}\right)/4$, then the odd branch of the bounds (19) *and* (20) *can be improved for every $s \in \left(t_{k-1}^{1,1}, t_k^{1,0}\right)$.*

5.3. **Bounds for codes and designs with inner products in a subinterval of $[-1,1]$.** Some classes of codes or designs are known to have inner products in proper subinterval of $[-1,1]$ and this implies restrictions on their structure. We proceed with examples for such situations with τ-designs in the binary case $\mathbb{H}(n,2)$ with even $\tau = 2k$ and of cardinality $M \in (R(n,2k), R(n,2k+1))$.

The next assertion is implicit in [8, Section 4] (see also Corollary 5.49 and Remark 5.58 in [24]).

Lemma 5.3. *If $C \subset \mathbb{H}(n,2)$ is a $(2k)$-design of cardinality $M \in (R(n,2k), R(n,2k+1))$ then $\gamma_0 M \in (0,1)$.*

Proof. We have the formulas (Equation (5.113) in [24])

$$\gamma_0 = \frac{T_k(s,1)}{T_k(-1,-1)T_k(s,1) - T_k(-1,1)T_k(s,-1)}$$

and (the equation in the last line of page 488 in [24])

$$M = L_{2k}(n,s) = \frac{T_k(1,1)T_k(s,-1) - T_k(1,-1)T_k(s,1)}{T_k(s,-1)}.$$

A little algebra then shows that

$$\gamma_0 M = \frac{T - A(s)}{T - 1/A(s)},$$

where $A(s) = T_k(s,1)/T_k(s,-1)$ as in [7] and $T = T_k(1,1)/T_k(1,-1)$. Moreover, we have

$$A(s) = T \cdot \frac{Q_k^{(1,0,n,2)}(s)}{Q_k^{(0,1,n,2)}(s)}$$

from [24, Lemma 5.24], where $Q_k^{(1,0,n,2)}(s) > 0$ and $Q_k^{(0,1,n,2)}(s) < 0$ for every $s \in \left(t_k^{1,0}, t_k^{1,1}\right)$ (see Lemmas 5.29 and 5.30 in [24]). Therefore the signs of $A(s)$ and T are opposite. We conclude that

$$\gamma_0 M = \frac{|T| + |A(s)|}{|T| + 1/|A(s)|}.$$

By Lemma 5.31 from [24] the ratio $\frac{Q_k^{(1,0,n,2)}(s)}{Q_k^{(0,1,n,2)}(s)}$ is decreasing in s in the interval $\left(t_k^{1,0}, t_k^{1,1}\right)$. Therefore $|A(s)|$ is increasing in $s \in \left(t_k^{1,0}, t_k^{1,1}\right)$. Since $\gamma_0 M = 0$ and 1 for $s = t_k^{1,0}$ and $t_k^{1,1}$, respectively, we obtain that $\gamma_0 M$ increases from 0 to 1 when s increases from $s = t_k^{1,0}$ to $t_k^{1,1}$. $\square$

Remark 5.4. We used in the proof the fact that the space $\mathbb{H}(n,2)$ is antipodal (i.e. for every $x \in \mathbb{H}(n,2)$ there exists a unique antipodal point $y \in \mathbb{H}(n,2)$ such that $d(x,y) = n \iff \langle x,y \rangle = -1$). Indeed, it follows from the definition of the Krawtchouk polynomials for $q=2$ that the kernel $T_k(u,v)$ is symmetric (we need $T_k(1,1) = T_k(-1,-1)$ and $T_k(1,-1) = T_k(-1,1)$ that is not true for $q \geq 3$).

Lemma 5.5. *(a part of Lemma 4.1 in* [6]*) Let $C \subset \mathbb{H}(n,2)$ be a $(2k)$-design of cardinality $M \in (R(n,2k), R(n,2k+1))$ and ξ be the smallest root of the equation $f(t) = \gamma_0 M f(-1)$, where $f(t) = (t-\beta_1)^2 \dots (t-\beta_k)^2$. Then $\ell(n,M,2k) \geq \xi$.*

Since $\gamma_0 M < 1$ from Lemma 5.3 implies $f(\xi) = \gamma_0 M f(-1) < f(-1)$ in Lemma 5.5, it follows that $-1 < \xi \leq \ell(n,M,2k)$ for every $M \in (R(n,2k), R(n,2k+1))$. We now modify Theorem 3.2 to use this fact.

Theorem 5.6. *Let $q=2$, n, h and $\tau = 2k$ be fixed and $f(t)$ be a real polynomial that satisfies* (A2$'$) *and*

(A1$''$) *$f(t) \leq h(t)$ for every $t \in T_n \cap [\ell(n,M,2k), s(n,M,2k)]$.*

Then $\mathcal{L}(n,M,2k;h) \geq f_0 M - f(1)$ for every $M \geq R(n,2k)$.

It follows that Theorem 5.6 gives strict improvements of the bound (19) in the whole range $M \in (R(n,2k), R(n,2k+1))$.

Theorem 5.7. *Let $\ell := \ell(n,M,2k)$ and $G(t)$ be the Hermite interpolant of $h(t)$*

$$G(\ell) = h(\ell),\ G(\beta_i) = h(\beta_i),\ G'(\beta_i) = h'(\beta_i),\ i = 1, \dots, k,$$

$G(t) = \sum_{i=0}^{2k} G_i Q_i^{(n,2)}(t)$*. Then $\mathcal{L}(n,M,2k;h) \geq G_0 M - G(1)) > M \sum_{i=0}^{k} \gamma_i h(\beta_i)$.*

Proof. It follows from Theorem 5.6 with $G(t)$ that $\mathcal{L}(n,M,2k;h) \geq G_0 M - G(1))$. The degree of $G(t)$ is $2k$ and we can apply the quadrature formula (15). We obtain

$$\begin{aligned} G_0 M - G(1) &= M \sum_{i=0}^{k} \gamma_i G(\beta_i) = M \left(\gamma_0 G(-1) + \sum_{i=1}^{k} \gamma_i h(\beta_i) \right) \\ &= M \left(\gamma_0 \left(G(-1) - h(-1) \right) + \sum_{i=0}^{k} \gamma_i h(\beta_i) \right) \\ &> M \sum_{i=0}^{k} \gamma_i h(\beta_i), \end{aligned}$$

(the inequality $G(-1) > h(-1)$ follows from the interpolation since $-1 < \ell(n,M,2k)$). $\square$

We show as example how the better linear programming on subintervals works in the binary ($q = 2$) case for $\tau = 2$. We first derive lower bounds on the quantity $\ell(n, M, 2)$. Note that the approach here is different than the one in Lemma 5.5.

Let $C \subset \mathbb{H}(n, 2)$ be a 2-design of minimum distance d and cardinality $M = |C| \in (R(n, 2), R(n, 3)) = (n + 1, 2n)$. Since the space $\mathbb{H}(n, 2)$ is antipodal, C does not possess pairs of antipodal points [19, Lemma 6.1], [6, Theorem 5.5(iv)]. Let $x, y \in C$ be at maximum possible distance $d(x, y) = \tilde{d} < n$.

Lemma 5.8. *We have*

$$\ell(n, M, 2) \geq 1 - \sqrt{\frac{2M}{n}} \tag{22}$$

for all 2-designs $C \subset \mathbb{H}(n, 2)$ with even $n - \tilde{d}$, and

$$\ell(n, M, 2) \geq 1 - \frac{\sqrt{2(nM - 2)}}{n} \tag{23}$$

for all 2-designs $C \subset \mathbb{H}(n, 2)$ with odd $n - \tilde{d} \geq 3$.

Proof. If $n - \tilde{d} = 2d'$ is even, we consider a point $u \in \mathbb{H}(n, 2)$ such that $d(-x, u) = d(y, u) = d'$, and if $n - \tilde{d} = 2d' + 1$ is odd, we take a point $v \in \mathbb{H}(n, 2)$ such that $d(-x, v) = d(y, v) - 1 = d'$. In both cases we apply Definition 2.2 with the polynomial $f(t) = t^2$.

In the even case we have

$$f_0|C| = \sum_{z \in C} f(\langle z, u\rangle) \geq 2f(\langle u, x\rangle) = 2\left(1 - \frac{2d'}{n}\right)^2$$

Then $2d' \geq n - \sqrt{\frac{Mn}{2}}$ whence we get (22):

$$\langle x, y\rangle = -\langle -x, y\rangle = \frac{4d'}{n} - 1 \geq 1 - \sqrt{\frac{2M}{n}}.$$

The bound (23) in the case of odd $n - \tilde{d} \geq 3$ similarly follows by using

$$f_0|C| = \sum_{z \in C} f(\langle z, v\rangle) \geq f(\langle v, x\rangle) + f(\langle v, y\rangle) = \left(1 - \frac{2d'}{n}\right)^2 + \left(1 - \frac{2d' + 2}{n}\right)^2.$$

□

We can now find the optimal bound for $\mathcal{L}(n, M, 2; h)$ for second degree polynomials which satisfy $f(t) \leq h(t)$ for every $t \in [\ell, 1)$, where ℓ is the lower bound for $\ell(n, M, 2)$ from Lemma 5.8.

Theorem 5.9. *Let $q=2$, n, $M \in [R(n,2),R(n,3)] = [n+1,2n]$ and h be fixed. Let ℓ be the lower bound for $\ell(n,M,2)$ from Lemma 5.8. Then*

$$\mathcal{L}(n,M,2;h) \geq \frac{n(M\ell+1-\ell)^2 h(a_0) + M(M-n-1)h(\ell)}{M(1+n\ell^2)-n(1-\ell)^2}, \tag{24}$$

where $a_0 = \frac{n(1-\ell)-M}{n(M\ell+1-\ell)}$.

Proof. The second degree polynomial which graph passes through the point $(\ell, h(\ell))$ and touches the graph if $h(t)$ at the point $(a_0, h(a_0))$ satisfies the conditions of Theorem 5.6 and gives the desired bound. ☐

Corollary 5.10. *If $q=2$ and $M/n \to \xi$, $\xi \in (1,2)$, as n and M tend to infinity simultaneously, then*

$$\liminf_{n\to\infty} \frac{\mathcal{L}(n,M,2;h)}{n} \geq h(0)\xi. \tag{25}$$

Proof. The asymptotic of the bounds from Lemma 5.8 is $1-\sqrt{2\xi} := \ell$. We plug this in (24) to obtain (25). ☐

6. Upper bounds for $\mathcal{U}(n,M,\tau;h)$

To apply Theorem 3.4 we need upper bounds on $s(n,M,\tau)$. Such bounds can be obtained as in Lemma 5.5 (see [6, Lemma 4.1]). We apply here different approach analogous to Lemma 5.8 in the case $q=2$.

Lemma 6.1. *We have*

$$s(n,M,2) \leq -1 + \sqrt{\frac{2(M-2)}{n}} \tag{26}$$

for all 2-designs $C \subset \mathbb{H}(n,2)$ with even minimum distance, and

$$s(n,M,2) \leq -1 + \frac{1}{n}\sqrt{\frac{2(M-2)(nM-2)}{M}} \tag{27}$$

for all 2-designs $C \subset \mathbb{H}(n,2)$ with odd minimum distance $d \geq 3$.

Proof. Let $C \subset \mathbb{H}(n,2)$ be a 2-design of minimum distance d and cardinality $M = |C| \in (R(n,2),R(n,3)) = (n+1,2n)$. Let $x,y \in C$ be such that $d(x,y)=d$. If $d=2d'$ is even, we consider a point $u \in \mathbb{H}(n,2)$ such that $d(x,u)=d(y,u)=d'$, and if $d=2d'+1$ is odd, we take a point $v \in \mathbb{H}(n,2)$ such that $d(x,v)=d(y,v)-1=d'$. In both cases we apply Definition 2.2 with a polynomial $f(t)=(t-a)^2$, where $a \in [-1,\beta_1]$ will be chosen to give the best possible upper bound on $s(n,M,2)$. Note that $\beta_1 = s = -\frac{2n-M}{n(M-2)}$ in this case.

In the even case we have

$$f_0|C| = \sum_{z\in C} f(\langle z,u\rangle) \geq 2f(\langle u,x\rangle)$$

for any $a < 1-\frac{d}{n}$, whence $M(a^2+1/n) \geq 2\left(\langle u,x\rangle - a\right)^2$ (we used $f_0 = a^2+1/n$). The optimization over a gives

$$\langle u,x\rangle = 1-\frac{d}{n} \leq \sqrt{\frac{M-2}{2n}}$$

(attained for $a = -\sqrt{\frac{2}{n(M-2)}}$). Then the equality $\langle x,y\rangle = 2\langle u,x\rangle - 1$ implies (26).

The bound (27) in the case of odd d similarly follows by using

$$f_0|C| = \sum_{z\in C} f(\langle z,v\rangle) \geq f(\langle v,x\rangle) + f(\langle v,y\rangle) = f\left(1-\frac{2d'}{n}\right) + f\left(1-\frac{2d'+2}{n}\right),$$

for any $a < 1-\frac{d+1}{n}$. $\square$

The bounds for $\ell(n,M,2)$ and $s(n,M,2)$ from Lemmas 5.8 and 6.1, respectively, imply an easy upper bound on $\mathcal{U}(n,M,2;h)$ by Theorem 3.4 with a first degree polynomial.

Theorem 6.2. *Let $q=2$, n, $M \in [R(n,2),R(n,3)] = [n+1,2n]$ and h be fixed. Let ℓ and s be the lower and upper bounds for $\ell(n,M,2)$ and $s(n,M,2)$ from Lemmas 5.8 and 6.1, respectively. Then*

$$\mathcal{U}(n,M,2;h) \leq \frac{(M-1)(sh(\ell)-\ell h(s)) + h(\ell) - h(s)}{s-\ell}. \tag{28}$$

Proof. The linear polynomial which graph passes through the points $(\ell,h(\ell))$ and $(s,h(s))$ satisfies the conditions of Theorem 3.4 and gives the desired bound. $\square$

Corollary 6.3. *If $q=2$ and n and $M=\xi n$, $\xi \in (1,2)$ is constant, tend simultaneously to infinity, then*

$$\mathcal{U}(n,M,2;h) \leq c_1 n + c_2 + o(1), \tag{29}$$

where $c_1 = \frac{\xi[(\sqrt{2\xi}-1)h(1-\sqrt{2\xi})-(1-\sqrt{2\xi})h(\sqrt{2\xi}-1)]}{2(\sqrt{2\xi}-1)}$ and $c_2 = \frac{(2-\sqrt{2\xi})h(1-\sqrt{2\xi})-\sqrt{2\xi}h(\sqrt{2\xi}-1)}{2(\sqrt{2\xi}-1)}$.

Proof. The asymptotics of the bounds from Lemmas 5.8 and 6.1 are $1-\sqrt{2\xi} := \ell$ and $-1+\sqrt{2\xi} := s$, respectively. We plug these in (28) to obtain (29). $\square$

Remark 6.4. Theorems 5.9 and 6.2 (or Corollaries 5.10 and 6.3, respectively) give a strip where the energies of all 2-designs in $\mathbb{H}(n,2)$ have their energies. Such strips can be obtained for higher strengths τ and in other spaces $\mathbb{H}(n,q)$ similar to [10, Theorem 3.7].

7. Asymptotics in the binary case

In this section we study the asymptotic behavior of the bounds (19) and (20) for the binary case $\mathbb{H}(n,2)$ in the following process:

$$\tau \text{ is fixed}, \ n \to \infty, \ M = R(n,\tau)(1+\delta) = \begin{cases} \frac{2n^{k-1}(1+\delta)}{(k-1)!}, & \tau = 2k-1, \\ \frac{n^k(1+\delta)}{k!}, & \tau = 2k, \end{cases} \tag{30}$$

where δ is nonnegative constant (note that $R(n,2k-1) = 2n^{k-1}/(k-1)! + o(n^{k-1})$ and $R(n,2k) = n^k/k! + o(n^k)$).

Lemma 7.1. *We have* $\lim_{n\to\infty} \beta_i = 0$ *for every* $i = 1,\dots,k$, $\lim_{n\to\infty} \alpha_i = 0$ *for every* $i = 1,\dots,k-1$, *and* $\lim_{n\to\infty} \alpha_0 = -1/(1+\delta(k-1)!)$ *in the process* (30).

Proof. The limits $\lim_{n\to\infty} \beta_i = 0$, $i = 1,\dots,k$, follow from the fact that the equation (16) behaves as $t^k = 0$ in (30).

The limits $\lim_{n\to\infty} \alpha_i = 0$, $i = 1,\dots,k-1$, follow from the inequalities

$$t_k^{1,1} > |\alpha_{k-1}| > |\alpha_1| > |\alpha_{k-2}| > |\alpha_2| > \cdots$$

(cf. [8, Corollary 3.9]). Now we use the Vieta formula

$$\sum_{i=0}^{k-1} \alpha_i = \frac{(n-k)Q_k^{(1,0,n,2)}(s)}{nQ_{k-1}^{(1,0,n,2)}(s)} - \frac{k}{n}$$

(follows directly from (14); can be seen in [7, Lemma 4.3a)]) to conclude that

$$\lim_{n\to\infty} \alpha_0 = \lim_{n\to\infty} \frac{Q_k^{(1,0,n,2)}(s)}{Q_{k-1}^{(1,0,n,2)}(s)}.$$

The behavior of the ratio $Q_k^{(1,0,n,2)}(s)/Q_{k-1}^{(1,0,n,2)}(s)$ can be found by using the identities (5.86) from [24]

$$|C| = L_{2k-1}(n,s) = \left(1 - \frac{Q_{k-1}^{(1,0,n,2)}(s)}{Q_k^{(n,2)}(s)}\right) R(n,2k-2) = \left(1 - \frac{Q_k^{(1,0,n,2)}(s)}{Q_k^{(n,2)}(s)}\right) R(n,2k).$$

These imply

$$\lim_{n\to\infty} \frac{Q_k^{(n,2)}(s)}{Q_{k-1}^{(1,0,n,2)}(s)} = -\frac{1}{1+\delta(k-1)!}, \quad \lim_{n\to\infty} \frac{Q_k^{(1,0,n,2)}(s)}{Q_k^{(n,2)}(s)} = 1,$$

correspondingly. Therefore

$$\lim_{n\to\infty} \alpha_0 = \lim_{n\to\infty} \frac{Q_k^{(1,0,n,2)}(s)}{Q_k^{(n,2)}(s)} \cdot \frac{Q_k^{(n,2)}(s)}{Q_{k-1}^{(1,0,n,2)}(s)} = -\frac{1}{1+\delta(k-1)!},$$

which completes the proof. □

Lemma 7.2. *If $\tau = 2k-1$ then $\lim_{n\to\infty} \rho_0 M = (1+\delta(k-1)!)^{2k-1}$ in the process* (30).

Proof. This follows from the formula

$$\rho_0 M = -\frac{(1-\alpha_1^2)(1-\alpha_2^2)\cdots(1-\alpha_{k-1}^2)}{\alpha_0(\alpha_0^2-\alpha_1^2)(\alpha_0^2-\alpha_2^2)\cdots(\alpha_0^2-\alpha_{k-1}^2)}$$

(cf. [8, Theorem 3.8]) and Lemma 7.1; follows from setting $f(t) = t, t^3, \dots, t^{2k-1}$ in (13)) are resolving the obtained linear system with respect to $\rho_0, \dots, \rho_{k-1}$) □

Theorem 7.3. *Let τ be fixed, n and M tend to infinity as in* (30). *Then we have*

$$\lim_{n,M\to\infty} \frac{\mathcal{L}(n,M,\tau;h)}{M} \geq h(0) \tag{31}$$

and

$$\lim_{n,M\to\infty} (\mathcal{L}(n,M,\tau;h) - Mh(0)) \geq C_\tau, \tag{32}$$

where

$$C_\tau = \begin{cases} \left((1+\gamma(k-1)!)^{2k-1}\right)\left(h\left(-\frac{1}{1+\gamma(k-1)!}\right) - h(0)\right) - h(0), & \tau = 2k-1, \\ \gamma_0 M(h(-1)-h(0)) - h(0), & \tau = 2k. \end{cases}$$

Proof. Let $\tau = 2k-1$. We use Lemmas 7.1 and 7.2 to calculate the odd branch of (19) and (20):

$$\begin{aligned} \mathcal{L}(n,M,2k-1;h) &\geq M\sum_{i=0}^{k-1} \rho_i h(\alpha_i) \\ &\sim M\left(\rho_0 h(\alpha_0) + h(0)\sum_{i=1}^{k-1}\rho_i\right) \\ &= M\left(\rho_0(h(\alpha_0)-h(0)) + h(0)\left(1-\frac{1}{M}\right)\right) \\ &\sim h(0)M + c_3, \end{aligned}$$

where

$$c_3 = \left((1+\gamma(k-1)!)^{2k-1}\right)\left(h\left(-\frac{1}{1+\gamma(k-1)!}\right) - h(0)\right) - h(0).$$

Similarly, in the even case we obtain

$$\begin{aligned} \mathcal{L}(n,M,2k;h) &\gtrsim M\left(\gamma_0(h(-1)-h(0)) + h(0)\left(1-\frac{1}{M}\right)\right) \\ &= h(0)M + c_4, \end{aligned}$$

where $c_4 = \gamma_0 M(h(-1)-h(0)) - h(0)$ (here $\gamma_0 M \in (0,1)$, see Lemma 5.3). □

8. Examples

Clearly, all codes which attain the Levenshtein bounds (11) achieve our bounds (19) and (20) and are therefore universally optimal (see Table 6.4 in [24]). We show here two other examples where our bounds are close.

There is a unique optimal (nonlinear) binary code of length 10 with 40 codewords and minimum distance 4. We have $q = 2$, $n = 10$, $M = 40$ and $\tau = 3$. Our bounds are very close, for example if $h = \frac{1}{5(1-t)}$, then the actual energy is 8.125, the universal bound is ≈ 8.0722, the pair-covering bound is ≈ 8.0857, obtained by

$$\begin{aligned} f(t) &= 0.111t^3 + 0.200t^2 + 0.205t + 0.2 \\ &= 0.220Q_0^{(10,2)}(t) + 0.236Q_1^{(10,2)}(t) + 0.180Q_2^{(10,2)}(t) + 0.080Q_3^{(10,2)}(t) \end{aligned}$$

(here and below all numbers are truncated after the fourth digit).

There is a 5-design of 128 points in $H(9,2)$ (here $q = 2$, $n = 9$, $M = 128$ and $\tau = 5$). For the potentials $h(t) = \left(\frac{2}{9(1-t)}\right)^s$ and $s = 0.1, 0.25, 0.5, 0.75, 1, 2.5$, respectively, the actual energy is 109.861, 88.593, 62.284, 44.143, 31.546, 5.029, the corresponding universal bound (2) is $\approx$ 109.853, 88.571, 62.236, 44.066, 31.440, 4.828, and the pair-covering bound is $\approx$ 109.858, 88.584, 62.264, 44.111, 31.503, 4.953. All these bounds are valid for binary $(9, 128)$ codes as well. For example, the above pair-covering bound of ≈ 31.503 (i.e. for $h(t) = \frac{2}{9(1-t)}$) is obtained by the polynomial

$$\begin{aligned} f(t) &= 0.183t^5 + 0.345t^4 + 0.284t^3 + 0.216t^2 + 0.216t + 0.221 \\ &= 0.257Q_0^{(9,2)}(t) + 0.330Q_1^{(9,2)}(t) + 0.366Q_2^{(9,2)}(t) + 0.306Q_3^{(9,2)}(t) \\ &\quad + 0.159Q_4^{(9,2)}(t) + 0.046Q_5^{(9,2)}(t) \end{aligned}$$

that satisfies the condition (A2) as well. Here, the best lower bound is 31.525 and can be obtained by a polynomial of degree 9

$$\begin{aligned} f(t) &= 0.540t^9 - 1.041t^7 + 0.773t^5 + 0.345t^4 + 0.171t^3 + 0.210t^2 + 0.222t + 0.222 \\ &= 0.257Q_0^{(9,2)}(t) + 0.330Q_1^{(9,2)}(t) + 0.361Q_2^{(9,2)}(t) + 0.296Q_3^{(9,2)}(t) \\ &\quad + 0.159Q_4^{(9,2)}(t) + 0.0394Q_5^{(9,2)}(t) + 0.005Q_9^{(9,2)}(t) \end{aligned}$$

that also satisfies (A2).

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