

NEXT ORDER ENERGY ASYMPTOTICS FOR RIESZ POTENTIALS ON FLAT TORI

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ABSTRACT. Let Λ be a lattice in \mathbb{R}^d with positive co-volume. Among Λ -periodic N -point configurations, we consider the minimal renormalized Riesz s -energy $\mathcal{E}_{s,\Lambda}(N)$. While the dominant term in the asymptotic expansion of $\mathcal{E}_{s,\Lambda}(N)$ as N goes to infinity in the long range case that $0 < s < d$ (or $s = \log$) can be obtained from classical potential theory, the next order term(s) require a different approach. Here we derive the form of the next order term or terms, namely for $s > 0$ they are of the form $C_{s,d}|\Lambda|^{-s/d}N^{1+s/d}$ and $-\frac{2}{d}N \log N + (C_{\log,d} - 2\zeta'_\Lambda(0))N$ where we show that the constant $C_{s,d}$ is independent of the lattice Λ .

1. PRELIMINARIES

Let $A = [v_1, \dots, v_d]$ be a $d \times d$ nonsingular matrix with j -th column v_j and let $\Lambda = \Lambda_A := A\mathbb{Z}^d$ denote the lattice generated by A . The set

$$\Omega = \Omega_\Lambda := \left\{ w : w = \sum_{j=1}^d \alpha_j v_j, \alpha_j \in [0, 1), j = 1, 2, \dots, d \right\}.$$

is a fundamental domain of the quotient space \mathbb{R}^d/Λ ; i.e., the collection of sets $\{\Omega + v : v \in \Lambda\}$ tiles \mathbb{R}^d . The volume of Ω_Λ , denoted by $|\Lambda|$, equals $|\det A|$ and is called the co-volume of Λ (in fact, any measurable fundamental domain of Λ has the same volume). We will let Λ^* denote the *dual lattice* of Λ which is the lattice generated by $(A^T)^{-1}$.

For an interaction potential $F : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$, we consider the F -energy of an N -tuple $\omega_N = (x_1, \dots, x_N) \in (\mathbb{R}^d)^N$

$$(1) \quad E_F(\omega_N) := \sum_{k=1}^N \sum_{\substack{j=1 \\ j \neq k}}^N F(x_k - x_j),$$

and for a subset $A \subset \mathbb{R}^d$, we consider the N -point minimal F -energy

$$(2) \quad \mathcal{E}_F(A, N) := \inf_{\omega_N \in A^N} E_F(\omega_N).$$

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In this paper we are mostly concerned with Λ -periodic potentials F , that is, $F(x+v) = F(x)$ for all $v \in \Lambda$. For such an F , the energy $E_F(\omega_N) = E_F(x_1, \dots, x_N)$ is Λ -periodic in each component x_k and so, without loss of generality, we may assume that $\omega_N \in (\Omega_\Lambda)^N$; i.e., $\mathcal{E}_F(\mathbb{R}^d, N) = \mathcal{E}_F(\Omega_\Lambda, N)$. Specifically, in this paper we consider periodized Riesz potentials and periodized logarithmic potentials and $A = \mathbb{R}^d$ (or, equivalently $A = \Omega_\Lambda$) as we next describe.

For $s > d$, we consider the periodic potential generated by the Riesz s -potential $f_s(x) = |x|^{-s}$ as follows

$$(3) \quad \zeta_\Lambda(s; x) := \sum_{v \in \Lambda} \frac{1}{|x+v|^s}, \quad s > d, x \in \mathbb{R}^d,$$

which is finite for $x \notin \Lambda$ and equals $+\infty$ when $x \in \Lambda$. Then $\zeta_\Lambda(s; x-y)$ can be considered to be the energy required to place a unit charge at location $x \in \mathbb{R}^d$ in the presence of unit charges placed at $y + \Lambda = \{y+v : v \in \Lambda\}$ with charges interacting through the Riesz s -potential. For $s \leq d$, the sum on the right side of (3) is infinite for all $x \in \mathbb{R}^d$. In [5], Λ -periodic energy problems for a class of long range potentials are considered and it is shown that for the case of the Riesz potential with $s \leq d$, the appropriate energy problem can be obtained through analytic continuation. Specifically, it follows from Theorems 1.1 and 3.1 of [5] that the potential $F_{s,\Lambda}(x)$ defined by

$$(4) \quad F_{s,\Lambda}(x) := \sum_{v \in \Lambda} \int_1^\infty e^{-|x+v|^2 t} \frac{t^{\frac{s}{2}-1}}{\Gamma(\frac{s}{2})} dt + \frac{1}{|\Lambda|} \sum_{w \in \Lambda^* \setminus \{0\}} e^{2\pi i w \cdot x} \int_0^1 \frac{\pi^{\frac{d}{2}}}{t^{\frac{d}{2}}} e^{-\frac{\pi^2 |w|^2}{t}} \frac{t^{\frac{s}{2}-1}}{\Gamma(\frac{s}{2})} dt,$$

is, for fixed $x \in \mathbb{R}^d \setminus \Lambda$, an entire function of s , and

$$(5) \quad F_{s,\Lambda}(x) = \zeta_\Lambda(s; x) + \frac{2\pi^{\frac{d}{2}} |\Lambda|^{-1}}{\Gamma(\frac{s}{2})(d-s)}, \quad s > d,$$

showing that (4) provides an analytic continuation of $\zeta_\Lambda(\cdot; x)$ to $\mathbb{C} \setminus \{d\}$ (note that $\zeta_\Lambda(s; x)$ has a simple pole at $s = d$ for $x \notin \Lambda$). We refer to (the analytically extended) $\zeta_\Lambda(s; x)$ as the *Epstein Hurwitz zeta function for the lattice Λ* . We shall also need the *Epstein zeta function* defined for $s > d$ by

$$(6) \quad \zeta_\Lambda(s) := \sum_{v \in \Lambda \setminus \{0\}} \frac{1}{|v|^s},$$

and continued analytically as above for $s \in \mathbb{C} \setminus \{d\}$. We remark that (4) is derived from the formula

$$(7) \quad r^{-s} = \int_0^\infty e^{-tr^2} \frac{t^{\frac{s}{2}-1}}{\Gamma(\frac{s}{2})} dt, \quad (r > 0),$$

together with the Poisson Summation Formula.

In [5], analytic continuation and periodized Riesz potentials are connected through the use of *convergence factors*; i.e., a parametrized family of functions $g_a : \mathbb{R}^d \rightarrow [0, \infty)$ such that

- (a) for $a > 0$, $f_s(x)g_a(x)$ decays sufficiently rapidly as $|x| \rightarrow \infty$ so that

$$F_{s,a,\Lambda}(x) := \sum_{v \in \Lambda} f_s(x+v)g_a(x+v)$$

converges to a finite value for all $x \notin \Lambda$, and

(b) $\lim_{a \rightarrow 0^+} g_a(x) = 1$ for all $x \in \mathbb{R}^d \setminus \{0\}$.

For example, the family of Gaussians $g_a(x) = e^{-a|x|^2}$ is a convergence factor for Riesz potentials. In [5], it is shown that for a large class of convergence factors $\{g_a\}_{a>0}$ (including the Gaussian convergence family) one may choose C_a (depending on the convergence factor $\{g_a\}_{a>0}$) such that

$$(8) \quad F_{s,\Lambda}(x) = \lim_{a \rightarrow 0^+} (F_{s,a,\Lambda}(x) - C_a).$$

Then, for $a > 0$, $F_{s,a,\Lambda}(x-y)$ represents the energy required to place a unit charge at location x in the presence of unit charges placed at $y+\Lambda = \{y+v : v \in \Lambda\}$ with charges interacting through the potential $f_s(x)g_a(x)$. This leads us to consider, for $s > 0$, the *periodic Riesz s -energy of ω_N associated with the lattice Λ* defined by

$$(9) \quad E_{s,\Lambda}(\omega_N) := \sum_{\substack{1 \leq k, j \leq N \\ k \neq j}} F_{s,\Lambda}(x_k - x_j),$$

as well as the *minimal N -point periodic Riesz s -energy*

$$(10) \quad \mathcal{E}_{s,\Lambda}(N) := \mathcal{E}_{F_{s,\Lambda}}(\mathbb{R}^d; N) = \inf_{\omega_N \in (\mathbb{R}^d)^N} E_{s,\Lambda}(\omega_N).$$

We shall also consider the *periodic logarithmic potential associated with Λ* generated from the logarithmic potential $F_{\log}(x) := \log(1/|x|)$ using convergence factors as above and resulting in the definition

$$(11) \quad F_{\log,\Lambda}(x) := \sum_{v \in \Lambda} \int_1^\infty e^{-|x+v|^2 t} \frac{dt}{t} + \frac{1}{|\Lambda|} \sum_{w \in \Lambda^* \setminus \{0\}} e^{2\pi i w \cdot x} \int_0^1 \frac{\pi^{\frac{d}{2}}}{t^{\frac{d}{2}}} e^{-\frac{\pi^2 |w|^2}{t}} \frac{dt}{t}.$$

Comparing (11) and (4), it is not difficult to obtain (cf. [5]) the relations

$$(12) \quad F_{\log,\Lambda}(x) = \lim_{s \rightarrow 0} \Gamma\left(\frac{s}{2}\right) F_{s,\Lambda}(x) = 2 \left(\frac{d}{ds} F_{s,\Lambda}(x) \right) \Big|_{s=0} = 2\zeta'_\Lambda(0; x) + \frac{2\pi^{d/2}}{d} |\Lambda|^{-1},$$

where the prime denotes differentiation with respect to the variable s . We then define the *periodic logarithmic energy of $\omega_N = (x_1, \dots, x_N)$,*

$$(13) \quad E_{\log,\Lambda}(\omega_N) := \sum_{\substack{1 \leq k, j \leq N \\ k \neq j}} F_{\log,\Lambda}(x_j - x_k),$$

and also the *N -point minimal periodic logarithmic energy for Λ ,*

$$(14) \quad \mathcal{E}_{\log,\Lambda}(N) := \inf_{\omega_N \in (\mathbb{R}^d)^N} E_{\log,\Lambda}(\omega_N).$$

For $0 < s < d$, the kernel $K_{s,\Lambda}(x, y) := F_{s,\Lambda}(x - y)$ is positive definite and integrable on $\Omega_\Lambda \times \Omega_\Lambda$ and so there is a unique probability measure μ_s (called the *Riesz s -equilibrium measure*) that minimizes the continuous Riesz s -energy

$$I_{s,\Lambda}(\mu) := \iint_{\Omega_\Lambda \times \Omega_\Lambda} K_{s,\Lambda}(x, y) d\mu(x) d\mu(y)$$

over all Borel probability measures μ on Ω_Λ . From the periodicity of $F_{s,\Lambda}$ and the uniqueness of the equilibrium measure, it follows that $\mu_s = \lambda_d$ where λ_d denotes Lebesgue measure restricted to Ω_Λ and normalized so that $\lambda_d(\Omega_\Lambda) = 1$; i.e., λ_d is the normalized Haar measure for $\Omega_\Lambda = \mathbb{R}^d/\Lambda$. The periodic logarithmic kernel $K_{\log,\Lambda}(x, y) := F_{\log,\Lambda}(x - y)$ is conditionally positive definite and integrable and it similarly follows that λ_d is the unique equilibrium measure minimizing the periodic logarithmic energy

$$I_{\log,\Lambda}(\mu) := \iint_{\Omega_\Lambda \times \Omega_\Lambda} K_{\log,\Lambda}(x, y) d\mu(x) d\mu(y)$$

over all Borel probability measures μ on Ω_Λ .

It is not difficult to verify (cf. [5]) that

$$(15) \quad \int_{\Omega_\Lambda} \zeta_\Lambda(s; x) d\lambda_d(x) = 0, \quad 0 < s < d,$$

and

$$(16) \quad \int_{\Omega_\Lambda} \zeta'_\Lambda(0; x) d\lambda_d(x) = 0,$$

from which we obtain

$$(17) \quad I_{s,\Lambda}(\lambda_d) = \frac{2\pi^{\frac{d}{2}}|\Lambda|^{-1}}{\Gamma(\frac{s}{2})(d-s)}, \quad 0 < s < d,$$

and

$$(18) \quad I_{\log,\Lambda}(\lambda_d) = \frac{2\pi^{d/2}}{d}|\Lambda|^{-1}.$$

It then follows (cf. [6]) that

$$(19) \quad \lim_{N \rightarrow \infty} \frac{\mathcal{E}_{s,\Lambda}(N)}{N^2} = \frac{2\pi^{\frac{d}{2}}|\Lambda|^{-1}}{\Gamma(\frac{s}{2})(d-s)}, \quad 0 < s < d,$$

and

$$(20) \quad \lim_{N \rightarrow \infty} \frac{\mathcal{E}_{\log,\Lambda}(N)}{N^2} = \frac{2\pi^{d/2}}{d}|\Lambda|^{-1}.$$

2. MAIN RESULTS

Our main result is the following asymptotic expansion of the periodic Riesz and logarithmic minimal energy as $N \rightarrow \infty$.

Theorem 1. *Let Λ be a lattice in \mathbb{R}^d with co-volume $|\Lambda| > 0$. Then, as $N \rightarrow \infty$,*

$$(21) \quad \mathcal{E}_{s,\Lambda}(N) = \frac{2\pi^{\frac{d}{2}}|\Lambda|^{-1}}{\Gamma(\frac{s}{2})(d-s)}N^2 + C_{s,d}|\Lambda|^{-s/d}N^{1+\frac{s}{d}} + o(N^{1+\frac{s}{d}}), \quad 0 < s < d,$$

$$(22) \quad \mathcal{E}_{\log,\Lambda}(N) = \frac{2\pi^{\frac{d}{2}}}{d}|\Lambda|^{-1}N(N-1) - \frac{2}{d}N \log N + (C_{\log,d} - 2\zeta'_\Lambda(0))N + o(N).$$

where $C_{\log,d}$ and $C_{s,d}$ are constants independent of Λ .

Petrache and Serfaty establish in [7] a result closely related to (21) for point configurations interacting through a Riesz s potential and confined by an external field for values of the Riesz parameter $d - 2 \leq s \leq d$ and Sandier and Serfaty prove in [8] a result closely related to (22) for the case that $s = \log$ and $d = 2$.

For comparison, when $s \geq d$ it is known that the leading order term of $\mathcal{E}_{s,\Lambda}(N)$ is the same as that of $\mathcal{E}_s(\Omega_\Lambda, N) := \mathcal{E}_{f_s}(\Omega_\Lambda, N)$.

Theorem 2 ([4], [5]). *Let Λ be a lattice in \mathbb{R}^d with co-volume $|\Lambda| > 0$. For $s > d$, there is a positive and finite constant $C_{s,d}$ such that*

$$(23) \quad \lim_{N \rightarrow \infty} \frac{\mathcal{E}_{s,\Lambda}(N)}{N^{1+s/d}} = \lim_{N \rightarrow \infty} \frac{\mathcal{E}_s(\Omega_\Lambda; N)}{N^{1+s/d}} = C_{s,d} |\Lambda|^{-s/d}, \quad s > d,$$

$$(24) \quad \lim_{N \rightarrow \infty} \frac{\mathcal{E}_{d,\Lambda}(N)}{N^2 \log N} = \lim_{N \rightarrow \infty} \frac{\mathcal{E}_d(\Omega_\Lambda, N)}{N^2 \log N} = \frac{2\pi^{d/2}}{d\Gamma(\frac{d}{2})} |\Lambda|^{-1},$$

By considering scaled lattice configurations (see Lemma 7) of the form $\omega_{m^d}^\Lambda := (1/m)\Lambda \cap \Omega_\Lambda$ for a lattice Λ of co-volume 1, we obtain the following upper bound for $C_{s,d}$ that holds both for $0 < s < d$ and $s = \log$ where $C_{s,d}$ is as in Theorem 1 as well as for $s > d$ where $C_{s,d}$ is as in Theorem 1.

Corollary 3. *Let Λ be a d -dimensional lattice with co-volume 1. Then,*

$$(25) \quad C_{s,d} \leq \begin{cases} \zeta_\Lambda(s), & s > 0, s \neq d, \\ 2\zeta'_\Lambda(0), & s = \log. \end{cases}$$

The constant $C_{s,d}$ for $s > d$ appearing in (23) is known only in the case $d = 1$ where $C_{s,1} = \zeta_{\mathbb{Z}}(s) = 2\zeta(s)$ and $\zeta(s)$ denotes the classical Riemann zeta function. For dimensions $d = 2, 4, 8$, and 24 , it has been conjectured (cf. [1, 2] and references therein) that $C_{s,d}$ for $s > d$ is also given by an Epstein zeta function, specifically, that $C_{s,d} = \zeta_{\Lambda_d}(s)$ for Λ_d denoting the equilateral triangular (or hexagonal) lattice, the D_4 lattice, the E_8 lattice, and the Leech lattice (all scaled to have co-volume 1) in the dimensions $d = 2, 4, 8$, and 24 , respectively. In [3], it is shown that periodized lattice configurations for these special lattices are *local* minima of the energy for a large class of energy potentials that includes periodic Riesz s -energy potentials for $s > d$.

Finally, we would like to say a little more about periodizing Riesz potentials. It is elementary to verify that

$$|x + v|^{-s} - |v|^{-s} = O\left(|v|^{-(s+1)}\right), \quad (|v| \rightarrow \infty).$$

Let C be a bounded set containing an open neighborhood of the origin and for $L > 0$ let $C_L = LC$ (the set C scaled by L). Then, we have

$$(26) \quad \zeta(s; x) - \zeta(s) = |x|^{-s} + (1/2) \lim_{L \rightarrow \infty} \sum_{v \in C_L \setminus \{0\}} |x + v|^{-s} - |v|^{-s}, \quad (x \notin \Lambda),$$

for $s > d - 1$. Further assuming that C is centrally symmetric (i.e., $v \in C \implies -v \in C$) and using the fact (see (53) Section 5) that

$$|x + v|^{-s} + |x - v|^{-s} - 2|v|^{-s} = O\left(|v|^{-(s+2)}\right), \quad (|v| \rightarrow \infty),$$

shows that (26) holds for $s > d - 2$ and so we obtain (up to a constant depending only on s) the same periodic Riesz potential given in (5); i.e., the convergence factors procedure and the limit in (26) give the same result. However, (26) breaks down for $0 < s < d - 2$. In this case the energy is dominated by long range contributions from translates near the boundary of C and (26) no longer holds. In fact, as a consequence of Theorem 4 below, the right hand side of (26) is $-\infty$ for all x . For the case that $C = \mathcal{B}_d$, the unit ball in \mathbb{R}^d centered at the origin, we find that (26) can be ‘renormalized’ by dividing by

$$(27) \quad D_L := \sum_{v \in C_L \setminus \{0\}} |v|^{-s},$$

but this leads to a non-periodic potential.

Theorem 4. *Let $C = \mathcal{B}_d$, $0 < s \leq d - 2$, and for $L > 0$, let D_L be given by (27). Then*

$$(28) \quad \lim_{L \rightarrow \infty} D_L^{-1} \left[|x|^{-s} + (1/2) \sum_{v \in C_L \setminus \{0\}} |x + v|^{-s} - |v|^{-s} \right] = -(s/d)(d - s - 2)|x|^2,$$

uniformly in x on compact subsets of $\mathbb{R}^d \setminus \Lambda$.

Remarks:

- Theorem 4 implies that for an N -point configuration $\{x_1, \dots, x_n\}$ the energy sum

$$(29) \quad \sum_{i \neq j} |x_i - x_j|^{-s} + \frac{1}{2} \sum_{v \in C_L \setminus \{0\}} (|x_i - x_j + v|^{-s} - |v|^{-s}) \\ = -(s/d)(d - s - 2)D_L \sum_{i \neq j} |x_i - x_j|^2 + o(D_L).$$

as $L \rightarrow \infty$.

- Since the first term on the right side of (29) vanishes if $s = d - 2$, the dominant term of the left hand side of (29) is not determined by (28) when $s = d - 2$; i.e., all we know is that this term is $o(D_L)$.

3. THE EPSTEIN HURWITZ ZETA FUNCTION

In this section, we will review some relevant terminology and notation involving special functions that will be crucial for our analysis in Section 4.

An argument utilizing the integral representation of the Riesz kernel given in (7) together with the Poisson Summation Formula can be used to establish the following lemma (cf., [9, Section 1.4]).

Lemma 5. *The Epstein zeta function $\zeta_\Lambda(s)$ can be analytically continued to $\mathbb{C} \setminus \{d\}$ through the following formula :*

$$\zeta_\Lambda(s) = \frac{2}{\Gamma(\frac{s}{2})} \left(\frac{2\pi^{\frac{d}{2}} |\Lambda|^{-1}}{s - d} - \frac{1}{s} \right) + \sum_{v \in \Lambda \setminus \{0\}} \int_1^\infty e^{-|v|^2 t} \frac{t^{\frac{s}{2}-1}}{\Gamma(\frac{s}{2})} dt + \frac{1}{|\Lambda|} \sum_{w \in \Lambda^* \setminus \{0\}} \int_0^1 \frac{\pi^{\frac{d}{2}}}{t^{\frac{d}{2}}} e^{-\frac{\pi^2 |w|^2}{t}} \frac{t^{\frac{s}{2}-1}}{\Gamma(\frac{s}{2})} dt.$$

Remark. From $\lim_{s \rightarrow 0^+} \Gamma(\frac{s}{2}) = \infty$ and $\lim_{s \rightarrow 0^+} \frac{s}{2} \Gamma(\frac{s}{2}) = \lim_{s \rightarrow 0^+} \Gamma(\frac{s}{2} + 1) = \Gamma(1) = 1$, it follows that $\zeta_\Lambda(0) = -1$ and $\zeta_\Lambda(0; x) \equiv 0$ for any lattice Λ .

Next we establish the following relation between the Epstein and Epstein Hurwitz zeta functions.

Lemma 6. *Let Λ be a sublattice of Λ' . Then for any $s \in \mathbb{C} \setminus \{d\}$, it holds that*

$$(30) \quad \sum_{x \in \Lambda' \cap \Omega_\Lambda \setminus \{0\}} \zeta_\Lambda(s; x) = \zeta_{\Lambda'}(s) - \zeta_\Lambda(s).$$

Proof. It is sufficient to prove that (30) holds for $s > d$, since the general result follows from the fact that both sides of this relation are analytic on $\mathbb{C} \setminus \{d\}$. For $s > d$, we have by definition

$$\begin{aligned} \zeta_{\Lambda'}(s) &= \sum_{x \in \Lambda' \setminus \{0\}} \frac{1}{|x|^s} = \sum_{x \in \Lambda' \cap \Omega_\Lambda \setminus \{0\}} \sum_{v \in \Lambda} \frac{1}{|x+v|^s} + \sum_{v \in \Lambda \setminus \{0\}} \frac{1}{|v|^s} \\ &= \sum_{x \in \Lambda' \cap \Omega_\Lambda \setminus \{0\}} \zeta_\Lambda(s; x) + \zeta_\Lambda(s), \end{aligned}$$

thus proving the lemma. □

Using the above lemma and scaling properties of Epstein zeta functions we obtain the following:

Lemma 7. *For every $m \in \mathbb{N}$ and $s \in \mathbb{C} \setminus \{d\}$, it holds that*

$$(31) \quad \sum_{x \in \frac{1}{m} \Lambda \cap \Omega_\Lambda \setminus \{0\}} \zeta_\Lambda(s; x) = (m^s - 1) \zeta_\Lambda(s).$$

Therefore,

$$(32) \quad \sum_{\substack{x, y \in \frac{1}{m} \Lambda \cap \Omega_\Lambda \\ x \neq y}} \zeta_\Lambda(s; x - y) = m^d (m^s - 1) \zeta_\Lambda(s).$$

We will also require the following two lemmas, which establish continuity properties of the Epstein Hurwitz Zeta function with respect to the lattice.

Lemma 8. *Let $\{P_m\}_{m \in \mathbb{N}}$ be a sequence of $d \times d$ matrices such that $P_m \rightarrow P$ in norm as $m \rightarrow \infty$. Fix any distinct x and y in Ω_Λ and suppose $\{x_m\}_{m \in \mathbb{N}}$ and $\{y_m\}_{m \in \mathbb{N}}$ are sequences in Ω_Λ converging to x and y , respectively. Then for any compact set $K \subset \mathbb{C} \setminus \{d\}$, $\zeta_{P_m \Lambda}(s; P_m(x_m - y_m))$ converges to $\zeta_{P \Lambda}(s; P(x - y))$ and $\zeta'_{P_m \Lambda}(s; P_m(x_m - y_m))$ converges to $\zeta'_{P \Lambda}(s; P(x - y))$ uniformly for s in K as $m \rightarrow \infty$.*

Proof. Let $R = \sup_{s \in K} \operatorname{Re}(s)$ and $r = \inf_{s \in K} \operatorname{Re}(s)$. Notice that $\sup_{s \in K} |1/\Gamma(\frac{s}{2})|$ is finite since $1/\Gamma(\frac{s}{2})$ is entire. Let m be large enough so that $x_m - y_m \notin \Lambda$. Using (4), we have

$$\begin{aligned}
(33) \quad & |\zeta_{P_m \Lambda}(s; P_m(x_m - y_m)) - \zeta_{P \Lambda}(s; P(x - y))| \\
&= |F_{s, P_m \Lambda}(P_m x_m - P_m y_m) - F_{s, P \Lambda}(P x - P y)| \\
&\leq \int_1^\infty \sum_{v \in \Lambda} \left| e^{-|P_m(x_m - y_m + v)|^2 t} - e^{-|P(x - y + v)|^2 t} \right| \frac{|t^{\frac{s}{2} - 1}|}{|\Gamma(\frac{s}{2})|} dt \\
&\quad + \int_0^1 \sum_{w \in \Lambda^* \setminus \{0\}} \left| e^{2\pi i w \cdot (x_m - y_m)} e^{-\frac{\pi^2 |(P_m^{-1})^T w|^2}{t}} - e^{2\pi i w \cdot (x - y)} e^{-\frac{\pi^2 |(P^{-1})^T w|^2}{t}} \right| \frac{\pi^{\frac{d}{2}} |t^{\frac{s-d}{2} - 1}|}{|\Gamma(\frac{s}{2})|} dt \\
&\leq \int_1^\infty \sum_{v \in \Lambda} \left| e^{-|P_m(x_m - y_m + v)|^2 t} - e^{-|P(x - y + v)|^2 t} \right| \frac{t^{\frac{R}{2} - 1}}{\inf_{s \in K} |\Gamma(\frac{s}{2})|} dt \\
&\quad + \int_0^1 \sum_{w \in \Lambda^* \setminus \{0\}} \left| e^{2\pi i w \cdot (x_m - y_m)} e^{-\frac{\pi^2 |(P_m^{-1})^T w|^2}{t}} - e^{2\pi i w \cdot (x - y)} e^{-\frac{\pi^2 |(P^{-1})^T w|^2}{t}} \right| \frac{\pi^{\frac{d}{2}} t^{\frac{r-d}{2} - 1}}{\inf_{s \in K} |\Gamma(\frac{s}{2})|} dt.
\end{aligned}$$

As in [5], it is elementary to establish that integrals of the form

$$\int_1^\infty \sum_{v \in \Lambda} e^{-|P(x - y + v)|^2 t} t^{\frac{R}{2} - 1} dt \quad \text{and} \quad \int_0^1 \sum_{w \in \Lambda^* \setminus \{0\}} e^{-\frac{\pi^2 |(P^{-1})^T w|^2}{t}} \pi^{\frac{d}{2}} t^{\frac{r-d}{2} - 1} dt$$

are finite and thus, by dominated convergence, it follows that the expressions in (33) tend to zero as $m \rightarrow \infty$.

Cauchy's integral formula for derivatives then implies that $\zeta'_{P_m \Lambda}(s; P_m(x_m - y_m)) \rightarrow \zeta'_{P \Lambda}(s; P(x - y))$ uniformly for $s \in K$ as $m \rightarrow \infty$. \square

We remark that the proof of Lemma 8 shows that $F_{s, P_m \Lambda}(P_m x_m - P_m y_m)$ converges to $F_{s, P \Lambda}(P x - P y)$ as $m \rightarrow \infty$ uniformly for s in any compact set of \mathbb{C} .

Corollary 9. *Let $\{P_m\}_{m \in \mathbb{N}}$ be a sequence of $d \times d$ matrices such that $P_m \rightarrow P$ in norm as $m \rightarrow \infty$ and suppose $s > 0$ or $s = \log$. Then, for all $N \geq 2$, we have $\mathcal{E}_{s, P_m \Lambda}(N) \rightarrow \mathcal{E}_{s, P \Lambda}(N)$ as $m \rightarrow \infty$.*

Proof. Let $\omega_N^* \subset \Omega_\Lambda$ be such that $P\omega_N^*$ is an $E_{s, P \Lambda}$ optimal N -point configuration. Then,

$$\limsup_{m \rightarrow \infty} \mathcal{E}_{s, P_m \Lambda}(N) \leq \limsup_{m \rightarrow \infty} E_{s, P_m \Lambda}(P_m \omega_N^*) = E_{s, P \Lambda}(P \omega_N^*) = \mathcal{E}_{s, P \Lambda}(N),$$

where the next to last equality follows from Lemma 8.

Next let $\omega_N^m = \{x_1^m, \dots, x_N^m\} \subset \Omega_\Lambda$ be such that $P_m \omega_N^m$ is an optimal N -point configuration for $F_{s, P_m \Lambda}$. Let $\{\omega_N^{m_k}\}_{k \in \mathbb{N}}$ be a subsequence such that

$$\lim_{k \rightarrow \infty} E_{s, P_{m_k} \Lambda}(P_{m_k} \omega_N^{m_k}) = \liminf_{m \rightarrow \infty} \mathcal{E}_{s, P_m \Lambda}(N).$$

Using the compactness of Ω_Λ in the 'flat torus' topology, we may assume without loss of generality that $\{\omega_N^{m_k}\}_{k \in \mathbb{N}}$ converges to some N -point configuration $\tilde{\omega}_N = \{\tilde{x}_1, \dots, \tilde{x}_N\}$; i.e.,

$x_j^{m_k} \rightarrow \tilde{x}_j$ as $k \rightarrow \infty$ for each $j = 1, \dots, N$. Then we have

$$\liminf_{m \rightarrow \infty} \mathcal{E}_{s, P_m \Lambda}(N) = \lim_{k \rightarrow \infty} E_{s, P_{m_k} \Lambda}(P_{m_k} \omega_N^{m_k}) = E_{s, P \Lambda}(P \tilde{\omega}_N) \geq \mathcal{E}_{s, P \Lambda}(N),$$

where the next to last equality follows from Lemma 8. \square

Finally, the following result expresses continuity properties of the Epstein zeta function with respect to the lattice similar to the results in Lemma 8 for the Epstein Hurwitz zeta function.

Lemma 10. *Let $\{P_m\}_{m \in \mathbb{N}}$ be a sequence of $d \times d$ matrices such that $P_m \rightarrow P$ in norm as $m \rightarrow \infty$. Then for any compact set $K \subset \mathbb{C} \setminus \{d\}$, $\zeta_{P_m \Lambda}(s)$ converges to $\zeta_{P \Lambda}(s)$ uniformly in K and hence $\zeta'_{P_m \Lambda}(s) \rightarrow \zeta'_{P \Lambda}(s)$ for all $s \in \mathbb{C} \setminus \{d\}$ as $m \rightarrow \infty$.*

Proof. Using Lemma 5, a similar argument as in the proof of Lemma 8 implies that $\zeta_{P_m \Lambda}(s)$ converges uniformly to $\zeta_{P \Lambda}(s)$ on compact sets $K \subset \mathbb{C} \setminus \{d\}$. The convergence of the derivatives then follows from Cauchy's integral formula for derivatives. \square

4. PROOF OF THEOREM 1

Throughout this section and the next we shall assume that $\Lambda = A\mathbb{Z}^d$ denotes a d -dimensional lattice in \mathbb{R}^d with fundamental domain $\Omega = \Omega_\Lambda$, co-volume 1, and generating matrix A . Then Theorem 1 follows from a simple rescaling. We shall find it convenient to use what we call the *classical periodic Riesz s -potential* $F_{s, \Lambda}^{\text{cp}}(x) := \zeta_\Lambda(s; x)$ which, for $s \neq d$, differs from $F_{s, \Lambda}$ only by the constant $\frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{s}{2})(d-s)}$. Similarly, we call $F_{\log, \Lambda}^{\text{cp}}(x) := 2\zeta'_\Lambda(0; x)$ the *classical periodic logarithmic potential*. The energies associated with these potentials are given by

$$(34) \quad E_{s, \Lambda}^{\text{cp}}(\omega_N) := \sum_{j \neq k} \zeta_\Lambda(s; x_j - x_k), \quad (s > 0),$$

and, similarly,

$$(35) \quad E_{\log, \Lambda}^{\text{cp}}(\omega_N) := 2 \sum_{j \neq k} \zeta'_\Lambda(0; x_j - x_k),$$

and we denote the respective minimal N -point energies by $\mathcal{E}_{s, \Lambda}^{\text{cp}}(N)$ and $\mathcal{E}_{\log, \Lambda}^{\text{cp}}(N)$.

From (5), we obtain

$$(36) \quad \mathcal{E}_{s, \Lambda}(N) = \mathcal{E}_{s, \Lambda}^{\text{cp}}(N) + \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{s}{2})(d-s)} N(N-1),$$

and

$$(37) \quad \mathcal{E}_{\log, \Lambda}(N) = \mathcal{E}_{\log, \Lambda}^{\text{cp}}(N) + \frac{2\pi^{\frac{d}{2}}}{d} N(N-1),$$

Define

$$\begin{aligned} \underline{g}_{s,d}(\Lambda) &:= \liminf_{N \rightarrow \infty} \frac{\mathcal{E}_{s,\Lambda}^{\text{cp}}(N)}{N^{1+s/d}}, \\ \overline{g}_{s,d}(\Lambda) &:= \limsup_{N \rightarrow \infty} \frac{\mathcal{E}_{s,\Lambda}^{\text{cp}}(N)}{N^{1+s/d}}, \\ \underline{g}_{\log,d}(\Lambda) &:= \liminf_{N \rightarrow \infty} \frac{\mathcal{E}_{\log,\Lambda}^{\text{cp}}(N) + \frac{2}{d}N \log N}{N}, \\ \overline{g}_{\log,d}(\Lambda) &:= \limsup_{N \rightarrow \infty} \frac{\mathcal{E}_{\log,\Lambda}^{\text{cp}}(N) + \frac{2}{d}N \log N}{N}. \end{aligned}$$

Our use of these quantities is motivated by the proof of the main results in [4], and indeed the general strategy of our proofs is similar to that of [4]. More precisely, we shall prove $\underline{g}_{s,d}(\Lambda) = \overline{g}_{s,d}(\Lambda)$ and $\underline{g}_{\log,d}(\Lambda) = \overline{g}_{\log,d}(\Lambda)$ and that these limits are finite. We first need estimates on quantities appearing in (4) and (11).

Lemma 11. *Let $s > 0$ and Λ be a d -dimensional lattice with co-volume 1 and $l_0 := \min_{0 \neq v \in \Lambda} \{|v|\}$. The following relations hold.*

$$(38) \quad \sum_{w \in \Lambda^*} e^{-\frac{\pi^2 |w|^2}{t}} t^{-\frac{d}{2}} = \pi^{-\frac{d}{2}} + O(e^{-l_0^2 t}), \quad \text{as } t \rightarrow \infty,$$

$$(39) \quad \sum_{w \in \Lambda^*} \int_1^{\frac{1}{\delta}} e^{-\frac{\pi^2 |w|^2}{t}} t^{\frac{s-d}{2}-1} dt = \frac{2\pi^{-\frac{d}{2}}}{s} \delta^{-\frac{s}{2}} + O(1), \quad \text{as } \delta \rightarrow 0^+,$$

$$(40) \quad \sum_{w \in \Lambda^*} \int_1^{\frac{1}{\delta}} e^{-\frac{\pi^2 |w|^2}{t}} t^{-\frac{d}{2}-1} dt = \pi^{-\frac{d}{2}} \log \delta^{-1} + O(1), \quad \text{as } \delta \rightarrow 0^+$$

Proof. Applying Poisson Summation, we obtain

$$\sum_{w \in \Lambda^*} e^{-\frac{\pi^2 |w|^2}{t}} t^{-\frac{d}{2}} = \pi^{-d/2} \sum_{v \in \Lambda} e^{-|v|^2 t} = \pi^{-d/2} + \pi^{-d/2} e^{-l_0^2 t} \sum_{v \in \Lambda \setminus \{0\}} e^{-(|v|^2 - l_0^2)t} = \pi^{-d/2} + O(e^{-l_0^2 t}),$$

proving (38). Hence, there exists a constant C_1 such that

$$\left| \sum_{w \in \Lambda^*} e^{-\frac{\pi^2 |w|^2}{t}} t^{-\frac{d}{2}} - \pi^{-\frac{d}{2}} \right| \leq C_1 e^{-l_0^2 t},$$

and so, multiplying both sides of the above by $t^{\frac{s}{2}-1}$, we have

$$\left| \sum_{w \in \Lambda^*} e^{-\frac{\pi^2 |w|^2}{t}} t^{\frac{s-d}{2}-1} - \pi^{-\frac{d}{2}} t^{\frac{s}{2}-1} \right| \leq C_1 t^{\frac{s}{2}-1} e^{-l_0^2 t}$$

and so

$$(41) \quad \left| \int_1^{\frac{1}{\delta}} \left(\sum_{w \in \Lambda^*} e^{-\frac{\pi^2 |w|^2}{t}} t^{\frac{s-d}{2}-1} - \pi^{-\frac{d}{2}} t^{\frac{s}{2}-1} \right) dt \right| \leq \int_1^{\frac{1}{\delta}} C_1 t^{\frac{s}{2}-1} e^{-l_0^2 t} dt$$

$$\leq \int_1^{\infty} C_1 t^{\frac{s}{2}-1} e^{-l_0^2 t} dt =: C_2(s).$$

Therefore,

$$\left| \sum_{w \in \Lambda^*} \int_1^{\frac{1}{\delta}} e^{-\frac{\pi^2 |w|^2}{t}} t^{\frac{s-d}{2}-1} dt - \frac{2\pi^{-\frac{d}{2}}}{s} (\delta^{-\frac{s}{2}} - 1) \right| \leq C_2(s), \quad s > 0$$

proving (39), while substituting $s = 0$ into (41) yields

$$\left| \sum_{w \in \Lambda^*} \int_1^{\frac{1}{\delta}} e^{-\frac{\pi^2 |w|^2}{t}} t^{-\frac{d}{2}-1} dt - \pi^{-\frac{d}{2}} \log \delta^{-1} \right| \leq C_2(0), \quad s = 0.$$

proving (40). \square

The following lemma is the key calculation that allows us to apply the method of [4]. Once we have established this lemma, the only remaining technical difficulty will be to establish the fact that the constants $C_{s,d}$ and $C_{\log,d}$ are independent of the lattice Λ .

Lemma 12. *With Λ as in Lemma 11 and $s > 0$, the following inequalities hold:*

$$-\infty < \underline{g}_{s,d}(\Lambda) \leq \overline{g}_{s,d}(\Lambda) \leq \zeta_{\Lambda}(s) < \infty,$$

$$-\infty < \underline{g}_{\log,d}(\Lambda) \leq \overline{g}_{\log,d}(\Lambda) \leq 0.$$

Proof. Let us first consider the case $s > 0$. For any configuration $\omega_N = (x_j)_{j=1}^N$ in Ω_{Λ} and any $\delta \in (0, 1]$,

$$E_{s,\Lambda}(\omega_N) = \sum_{j \neq k} K_{s,\Lambda}(x_j, x_k) =: I_1 + I_2.$$

where

$$I_1 = \sum_{j \neq k} \sum_{v \in \Lambda} \int_1^{\infty} e^{-|x_j - x_k + v|^2 t} \frac{t^{\frac{s}{2}-1}}{\Gamma(\frac{s}{2})} dt,$$

$$I_2 = \sum_{j \neq k} \sum_{w \in \Lambda^* \setminus \{0\}} e^{2\pi i w \cdot (x_j - x_k)} \int_0^1 \frac{\pi^{\frac{d}{2}}}{t^{\frac{d}{2}}} e^{-\frac{\pi^2 |w|^2}{t}} \frac{t^{\frac{s}{2}-1}}{\Gamma(\frac{s}{2})} dt.$$

Let

$$h_{\delta}(x) := \int_1^{\frac{1}{\delta}} e^{-|x|^2 t} \frac{t^{\frac{s}{2}-1}}{\Gamma(\frac{s}{2})} dt.$$

then

$$\hat{h}_{\delta}(\xi) = \int_1^{\frac{1}{\delta}} \left(\frac{\pi}{t} \right)^{\frac{d}{2}} e^{-\frac{\pi^2 |\xi|^2}{t}} \frac{t^{\frac{s}{2}-1}}{\Gamma(\frac{s}{2})} dt \geq 0, \quad \hat{h}_{\delta}(0) = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{s}{2})(d-s)} \left(1 - \delta^{\frac{d-s}{2}} \right).$$

Notice that since the upper limit of the integral defining h_δ is finite, it is easy to verify that h_δ satisfies the hypotheses of Poisson Summation. Applying it gives us the following inequalities:

$$\begin{aligned}
I_1 &\geq \sum_{j \neq k} \sum_{v \in \Lambda} h_\delta(x_j - x_k + v) \\
&= \sum_{j \neq k} \sum_{w \in \Lambda^*} \hat{h}_\delta(w) e^{2\pi i w \cdot (x_j - x_k)} \\
&= \sum_{w \in \Lambda^*} \hat{h}_\delta(w) \left(\sum_{j,k} e^{2\pi i w \cdot (x_j - x_k)} - N \right) \\
(42) \quad &= \sum_{w \in \Lambda^*} \hat{h}_\delta(w) \left(\left| \sum_j e^{2\pi i w \cdot x_j} \right|^2 - N \right) \\
&\geq N^2 \hat{h}_\delta(0) - N \sum_{w \in \Lambda^*} \hat{h}_\delta(w) \\
&= N^2 \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{s}{2})(d-s)} \left(1 - \delta^{\frac{d-s}{2}}\right) - N \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{s}{2})} \sum_{w \in \Lambda^*} \int_1^{\frac{1}{\delta}} e^{-\frac{\pi^2 |w|^2}{t}} t^{\frac{s-d}{2}-1} dt
\end{aligned}$$

By Lemma 11, we conclude

$$\begin{aligned}
(43) \quad I_1 &\geq N^2 \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{s}{2})(d-s)} \left(1 - \delta^{\frac{d-s}{2}}\right) - N \frac{2\pi^{\frac{d}{2}}}{s\Gamma(\frac{s}{2})} \left(\pi^{-\frac{d}{2}} \delta^{-\frac{s}{2}} + O(1)\right) \\
&= \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{s}{2})(d-s)} N^2 - \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{s}{2})(d-s)} N^2 \delta^{\frac{d-s}{2}} - \frac{2}{s\Gamma(\frac{s}{2})} N \delta^{-\frac{s}{2}} - O(N),
\end{aligned}$$

To obtain lower bounds on I_2 , we calculate

$$\begin{aligned}
(44) \quad I_2 &= \sum_{w \in \Lambda^* \setminus \{0\}} \left(\sum_{j,k} e^{2\pi i w \cdot (x_j - x_k)} - N \right) \int_0^1 \frac{\pi^{\frac{d}{2}}}{t^{\frac{d}{2}}} e^{-\frac{\pi^2 |w|^2}{t}} \frac{t^{\frac{s}{2}-1}}{\Gamma(\frac{s}{2})} dt \\
&= \sum_{w \in \Lambda^* \setminus \{0\}} \left(\left| \sum_j e^{2\pi i w \cdot x_j} \right|^2 - N \right) \int_0^1 \frac{\pi^{\frac{d}{2}}}{t^{\frac{d}{2}}} e^{-\frac{\pi^2 |w|^2}{t}} \frac{t^{\frac{s}{2}-1}}{\Gamma(\frac{s}{2})} dt \\
&\geq -N \cdot \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{s}{2})} \sum_{w \in \Lambda^* \setminus \{0\}} \int_0^1 e^{-\frac{\pi^2 |w|^2}{t}} t^{\frac{s-d}{2}-1} dt \\
&= O(N).
\end{aligned}$$

Therefore

$$E_{s,\Lambda}(\omega_N) = I_1 + I_2 \geq \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{s}{2})(d-s)} N^2 - \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{s}{2})(d-s)} N^2 \delta^{\frac{d-s}{2}} - \frac{2}{s\Gamma(\frac{s}{2})} N \delta^{-\frac{s}{2}} - O(N).$$

If we let $\delta = \pi^{-1}N^{-\frac{2}{d}}$, then this lower bound becomes

$$(45) \quad E_{s,\Lambda}(\omega_N) \geq \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{s}{2})(d-s)}N^2 + C^*N^{1+\frac{s}{d}} + O(N).$$

where

$$C^* = -\frac{2\pi^{\frac{s}{2}}d}{\Gamma(\frac{s}{2})s(d-s)}.$$

The right hand side of (45) is independent of ω_N and thus

$$\begin{aligned} \mathcal{E}_{s,\Lambda}(N) &\geq \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{s}{2})(d-s)}N^2 + C^*N^{1+\frac{s}{d}} + O(N), \\ \mathcal{E}_{s,\Lambda}^{\text{cp}}(N) &\geq C^*N^{1+\frac{s}{d}} + O(N). \end{aligned}$$

We conclude that $\underline{g}_{s,d}(\Lambda) \geq C^*$.

To establish the finiteness of $\bar{g}_{s,d}$, we will use the same method as was used in [4]. For any natural number N , let $m = m_N$ be a positive integer such that $(m-1)^d < N \leq m^d$. Let $\omega^m = \frac{1}{m}\Lambda \cap \Omega_\Lambda$. Then

$$(46) \quad \mathcal{E}_{s,\Lambda}^{\text{cp}}(m^d) \leq E_{s,\Lambda}^{\text{cp}}(\omega^m) = \sum_{\substack{x_j, x_k \in \frac{1}{m}\Lambda \cap \Omega_\Lambda \\ x_j \neq x_k}} \zeta_\Lambda(s; x_j - x_k) = m^d(m^s - 1)\zeta_\Lambda(s),$$

where we used Lemma 7

As $\left\{ \frac{\mathcal{E}_{s,\Lambda}^{\text{cp}}(N)}{N(N-1)} \right\}_{N=2}^\infty$ is an increasing sequence (see, e.g., [6, Chapter II §3.12, page 160]) we arrive at the following:

$$\begin{aligned} \bar{g}_{s,d}(\Lambda) &= \limsup_{N \rightarrow \infty} \frac{\mathcal{E}_{s,\Lambda}^{\text{cp}}(N)}{N^{1+s/d}} = \limsup_{N \rightarrow \infty} \frac{\mathcal{E}_{s,\Lambda}^{\text{cp}}(N)}{N(N-1)} \cdot \frac{N-1}{N^{\frac{s}{d}}} \\ &\leq \limsup_{N \rightarrow \infty} \frac{\mathcal{E}_{s,\Lambda}^{\text{cp}}(\Omega_\Lambda, m^d)}{m^d(m^d-1)} \cdot \frac{N-1}{N^{\frac{s}{d}}} \\ &\leq \limsup_{N \rightarrow \infty} \frac{m^d(m^s-1)\zeta_\Lambda(s)}{m^d(m^d-1)} \cdot \frac{N-1}{N^{\frac{s}{d}}} = \zeta_\Lambda(s) < \infty. \end{aligned}$$

Now we turn our attention to the classical periodic logarithmic energy. Using (12), (43), and (44), we obtain

$$\begin{aligned}
E_{\log, \Lambda}(\omega_N) &= \lim_{s \rightarrow 0^+} \Gamma\left(\frac{s}{2}\right) E_{s, \Lambda}(\omega_N) = \lim_{s \rightarrow 0^+} \Gamma\left(\frac{s}{2}\right) (I_1 + I_2) \\
&\geq N^2 \frac{2\pi^{\frac{d}{2}}}{d} \left(1 - \delta^{\frac{d}{2}}\right) - N\pi^{\frac{d}{2}} \sum_{w \in \Lambda^*} \int_1^{\frac{1}{\delta}} e^{-\frac{\pi^2 |w|^2}{t}} t^{-\frac{d}{2}-1} dt + O(N) \\
&= N^2 \frac{2\pi^{\frac{d}{2}}}{d} \left(1 - \delta^{\frac{d}{2}}\right) - N\pi^{\frac{d}{2}} \left(\pi^{-\frac{d}{2}} \log \delta^{-1} + O(1)\right) + O(N) \\
&= \frac{2\pi^{\frac{d}{2}}}{d} N^2 - \frac{2\pi^{\frac{d}{2}}}{d} N^2 \delta^{\frac{d}{2}} - N \log \delta^{-1} + O(N).
\end{aligned}$$

If we let $\delta = N^{-\frac{2}{d}}$, then we get

$$E_{\log, \Lambda}(\omega_N) = \frac{2\pi^{\frac{d}{2}}}{d} N^2 - \frac{2}{d} N \log N + O(N).$$

Thus

$$\begin{aligned}
\mathcal{E}_{\log, \Lambda}(N) &\geq \frac{2\pi^{\frac{d}{2}}}{d} N^2 - \frac{2}{d} N \log N + O(N), \\
\mathcal{E}_{\log, \Lambda}^{\text{cp}}(N) + \frac{2}{d} N \log N &\geq O(N),
\end{aligned}$$

and we conclude that $\underline{g}_{\log, d}(\Lambda) > -\infty$.

To establish the finiteness of $\bar{g}_{s, d}$, let $m = m_N$ be a positive integer such that $(m-1)^d < N \leq m^d$. Let $\omega^m = \frac{1}{m} \Lambda \cap \Omega_\Lambda$. Then by (46)

$$E_{s, \Lambda}^{\text{cp}}(\omega^m) = m^d (m^s - 1) \zeta_\Lambda(s).$$

By definition,

$$\begin{aligned}
\mathcal{E}_{\log, \Lambda}^{\text{cp}}(m^d) &\leq E_{\log, \Lambda}^{\text{cp}}(\omega^m) = 2 \left. \frac{d}{ds} E_{s, \Lambda}^{\text{cp}}(\omega_N) \right|_{s=0} \\
&= 2 m^d \left(m^s \log m \cdot \zeta_\Lambda(s) + (m^s - 1) \zeta'_\Lambda(s) \right) \Big|_{s=0} \\
&= 2 m^d \log m \cdot \zeta_\Lambda(0) = -2 m^d \log m = -\frac{2}{d} m^d \log m^d
\end{aligned}$$

Here we use the fact that $\zeta_\Lambda(0) = -1$ for every lattice Λ (see the remark following Lemma 5). We conclude that

$$\frac{\mathcal{E}_{s, \Lambda}^{\text{cp}}(N)}{N} = \frac{\mathcal{E}_{s, \Lambda}^{\text{cp}}(N)}{N(N-1)} \cdot (N-1) \leq \frac{\mathcal{E}_{s, \Lambda}^{\text{cp}}(m^d)}{m^d(m^d-1)} \cdot (N-1) \leq -\frac{2}{d} \log m^d \cdot \frac{N-1}{m^d-1}$$

This implies

$$\frac{\mathcal{E}_{s, \Lambda}^{\text{cp}}(N) + \frac{2}{d} N \log N}{N} \leq -\frac{2}{d} \log m^d \cdot \frac{N-1}{m^d-1} + \frac{2}{d} \log N,$$

which tends to 0 as $N \rightarrow \infty$, and hence $\bar{g}_{\log, d}(\Lambda) \leq 0$. \square

The following lemma establishes scaling properties of the classical periodic energy and will be helpful in establishing independence of the constants $C_{s,d}$ and $C_{\log,d}$ of the lattice Λ .

Lemma 13. *Let Λ be a lattice and $\Lambda' = B\Lambda$ be a sublattice of Λ (i.e. $B \in GL(d, \mathbb{Z})$), then for any $N > 0$,*

$$\begin{aligned}\mathcal{E}_{s,\Lambda'}^{\text{cp}}(N|\det B|) &\leq |\det B| \mathcal{E}_{s,\Lambda}^{\text{cp}}(N) + N|\det B|(\zeta_\Lambda(s) - \zeta_{\Lambda'}(s)), \\ \mathcal{E}_{\log,\Lambda'}^{\text{cp}}(N|\det B|) &\leq |\det B| \mathcal{E}_{\log,\Lambda}^{\text{cp}}(N) + 2N|\det B|(\zeta'_\Lambda(0) - \zeta'_{\Lambda'}(0)).\end{aligned}$$

Proof. For any $\omega_N = (x_j)_{j=1}^N \in (\Omega_\Lambda)^N$, let $S(\omega_N) = (\omega_N + \Lambda) \cap \Omega_{\Lambda'}$. Then $S(\omega_N)$ is a $(N|\det B|)$ -point configuration in $\Omega_{\Lambda'}$ and

$$\begin{aligned}E_{s,\Lambda'}^{\text{cp}}(S(\omega_N)) &= \sum_{\substack{x,y \in S(\omega_N) \\ x \neq y}} \zeta_{\Lambda'}(s, x-y) = \sum_{j,k} \sum_{\substack{r \in \Lambda, x_j+r \in \Omega_{\Lambda'} \\ t \in \Lambda, x_k+t \in \Omega_{\Lambda'} \\ x_j+r \neq x_k+t}} \zeta_{\Lambda'}(s; x_j+r-x_k-t) \\ &= \sum_{j \neq k} \sum_{\substack{r \in \Lambda \cap \Omega_{\Lambda'} \\ t \in \Lambda \cap \Omega_{\Lambda'}}} \zeta_{\Lambda'}(s; x_j-x_k+r-t) + \sum_{j=1}^N \sum_{\substack{r \in \Lambda \cap \Omega_{\Lambda'} \\ t \in \Lambda \cap \Omega_{\Lambda'} \\ r \neq t}} \zeta_{\Lambda'}(s; r-t) \\ &= \sum_{j \neq k} \sum_{r \in \Lambda \cap \Omega_{\Lambda'}} \zeta_\Lambda(s; x_j-x_k) + N \sum_{r \in \Lambda \cap \Omega_{\Lambda'}} (\zeta_\Lambda(s) - \zeta_{\Lambda'}(s)) \\ (47) \quad &= |\det B| \cdot E_{s,\Lambda}^{\text{cp}}(\omega_N) + N|\det B|(\zeta_\Lambda(s) - \zeta_{\Lambda'}(s)),\end{aligned}$$

where we used Lemma 6. Taking the infimum over all configurations $(x_j)_{j=1}^N \in (\Omega_\Lambda)^N$, we conclude that

$$\mathcal{E}_{s,\Lambda'}^{\text{cp}}(N|\det B|) \leq \inf_{\omega_N \in (\Omega_\Lambda)^N} E_{s,\Lambda'}^{\text{cp}}(S(\omega_N)) = |\det B| \cdot \mathcal{E}_{s,\Lambda}^{\text{cp}}(N) + N|\det B|(\zeta_\Lambda(s) - \zeta_{\Lambda'}(s)).$$

The logarithmic case follows from this by differentiating (47) and evaluating at $s = 0$. \square

Corollary 14. *For any positive integers m and N , we have*

$$(48) \quad \frac{\mathcal{E}_{s,\Lambda}^{\text{cp}}(m^d N)}{(m^d N)^{1+\frac{s}{d}}} \leq \frac{\mathcal{E}_{s,\Lambda}^{\text{cp}}(N)}{N^{1+\frac{s}{d}}} + \frac{(1-m^{-s})\zeta_\Lambda(s)}{N^{\frac{s}{d}}},$$

$$(49) \quad \frac{\mathcal{E}_{\log,\Lambda}^{\text{cp}}(m^d N) + \frac{2}{d}m^d N \log(m^d N)}{m^d N} \leq \frac{\mathcal{E}_{\log,\Lambda}^{\text{cp}}(N) + \frac{2}{d}N \log N}{N}.$$

Proof. Using Lemma 13, we obtain

$$\begin{aligned}\mathcal{E}_{s,m\Lambda}^{\text{cp}}(m^d N) &\leq m^d \cdot \mathcal{E}_{s,\Lambda}^{\text{cp}}(N) + m^d N (\zeta_\Lambda(s) - \zeta_{m\Lambda}(s)), \\ \mathcal{E}_{\log,m\Lambda}^{\text{cp}}(m^d N) &\leq m^d \cdot \mathcal{E}_{\log,\Lambda}^{\text{cp}}(N) + 2m^d N (\zeta'_\Lambda(0) - \zeta'_{m\Lambda}(0)).\end{aligned}$$

Then (48) and (49) follow from

$$(50) \quad \mathcal{E}_{s,m\Lambda}^{\text{cp}}(m^d N) = m^{-s} \mathcal{E}_{s,\Lambda}^{\text{cp}}(m^d N), \quad \zeta_{m\Lambda}(s) = m^{-s} \zeta_\Lambda(s),$$

$$(51) \quad \mathcal{E}_{\log,m\Lambda}^{\text{cp}}(m^d N) = \mathcal{E}_{\log,\Lambda}^{\text{cp}}(m^d N), \quad \zeta'_{m\Lambda}(0) = \log m + \zeta'_\Lambda(0),$$

where the first identity in (51) is obtained from the first identity in (50) using (12) while the second identity in (51) follows by differentiating the second identity in (50) and evaluating at $s = 0$. \square

We are now ready to prove our main result.

Proof of Theorem 1. By (36) and (37) it suffices to show

$$\begin{aligned}\bar{g}_{s,d}(\Lambda) &= \underline{g}_{s,d}(\Lambda) = C_{s,d}, \\ \bar{g}_{\log,d}(\Lambda) &= \underline{g}_{\log,d}(\Lambda) = C_{\log,d} - 2\zeta'_\Lambda(0).\end{aligned}$$

Fix some positive integer N_0 . For any $N > N_0$ there exists $m \in \mathbb{N}$ such that $(m-1)^d N_0 \leq N < m^d N_0$, using Corollary 14 and the fact that $\{\frac{\mathcal{E}_{s,\Lambda}^{\text{cp}}(N)}{N(N-1)}\}_{N=2}^\infty$ is an increasing sequence we obtain

$$\begin{aligned}\frac{\mathcal{E}_{s,\Lambda}^{\text{cp}}(N)}{N^{1+\frac{s}{d}}} &= \frac{\mathcal{E}_{s,\Lambda}^{\text{cp}}(N)}{N(N-1)} \cdot \frac{N-1}{N^{\frac{s}{d}}} \leq \frac{\mathcal{E}_{s,\Lambda}^{\text{cp}}(m^d N_0)}{m^d N_0(m^d N_0 - 1)} \cdot \frac{N-1}{N^{\frac{s}{d}}} \\ &= \frac{\mathcal{E}_{s,\Lambda}^{\text{cp}}(m^d N_0)}{(m^d N_0)^{1+\frac{s}{d}}} \cdot \frac{(m^d N_0)^{\frac{s}{d}}}{(m^d N_0 - 1)} \cdot \frac{N-1}{N^{\frac{s}{d}}} \\ &\leq \left(\frac{\mathcal{E}_{s,\Lambda}^{\text{cp}}(N_0)}{N_0^{1+\frac{s}{d}}} + \frac{(1-m^{-s})\zeta_\Lambda(s)}{N_0^{\frac{s}{d}}} \right) \cdot \frac{(m^d N_0)^{\frac{s}{d}}}{N^{\frac{s}{d}}} \cdot \frac{N-1}{(m^d N_0 - 1)}.\end{aligned}$$

Similarly

$$\begin{aligned}\frac{\mathcal{E}_{\log,\Lambda}^{\text{cp}}(N) + \frac{2}{d}N \log N}{N} &= \frac{\mathcal{E}_{\log,\Lambda}^{\text{cp}}(N)}{N(N-1)} \cdot (N-1) + \frac{2}{d} \log N \\ &\leq \frac{\mathcal{E}_{\log,\Lambda}^{\text{cp}}(m^d N_0)}{m^d N_0(m^d N_0 - 1)} \cdot (N-1) + \frac{2}{d} \log(m^d N_0) \\ &= \left(\frac{\mathcal{E}_{\log,\Lambda}^{\text{cp}}(m^d N_0) + \frac{2}{d}m^d N_0 \log(m^d N_0)}{m^d N_0} - \frac{2}{d} \log(m^d N_0) \right) \cdot \frac{N-1}{m^d N_0 - 1} + \frac{2}{d} \log(m^d N_0) \\ &\leq \left(\frac{\mathcal{E}_{\log,\Lambda}^{\text{cp}}(N_0) + \frac{2}{d}N_0 \log N_0}{N_0} - \frac{2}{d} \log(m^d N_0) \right) \cdot \frac{N-1}{m^d N_0 - 1} + \frac{2}{d} \log(m^d N_0).\end{aligned}$$

Letting $N \rightarrow \infty$ yields

$$\begin{aligned}\bar{g}_{s,d}(\Lambda) &= \limsup_{N \rightarrow \infty} \frac{\mathcal{E}_{s,\Lambda}^{\text{cp}}(N)}{N^{1+s/d}} \leq \left(\frac{\mathcal{E}_{s,\Lambda}^{\text{cp}}(N_0)}{N_0^{1+\frac{s}{d}}} + \frac{\zeta_\Lambda(s)}{N_0^{\frac{s}{d}}} \right), \\ \bar{g}_{\log,d}(\Lambda) &= \limsup_{N \rightarrow \infty} \frac{\mathcal{E}_{\log,\Lambda}^{\text{cp}}(N) + \frac{2}{d}N \log N}{N} \leq \frac{\mathcal{E}_{\log,\Lambda}^{\text{cp}}(N_0) + \frac{2}{d}N_0 \log N_0}{N_0}.\end{aligned}$$

Letting $N_0 \rightarrow \infty$ through an appropriate subsequence yields

$$\begin{aligned}\bar{g}_{s,d}(\Lambda) &\leq \liminf_{N_0 \rightarrow \infty} \frac{\mathcal{E}_{s,\Lambda}^{\text{cp}}(N_0)}{N_0^{1+\frac{s}{d}}} = \underline{g}_{s,d}(\Lambda), \\ \bar{g}_{\log,d}(\Lambda) &\leq \liminf_{N_0 \rightarrow \infty} \frac{\mathcal{E}_{\log,\Lambda}^{\text{cp}}(N_0) + \frac{2}{d}N_0 \log N_0}{N_0} = \underline{g}_{\log,d}(\Lambda).\end{aligned}$$

Therefore $\bar{g}_{s,d}(\Lambda) = \underline{g}_{s,d}(\Lambda) =: C_{s,d}(\Lambda)$ and $\bar{g}_{\log,d}(\Lambda) = \underline{g}_{\log,d}(\Lambda) =: C_{\log,d}(\Lambda)$.

To show $C_{s,d}(\Lambda)$ is independent of Λ , let $\Lambda_1 = A_1\mathbb{Z}^d$ and $\Lambda_2 = A_2\mathbb{Z}^d$ be any two lattices with co-volume 1. Then $\Lambda_2 = Q\Lambda_1$ where $Q = A_2A_1^{-1}$. We can use rational matrices to approximate Q , namely, there exists a sequence $Q_m \in \frac{1}{m}GL(d; \mathbb{Z})$ such that $Q_m \rightarrow Q$.

For any lattice Λ , $mQ_m\Lambda = (mQ_m)\Lambda$ is a sublattice of Λ since $mQ_m \in GL(d; \mathbb{Z})$. Applying Lemma 13 to $mQ_m\Lambda$ and Λ we get

$$\mathcal{E}_{s,mQ_m\Lambda}^{\text{cp}}(Nm^d |\det Q_m|) \leq m^d |\det Q_m| \mathcal{E}_{s,\Lambda}^{\text{cp}}(N) + Nm^d |\det Q_m| (\zeta_\Lambda(s) - \zeta_{mQ_m\Lambda}(s)).$$

Now if we let $\Lambda = Q_m^{-1}\Lambda_2$ we get

$$\mathcal{E}_{s,m\Lambda_2}^{\text{cp}}(Nm^d |\det Q_m|) \leq m^d |\det Q_m| \mathcal{E}_{s,Q_m^{-1}\Lambda_2}^{\text{cp}}(N) + Nm^d |\det Q_m| (\zeta_{Q_m^{-1}\Lambda_2}(s) - \zeta_{m\Lambda_2}(s)).$$

Using relation (50) again implies

$$m^{-s} \mathcal{E}_{s,\Lambda_2}^{\text{cp}}(Nm^d |\det Q_m|) \leq m^d |\det Q_m| \mathcal{E}_{s,Q_m^{-1}\Lambda_2}^{\text{cp}}(N) + Nm^d |\det Q_m| (\zeta_{Q_m^{-1}\Lambda_2}(s) - m^{-s} \zeta_{\Lambda_2}(s)),$$

which can be rewritten as

$$\frac{\mathcal{E}_{s,\Lambda_2}^{\text{cp}}(Nm^d |\det Q_m|)}{(Nm^d |\det Q_m|)^{1+\frac{s}{d}}} \leq \frac{\mathcal{E}_{s,Q_m^{-1}\Lambda_2}^{\text{cp}}(N)}{N^{1+\frac{s}{d}} |\det Q_m|^{\frac{s}{d}}} + \frac{\zeta_{Q_m^{-1}\Lambda_2}(s) - m^{-s} \zeta_{\Lambda_2}(s)}{N^{\frac{s}{d}} |\det Q_m|^{\frac{s}{d}}}.$$

Letting $m \rightarrow \infty$ and using Corollary 9 and Lemma 10, we obtain

$$C_{s,d}(\Lambda_2) \leq \frac{\mathcal{E}_{s,Q^{-1}\Lambda_2}^{\text{cp}}(N)}{N^{1+\frac{s}{d}}} + \frac{\zeta_{Q^{-1}\Lambda_2}(s)}{N^{\frac{s}{d}}} = \frac{\mathcal{E}_{s,\Lambda_1}^{\text{cp}}(N)}{N^{1+\frac{s}{d}}} + \frac{\zeta_{\Lambda_1}(s)}{N^{\frac{s}{d}}}.$$

Taking $N \rightarrow \infty$ implies

$$C_{s,d}(\Lambda_2) \leq C_{s,d}(\Lambda_1).$$

By the arbitrariness of Λ_1 and Λ_2 we must have $C_{s,d}(\Lambda) \equiv C_{s,d}$ which is independent of Λ .

For the logarithmic case, we apply Lemma 13 to $mQ_m\Lambda$ and Λ to deduce

$$\mathcal{E}_{\log,mQ_m\Lambda}^{\text{cp}}(Nm^d |\det Q_m|) \leq m^d |\det Q_m| \mathcal{E}_{\log,\Lambda}^{\text{cp}}(N) + 2Nm^d |\det Q_m| (\zeta'_\Lambda(0) - \zeta'_{mQ_m\Lambda}(0)).$$

Now if we let $\Lambda = Q_m^{-1}\Lambda_2$ we have

$$\mathcal{E}_{\log,m\Lambda_2}^{\text{cp}}(Nm^d |\det Q_m|) \leq m^d |\det Q_m| \mathcal{E}_{\log,Q_m^{-1}\Lambda_2}^{\text{cp}}(N) + 2Nm^d |\det Q_m| (\zeta'_{Q_m^{-1}\Lambda_2}(0) - \zeta'_{m\Lambda_2}(0)).$$

Using relation (50) again implies

$$\mathcal{E}_{\log,\Lambda_2}^{\text{cp}}(Nm^d |\det Q_m|) \leq m^d |\det Q_m| \mathcal{E}_{\log,Q_m^{-1}\Lambda_2}^{\text{cp}}(N) + 2Nm^d |\det Q_m| (\zeta'_{Q_m^{-1}\Lambda_2}(0) - \log m - \zeta'_{\Lambda_2}(0)),$$

which can be rewritten as

$$\frac{\mathcal{E}_{\log, \Lambda_2}^{\text{cp}}(Nm^d |\det Q_m|)}{Nm^d |\det Q_m|} \leq \frac{\mathcal{E}_{\log, Q_m^{-1} \Lambda_2}^{\text{cp}}(N)}{N} + 2 \left(\zeta'_{Q_m^{-1} \Lambda_2}(0) - \log m - \zeta'_{\Lambda_2}(0) \right).$$

Therefore,

$$\begin{aligned} & \frac{\mathcal{E}_{\log, \Lambda_2}^{\text{cp}}(Nm^d |\det Q_m|) + \frac{2}{d} Nm^d |\det Q_m| \log(Nm^d |\det Q_m|)}{Nm^d |\det Q_m|} \\ & \leq \frac{\mathcal{E}_{\log, Q_m^{-1} \Lambda_2}^{\text{cp}}(N) + \frac{2}{d} N \log N}{N} + 2 \left(\zeta'_{Q_m^{-1} \Lambda_2}(0) - \zeta'_{\Lambda_2}(0) \right) + \frac{2}{d} \log |\det Q_m|. \end{aligned}$$

Now let $m \rightarrow \infty$ and using Corollary 9 and Lemma 10 we obtain

$$C_{\log, d}(\Lambda_2) \leq \frac{\mathcal{E}_{\log, Q^{-1} \Lambda_2}^{\text{cp}}(N) + \frac{2}{d} N \log N}{N} + 2 \left(\zeta'_{Q^{-1} \Lambda_2}(0) - \zeta'_{\Lambda_2}(0) \right).$$

Taking $N \rightarrow \infty$ implies

$$C_{\log, d}(\Lambda_2) \leq C_{\log, d}(\Lambda_1) + 2 \left(\zeta'_{\Lambda_1}(0) - \zeta'_{\Lambda_2}(0) \right).$$

By symmetry

$$C_{\log, d}(\Lambda_1) \leq C_{\log, d}(\Lambda_2) + 2 \left(\zeta'_{\Lambda_2}(0) - \zeta'_{\Lambda_1}(0) \right).$$

It follows that

$$C_{\log, d}(\Lambda_1) + 2\zeta'_{\Lambda_1}(0) = C_{\log, d}(\Lambda_2) + 2\zeta'_{\Lambda_2}(0).$$

Hence, if we define $C_{\log, d} := C_{\log, d}(\Lambda) + 2\zeta'_{\Lambda}(0)$ for any lattice Λ of co-volume 1, then this quantity is in fact independent of the choice of lattice Λ , which is what we wanted to show. \square

5. PROOF OF THEOREM 4

Recall that throughout this section $\Lambda = A\mathbb{Z}^d$ denotes a d -dimensional lattice in \mathbb{R}^d with fundamental domain $\Omega = \Omega_\Lambda$ and co-volume 1. The j th column of the matrix A is denoted by v_j . First we establish the following lemma that will be used in the proof of Theorem 4.

Lemma 15. *If $s \leq d$ and $M \in \mathbb{N}$, then the unique weak limit of the measures*

$$\mu_L := \frac{\sum_{v \in \Lambda: M < |v| \leq L} |v|^{-s} \delta_{\frac{v}{|v|}}}{\sum_{v \in \Lambda: M < |v| \leq L} |v|^{-s}}$$

as $L \rightarrow \infty$ is volume measure on the sphere $\mathbb{S}^{d-1} \subseteq \mathbb{R}^d$.

Proof. First note that as $|v| \rightarrow \infty$

$$\begin{aligned} (52) \quad |x + v|^{-s} &= |v|^{-s} \left(1 + 2 \frac{x \cdot v}{|v|^2} + \frac{|x|^2}{|v|^2} \right)^{-s/2} \\ &= |v|^{-s} \left[1 - s \left(\frac{x \cdot v}{|v|^2} + \left(\frac{1}{2} \right) \frac{|x|^2}{|v|^2} \right) + s(s+2) \left(\frac{x \cdot v}{|v|^2} \right)^2 \right] + O(|v|^{-s-3}), \end{aligned}$$

which implies:

$$(53) \quad |x+v|^{-s} + |x-v|^{-s} - 2|v|^{-s} = -s \frac{|x|^2}{|v|^{s+2}} + s(s+2) \left(\frac{(x \cdot v/|v|)^2}{|v|^{s+2}} \right) + O(|v|^{-s-3}), \quad (|v| \rightarrow \infty).$$

Let $[\cdot]_\Lambda$ denote *lattice greatest integer function* defined by

$$[x]_\Lambda = A \cdot \sup \left\{ k \in \mathbb{Z}^d : x - Ak = \sum_{j=1}^d a_j v_j, a_j \geq 0 \right\},$$

where we take the supremum with respect to the dictionary order on \mathbb{Z}^d .

Using (53) and writing $x = [x]_\Lambda + \{x\}_\Lambda$, where $\{x\}_\Lambda \in \Omega$ we obtain

$$(54) \quad |x|^{-s} - |[x]_\Lambda|^{-s} = O(|x|^{-s-1}), \quad |x| \rightarrow \infty.$$

We shall also need the following result. Claim: Let m_d denote Lebesgue measure on \mathbb{R}^d and $0 < s \leq d$. Then

$$(55) \quad \lim_{L \rightarrow \infty} \frac{\sum_{M < |v| \leq L} |v|^{-s}}{\int_{M < |x| \leq L} |x|^{-s} dm_d(x)} = |\Omega|^{-1},$$

where the sum in the numerator of the left hand side of (55) is over $v \in \Lambda$ such that $M < |v| \leq L$.

To prove this claim, we first consider the case $s < d - 1$. We write

(56)

$$|\Omega|^{-1} \int_{M < |x| \leq L} |x|^{-s} dm_d(x) - \sum_{M < |v| \leq L} |v|^{-s} = |\Omega|^{-1} \int_{M < |x| \leq L} |x|^{-s} - |[x]_\Lambda|^{-s} dm_d(x) + \epsilon_L,$$

where

$$(57) \quad \epsilon_L = |\Omega|^{-1} \int_{M < |x| \leq L} |[x]_\Lambda|^{-s} dm_d(x) - \sum_{M < |v| \leq L} |v|^{-s},$$

is an error term. Suppose $(\Omega + v) \cap \{x : |x| = L\} = \emptyset$. Then the contribution to (57) from v is zero. Therefore, to estimate this error term, we need only consider vectors v for which $(\Omega + v) \cap \{x : |x| = L\} \neq \emptyset$ in the sum and vectors x for which $[x]_\Lambda = v$ and $(\Omega + v) \cap \{x : |x| = L\} \neq \emptyset$ in the integral. This implies that $\epsilon_L = O(L^{d-1-s})$ as $L \rightarrow \infty$. Furthermore, (54) implies that the integral on the right-hand side of (56) is $O(L^{d-1-s})$ as $L \rightarrow \infty$. Therefore, since $\sum_{M < |v| \leq L} |v|^{-s}$ grows like L^{d-s} as $L \rightarrow \infty$, we have proven (55) in the case $s < d - 1$.

In the case $s = d - 1$, the same reasoning shows that the integral on the right-hand side of (56) is $O(\log(L))$ as $L \rightarrow \infty$ and ϵ_L is bounded as $L \rightarrow \infty$, while the sum $\sum_{M < |v| \leq L} |v|^{-s}$ grows like L as $L \rightarrow \infty$. Therefore, (55) holds in this case as well.

If $d - 1 < s < d$, then the integral on the right-hand side of (56) is $O(1)$ as $L \rightarrow \infty$ and ϵ_L is $O(L^{d-1-s})$ as $L \rightarrow \infty$, while the sum $\sum_{M < |v| \leq L} |v|^{-s}$ grows like L^{d-s} as $L \rightarrow \infty$ showing (55) is true in this case as well.

Finally, if $s = d$, then the integral on the right-hand side of (56) is $O(1)$ as $L \rightarrow \infty$ and ϵ_L is $O(L^{-1})$ as $L \rightarrow \infty$, while the sum $\sum_{M < |v| \leq L} |v|^{-s}$ grows like $\log(L)$ as $L \rightarrow \infty$. Therefore, (55) holds for all $s \in (0, d]$.

Now we are ready to finish the proof of the lemma. Let f be any continuous function on \mathbb{S}^{d-1} and consider

$$(58) \quad \begin{aligned} \lim_{L \rightarrow \infty} \int_{\mathbb{S}^{d-1}} f(x) d\mu_L(x) &= \lim_{L \rightarrow \infty} \frac{\sum_{M < |v| \leq L} f\left(\frac{v}{|v|}\right) |v|^{-s}}{\sum_{M < |v| \leq L} |v|^{-s}} \\ &= (1 + o(1)) \lim_{L \rightarrow \infty} \frac{|\Omega| \sum_{M < |v| \leq L} f\left(\frac{v}{|v|}\right) |v|^{-s}}{\int_{M < |x| \leq L} |x|^{-s} dm_d(x)}, \end{aligned}$$

where we used (55). Notice that the denominator of this last expression grows like L^{d-s} (or $\log(L)$ in the case $s = d$) as $L \rightarrow \infty$. Now consider the expression

$$(59) \quad \begin{aligned} &|\Omega| \sum_{M < |v| \leq L} f\left(\frac{v}{|v|}\right) |v|^{-s} - \int_{M < |x| \leq L} f\left(\frac{x}{|x|}\right) |x|^{-s} dm_d(x) \\ &= \int_{M < |x| \leq L} f\left(\frac{\lfloor x \rfloor_\Lambda}{|\lfloor x \rfloor_\Lambda|}\right) |\lfloor x \rfloor_\Lambda|^{-s} dm_d(x) - \int_{M < |x| \leq L} f\left(\frac{x}{|x|}\right) |x|^{-s} dm_d(x) + \epsilon'_L, \end{aligned}$$

where ϵ'_L is an error term. An argument similar to the proof of the claim shows that ϵ'_L is negligibly small compared to the denominator in (58) as $L \rightarrow \infty$, so we may ignore this term when calculating the limit (57). We may then write

$$(60) \quad \begin{aligned} &\int_{M < |x| \leq L} f\left(\frac{\lfloor x \rfloor_\Lambda}{|\lfloor x \rfloor_\Lambda|}\right) |\lfloor x \rfloor_\Lambda|^{-s} dm_d(x) - \int_{M < |x| \leq L} f\left(\frac{x}{|x|}\right) |x|^{-s} dm_d(x) \\ &= \int_{M < |x| \leq L} \left(f\left(\frac{\lfloor x \rfloor_\Lambda}{|\lfloor x \rfloor_\Lambda|}\right) - f\left(\frac{x}{|x|}\right) \right) |\lfloor x \rfloor_\Lambda|^{-s} dm_d(x) \\ &\quad + \int_{M < |x| \leq L} f\left(\frac{x}{|x|}\right) (|\lfloor x \rfloor_\Lambda|^{-s} - |x|^{-s}) dm_d(x). \end{aligned}$$

Let us examine the first expression on the right-hand side of (60). Since the denominator on the right-hand side of (58) tends to infinity as $L \rightarrow \infty$, we may replace M by M_L tending to infinity very slowly as $L \rightarrow \infty$ when calculating the first integral on the right-hand side of (60). In so doing, we introduce an error term that is negligible compared to the denominator on the right-hand side of (58) as $L \rightarrow \infty$. Since $M_L \rightarrow \infty$ as $L \rightarrow \infty$, the uniform continuity of f implies that

$$\lim_{L \rightarrow \infty} \left| \frac{\int_{M < |x| \leq L} \left(f\left(\frac{\lfloor x \rfloor_\Lambda}{|\lfloor x \rfloor_\Lambda|}\right) - f\left(\frac{x}{|x|}\right) \right) |\lfloor x \rfloor_\Lambda|^{-s} dm_d(x)}{\int_{M < |x| \leq L} |x|^{-s} dm_d(x)} \right| = 0.$$

Furthermore, (54) implies that the second integral on the right-hand side of (60) is $O(L^{d-1-s})$ (or $O(\log(L))$ in the case $s = d - 1$, or $O(1)$ if $s \in (d - 1, d]$) as $L \rightarrow \infty$. We conclude that the

limit on the right-hand side of (58) is the same as

$$\lim_{L \rightarrow \infty} \frac{\int_{M < |x| \leq L} f\left(\frac{x}{|x|}\right) |x|^{-s} dm_d(x)}{\int_{M < |x| \leq L} |x|^{-s} dm_d(x)},$$

which is clearly invariant under any symmetry of the sphere \mathbb{S}^{d-1} , so it must be equal to

$$\int_{\mathbb{S}^{d-1}} f(z) d\sigma_{d-1}(z)$$

as desired. \square

Now we are ready to prove Theorem 4.

Proof of Theorem 4. Let $x \notin \Lambda$ be fixed.

$$\begin{aligned} & \frac{|x|^{-s} + \frac{1}{2} \sum_{v \in \Lambda_L \setminus \{0\}} (|x+v|^{-s} + |x-v|^{-s} - 2|v|^{-s})}{\sum_{v \in \Lambda_L \setminus \{0\}} |v|^{-(s+2)}} \\ &= \frac{\frac{1}{2} \sum_{v \in \Lambda_L \setminus \{0\}} -s \frac{|x|^2}{|v|^{s+2}} + s(s+2) \left(\frac{(x \cdot v/|v|)^2}{|v|^{s+2}} \right)}{\sum_{v \in \Lambda_L \setminus \{0\}} |v|^{-(s+2)}} + O(|v|^{-1}), \end{aligned}$$

as $|v| \rightarrow \infty$ where the implied constants depend only on $|x|$ and the distance from x to Λ and so the above holds uniformly on compact subsets of $\mathbb{R}^d \setminus \Lambda$. Therefore, we have

$$\begin{aligned} & \lim_{L \rightarrow \infty} \frac{|x|^{-s} + \frac{1}{2} \sum_{v \in \Lambda_L \setminus \{0\}} (|x+v|^{-s} + |x-v|^{-s} - 2|v|^{-s})}{\sum_{v \in \Lambda_L \setminus \{0\}} |v|^{-(s+2)}} \\ &= \lim_{L \rightarrow \infty} \int_{\mathbb{S}^{d-1}} (-s|x|^2 + s(s+2)\langle x, z \rangle^2) d\mu_L(z) \\ &= \int_{\mathbb{S}^{d-1}} (-s|x|^2 + s(s+2)\langle x, z \rangle^2) d\sigma_{d-1}(z), \\ &= s \left(\frac{s+2}{d} - 1 \right) |x|^2, \end{aligned}$$

where we used Lemma 15, but with s replaced by $s+2$. \square

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