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Minimum Riesz energy problems for a condenser with ”touching plates”

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Abstract We consider minimum energy problems in the presence of an external field for a condenser with ”touching plates” A_1 and A_2 in $\mathbb{R}^n$, $n \geqslant 3$, relative to the α-Riesz kernel $|x-y|^{\alpha-n}$, $0 < \alpha \leqslant 2$. An intimate relationship between such problems and minimal α-Green energy problems for positive measures on A_1 is shown. We obtain sufficient and/or necessary conditions for the solvability of these problems in both the unconstrained and the constrained settings, investigate the properties of minimizers, and prove their uniqueness. Furthermore, characterization theorems in terms of variational inequalities for the weighted potentials are established. The approach applied is mainly based on the establishment of

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a perfectness-type property for the α-Green kernel with $0 < \alpha \leqslant 2$ which enables us, in particular, to analyze the existence of the α-Green equilibrium measure of a set. The results obtained are illustrated by several examples.

Keywords Minimum energy problems · α-Riesz kernels · α-Green kernels · External fields · Constraints · Condensers with touching plates · α-Green equilibrium measures

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1 Introduction

The purpose of the paper is to study minimum energy problems in the presence of an external field for a condenser $\mathbf{A}$ with touching oppositely-charged plates A_1 and A_2 in $\mathbb{R}^n$, $n \geqslant 3$, relative to the α-Riesz kernel $|x-y|^{\alpha-n}$, $0 < \alpha \leqslant 2$. The difficulties appearing in the course of our investigation are caused by the fact that a short-circuit between A_1 and A_2 might occur, for the Euclidean distance between these conductors is zero.

Therefore, it is meaningful to ask what kind of conditions on the objects in question could prevent such a phenomenon so that a minimizer for the corresponding α-Riesz energy problem might exist. One of the ideas, to be discussed for this purpose, is to impose upper constraints on the charges of the touching conductors.

We establish sufficient and/or necessary conditions for the existence of minimizing measures for both the unconstrained and the constrained problems, and prove their uniqueness. The conditions obtained are expressed in geometric-potential terms for A_1 and A_2, or in measure theory terms for the constraints under consideration, or in terms of variational inequalities for the weighted potentials. We also provide a detailed analysis of the supports of the minimizers.

The approach developed in the paper is based on a newly observed important relationship between, on the one hand, minimum α-Riesz energy problems over signed measures associated with a condenser $\mathbf{A}$ and, on the other hand, minimum energy problems for nonnegative measures on A_1 relative to the α-Green function g^α_D of the domain $D := \mathbb{R}^n \setminus A_2$.

Regarding the latter problems, crucial to the arguments applied in their investigation is a pre-Hilbert structure on the linear space $\mathscr{E}_{g^\alpha_D}(D)$ of all (signed) Radon measures on D with finite g^α_D-Green energy, which can be introduced due to the strict positive definiteness of the kernel g^α_D, and, most importantly, a completeness theorem for certain metric subspaces of $\mathscr{E}_{g^\alpha_D}(D)$ with nonnegative elements. This completeness theorem enables us, in particular, to analyze the existence of the α-Green equilibrium measure of a set.

To formulate precisely the problems in question, we first need to introduce several notions, to discuss relations between them, and to establish some preliminary results; this is the purpose of the next section. The scheme of the rest of the paper is described at the end of Section 3, after the formulations of the problems (see Problems 3.1 and 3.2).

2 Basic notions; relations between them. Preliminary results

2.1 Measures, energies, potentials

Let X be a locally compact (Hausdorff) space, to be specified below, and $\mathfrak{M} = \mathfrak{M}(\mathrm{X})$ the linear space of all real-valued Radon measures μ on X, equipped with the *vague* (=*weak**) topology, i.e. the topology of pointwise convergence on the class $\mathrm{C}_0(\mathrm{X})$ of all real-valued continuous functions on X with compact support. The vague topology on $\mathfrak{M}$ is Hausdorff; hence, a vague limit of any sequence (net) in $\mathfrak{M}$ is unique (whenever exists). These and other notions and results of the theory of measures and integration on a locally compact space, to be used throughout the paper, can be found in [3, 10] (see also [11] for a short survey).

Let μ^+ and μ^- denote the positive and the negative parts in the Hahn–Jordan decomposition of a measure $\mu \in \mathfrak{M}(\mathrm{X})$, respectively, $|\mu| := \mu^+ + \mu^-$ its total variation, and S^μ_X its support. A measure μ is said to be *bounded* (*finite*) if $|\mu|(\mathrm{X}) < \infty$. Given μ and a μ-measurable function u, for the sake of brevity we shall write $\langle u, \mu\rangle := \int u\, d\mu$.[1]

The following well-known fact (see, e.g., [11, Section 1.1]) will often be used.

Lemma 2.1 *Let $\psi : \mathrm{X} \to (-\infty, \infty]$ be lower semicontinuous function that is $\geqslant 0$ unless X is compact. Then $\mu \mapsto \langle \psi, \mu\rangle$ is vaguely lower semicontinuous on nonnegative $\mu \in \mathfrak{M}(\mathrm{X})$.*

We define a *kernel* $\kappa(x,y)$ on X as a symmetric, lower semicontinuous function $\kappa : \mathrm{X}\times\mathrm{X} \to [0,\infty]$. Given $\mu, \nu \in \mathfrak{M}$, let $E_\kappa(\mu,\nu)$ and $U^\mu_\kappa(\cdot)$ denote the *mutual energy* and the *potential* relative to the kernel κ, respectively, i.e.

$$\begin{aligned} E_\kappa(\mu,\nu) &:= \int \kappa(x,y)\, d(\mu\otimes\nu)(x,y),\\ U^\mu_\kappa(x) &:= \int \kappa(x,y)\, d\mu(y), \quad x \in \mathrm{X}. \end{aligned}$$

Observe that $U^\mu_\kappa(x)$, $\mu \in \mathfrak{M}$, is well defined provided $U^{\mu^+}_\kappa(x)$ and $U^{\mu^-}_\kappa(x)$ are not both infinite, and then $U^\mu_\kappa(x) = U^{\mu^+}_\kappa(x) - U^{\mu^-}_\kappa(x)$. In particular, if $\mu \geqslant 0$, then U^μ_κ is defined everywhere and represents a nonnegative lower semicontinuous function on X.

A kernel κ is called *regular* if, for any $\mu \geqslant 0$ with compact S^μ_X, the potential U^μ_κ is continuous throughout X whenever the restriction of U^μ_κ to S^μ_X is continuous.[2] Furthermore, κ is said to satisfy *Frostman's maximum principle* if, for any $\mu \geqslant 0$ with compact support,

$$\sup_{x\in\mathrm{X}} U^\mu_\kappa(x) = \sup_{x\in S^\mu_\mathrm{X}} U^\mu_\kappa(x).$$

Also note that $E_\kappa(\mu,\nu)$, $\mu,\nu \in \mathfrak{M}$, is well defined provided $E_\kappa(\mu^+,\nu^+) + E_\kappa(\mu^-,\nu^-)$ or $E_\kappa(\mu^+,\nu^-) + E_\kappa(\mu^-,\nu^+)$ is finite. For $\mu = \nu$, $E_\kappa(\mu,\nu)$ defines the *energy* $E_\kappa(\mu) := E_\kappa(\mu,\mu)$. Let $\mathscr{E}_\kappa = \mathscr{E}_\kappa(\mathrm{X})$ consist of all $\mu \in \mathfrak{M}$ with $-\infty < E_\kappa(\mu) < \infty$, the latter by definition means that $E_\kappa(\mu^+)$, $E_\kappa(\mu^-)$ and $E_\kappa(\mu^+,\mu^-)$ are all finite. See [11, Section 2.1].

If $f : \mathrm{X} \to [-\infty,\infty]$ is an *external field*, then the *f-weighted potential* $W^\mu_{\kappa,f}$ and the *f-weighted energy* $G_{\kappa,f}(\mu)$ of $\mu \in \mathscr{E}_\kappa(\mathrm{X})$ are given by

$$\begin{aligned} W^\mu_{\kappa,f}(x) &:= U^\mu_\kappa(x) + f(x), \quad x \in \mathrm{X},\\ G_{\kappa,f}(\mu) &:= E_\kappa(\mu) + 2\langle f,\mu\rangle = \langle W^\mu_{\kappa,f} + f, \mu\rangle, \end{aligned}$$

respectively. We also define

$$\mathscr{E}_{\kappa,f}(\mathrm{X}) := \bigl\{\mu \in \mathscr{E}_\kappa(\mathrm{X}) :\ G_{\kappa,f}(\mu) < \infty\bigr\}.$$

[1] When introducing notation, we assume the corresponding object on the right to be well-defined.

[2] When speaking of a continuous function, we understand that the values are *finite* real numbers.

2.2 Strictly positive definite and perfect kernels. Capacities

A kernel κ is called *positive definite* if $E_\kappa(\mu)$, $\mu \in \mathfrak{M}$, is $\geqslant 0$ provided defined. Then $\mathcal{E}_\kappa$ forms a pre-Hilbert space with the scalar product $E_\kappa(\mu,\mu_1)$ and the seminorm $\|\mu\|_\kappa := \sqrt{E_\kappa(\mu)}$ (see [11]). The topology on $\mathcal{E}_\kappa$ defined by $\|\cdot\|_\kappa$ is called *strong*.

In the rest of Section 2.2, the kernel κ is assumed to be *strictly positive definite*, which means that the seminorm $\|\mu\|_\kappa$, $\mu \in \mathcal{E}_\kappa$, is a norm. The following lemma from the geometry of the pre-Hilbert space $\mathcal{E}_\kappa$ is often useful (see [11, Lemma 4.1.1]).

Lemma 2.2 *Let Γ be a convex subset of $\mathcal{E}_\kappa$. If there exists $\mu_0 \in \Gamma$ with minimal norm*

$$\|\mu_0\|_\kappa = \inf_{\mu\in\Gamma} \|\mu\|_\kappa,$$

then such a minimal element is unique. Moreover,

$$\|\mu-\mu_0\|_\kappa^2 \leqslant \|\mu\|_\kappa^2 - \|\mu_0\|_\kappa^2 \quad \textit{for all} \;\; \mu \in \Gamma.$$

Given a set $B \subset \mathrm{X}$, let $\mathfrak{M}^+(B)$ be the convex cone of all nonnegative measures concentrated in B, and let $\mathfrak{M}^+(B,b)$, $b>0$, consist of all $\mu \in \mathfrak{M}^+(B)$ with $\mu(B)=b$. We also write $\mathfrak{M}^+ := \mathfrak{M}^+(\mathrm{X})$, $\mathcal{E}_\kappa^+(B) := \mathcal{E}_\kappa \cap \mathfrak{M}^+(B)$, $\mathcal{E}_\kappa^+ := \mathcal{E}_\kappa^+(\mathrm{X})$, and $\mathcal{E}_\kappa^+(B,b) := \mathcal{E}_\kappa \cap \mathfrak{M}^+(B,b)$.

Let $C_\kappa(B)$ denote the *interior capacity* of B relative to a kernel κ, defined by[3]

$$1/C_\kappa(B) := w_\kappa(B) := \inf_{\mu\in\mathcal{E}_\kappa^+(B,1)} E_\kappa(\mu). \tag{2.1}$$

Note that, in consequence of Lemma 2.2, a measure $\lambda_B = \lambda_B^\kappa \in \mathcal{E}_\kappa^+(B,1)$ with minimal energy $\|\lambda_B\|_\kappa^2 = w_\kappa(B)$ is unique (provided it exists).

Following [11], we call a kernel κ *perfect* if any strong Cauchy sequence in $\mathcal{E}_\kappa^+$ converges strongly and, in addition, the strong topology on $\mathcal{E}_\kappa^+$ is finer than the induced vague topology on $\mathcal{E}_\kappa^+$. Note that then the metric space $\mathcal{E}_\kappa^+$ is strongly complete. What is also important is that the solution λ_B^κ to the minimum energy problem appeared in (2.1) exists, provided that κ is perfect, B is closed, and $0 < C_\kappa(B) < \infty$ (see [11, Theorem 4.1]).

Remark 2.1 When speaking of the vague topology, one has to consider *nets* or *filters* in $\mathfrak{M}^+$ instead of sequences, since the vague topology in general does not satisfy the first axiom of countability. We follow Moore's and Smith's theory of convergence, based on the concept of nets (see [17]; cf. also [10, Chapter 0] and [15, Chapter 2]). However, if X is metrizable and can be written as a countable union of compact sets, then $\mathfrak{M}^+$ satisfies the first axiom of countability (see [11, Lemma 1.2.1]) and the use of nets may be avoided.

Remark 2.2 Let $\mathrm{X} = \mathbb{R}^n$, $n \geqslant 2$. It is well known that, for any $\alpha \in (0,n)$, the *α-Riesz kernel* $\kappa_\alpha(x,y) := |x-y|^{\alpha-n}$ (where $|x-y|$ is the Euclidean distance in $\mathbb{R}^n$ between x and y) is *perfect* (see, e.g., [5–7, 11, 16]); therefore, the metric space $\mathcal{E}_{\kappa_\alpha}^+(\mathbb{R}^n)$ is strongly complete. However, by Cartan [5], the whole pre-Hilbert space $\mathcal{E}_{\kappa_\alpha}(\mathbb{R}^n)$ is, in general, strongly *incomplete*, and this is the case even for the Coulomb kernel $\kappa_2(x,y) = |x-y|^{-1}$ on $\mathbb{R}^3$ (compare with Theorem 2.3 below).

From now on we shall often write simply α instead of κ_α if it serves as an index. E.g., $C_\alpha(\cdot) = C_{\kappa_\alpha}(\cdot)$ denote the interior α-Riesz capacity of a set.

[3] As usual, the infimum over the empty set is taken to be $+\infty$. We put $1/(+\infty) = 0$ and $1/0 = +\infty$.

2.3 α-Riesz balayage

Let a closed set $Q \subset \mathbb{R}^n$, $n \geqslant 3$, a bounded measure $\mu \in \mathfrak{M}(\mathbb{R}^n)$, and $\alpha \in (0,2]$ be fixed. By definition (cf. [16, Chapter IV, Section 5]), the α-Riesz *balayage* measure $\beta_Q^\alpha \mu$ of μ onto Q is supported by Q and satisfies the relation

$$U_\alpha^{\beta_Q^\alpha \mu}(x) = U_\alpha^\mu(x) \quad \text{n.e. in } Q, \tag{2.2}$$

where "*n.e.*" (*nearly everywhere*) means that the equality holds everywhere in Q except for a subset with α-Riesz capacity zero. Such $\beta_Q^\alpha \mu$ exists and it is unique among the *C-absolutely continuous* measures $\nu \in \mathfrak{M}(\mathbb{R}^n)$, namely those that $\nu(K) = 0$ for every compact $K \subset \mathbb{R}^n$ with $C_\alpha(K) = 0$. Throughout the paper, when speaking of the α-Riesz balayage measure, we always mean exactly this one. Then, by [16, Chapter IV, Section 5],

$$\beta_Q^\alpha \mu = \int \beta_Q^\alpha \varepsilon_y \, d\mu(y). \tag{2.3}$$

If $\mu \geqslant 0$, then also $U_\alpha^{\beta_Q^\alpha \mu}(x) \leqslant U_\alpha^\mu(x)$ for all $x \in \mathbb{R}^n$. If, moreover, $\mu \in \mathscr{E}_\alpha^+(\mathbb{R}^n)$, then

$$E_\alpha(\beta_Q^\alpha \mu) \leqslant E_\alpha(\mu)$$

and also

$$\left\|\mu - \beta_Q^\alpha \mu\right\|_\alpha < \left\|\mu - \nu\right\|_\alpha \quad \text{for all } \nu \in \mathscr{E}_\alpha^+(Q),\ \nu \neq \beta_Q^\alpha \mu, \tag{2.4}$$

so that $\beta_Q^\alpha \mu$ is, in fact, the orthogonal projection of μ in the pre-Hilbert space $\mathscr{E}_\alpha(\mathbb{R}^n)$ onto the convex cone $\mathscr{E}_\alpha^+(Q)$.

It is well known (see [16]) that, for any bounded $\mu \geqslant 0$, there holds $\beta_Q^\alpha \mu(\mathbb{R}^n) \leqslant \mu(\mathbb{R}^n)$. This general fact is specified by the following assertion (see [22, Theorem 4]; for $\alpha = 2$, see also [21, Theorem B]).

Theorem 2.1 *Q is not α-thin at the Alexandroff point $\infty_{\mathbb{R}^n}$ of $\mathbb{R}^n$ if and only if, for every bounded $\mu \geqslant 0$,*

$$\beta_Q^\alpha \mu(\mathbb{R}^n) = \mu(\mathbb{R}^n). \tag{2.5}$$

By definition, Q is *not α-thin at $\infty_{\mathbb{R}^n}$* if Q^*, the inverse of Q relative to the unit sphere $S(0,1) := \{x \in \mathbb{R}^n : |x| = 1\}$, is not α-thin at $x = 0$, or equivalently (see [16, Theorem 5.10]), if $x = 0$ is an α-regular point for Q^*.[4]

2.4 α-Green kernels

Fix $n \geqslant 3$, a domain $D \subset \mathbb{R}^n$, $D \neq \mathbb{R}^n$, and $\alpha \in (0,2]$. In the rest of the paper, unless stated otherwise, one of the following two cases is assumed to hold: either $\mathrm{X} = \mathbb{R}^n$ and κ is the α-Riesz kernel κ_α, or $\mathrm{X} = D$ and κ is the generalized α-Green function g_D^α of D, defined by

$$g_D^\alpha(x,y) = U_\alpha^{\varepsilon_x}(y) - U_\alpha^{\beta_{D^c}^\alpha \varepsilon_x}(y) \quad \text{for all } x, y \in D, \tag{2.6}$$

where ε_x denotes the unit Dirac measure at a point x and $\beta_{D^c}^\alpha$ the α-Riesz balayage onto the set $D^c := \mathbb{R}^n \setminus D$.

[4] For $\alpha = 2$, this definition is due to Brelot (see [4]; cf. also [12,14]). For $\alpha \in (0,2)$, such a notion has been introduced in [22].

The function $g_D^\alpha(x,y)$, $x,y\in D$, is nonnegative and symmetric (see [13] and [16, Chapter IV, Section 5]). To show that g_D^α can be treated as a kernel on the locally compact space D, we observe that the first term on the right in (2.6) is lower semicontinuous on $D\times D$, while the second one is continuous. To verify the latter, let $\varepsilon>0$ and $x_0,y_0\in D$ be fixed. Denote $2u_0:=\min\bigl\{\operatorname{dist}(x_0,D^c),\operatorname{dist}(y_0,D^c)\bigr\}>0$. The function $t^{\alpha-n}$, $t\in[u_0,\infty)$, is uniformly continuous, hence there is $\delta\in(0,u_0]$ such that $|t^{\alpha-n}-u^{\alpha-n}|<\varepsilon$ whenever $|t-u|<\delta$, $t,u\in[u_0,\infty)$. This implies that, for all $x,y\in D$ with $|x-x_0|<\delta$ and $|y-y_0|<\delta$,

$$\bigl|U_\alpha^{\beta_{D^c}^\alpha\varepsilon_y}(x)-U_\alpha^{\beta_{D^c}^\alpha\varepsilon_y}(x_0)\bigr|\leqslant\int_{D^c}\bigl||z-x|^{\alpha-n}-|z-x_0|^{\alpha-n}\bigr|d\beta_{D^c}^\alpha\varepsilon_y(z)<\varepsilon\beta_{D^c}^\alpha\varepsilon_y(D^c)\leqslant\varepsilon.$$

Similarly,

$$\bigl|U_\alpha^{\beta_{D^c}^\alpha\varepsilon_{x_0}}(y)-U_\alpha^{\beta_{D^c}^\alpha\varepsilon_{x_0}}(y_0)\bigr|<\varepsilon.$$

As $U_\alpha^{\beta_{D^c}^\alpha\varepsilon_{x_0}}(y)=U_\alpha^{\beta_{D^c}^\alpha\varepsilon_y}(x_0)$ for all $x_0,y\in D$, $\bigl|U_\alpha^{\beta_{D^c}^\alpha\varepsilon_y}(x)-U_\alpha^{\beta_{D^c}^\alpha\varepsilon_{y_0}}(x_0)\bigr|<2\varepsilon$ follows. Also notice that, since for any $y\in D$ the second term on the right in (2.6) takes finite values at every $x\in D$, the function $g_D^\alpha(x,y)$ is infinite on the diagonal in $D\times D$ and finite elsewhere.

To avoid triviality, assume each component of D^c to have nonzero α-Riesz capacity. Note that, if $\alpha=2$ and D is regular in the sense of the solvability of the (classical) Dirichlet problem, then g_D^2 is, in fact, the classical Green function of D.

It is often useful to consider the extension $\hat g_D^\alpha(x,y)$ of $g_D^\alpha(x,y)$ from $D\times D$ to $D\times\mathbb{R}^n$, defined for $x\in D$ and $y\in D^c$ by the same formula (2.6) (see [16, Chapter IV, Section 5]). If $x\in D$ is fixed, then $\hat g_D^\alpha(x,y)\geqslant 0$ for all $y\in D^c$, where the strict inequality holds if and only if $y\in D^c$ is an *α-irregular* point of D^c. The notion of an α-irregular point of D^c does not depend on the choice of $x\in D$; the collection I_{D^c} of all these points is a subset of $\partial_{\mathbb{R}^n}D$ and its α-Riesz capacity equals zero. If $\mu\in\mathfrak{M}(D)$ is given, then for the sake of brevity we write

$$U_{\hat g_D^\alpha}^\mu(y):=\int\hat g_D^\alpha(x,y)\,d\mu(x),\quad y\in\mathbb{R}^n,$$

provided the value on the right is well-defined. Then $U_{\hat g_D^\alpha}^\mu(y)=U_{g_D^\alpha}^\mu(y)$ for all $y\in D$.

2.5 Energy principle and Frostman's maximum principle for the α-Green kernel

A measure $\mu\in\mathfrak{M}(D)$ will be tacitly extended to $\mathbb{R}^n$ by 0 off D whenever such an extension (denoted by the same symbol μ) is an element of $\mathfrak{M}(\mathbb{R}^n)$. This can be done, in particular, if μ is bounded.

Lemma 2.3 *Fix $y\in\mathbb{R}^n$ and a bounded $\mu\in\mathfrak{M}(D)$. If $U_\alpha^\mu(y)$ or $U_\alpha^{\beta_{D^c}^\alpha\mu}(y)$ is finite,*[5] *then $U_{\hat g_D^\alpha}^\mu(y)$ is well defined and given by*

$$U_{\hat g_D^\alpha}^\mu(y)=U_\alpha^{\mu-\beta_{D^c}^\alpha\mu}(y).\tag{2.7}$$

Proof Under the stated assumptions, in view of (2.3) and (2.6) we have

$$U_{\hat g_D^\alpha}^\mu(y)=\int\hat g_D^\alpha(x,y)\,d\mu(x)=\int\bigl[U_\alpha^{\varepsilon_x}(y)-U_\alpha^{\beta_{D^c}^\alpha\varepsilon_x}(y)\bigr]\,d\mu(x)=U_\alpha^\mu(y)-U_\alpha^{\beta_{D^c}^\alpha\mu}(y),$$

and the lemma follows. □

[5] This holds for any $y\in D$; and also for any $y\in D^c$, the latter provided μ is compactly supported in D.

Lemma 2.4 *Fix a bounded measure $\mu \in \mathcal{E}_{g_D^\alpha}(D)$, $0<\alpha\leqslant 2$. Then the measure μ (extended to $\mathbb{R}^n$ by 0 off D) has finite α-Riesz energy, while $E_{g_D^\alpha}(\mu)$ can be written in the form*

$$E_{g_D^\alpha}(\mu) = E_\alpha\bigl(\mu - \beta_{D^c}^\alpha\mu\bigr) = E_\alpha(\mu) - E_\alpha\bigl(\beta_{D^c}^\alpha\mu\bigr). \tag{2.8}$$

Proof It is seen from Lemma 2.3 that the potential $U_{g_D^\alpha}^\mu(y)$ is well defined for all $y\in D$ and given by (2.7). Besides, since $E_{g_D^\alpha}(\mu)$ is finite, $U_{g_D^\alpha}^\mu$ is finite μ-almost everywhere (μ-a.e.). Integrating (2.7) with respect to μ, we therefore obtain

$$E_{g_D^\alpha}(\mu) = E_\alpha\bigl(\mu - \beta_{D^c}^\alpha\mu, \mu\bigr).$$

As $\beta_{D^c}^\alpha\mu$ is C-absolutely continuous, while $U_\alpha^{\mu-\beta_{D^c}^\alpha\mu}(x)=0$ n.e. in D^c, this yields

$$\infty > E_{g_D^\alpha}(\mu) = E_\alpha\bigl(\mu - \beta_{D^c}^\alpha\mu\bigr). \tag{2.9}$$

Likewise, (2.9) holds true for $|\mu|$ in place of μ, which in view of the pre-Hilbert structure on $\mathcal{E}_\alpha(\mathbb{R}^n)$ implies that the measure $|\mu|$ extended by 0 off D belongs to $\mathcal{E}_\alpha(\mathbb{R}^n)$; hence, so does μ. Since $E_\alpha(\mu,\beta_{D^c}^\alpha\mu) = E_\alpha(\beta_{D^c}^\alpha\mu)$, the lemma follows. $\square$

Corollary 2.1 *Fix a bounded $\mu \in \mathcal{E}_{g_D^\alpha}(D)$, $0<\alpha\leqslant 2$. For nearly all $y\in\mathbb{R}^n$, the function $U_{\hat{g}_D^\alpha}^\mu(y)$ is finite and given by relation (2.7).*

Proof By Lemma 2.4, $\mu\in\mathcal{E}_\alpha(\mathbb{R}^n)$ and, consequently, $U_\alpha^\mu(y)$ is finite n.e. in $\mathbb{R}^n$ (see, e.g., [11, p. 164]). Since so is $U_\alpha^{\beta_{D^c}^\mu}(y)$, the corollary follows. $\square$

Lemma 2.5 *Fix a bounded measure $\mu \in \mathcal{E}^+_{g_D^\alpha}(D)$, $0<\alpha\leqslant 2$, possessing the property*[6]

$$U_{g_D^\alpha}^\mu(y) \leqslant M \quad \mu\text{-a.e.}, \tag{2.10}$$

where $M\in(0,\infty)$. Then

$$U_\alpha^\mu(y) \leqslant M + U_\alpha^{\beta_{D^c}^\alpha\mu}(y) \quad \textit{for all } \ y\in\mathbb{R}^n. \tag{2.11}$$

Proof According to Lemma 2.4, $\mu\in\mathcal{E}_\alpha^+(\mathbb{R}^n)$. Therefore, by Corollary 2.1 and (2.10), we have $U_\alpha^\mu(y)\leqslant M + U_\alpha^{\beta_{D^c}^\alpha\mu}(y)$ μ-a.e. As $M>0$, it is seen from [16, Theorems 1.24, 1.30] that the function on the right-hand side of this inequality is nonnegative and α-superharmonic on $\mathbb{R}^n$, which in view of [16, Theorems 1.27, 1.29] yields (2.11). $\square$

Corollary 2.2 *The kernel g_D^α, $0<\alpha\leqslant 2$, satisfies Frostman's maximum principle.*

Proof Consider a compactly supported $\mu\in\mathfrak{M}^+(D)$ such that $U_{g_D^\alpha}^\mu(y)\leqslant M$ on S_D^μ. Since then both (2.7) and Lemma 2.5 can be applied, we get $U_{g_D^\alpha}^\mu(y)\leqslant M$ on D, as was to be proved. $\square$

Next we prove the strict positive definiteness of the α-Green kernel, also referred to as the *energy principle* [20, p. 144].

Theorem 2.2 *The kernel g_D^α, $0<\alpha\leqslant 2$, is strictly positive definite.*

[6] Since $U_{g_D^\alpha}^\mu$, where $\mu\in\mathfrak{M}^+(D)$, is lower semicontinuous on D, inequality (2.10) actually holds on S_D^μ.

Proof It is enough to consider the case $\alpha \neq 2$, for the 2-Green kernel is strictly positive definite by [8, Chapter XIII, Section 7].

We start by showing that g_D^α is positive definite. Although this follows immediately from [18, Theorem 3] in view of Corollary 2.2, we present here a direct proof.

To this end, note that for any increasing sequence (or net) of compact subsets $K \subset D$ with the union D,

$$\lim_{K\uparrow D} E_{g_D^\alpha}(\nu,\mu_K) = \lim_{K\uparrow D} \langle U^\nu_{g_D^\alpha},\mu_K\rangle = E_{g_D^\alpha}(\nu,\mu) \quad \text{for any } \ \nu,\mu \in \mathscr{E}_{g_D^\alpha}(D), \tag{2.12}$$

where ω_K is the trace of a Radon measure ω on K. Indeed, assume first that $\nu,\mu \in \mathscr{E}^+_{g_D^\alpha}(D)$. Since then $U^\nu_{g_D^\alpha}$ is nonnegative and lower semicontinuous on D, while $\mu_K \to \mu$ vaguely as $K \uparrow D$, (2.12) is obtained directly from Lemma 2.1. On account of the definition of finite energy for signed measures, (2.12) extends right-away to any $\mu,\nu \in \mathscr{E}_{g_D^\alpha}(D)$, as claimed.

Suppose $\mu \in \mathscr{E}_{g_D^\alpha}(D)$. Then by (2.12) with $\mu = \nu$ and Lemma 2.4 we have

$$E_{g_D^\alpha}(\mu) = \lim_{K\uparrow D} E_{g_D^\alpha}(\mu_K) = \lim_{K\uparrow D} E_\alpha(\mu_K - \beta^\alpha_{D^c}\mu_K) \geqslant 0,$$

for the α-Riesz kernel is strictly positive definite, and so g_D^α is indeed positive definite.

Having chosen $\mu \in \mathscr{E}_{g_D^\alpha}(D)$ with $\|\mu\|_{g_D^\alpha} = 0$, we next proceed to show that $\mu = 0$. By the Cauchy–Schwarz (Bunyakovski) inequality in the pre-Hilbert space $\mathscr{E}_{g_D^\alpha}(D)$,

$$E_{g_D^\alpha}(\mu,\nu) = 0 \quad \text{for any } \ \nu \in \mathscr{E}_{g_D^\alpha}(D). \tag{2.13}$$

To establish the claimed assertion $\mu = 0$, fix an arbitrary $\varphi \in C_0^\infty(D)\ \big(\subset C_0^\infty(\mathbb{R}^n)\big)$. According to [16, Lemma 1.1], there exists a (signed Radon) absolutely continuous (with respect to the n-dimensional Lebesgue measure) measure $\psi \in \mathfrak{M}(\mathbb{R}^n)$ such that $\varphi(x) = U^\psi_\alpha(x)$ for all $x \in \mathbb{R}^n$. Furthermore, according to [16, Eq. (1.3.16)], its density satisfies the assumption

$$\psi(x) = O\Big(\frac{1}{|x|^{n+\alpha}}\Big),$$

and consequently $\psi \in \mathscr{E}_\alpha(\mathbb{R}^n)$. For any compact set $K \subset D$ we then have, by (2.7) for μ_K instead of μ and Corollary 2.1,

$$\begin{aligned}\langle \varphi,\mu_K\rangle = \langle U^\psi_\alpha,\mu_K\rangle = \langle U^{\mu_K}_\alpha,\psi\rangle &= \langle U^{\beta^\alpha_{D^c}\mu_K}_\alpha,\psi\rangle + \langle U^{\mu_K}_{\hat{g}_D^\alpha},\psi\rangle\\ &= \langle U^\psi_\alpha,\beta^\alpha_{D^c}\mu_K\rangle + \langle U^{\mu_K}_{\hat{g}_D^\alpha},\psi\rangle = \langle U^{\mu_K}_{g_D^\alpha},\psi_D\rangle,\end{aligned}$$

because $U^\psi_\alpha(x) = \varphi(x) = 0$ for all $x \in S^{\beta^\alpha_{D^c}\mu_K}_{\mathbb{R}^n} \subset D^c$, while $U^{\mu_K}_{\hat{g}_D^\alpha}(x) = 0$ n.e. in D^c by (2.7).

Observe that ψ_D, being of finite α-Riesz energy, has finite α-Green energy as well. Therefore, by the preceding display,

$$\langle \varphi,\mu_K\rangle = \langle U^{\psi_D}_{g_D^\alpha},\mu_K\rangle.$$

Since φ is $|\mu|$-integrable, we thus get, by (2.12),

$$\langle \varphi,\mu\rangle = \lim_{K\uparrow D}\langle \varphi,\mu_K\rangle = \lim_{K\uparrow D}\langle U^{\psi_D}_{g_D^\alpha},\mu_K\rangle = \lim_{K\uparrow D} E_{g_D^\alpha}(\psi_D,\mu_K) = E_{g_D^\alpha}(\psi_D,\mu) = 0,$$

where the very last equality being valid because of (2.13) for $\nu = \psi_D$.

In view of the arbitrary choice of $\varphi \in C_0^\infty(D)$, $\mu = 0$ as a distribution on D. Since $C_0^\infty(D)$ is dense in $C_0(D)$, μ equals 0 as a Radon measure on D as well. □

Remark 2.3 If $\alpha = 2$ and D is regular, then the (classical) Green kernel g^2_D is known to be perfect (see [5, 11, 16]). For a similar perfectness-type result related to the case where either $\alpha < 2$, or $\alpha = 2$ but D is not regular, see Theorem 10.1 below. At the moment we can only assert that then the restriction of g^α_D to any compact subset of D is perfect, which is seen from [11, Theorems 3.3, 3.4.1] and Theorem 2.2 in view of the regularity of g^α_D, the latter being obvious from (2.7) and the regularity of κ_α (see [16, Theorem 1.7]).

Lemma 2.6 *For any $B \subset D$, $C_\alpha(B) = 0$ if and only if $C_{g^\alpha_D}(B) = 0$.*

Proof We need the following general facts related to an arbitrary strictly positive definite kernel κ on a locally compact space X. First of all, for any $B \subset \mathrm{X}$,

$$C_\kappa(B) = \sup_{K \in \{K\}_B} C_\kappa(K), \tag{2.14}$$

where $\{K\}_B$ consists of all compact subsets of B (see [11]). Further, for any compact set $K \subset \mathrm{X}$ one has $C_\kappa(K) < \infty$ and hence, by [11, Theorem 2.5],

$$C_\kappa(K) = \sup \mu(K), \tag{2.15}$$

where μ ranges over all nonnegative measures supported by K with the additional property

$$U^\mu_\kappa(x) \leqslant 1 \quad \text{for all } \ x \in S^\mu_{\mathrm{X}}.$$

Now we apply representation (2.15) to a compact set $K \subset D$ and each of the α-Riesz and the α-Green kernels, which is possible in view of their strict positive definiteness. Since for every $\mu \in \mathfrak{M}^+(K)$, $U^\mu_\alpha(x) - U^\mu_{g^\alpha_D}(x)$ is bounded on K in consequence of (2.7), the lemma for $B = K$ follows. To prove the lemma for any $B \subset D$, it is thus left to apply (2.14). $\square$

Let $B \subset D$. By Lemma 2.6, if some expression $\mathscr{U}(x)$ is valid n.e. in B, then $C_{g^\alpha_D}(N) = 0$, N being the set of all $x \in B$ with $\mathscr{U}(x)$ not to hold; and also the other way around.

2.6 Condensers. Existence of minimizers

By a *condenser* in $\mathbb{R}^n$ we mean an ordered pair $\mathbf{B} = (B_1, B_2)$ of nonintersecting sets $B_1, B_2 \subset \mathbb{R}^n$ (so far of arbitrary topological structure) treated as the *positive* and the *negative* plates of $\mathbf{B}$, respectively. Define

$$\mathfrak{M}(\mathbf{B}) := \bigl\{\nu \in \mathfrak{M}(\mathbb{R}^n) : \ \nu^+ \in \mathfrak{M}^+(B_1), \ \nu^- \in \mathfrak{M}^+(B_2)\bigr\},$$

$$\mathscr{E}_\alpha(\mathbf{B}) := \mathfrak{M}(\mathbf{B}) \cap \mathscr{E}_\alpha(\mathbb{R}^n).$$

Then the following theorem on the strong completeness is true (see [23, Theorem 1]; compare with [5] or [16, Theorem 1.19]).[7]

Theorem 2.3 *Assume B_1 and B_2 to be closed in $\mathbb{R}^n$. Then the metric space $\mathscr{E}_\alpha(\mathbf{B})$ is complete in the induced strong topology, and the strong convergence in this space implies the vague convergence to the same limit.*

[7] In fact, Theorem 2.3 holds true also for $\alpha \in (2, n)$; see [23, Theorem 1]. Its proof is based on Deny's theorem [6] stating that, for the Riesz kernels, $\mathscr{E}_\alpha$ can be completed by making use of distributions with finite energy. Regarding the history of the question, see [21, Theorem A] and [22, Theorem 1].

From now on we fix a (particular) condenser $\mathbf{A} = (A_1, A_2)$ in $\mathbb{R}^n$ with the plates $A_2 := D^c$ and $A_1 := F$, where $F \subsetneqq D$ is closed in the relative topology of D and $C_\alpha(F) > 0$. Recall that each component of D^c has been assumed in Section 2.4 to have nonzero α-Riesz capacity as well; and therefore

$$C_\alpha(A_i) > 0 \quad \text{for all } \ i = 1, 2. \tag{2.16}$$

Also fix a unit two-dimensional numerical vector $\mathbf{1} = (1, 1)$, and define

$$\mathscr{E}_\alpha(\mathbf{A}, \mathbf{1}) := \bigl\{\mu \in \mathscr{E}_\alpha(\mathbf{A}) : \ \mu^+(A_1) = \mu^-(A_2) = 1\bigr\}.$$

In view of (2.16), the class $\mathscr{E}_\alpha(\mathbf{A}, \mathbf{1})$ is nonempty and, hence, it makes sense to consider the variational problem on the existence of $\lambda_{\mathbf{A}} \in \mathscr{E}_\alpha(\mathbf{A}, \mathbf{1})$ with

$$E_\alpha(\lambda_{\mathbf{A}}) = \inf_{\mu \in \mathscr{E}_\alpha(\mathbf{A}, \mathbf{1})} E_\alpha(\mu) \quad \bigl(=: E_\alpha(\mathbf{A}, \mathbf{1})\bigr). \tag{2.17}$$

In particular, the following theorem on the solvability holds (see [22, Theorems 5, 7]).[8]

Theorem 2.4 *Let*

$$\inf_{x \in F,\ y \in D^c} |x - y| > 0.$$

If, moreover, $C_\alpha(F) < \infty$ then, for a solution to the minimum energy problem (2.17) *to exist, it is necessary and sufficient that either $C_\alpha(D^c) < \infty$ or D^c be not α-thin at $\infty_{\mathbb{R}^n}$.*[9]

In the paper, we are mainly interested in the case

$$\overline{F} \cap \partial_{\overline{\mathbb{R}^n}} D \neq \varnothing,$$

where $\overline{\mathbb{R}^n} := \mathbb{R}^n \cup \{\infty_{\mathbb{R}^n}\}$ is the one-point compactification of $\mathbb{R}^n$ and $\overline{F} := \mathrm{C}\ell_{\overline{\mathbb{R}^n}} F$. Then, in general, the infimum in (2.17) can not be achieved among $\mu \in \mathscr{E}_\alpha(\mathbf{A}, \mathbf{1})$. Using the physical interpretation, which is possible for the Coulomb kernel, a short-circuit between the plates of the condenser might occur.

Therefore, it is meaningful to investigate what kind of conditions on the objects under consideration would prevent such a phenomenon, and a minimizer in the corresponding minimum α-Riesz energy problem for the condenser $\mathbf{A}$ would, nevertheless, exist. One of the ideas, to be discussed, is to find out such an upper constraint on the measures (charges) from $\mathscr{E}^+_\alpha(F, 1)$ which would not allow the "blow-up" effect between F and D^c. Note that we do not intend to impose any constraint on the measures on D^c.

Assume also the measures from $\mathscr{E}^+_\alpha(F)$ to be influenced by some external field f, while $\langle f, \mu\rangle = 0$ for all $\mu \in \mathscr{E}^+_\alpha(D^c)$. Then, what kind of external fields, acting on the charges on F only, would still guarantee the existence of minimizers?

Recently a similar problem for the logarithmic kernel in $\mathbb{R}^2$ has been investigated by Beckermann and Gryson [1, Theorem 2.2]. Our study is related to the Riesz kernels and the results obtained and the approaches applied are rather different from those in [1].

[8] The first result of this type was obtained in [21, Theorem 1], where $\alpha = 2$ and A_1 was assumed to be compact in D. See also [23, Theorem 12] where Theorem 2.4 has been generalized to any $\alpha \in (0, n)$ and any $\mathbf{a} = (a_1, a_2)$ with $a_i > 0$, $i = 1, 2$. Instead of the balayage technique, which implicitly appears in Theorem 2.4, for $\alpha \in (2, n)$ one should use the operator of orthogonal projection in the pre-Hilbert space $\mathscr{E}_\alpha$ onto the convex cone $\mathscr{E}^+_\alpha(D^c)$.

[9] We refer to [21, 27] for an example of a set with infinite Newtonian capacity, though 2-thin at $\infty_{\mathbb{R}^n}$.

3 Constrained and unconstrained minimum energy problems

When speaking of the external field f, we shall tacitly assume that at least one of the following Cases I or II holds:

I. $f\big|_F$ *is lower semicontinuous, and it is* $\geqslant 0$ *unless* F *is compact, while* $f(x)=0$ *n.e. in* D^c;
II. $f(x)=U_\alpha^{\zeta-\beta_{D^c}^\alpha\zeta}(x)$, $x\in\mathbb{R}^n$, *where a (signed) bounded measure* $\zeta\in\mathscr{E}_{g_D^\alpha}(D)$ *is given.*

Applying relations (2.2) and (2.7) to ζ from Case II, on account of Corollary 2.1 we get

$$U_\alpha^{\zeta-\beta_{D^c}^\alpha\zeta}(x)=U_{\hat{g}_{D^c}^\alpha}^{\zeta}(x)=\begin{cases}U_{g_{D^c}^\alpha}^{\zeta}(x) & \text{for all } x\in D,\\ 0 & \text{n.e. in } D^c.\end{cases}\tag{3.1}$$

Thus, Case II can be reduced to Case I provided $\zeta^-=0$.

In both Cases I and II, the values $G_{\alpha,f}(\mu)$ and $G_{g_D^\alpha,f}(\nu)$ are well defined for all $\mu\in\mathscr{E}_\alpha(\mathbf{A})$ and $\nu\in\mathscr{E}^+_{g_D^\alpha}(F)$, respectively, and

$$G_{\alpha,f}(\mu)=\|\mu\|_\alpha^2+2\langle f,\mu^+\rangle>-\infty,\tag{3.2}$$
$$G_{g_D^\alpha,f}(\nu)=\|\nu\|_{g_D^\alpha}^2+2\langle f,\nu\rangle>-\infty.\tag{3.3}$$

If Case II takes place then, in consequence of (3.1) and the pre-Hilbert structures on $\mathscr{E}_\alpha(\mathbb{R}^n)$ and $\mathscr{E}_{g_D^\alpha}(D)$, these values can alternatively be expressed in the form

$$G_{\alpha,f}(\mu)=\|\mu\|_\alpha^2+2E_\alpha(\zeta-\beta_{D^c}^\alpha\zeta,\mu)=\|\mu+\zeta-\beta_{D^c}^\alpha\zeta\|_\alpha^2-\|\zeta-\beta_{D^c}^\alpha\zeta\|_\alpha^2,\tag{3.4}$$
$$G_{g_D^\alpha,f}(\nu)=\|\nu\|_{g_D^\alpha}^2+2E_{g_D^\alpha}(\zeta,\nu)=\|\nu+\zeta\|_{g_D^\alpha}^2-\|\zeta\|_{g_D^\alpha}^2.\tag{3.5}$$

Write

$$\mathscr{E}_{\alpha,f}(\mathbf{A},\mathbf{1}):=\mathscr{E}_\alpha(\mathbf{A},\mathbf{1})\cap\mathscr{E}_{\alpha,f}(\mathbb{R}^n)\quad\text{and}\quad\mathscr{E}^+_{g_D^\alpha,f}(F,1):=\mathscr{E}^+_{g_D^\alpha}(F,1)\cap\mathscr{E}_{g_D^\alpha,f}(D);$$

these classes of measures are both convex. It is seen from (3.4) and (3.5) that, in Case II,

$$\mathscr{E}_{\alpha,f}(\mathbf{A},\mathbf{1})=\mathscr{E}_\alpha(\mathbf{A},\mathbf{1})\quad\text{and}\quad\mathscr{E}^+_{g_D^\alpha,f}(F,1)=\mathscr{E}^+_{g_D^\alpha}(F,1).\tag{3.6}$$

We denote by $\mathfrak{C}(F)$ the collection of all $\xi\in\mathfrak{M}^+(D)$ with the properties

$$S_D^\xi=F\quad\text{and}\quad\xi(F)>1;$$

such ξ will be treated as *constraints* for measures from the class $\mathfrak{M}^+(F,1)$. Let $\mathfrak{C}_0(F)$ consist of all bounded $\xi\in\mathfrak{C}(F)$. Given $\xi\in\mathfrak{C}(F)$, write

$$\mathscr{E}^\xi_{\alpha,f}(\mathbf{A},\mathbf{1}):=\bigl\{\mu\in\mathscr{E}_{\alpha,f}(\mathbf{A},\mathbf{1}):\ \mu^+\leqslant\xi\bigr\},$$
$$\mathscr{E}^\xi_{g_D^\alpha,f}(F,1):=\bigl\{\nu\in\mathscr{E}^+_{g_D^\alpha,f}(F,1):\ \nu\leqslant\xi\bigr\},$$

where $\nu_1\leqslant\nu_2$ means that $\nu_2-\nu_1$ is a nonnegative Radon measure.

To combine (where this is possible) assertions related to extremal problems in both the constrained and unconstrained settings, we accept the notations

$$\mathscr{E}^\infty_{\alpha,f}(\mathbf{A},\mathbf{1}):=\mathscr{E}_{\alpha,f}(\mathbf{A},\mathbf{1})\quad\text{and}\quad\mathscr{E}^\infty_{g_D^\alpha,f}(F,1):=\mathscr{E}^+_{g_D^\alpha,f}(F,1).$$

In all that follows, we consider a fixed $\sigma \in \mathfrak{C}(F) \cup \{\infty\}$, where the formal formula $\sigma = \infty$ means that *no* upper constraint is allowed,[10] and we define

$$G^{\sigma}_{\alpha,f}(\mathbf{A},\mathbf{1}) := \inf_{\mu \in \mathscr{E}^{\sigma}_{\alpha,f}(\mathbf{A},\mathbf{1})} G_{\alpha,f}(\mu),$$
$$G^{\sigma}_{g^{\alpha}_{D},f}(F,1) := \inf_{\nu \in \mathscr{E}^{\sigma}_{g^{\alpha}_{D},f}(F,1)} G_{g^{\alpha}_{D},f}(\nu).$$

In the case $\sigma = \infty$, the upper index ∞ will often be omitted, i.e., we shall write

$$G_{\alpha,f}(\mathbf{A},\mathbf{1}) := G^{\infty}_{\alpha,f}(\mathbf{A},\mathbf{1}) \quad \text{and} \quad G_{g^{\alpha}_{D},f}(F,1) := G^{\infty}_{g^{\alpha}_{D},f}(F,1).$$

Note that each of $G^{\sigma}_{\alpha,f}(\mathbf{A},\mathbf{1})$ and $G^{\sigma}_{g^{\alpha}_{D},f}(F,1)$ is $> -\infty$. Indeed, in Case I this follows from the fact that a lower semicontinuous function is bounded from below on a compact set, while in Case II it is obtained directly from (3.4) and (3.5). One can also see from (2.16) and Lemma 2.6 that any of the classes $\mathscr{E}^{\sigma}_{\alpha,f}(\mathbf{A},\mathbf{1})$ and $\mathscr{E}^{\sigma}_{g^{\alpha}_{D},f}(F,1)$ is nonempty if and only if so is the other, and therefore the following two assumptions are equivalent:[11]

$$G^{\sigma}_{\alpha,f}(\mathbf{A},\mathbf{1}) < \infty, \tag{3.7}$$
$$G^{\sigma}_{g^{\alpha}_{D},f}(F,1) < \infty. \tag{3.8}$$

Problem 3.1 Under condition (3.7), does there exist $\lambda^{\sigma}_{\mathbf{A}}$ from $\mathscr{E}^{\sigma}_{\alpha,f}(\mathbf{A},\mathbf{1})$ whose f-weighted α-Riesz energy is minimal in this class, i.e.

$$G_{\alpha,f}(\lambda^{\sigma}_{\mathbf{A}}) = G^{\sigma}_{\alpha,f}(\mathbf{A},\mathbf{1})\,? \tag{3.9}$$

Problem 3.1 turns out to be intricately related to the following one.

Problem 3.2 Under condition (3.8), does there exist λ^{σ}_{F} from $\mathscr{E}^{\sigma}_{g^{\alpha}_{D},f}(F,1)$ whose f-weighted α-Green energy is minimal in this class, i.e.

$$G_{g^{\alpha}_{D},f}(\lambda^{\sigma}_{F}) = G^{\sigma}_{g^{\alpha}_{D},f}(F,1)\,? \tag{3.10}$$

Note that, if $\sigma = \infty$ and $f = 0$, then Problem 3.1 is in fact reduced to the minimum energy problem (2.17), while Problem 3.2 to that appeared in (2.1) for $B = F$ and $\kappa = g^{\alpha}_{D}$.

The rest of the paper is organized as follows. Sufficient and/or necessary conditions for Problems 3.1 and 3.2 to be solvable are established in Sections 5, 6 and 7. In Section 5 they are formulated either in measure theory terms for the constraints under consideration, or in geometric-potential terms for F and D^c, while in Sections 6 and 7 they are given in terms of variational inequalities for the f-weighted potentials. Sections 6 and 7 provide also a detailed analysis of the supports of the minimizers. The results obtained are proved in Sections 8, 9, 12 and 13, and they are illustrated by the examples in Section 14.

E.g., by Theorem 5.2, in both Cases I and II, the condition $C_{g^{\alpha}_{D}}(F) < \infty$ is close to be sufficient for Problem 3.2 to be solvable for every $\sigma \in \mathfrak{C}(F) \cup \{\infty\}$, while according to Theorem 5.3, it is also necessary for this to happen provided Case II with $\zeta \geqslant 0$ holds. However, if we restricted our analysis to the constraints from the class $\mathfrak{C}_0(F)$ then, in Case I, Problem 3.2 would already be *always* solvable (see Theorem 5.1). Further, if we assume D^c

[10] It is natural to set $\nu \leqslant \infty$ for all $\nu \in \mathfrak{M}^+(F)$.

[11] See Lemmas 4.5, 4.6 and Remark 4.1 below, providing necessary and/or sufficient conditions for (3.7) and (3.8) to hold.

to be not α-thin at $\infty_{\mathbb{R}^n}$, then all this remains true for Problem 3.1 as well. See Section 5 for the strict formulations of the results just described.

It is seen from the results obtained that the classes of condensers for which the problems under discussion admit solutions in the constrained or the unconstrained settings, respectively, are drastically different from each other. In fact, under quite general assumptions on the external field, the solvability of these problems with a proper active constraint holds (and, hence, no blow-up effect occurs) even if F touches the boundary of D over a set with nonzero α-Riesz capacity, while this is no longer the case for $\sigma = \infty$ (see Remark 6.1).

A crucial key in the proofs is Theorem 10.1, which provides us with a perfectness-type result for the kernel g^α_D, $\alpha \leqslant 2$. It makes it possible, in particular, to establish Theorem 11.1 on the existence of the α-Green equilibrium measure of a set.

The uniqueness of solutions to Problems 3.1 and 3.2 is shown by Lemma 4.1 in the next section. Another assertion of this section, Lemma 4.2, discovers an intimate relationship between their solvability (or unsolvability), as well as their minimizers (provided they exist), which turns out to be a powerful tool in the proofs of the above-mentioned results.

4 Auxiliary assertions

Lemma 4.1 *A solution to Problem* 3.1*, as well as that to Problem* 3.2*, is unique* (*provided it exists*)*.*

Proof We shall verify the latter part of the lemma. Assume there exist two solutions to Problem 3.2, λ^σ_F and $\widehat{\lambda}^\sigma_F$. Since the class $\mathscr{E}^\sigma_{g^\alpha_D,f}(F,1)$ is convex, from (3.3) we get

$$4G^\sigma_{g^\alpha_D,f}(F,1) \leqslant 4G_{g^\alpha_D,f}\Bigl(\frac{\lambda^\sigma_F+\widehat{\lambda}^\sigma_F}{2}\Bigr) = \|\lambda^\sigma_F+\widehat{\lambda}^\sigma_F\|^2_{g^\alpha_D} + 4\langle f,\lambda^\sigma_F+\widehat{\lambda}^\sigma_F\rangle.$$

Applying the parallelogram identity in the pre-Hilbert space $\mathscr{E}_{g^\alpha_D}(D)$, we obtain

$$0 \leqslant \|\lambda^\sigma_F-\widehat{\lambda}^\sigma_F\|^2_{g^\alpha_D} \leqslant -4G^\sigma_{g^\alpha_D,f}(F,1) + 2G_{g^\alpha_D,f}(\lambda^\sigma_F) + 2G_{g^\alpha_D,f}(\widehat{\lambda}^\sigma_F),$$

so that $\|\lambda^\sigma_F-\widehat{\lambda}^\sigma_F\|_{g^\alpha_D} = 0$ by (3.10). As g^α_D is strictly positive definite, $\lambda^\sigma_F = \widehat{\lambda}^\sigma_F$.

Likewise, the former part of the lemma can be proved based on the convexity of the class $\mathscr{E}^\sigma_{\alpha,f}(\mathbf{A},\mathbf{1})$, the pre-Hilbert structure in the space $\mathscr{E}_\alpha(\mathbb{R}^n)$, and the strict positive definiteness of the kernel κ_α. □

Lemma 4.2 *Assume D^c to be not α-thin at $\infty_{\mathbb{R}^n}$. Then for every $\sigma \in \mathfrak{C}(F)\cup\{\infty\}$,*

$$G^\sigma_{g^\alpha_D,f}(F,1) = G^\sigma_{\alpha,f}(\mathbf{A},\mathbf{1}). \tag{4.1}$$

In addition, the solution to Problem 3.1 *exists if and only if so does that to Problem* 3.2*, and then they are related to each other by the formula*

$$\lambda^\sigma_{\mathbf{A}} = \lambda^\sigma_F - \beta^\alpha_{D^c}\lambda^\sigma_F. \tag{4.2}$$

Proof We begin by establishing the inequality

$$G^\sigma_{g^\alpha_D,f}(F,1) \geqslant G^\sigma_{\alpha,f}(\mathbf{A},\mathbf{1}). \tag{4.3}$$

Having assumed $G^{\sigma}_{g^{\alpha}_D,f}(F,1)<\infty$, we fix $\nu\in\mathscr{E}^{\sigma}_{g^{\alpha}_D,f}(F,1)$; then, by Lemma 2.4 for ν in place of μ and (2.5) for $Q=D^c$,

$$\nu-\beta^{\alpha}_{D^c}\nu\in\mathscr{E}^{\sigma}_{\alpha,f}(\mathbf{A},\mathbf{1}).$$

On account of (2.8), (3.2) and (3.3), we therefore get

$$G_{g^{\alpha}_D,f}(\nu)=\|\nu\|^2_{g^{\alpha}_D}+2\langle f,\nu\rangle=\|\nu-\beta^{\alpha}_{D^c}\nu\|^2_{\alpha}+2\langle f,\nu\rangle=G_{\alpha,f}(\nu-\beta^{\alpha}_{D^c}\nu)\geqslant G^{\sigma}_{\alpha,f}(\mathbf{A},\mathbf{1}).$$

Since $\nu\in\mathscr{E}^{\sigma}_{g^{\alpha}_D,f}(F,1)$ has been chosen arbitrarily, this yields (4.3).

On the other hand, for any $\mu\in\mathscr{E}^{\sigma}_{\alpha,f}(\mathbf{A},\mathbf{1})$ we have $\mu^+\in\mathscr{E}^{\sigma}_{g^{\alpha}_D,f}(F,1)$. Thus, due to (2.4) and (2.8),

$$\begin{aligned}G_{\alpha,f}(\mu)&=\|\mu\|^2_{\alpha}+2\langle f,\mu^+\rangle\geqslant\|\mu^+-\beta^{\alpha}_{D^c}\mu^+\|^2_{\alpha}+2\langle f,\mu^+\rangle\\&=\|\mu^+\|^2_{g^{\alpha}_{D^c}}+2\langle f,\mu^+\rangle=G_{g^{\alpha}_D,f}(\mu^+)\geqslant G^{\sigma}_{g^{\alpha}_D,f}(F,1).\end{aligned}\tag{4.4}$$

In view of the arbitrary choice of $\mu\in\mathscr{E}^{\sigma}_{\alpha,f}(\mathbf{A},\mathbf{1})$, this proves (4.1) when combined with (4.3).

Let now $\lambda^{\sigma}_F\in\mathscr{E}^{\sigma}_{g^{\alpha}_D,f}(F,1)$ satisfy (3.10). Then, in consequence of (2.5) and Lemma 2.4,

$$\widehat{\mu}:=\lambda^{\sigma}_F-\beta^{\alpha}_{D^c}\lambda^{\sigma}_F\in\mathscr{E}^{\sigma}_{\alpha,f}(\mathbf{A},\mathbf{1}).$$

Substituting $\widehat{\mu}$ instead of μ in relation (4.4), we see that all the inequalities therein are, in fact, equalities. Therefore, by (4.1),

$$G_{\alpha,f}(\widehat{\mu})=G^{\sigma}_{g^{\alpha}_{D^c},f}(F,1)=G^{\sigma}_{\alpha,f}(\mathbf{A},\mathbf{1}),$$

so that the measure $\lambda^{\sigma}_{\mathbf{A}}$, defined by (4.2), solves Problem 3.1.

To complete the proof, assume further that $\lambda^{\sigma}_{\mathbf{A}}=\lambda^+-\lambda^-\in\mathscr{E}^{\sigma}_{\alpha,f}(\mathbf{A},\mathbf{1})$ satisfies (3.9). Then, by (4.1) and (4.4), the latter with $\lambda^{\sigma}_{\mathbf{A}}$ instead of μ,

$$\begin{aligned}G^{\sigma}_{g^{\alpha}_D,f}(F,1)=G_{\alpha,f}(\lambda^{\sigma}_{\mathbf{A}})&\geqslant\|\lambda^+-\beta^{\alpha}_{D^c}\lambda^+\|^2_{\alpha}+2\langle f,\lambda^+\rangle\\&=\|\lambda^+\|^2_{g^{\alpha}_{D^c}}+2\langle f,\lambda^+\rangle=G_{g^{\alpha}_D,f}(\lambda^+)\geqslant G^{\sigma}_{g^{\alpha}_D,f}(F,1).\end{aligned}$$

Hence, all the inequalities here are, in fact, equalities. This shows that $\lambda^{\sigma}_F:=\lambda^+$ solves Problem 3.2 and, on account of (2.4), also that $\lambda^-=\beta^{\alpha}_{D^c}\lambda^+=\beta^{\alpha}_{D^c}\lambda^{\sigma}_F$. $\square$

Lemma 4.3 *Assume* (3.8) *holds. For a measure $\lambda=\lambda^{\sigma}_F\in\mathscr{E}^{\sigma}_{g^{\alpha}_D,f}(F,1)$ to solve Problem* 3.2*, it is necessary and sufficient that*

$$\bigl\langle W^{\lambda}_{g^{\alpha}_D,f},\nu-\lambda\bigr\rangle\geqslant 0\quad\text{for all}\ \ \nu\in\mathscr{E}^{\sigma}_{g^{\alpha}_D,f}(F,1).\tag{4.5}$$

Proof By direct calculation, for any $\nu,\mu\in\mathscr{E}^{\sigma}_{g^{\alpha}_D,f}(F,1)$ and any $h\in(0,1]$ we obtain

$$G_{g^{\alpha}_D,f}\bigl(h\nu+(1-h)\mu\bigr)-G_{g^{\alpha}_D,f}\bigl(\mu\bigr)=2h\bigl\langle W^{\mu}_{g^{\alpha}_D,f},\nu-\mu\bigr\rangle+h^2\bigl\|\nu-\mu\bigr\|^2_{g^{\alpha}_D}.\tag{4.6}$$

If $\mu=\lambda^{\sigma}_F$ solves Problem 3.2, then the left (hence, the right) side of (4.6) is $\geqslant 0$, for the class $\mathscr{E}^{\sigma}_{g^{\alpha}_D,f}(F,1)$ is convex, which leads to (4.5) by letting $h\to 0$.

Conversely, if (4.5) holds, then (4.6) with $\mu=\lambda$ and $h=1$ gives $G_{g^{\alpha}_D,f}(\nu)\geqslant G_{g^{\alpha}_D,f}(\lambda)$ for all $\nu\in\mathscr{E}^{\sigma}_{g^{\alpha}_D,f}(F,1)$, which means that $\lambda=\lambda^{\sigma}_F$ solves Problem 3.2. $\square$

Lemma 4.4 $G_{g^\alpha_D,f}(\cdot)$ *is vaguely lower semicontinuous on* $\mathscr{E}^+_{g^\alpha_D}(D)$ *if Case* I *takes place, and otherwise, it is strongly continuous.*

Proof If Case I holds, then the lemma follows from Lemma 2.1 and [11, Lemma 2.2.1, (e)], while otherwise it is a direct consequence of relation (3.5). $\square$

Lemmas 4.5 and 4.6 below provide sufficient and/or necessary conditions that guarantee (3.7) and (3.8) (compare with Lemmas 4 and 5 from [25]). From now on we write

$$F_0 := \bigl\{x \in F : \ f(x) < \infty\bigr\}. \tag{4.7}$$

Lemma 4.5 *Let* $\sigma = \infty$*. Then* (3.8) (*hence, also* (3.7)) *holds if and only if* $C_\alpha(F_0) > 0$.

Proof Suppose first that $C_\alpha(F_0) > 0$. On account of [11, Lemma 2.3.3], then one can choose a compact set $K \subset F_0$ with $C_\alpha(K) > 0$ so that $f(x) \leqslant M < \infty$ for all $x \in K$. In turn, this yields that there exists $\nu \in \mathscr{E}^+_{g^\alpha_D}(K,1)$ with $G_{g^\alpha_D,f}(\nu) < \infty$, and (3.8) follows.

To prove the necessary part, assume, on the contrary, that $C_\alpha(F_0) = 0$. Since then $C_{g^\alpha_D}(F_0) = 0$ by Lemma 2.6, [11, Lemma 2.3.1] implies $\mathscr{E}^+_{g^\alpha_D,f}(F,1) = \varnothing$, which contradicts (3.8). $\square$

Definition 4.1 $\xi \in \mathfrak{C}(F)$ is called *admissible* if its restriction to any compact subset of F has finite α-Riesz (hence, α-Green) energy. Let $\mathscr{A}(F)$ consist of all admissible constraints.

When considering admissibility of a constraint, the parameter α and the set F should be clear in each context. Observe that, for a constraint $\xi \in \mathfrak{C}(F)$ to be admissible, it is sufficient that its α-Green potential be continuous. Also note that any $\xi \in \mathscr{A}(F)$ is C-absolutely continuous.

Lemma 4.6 *If* $\xi \in \mathscr{A}(F)$*, then* (3.8) (*hence, also* (3.7)) *holds provided that* $\xi(F_0) > 1$.

Proof Choose a compact set $K \subset F_0$ so that $\xi(K) > 1$ and $f(x) \leqslant M < \infty$ for all $x \in K$. Then $\xi|_K/\xi(K) \in \mathscr{E}^\xi_{g^\alpha_D,f}(F,1)$, which results in (3.8). $\square$

Remark 4.1 If Case II takes place, then $f(x)$ is finite n.e. in F and, hence, Lemma 4.5 (similarly, Lemma 4.6) remains true with $C_\alpha(F_0) > 0$ (respectively, $\xi(F_0) > 1$) dropped.

5 Criteria of the solvability given either in measure theory terms for σ, or in geometric-potential terms for F and D^c

Throughout this section and Sections 6 and 7, assume (3.7) or, equivalently, (3.8) to be satisfied. See Lemmas 4.5, 4.6 and Remark 4.1 above, providing necessary and/or sufficient conditions for these to hold.

Theorem 5.1 *If Case* I *takes place, then Problem* 3.2 *is* (*uniquely*) *solvable for every constraint* $\xi \in \mathfrak{C}_0(F)$.

Theorem 5.2 *Assume that*

$$C_{g^\alpha_D}(F) < \infty \tag{5.1}$$

and

($*$) $\alpha = 2$ *and D is regular, or* $\overline{F} \cap \partial_{\overline{\mathbb{R}^n}} D$ *consists of at most one point.*

Then, in both Cases I *and* II*, Problem* 3.2 *is* (*uniquely*) *solvable for every* $\sigma \in \mathfrak{C}(F) \cup \{\infty\}$.[12]

Theorem 5.3 *Suppose that Case* II *with* $\zeta \geqslant 0$ *takes place. If, moreover,* $C_{g_D^\alpha}(F) = \infty$*, then Problem* 3.2 *is unsolvable for every* $\sigma \in \mathfrak{C}(F) \cup \{\infty\}$ *such that* $\sigma \geqslant \xi_0$*, where* $\xi_0 \in \mathfrak{C}(F) \setminus \mathfrak{C}_0(F)$ *is properly chosen.*

Combining Theorems 5.2 and 5.3 shows that, if assumption $(*)$ and Case II with $\zeta \geqslant 0$ both hold, then (5.1) is necessary and sufficient for Problem 3.2 to be solvable for every $\sigma \in \mathfrak{C}(F) \cup \{\infty\}$.

Theorems 5.1 and 5.3 are proved in Sections 8 and 9, respectively. The proof of Theorem 5.2, to be given is Section 12, is based on the auxiliary Theorems 10.1 and 11.1 (see Sections 10 and 11, respectively) which present also independent interest.

Theorem 5.4 *Assume* D^c *to be not* α*-thin at* $\infty_{\mathbb{R}^n}$*. Under the hypotheses of Theorem* 5.1 (*similarly, Theorems* 5.2 *or* 5.3)*, its conclusion remains true for Problem* 3.1 *as well.*

Indeed, Theorem 5.4 is obtained from Theorems 5.1–5.3 with the help of Lemma 4.2.

In the next two sections we shall examine properties of the f-weighted potentials and the supports of the minimizers λ_F^σ and $\lambda_{\mathbf{A}}^\sigma$, whose existence has been ensured, e.g., by Theorems 5.1, 5.2, and 5.4.

6 Variational inequalities for the f-weighted α-Green potentials

This section provides necessary and/or sufficient conditions for the solvability of Problem 3.2 in terms of variational inequalities for the f-weighted α-Green potentials. It also presents a detailed analysis of properties of the supports of the minimizers. The proofs of the results formulated in this section are given in Section 13.

Following [16, p. 164], we denote by $\breve{F}$ the *reduced kernel* of F, i.e.

$$\breve{F} := \bigl\{x \in F : \ C_\alpha\bigl(B(x,\varepsilon) \cap F\bigr) > 0 \ \text{ for every } \varepsilon > 0\bigr\}. \tag{6.1}$$

Here $B(x,\varepsilon) := \{y \in \mathbb{R}^n : \ |y-x| < \varepsilon\}$. Observe that, if the constraint under consideration is admissible, then necessarily $F = \breve{F}$.

To simplify the formulations of the results obtained, throughout this section and Section 7 we assume ∂D to be simultaneously the boundary of the (open) set $\operatorname{Int} D^c$. Here, the boundary and the interior are considered relative to $\mathbb{R}^n$. Notice that then $m_n(D^c) > 0$, where m_n is the n-dimensional Lebesgue measure.

6.1 Variational inequalities in the constrained α-Green minimum energy problems

We start by studying Problem 3.2 in the constrained case (i.e., for $\sigma \neq \infty$). In this section, we consider $\xi \in \mathscr{A}(F)$ and assume that $\xi(F_0) > 1$. Note that, for any $\nu \in \mathscr{E}^+_{g_D^\alpha}(F)$, $W^\nu_{g_D^\alpha,f}(x)$ is well defined and $\neq -\infty$ n.e. in F, while it is finite n.e. in F_0 (see (4.7)).

[12] Compare with [26, Theorem 2.2] and [27, Theorem 8.1]. Also notice that the requirement $(*)$ guarantees the completeness of a proper metric subspace of $\mathscr{E}^+_{g_{D^c}^\alpha}(F)$ (see Theorem 10.1), and it could be omitted if one would establish the perfectness of the kernel $g_{D^c}^\alpha$, $\alpha \in (0,2]$, in its entire generality.

Theorem 6.1 *Let Case* I *take place. Then a measure $\lambda\in\mathscr{E}^{\xi}_{g^\alpha_D,f}(F,1)$ solves Problem* 3.2 *if and only if there exists $w_\lambda\in\mathbb{R}$ possessing the following two properties:*[13]

$$W^{\lambda}_{g^\alpha_D,f}(x)\geqslant w_\lambda\quad(\xi-\lambda)\text{-a.e. in } F,\tag{6.2}$$

$$W^{\lambda}_{g^\alpha_D,f}(x)\leqslant w_\lambda\quad\text{for all } x\in S^{\lambda}_D.\tag{6.3}$$

Corollary 6.1 *Let $f\big|_D=U^{\chi}_{g^\alpha_D}$, where $\chi\in\mathscr{E}^{+}_{g^\alpha_{D^c}}(D)$ is bounded. If λ solves Problem* 3.2*, then*

$$C_\alpha\bigl(\partial D\cap \mathrm{C}\ell_{\mathbb{R}^n}S^{\xi-\lambda}_D\bigr)=0.$$

When speaking of the *non-weighted case* $f=0$, we simply write

$$\mathscr{E}^{\xi}_{g^\alpha_D}(F,1):=\bigl\{\nu\in\mathscr{E}^{+}_{g^\alpha_D}(F,1):\ \nu\leqslant\xi\bigr\}.$$

Then Problem 3.2 is in fact reduced to that on the existence of $\lambda_0\in\mathscr{E}^{\xi}_{g^\alpha_D}(F,1)$ with

$$E_{g^\alpha_D}(\lambda_0)=\inf_{\nu\in\mathscr{E}^{\xi}_{g^\alpha_D}(F,1)}E_{g^\alpha_D}(\nu).\tag{6.4}$$

Corollary 6.2 *Let $f=0$. A measure $\lambda_0\in\mathscr{E}^{\xi}_{g^\alpha_D}(F,1)$ solves Problem* 3.2 *if and only if there exists $w'_{\lambda_0}\in(0,\infty)$ such that*

$$U^{\lambda_0}_{g^\alpha_D}(x)=w'_{\lambda_0}\quad(\xi-\lambda_0)\text{-a.e. in } F,\tag{6.5}$$

$$U^{\lambda_0}_{g^\alpha_D}(x)\leqslant w'_{\lambda_0}\quad\text{for all } x\in D.\tag{6.6}$$

If, moreover, $\alpha<2$, then also

$$S^{\lambda_0}_D=F.\tag{6.7}$$

On account of the uniqueness of a solution to Problem 3.2, such w'_{λ_0} is unique (provided it exists). If the constraint ξ is bounded, then integration (6.5) with respect to $\xi-\lambda_0$ gives

$$w'_{\lambda_0}=\frac{E_{g^\alpha_D}(\lambda_0,\xi-\lambda_0)}{(\xi-\lambda_0)(D)}.\tag{6.8}$$

6.2 Variational inequalities in the unconstrained α-Green minimum energy problems

Throughout this section, it is assumed that $\sigma=\infty$. We proceed with criteria of the solvability of Problem 3.2, given in terms of variational inequalities for the f-weighted α-Green potentials. In the unconstrained case, the results obtained take a simpler form if compare with those in the constrained case, while provide us with much more detailed information about the potentials and the supports of the minimizers.

[13] The first results of such kind have been established by [9, Theorem 2.1] and [19, Theorem 3] for the logarithmic kernel on the plane; see also [26, Theorem 2.3] pertaining to a positive definite kernel on a locally compact space.

Theorem 6.2 *Suppose that Case* I *takes place. For $\lambda \in \mathscr{E}^+_{g^\alpha_D,f}(F,1)$ to solve Problem* 3.2*, it is necessary and sufficient that there exist $w_f \in \mathbb{R}$ possessing the properties*

$$W^\lambda_{g^\alpha_D,f}(x) \geqslant w_f \quad \text{n.e. in } F, \tag{6.9}$$
$$W^\lambda_{g^\alpha_D,f}(x) \leqslant w_f \quad \text{for all } x \in S^\lambda_D.$$

Such a number w_f is unique (*provided it exists*) *and can be given by the formula*

$$w_f = \langle W^\lambda_{g^\alpha_D,f}, \lambda \rangle.$$

Recall that $\breve{F}$, the reduced kernel of F, has been defined by (6.1).

Corollary 6.3 *Let Problem* 3.2 *be solvable. Then the following two assertions hold:*

(a) *If Case* II *with $\zeta \geqslant 0$ takes place, then $C_{g^\alpha_D}(F) < \infty$;*
(b) *If $f\big|_D = U^\chi_{g^\alpha_D}$, where $\chi \in \mathscr{E}^+_{g^\alpha_{D^c}}(D)$ is bounded, then $C_\alpha\big(\partial D \cap \mathrm{C}\ell_{\mathbb{R}^n}\breve{F}\big) = 0$.*

Corollary 6.4 *Let $f = 0$. Then $\lambda_F \in \mathscr{E}^+_{g^\alpha_D}(F,1)$ solves Problem* 3.2 *if and only if there exists a number $w \in (0,\infty)$ admitting the properties*

$$U^{\lambda_F}_{g^\alpha_D}(x) = w \quad \text{n.e. in } F, \tag{6.10}$$

$$U^{\lambda_F}_{g^\alpha_D}(x) \leqslant w \quad \text{for all } x \in D. \tag{6.11}$$

Such a number w is unique (*provided it exists*) *and can be written in the form*

$$w = E_{g^\alpha_D}(\lambda_F) = w_{g^\alpha_D}(F) = \big[C_{g^\alpha_D}(F)\big]^{-1}. \tag{6.12}$$

Furthermore, if the minimizer λ_F exists, then it is the unique measure in the class $\mathscr{E}^+_{g^\alpha_D}(F,1)$ whose α-Green potential is constant n.e. in F. Namely, if $\nu \in \mathscr{E}^+_{g^\alpha_D}(F,1)$ and $U^\nu_{g^\alpha_D}(x) = c$ n.e. in F, where $c \in \mathbb{R}$, then $\nu = \lambda_F$.

For the sake of simplicity, in the following assertion we assume that, if $\alpha = 2$, then $D \setminus F$ is connected.

Corollary 6.5 *Let $f = 0$. If λ_F solves Problem* 3.2*, then, in addition to* (6.10) *and* (6.11)*, we have*

$$U^{\lambda_F}_{g^\alpha_D}(x) < w \quad \text{for all } x \in D \setminus \breve{F}. \tag{6.13}$$

Furthermore,

$$S^{\lambda_F}_D = \begin{cases} \breve{F} & \text{if } \alpha < 2, \\ \partial_D \breve{F} & \text{if } \alpha = 2. \end{cases} \tag{6.14}$$

Remark 6.1 It follows from the above-mentioned results that the classes of condensers for which Problem 3.2 is solvable in the constrained or the unconstrained settings, respectively, are drastically different from each other. To be specific, consider $f\big|_D = U^\chi_{g^\alpha_D}$ where $\chi \in \mathscr{E}^+_{g^\alpha_{D^c}}(D)$ is bounded. Then f is finite n.e. in D and, hence, by Lemmas 4.5 and 4.6, assumption (3.8) holds automatically for any $\sigma \in \mathscr{A}(F) \cup \{\infty\}$. If now $\sigma \in \mathscr{A}(F)$ is bounded then, according to Theorem 5.1, Problem 3.2 with the active constraint σ is solvable for *any* $F \subsetneqq D$ (e.g., that touches ∂D over a set with nonzero α-Riesz capacity or even over the whole ∂D). But if Problem 3.2 admits a solution for $\sigma = \infty$, then, by Corollary 6.3, (b), $\breve{F}$ has to touch ∂D only over a set with α-Riesz capacity zero.

6.3 Duality relation between non-weighted constrained and weighted unconstrained minimum α-Green energy problems

Throughout this section, F is compact. Consider the non-weighted Problem 3.2 with a constraint $\xi\in\mathfrak{C}(F)$ whose potential $U^{\xi}_{g^\alpha_D}(x)$ is continuous. Note that then ξ is bounded and admissible. By either of Theorems 5.1 or 5.2, there exists the solution λ_0 to the problem, i.e. both $\lambda_0\in\mathscr{E}^{\xi}_{g^\alpha_D}(F,1)$ and (6.4) hold. Write

$$\theta:=q(\xi-\lambda_0),\quad\text{where}\;\; q:=\frac{1}{\xi(F)-1}.$$

Combining Corollary 6.2 and Theorem 6.2 allows us to formulate the following result.

Theorem 6.3 *The measure θ solves Problem 3.2 with the external field $f(x):=-qU^{\xi}_{g^\alpha_D}(x)$ in both the unconstrained and the $q\xi$-constrained settings, i.e.*

$$\theta\in\mathscr{E}^{q\xi}_{g^\alpha_D,f}(F,1)\subset\mathscr{E}^{+}_{g^\alpha_D,f}(F,1)\;\;\textit{and}\;\;G_{g^\alpha_D,f}(\theta)=G^{q\xi}_{g^\alpha_D,f}(F,1)=G_{g^\alpha_D,f}(F,1).$$

Moreover,

$$W^{\theta}_{g^\alpha_D,f}(x)=-qw'_{\lambda_0}\quad\textit{for all}\;\; x\in S^{\theta}_D, \tag{6.15}$$

$$W^{\theta}_{g^\alpha_D,f}(x)\geqslant-qw'_{\lambda_0}\quad\textit{for all}\;\; x\in D, \tag{6.16}$$

where w'_{λ_0} is the number determined uniquely by identity (6.8).

7 Variational inequalities for the f-weighted α-Riesz potentials

This section is devoted to necessary and/or sufficient conditions for the solvability of Problem 3.1 with $\sigma\in\mathfrak{C}(F)\cup\{\infty\}$, given in terms of variational inequalities for the f-weighted α-Riesz potentials. Throughout this section, we assume D^c to be not α-thin at $\infty_{\mathbb{R}^n}$.

Then, by Lemma 4.2, for $\lambda^{\sigma}_{\mathbf{A}}=\lambda^{+}-\lambda^{-}$ to solve Problem 3.1, it is necessary and sufficient that λ^{+} solve Problem 3.2 with the same σ. Furthermore, by (4.2),

$$\lambda^{-}=\beta^{\alpha}_{D^c}\lambda^{+}, \tag{7.1}$$

which yields

$$W^{\lambda^{\sigma}_{\mathbf{A}}}_{\alpha,f}(x)=U_{\alpha}^{\lambda^{+}-\beta^{\alpha}_{D^c}\lambda^{+}}(x)+f(x)=W^{\lambda^{+}}_{g^\alpha_D,f}(x)\quad\text{for all}\;\; x\in D.$$

For the sake of simplicity, in the next assertion we assume that in the case $\alpha=2$, D is simply connected.

Lemma 7.1 *If $\lambda^{\sigma}_{\mathbf{A}}=\lambda^{+}-\lambda^{-}$ solves Problem* 3.1*, then*

$$S^{\lambda^{-}}_{\mathbb{R}^n}=\begin{cases}D^c & \textit{if}\quad \alpha<2,\\ \partial D & \textit{if}\quad \alpha=2.\end{cases} \tag{7.2}$$

Indeed, Lemma 7.1 follows from (7.1) and the description of the supports of the α-Riesz balayage measures.

7.1 Variational inequalities in the constrained α-Riesz minimum energy problems

In this section, consider $\xi \in \mathscr{A}(F)$ and assume $\xi(F_0) > 1$, where F_0 is given by (4.7). Combining what has been noticed just above with the assertions of Section 6.1 (for λ^+ instead of λ or λ_0) results in the following Theorem 7.1 and Corollaries 7.1 and 7.2.

Theorem 7.1 *Let Case* I *take place. Then* $\lambda_{\mathbf{A}}^{\xi} = \lambda^+ - \lambda^- \in \mathscr{E}_{\alpha,f}^{\xi}(\mathbf{A}, \mathbf{1})$ *is the* (*unique*) *solution to Problem* 3.1 *if and only if* (7.1) *holds and, in addition, there exists* $w_{\lambda_{\mathbf{A}}^{\xi}} \in \mathbb{R}$ *possessing the following two properties:*

$$W_{\alpha,f}^{\lambda_{\mathbf{A}}^{\xi}}(x) \geqslant w_{\lambda_{\mathbf{A}}^{\xi}} \quad (\xi - \lambda^+)\text{-a.e. in } F,$$

$$W_{\alpha,f}^{\lambda_{\mathbf{A}}^{\xi}}(x) \leqslant w_{\lambda_{\mathbf{A}}^{\xi}} \quad \text{for all } x \in S_D^{\lambda^+}.$$

Corollary 7.1 *Assume that* $f\big|_D = U_{g_D^\alpha}^{\chi}$, *where* $\chi \in \mathscr{E}_{g_{D^c}^\alpha}^{+}(D)$ *is bounded. If* $\lambda_{\mathbf{A}}^{\xi} = \lambda^+ - \lambda^-$ *solves Problem* 3.1, *then* $C_\alpha\bigl(\partial D \cap \mathrm{C}\ell_{\mathbb{R}^n} S_D^{\xi-\lambda^+}\bigr) = 0$.

Corollary 7.2 *Let* $f = 0$. *A measure* $\lambda_{\mathbf{A}}^{\xi} = \lambda^+ - \lambda^- \in \mathscr{E}_{\alpha,f}^{\xi}(\mathbf{A}, \mathbf{1})$ *solves Problem* 3.1 *if and only if* (7.1) *holds and, in addition, there exists a number* $w'_{\lambda_{\mathbf{A}}^{\xi}} \in (0, \infty)$ *such that*

$$U_\alpha^{\lambda_{\mathbf{A}}^{\xi}}(x) = w'_{\lambda_{\mathbf{A}}^{\xi}} \quad (\xi - \lambda^+)\text{-a.e. in } F,$$

$$U_\alpha^{\lambda_{\mathbf{A}}^{\xi}}(x) \leqslant w'_{\lambda_{\mathbf{A}}^{\xi}} \quad \text{for all } x \in D.$$

Furthermore, if $\alpha < 2$, *then also* $S_D^{\lambda^+} = F$ *and* $S_{\mathbb{R}^n}^{\lambda^-} = D^c$.

A number $w'_{\lambda_{\mathbf{A}}^{\xi}}$ is unique (provided exists) and equal to w'_{λ^+}, where w'_{λ^+} is the number from Corollary 6.2 for $\lambda_0 = \lambda^+$.

7.2 Variational inequalities in the unconstrained α-Riesz minimum energy problems

In this section, $\sigma = \infty$. Similarly as it has been done just above, we derive the following corollaries from the assertions of Section 6.2.

Corollary 7.3 *Assume Case* I *takes place. A measure* $\lambda_{\mathbf{A}} = \lambda^+ - \lambda^- \in \mathscr{E}_{\alpha,f}(\mathbf{A}, \mathbf{1})$ *solves Problem* 3.1 *if and only if* (7.1) *holds and, in addition, there exists a* (*unique*) *number* $w'_f \in \mathbb{R}$ *possessing the properties*

$$W_{\alpha,f}^{\lambda_{\mathbf{A}}}(x) \geqslant w'_f \quad \text{n.e. in } F,$$

$$W_{\alpha,f}^{\lambda_{\mathbf{A}}}(x) \leqslant w'_f \quad \text{for all } x \in S_D^{\lambda^+}.$$

Furthermore, then $w'_f = w_f$, *where* w_f *is the number from Theorem* 6.2, *and assertions* (a) *and* (b) *of Corollary* 6.3 *both hold.*

For the sake of simplicity, in the following assertion we assume that in the case $\alpha = 2$, $D \setminus F$ is simply connected.

Corollary 7.4 *Let $f = 0$. A measure $\lambda_{\mathbf{A}} = \lambda^+ - \lambda^- \in \mathscr{E}_\alpha(\mathbf{A}, \mathbf{1})$ solves Problem* 3.1 *if and only if* (7.1) *holds and there exists a* (*unique*) *number $w' \in (0,\infty)$ such that*

$$\begin{aligned}
U_\alpha^{\lambda_{\mathbf{A}}}(x) &= w' \quad \text{n.e. in } F,\\
U_\alpha^{\lambda_{\mathbf{A}}}(x) &\leqslant w' \quad \text{for all } x \in D,\\
U_\alpha^{\lambda_{\mathbf{A}}}(x) &< w' \quad \text{for all } x \in D \setminus \breve{F}.
\end{aligned}$$

Furthermore, then $w' = w$, where w is the number from Corollary 6.4*, i.e.*

$$w' = E_\alpha(\lambda_{\mathbf{A}}, \lambda^+) = E_\alpha(\lambda_{\mathbf{A}}) = E_{g_D^\alpha}(\lambda^+) = w_{g_D^\alpha}(F) = \big[C_{g_D^\alpha}(F)\big]^{-1} = E_\alpha(\mathbf{A}, \mathbf{1}).$$

The descriptions of $S_D^{\lambda^+}$ and $S_{\mathbb{R}^n}^{\lambda^-}$ are given by (6.14) *for λ^+ in place of λ_F and* (7.2)*, respectively.*

8 Proof of Theorem 5.1

Consider an exhaustion of F by an increasing sequence of compact sets K_k, $k \in \mathbb{N}$. Since the constraint ξ is bounded, it holds

$$\lim_{k\to\infty} \xi(F \setminus K_k) = 0. \tag{8.1}$$

Because of assumption (3.8), there exists $\{\mu_\ell\}_{\ell\in\mathbb{N}} \subset \mathscr{E}^{\xi}_{g_D^\alpha,f}(F,1)$ such that

$$\lim_{\ell\to\infty} G_{g_D^\alpha,f}(\mu_\ell) = G^{\xi}_{g_D^\alpha,f}(F,1). \tag{8.2}$$

This sequence $\{\mu_\ell\}_{\ell\in\mathbb{N}}$ is vaguely bounded; hence, by [3, Chapter III, Section 2, Prop. 9], it has a vague cluster point μ_0. We assert that, in Case I, μ_0 is the solution to Problem 3.2.

Since $\mathfrak{M}^+(F)$ is vaguely closed in $\mathfrak{M}^+(D)$, we get $\mu_0 \in \mathfrak{M}^+(F)$ and $\mu_0 \leqslant \xi$. As, due to [11, Lemma 1.2.1], $\mathfrak{M}^+(F)$ is actually sequentially closed, there exists a subsequence $\{\mu_{\ell_m}\}_{m\in\mathbb{N}}$ of $\{\mu_\ell\}_{\ell\in\mathbb{N}}$ converging vaguely to μ_0. Then, in consequence of Lemma 2.1,

$$\begin{aligned}
1 = \lim_{m\to\infty} \mu_{\ell_m}(F) &\geqslant \mu_0(D) = \mu_0(F) = \lim_{k\to\infty} \mu_0(K_k)\\
&\geqslant \lim_{k\to\infty} \limsup_{m\to\infty} \mu_{\ell_m}(K_k) = 1 - \lim_{k\to\infty} \liminf_{m\to\infty} \mu_{\ell_m}(F \setminus K_k).
\end{aligned}$$

On account of the fact that $\mu_{\ell_m}(F\setminus K_k) \leqslant \xi(F\setminus K_k)$ for all $m,k \in \mathbb{N}$, combining the preceding chain of inequalities with (8.1) yields $\mu_0(F) = 1$.

To complete the proof, it thus remains to observe that $G_{g_D^\alpha,f}(\mu_0) \leqslant G^{\xi}_{g_D^\alpha,f}(F,1)$, which is seen from (8.2) in view of the lower semicontinuity of $G_{g_D^\alpha,f}$ on $\mathscr{E}^+_{g_D^\alpha}(D)$ (see Lemma 4.4).

□

9 Proof of Theorem 5.3

Under the assumptions of the theorem, Case II with $\zeta \geqslant 0$ takes place, and therefore

$$G_{g_D^\alpha,f}(\nu) = \|\nu\|^2_{g_D^\alpha} + 2E_{g_D^\alpha}(\zeta,\nu) \geqslant \|\nu\|^2_{g_D^\alpha} \geqslant 0 \quad \text{for all } \nu \in \mathscr{E}^+_{g_D^\alpha}(D). \tag{9.1}$$

Consider an exhaustion of F by an increasing sequence of compact sets K_k, $k \in \mathbb{N}$. Since $C_{g_D^\alpha}(F) = \infty$, the strict positive definiteness of the α-Green kernel and the subadditivity of $C_{g_D^\alpha}(\cdot)$ on the universally measurable sets yield $C_{g_D^\alpha}(F \setminus K_k) = \infty$ for all $k \in \mathbb{N}$. Hence, for every k one can choose a measure $\nu_k \in \mathscr{E}^+_{g_D^\alpha}(F \setminus K_k, 1)$ with compact support so that

$$\lim_{k\to\infty} \|\nu_k\|^2_{g_D^\alpha} = 0. \tag{9.2}$$

Certainly, there is no loss of generality in assuming $K_k \cup S_D^{\nu_k} \subset K_{k+1}$.

Fix $\xi \in \mathfrak{C}(F)$ and write $\xi_0 := \xi + \sum_{k\in\mathbb{N}} \nu_k$; then $\xi_0 \in \mathfrak{C}(F) \setminus \mathfrak{C}_0(F)$. Due to (3.6), for each $\sigma \in \mathfrak{C}(F) \cup \{\infty\}$ such that $\sigma \geqslant \xi_0$, it holds

$$\nu_k \in \mathscr{E}^\sigma_{g_D^\alpha,f}(F,1) \quad \text{for all } k \in \mathbb{N}.$$

From the Cauchy–Schwarz inequality in the pre-Hilbert space $\mathscr{E}_{g_D^\alpha}(D)$ and (9.2) we get

$$\lim_{k\to\infty} G_{g_D^\alpha,f}(\nu_k) = \lim_{k\to\infty} \bigl[\|\nu_k\|^2_{g_D^\alpha} + 2E_{g_D^\alpha}(\zeta,\nu_k)\bigr] \leqslant 2\|\zeta\|_{g_D^\alpha} \lim_{k\to\infty} \|\nu_k\|_{g_D^\alpha} = 0.$$

Combined with (9.1), this yields $G^\sigma_{g_D^\alpha,f}(F,1) = 0$. In view of the strict positive definiteness of g_D^α, repeated application of (9.1) shows also that such infimum value can be attained only at zero measure. As $0 \notin \mathscr{E}^\sigma_{g_D^\alpha,f}(F,1)$, Problem 3.2 with σ specified above is unsolvable. $\square$

10 Perfectness-type result for the α-Green kernel

A crucial point in our proof of Theorem 5.2, given in Section 12, is the following perfectness-type result for the α-Green kernel, $0 < \alpha \leqslant 2$.

Theorem 10.1 *Let $E \subset D$ be relatively closed. Suppose that $\alpha = 2$ and D is regular, or $\overline{E} \cap \partial_{\overline{\mathbb{R}^n}} D$ consists of at most one point. Then any strong Cauchy sequence $\{\nu_k\}_{k\in\mathbb{N}} \subset \mathscr{E}^+_{g_D^\alpha}(E)$ with*

$$\sup_{k\in\mathbb{N}} \nu_k(E) \leqslant M, \tag{10.1}$$

where $M \in (0,\infty)$, converges both strongly and vaguely to the unique $\nu_0 \in \mathscr{E}^+_{g_D^\alpha}(E)$.

Proof We can certainly assume that either $\alpha < 2$, or $\alpha = 2$ but D is not regular, since otherwise the theorem holds true due to the perfectness of the classical g_D^2-kernel, established in [5] (see also [16]).

The (strongly fundamental) sequence $\{\nu_k\}_{k\in\mathbb{N}} \subset \mathscr{E}^+_{g_D^\alpha}(E)$ is strongly bounded, i.e.

$$\sup_{k\in\mathbb{N}} \|\nu_k\|_{g_D^\alpha} < \infty.$$

Besides, by (10.1), $\{\nu_k\}_{k\in\mathbb{N}}$ is vaguely bounded and hence, according to [3, Chapter III, Section 2, Prop. 9], it has a vague cluster point ν_0. Since $\mathscr{E}^+_{g_D^\alpha}(E)$ is a vaguely closed subset of $\mathfrak{M}^+(D)$, we actually have $\nu_0 \in \mathscr{E}^+_{g^\alpha_D}(E)$. Also observe that ν_0 is bounded, which follows from (10.1) in view of Lemma 2.1.

Application of [11, Lemma 1.2.1] shows that one can choose a subsequence $\{\nu_{k_\ell}\}_{\ell\in\mathbb{N}}$ of $\{\nu_k\}_{k\in\mathbb{N}}$ so that

$$\nu_{k_\ell} \to \nu_0 \quad \text{vaguely as } \ell \to \infty. \tag{10.2}$$

We next proceed to show that $\nu_{k_\ell} \to \nu_0$ also strongly in $\mathscr{E}^+_{g^\alpha_D}(E)$, i.e.

$$\lim_{\ell\to\infty} \|\nu_{k_\ell} - \nu_0\|_{g^\alpha_D} = 0. \tag{10.3}$$

As $\{\nu_{k_\ell}\}_{\ell\in\mathbb{N}} \subset \mathscr{E}^+_{g^\alpha_D}(E)$, being a subsequence of the strong Cauchy sequence $\{\nu_k\}_{k\in\mathbb{N}}$, is strongly fundamental as well, we see from (2.8) that

$$\widetilde{\nu_{k_\ell}} := \nu_{k_\ell} - \beta^\alpha_{D^c}\nu_{k_\ell}, \quad \ell \in \mathbb{N}, \tag{10.4}$$

is strongly fundamental in $\mathscr{E}_\alpha(\mathbb{R}^n)$. The proof of (10.3) is given in two steps.

Step 1. Throughout this step, let $\overline{E} \cap \partial_{\overline{\mathbb{R}^n}} D$ either be empty or consist of only $\infty_{\mathbb{R}^n}$. Then ν_{k_ℓ} and $\beta^\alpha_{D^c}\nu_{k_\ell}$ are supported by the sets E and D^c, which due to the assumptions made are closed in $\mathbb{R}^n$ and nonintersecting. Consider the condenser $\mathbf{B} := (E, D^c)$. The strong completeness theorem from [23] (or see Theorem 2.3 above) yields that there exists the unique measure $\tilde{\nu} = \tilde{\nu}^+ - \tilde{\nu}^- \in \mathscr{E}_\alpha(\mathbf{B})$ such that

$$\lim_{\ell\to\infty} \|\widetilde{\nu_{k_\ell}} - \tilde{\nu}\|_\alpha = 0. \tag{10.5}$$

Furthermore, by this theorem, $\tilde{\nu}^+$ and $\tilde{\nu}^-$ are the vague limits of the positive and the negative parts of $\widetilde{\nu_{k_\ell}}$, $\ell \in \mathbb{N}$, respectively. In view of (10.2), we thus have

$$\tilde{\nu}^+ = \nu_0, \tag{10.6}$$

for the vague topology is Hausdorff.

By the remark in [11, p. 166], it follows from (10.5) that there exists a subsequence of the sequence $\{\widetilde{\nu_{k_\ell}}\}_{\ell\in\mathbb{N}}$ (denote it again by the same symbol) such that

$$U^{\tilde{\nu}}_\alpha(x) = \lim_{\ell\to\infty} U^{\widetilde{\nu_{k_\ell}}}_\alpha(x) \quad \text{n.e. in } \mathbb{R}^n.$$

On account of (10.4) and the countable subadditivity of $C_\alpha(\cdot)$ over Borel sets, we see from the preceding relation that $U^{\tilde{\nu}}_\alpha(x) = 0$ n.e. in D^c and, therefore, $\tilde{\nu}^- = \beta^\alpha_{D^c}\tilde{\nu}^+$. Combining this with (10.4), (10.5) and (10.6) implies

$$\lim_{\ell\to\infty} \left\|\left(\nu_{k_\ell} - \nu_0\right) - \beta^\alpha_{D^c}\left(\nu_{k_\ell} - \nu_0\right)\right\|_\alpha = 0,$$

which in view of (2.8), applied to the (bounded) measures $\nu_{k_\ell} - \nu_0 \in \mathscr{E}_{g^\alpha_D}(D)$, $\ell \in \mathbb{N}$, establishes (10.3).

Step 2. We next prove relation (10.3) in the case $\overline{E} \cap \partial_{\overline{\mathbb{R}^n}} D = \{x_0\}$, where $x_0 \neq \infty_{\mathbb{R}^n}$. Throughout this step, all the measures can be assumed to have zero mass at x_0, for we can restrict our consideration to those with finite energy.

Define the inversion with respect to $S(x_0, 1)$, namely, each point $x \neq x_0$ is mapped to the point x^* on the ray through x which issues from x_0, determined uniquely by

$$|x - x_0| \cdot |x^* - x_0| = 1.$$

This is a one-to-one, bicontinuous mapping of $\mathbb{R}^n \setminus \{x_0\}$ onto itself; furthermore,

$$|x^* - y^*| = \frac{|x-y|}{|x_0 - x||x_0 - y|}. \tag{10.7}$$

It can be extended to a one-to-one, bicontinuous map of $\overline{\mathbb{R}^n}$ onto itself by setting $x_0 \mapsto \infty_{\mathbb{R}^n}$.

To each $\nu \in \mathfrak{M}(\mathbb{R}^n)$ (with $\nu(\{x_0\}) = 0$) we correspond the Kelvin transform $\nu^* \in \mathfrak{M}(\mathbb{R}^n)$ by means of the formula

$$d\nu^*(x^*) = |x - x_0|^{\alpha - n}\, d\nu(x), \quad x^* \in \mathbb{R}^n.$$

Then, in view of (10.7),

$$U_\alpha^{\nu^*}(x^*) = |x - x_0|^{n-\alpha} U_\alpha^\nu(x), \quad x^* \in \mathbb{R}^n, \tag{10.8}$$

and therefore

$$E_\alpha(\nu^*) = E_\alpha(\nu) \tag{10.9}$$

(see [16, Chapter IV, Section 5, n° 19] and [16, Chapter V, Section 2, n° 8], respectively).

It is obvious that the Kelvin transformation is additive, i.e.

$$\big(\nu_1 + \nu_2\big)^* = \nu_1^* + \nu_2^*. \tag{10.10}$$

We also observe that

$$(\beta_{D^c}^\alpha \nu)^* = \beta_{(\overline{D^c})^*}^\alpha \nu^*, \tag{10.11}$$

where $(\overline{D^c})^*$ is the image of $\overline{D^c}$ under the inversion $x \mapsto x^*$. Indeed, in view of (10.8) and the definition of the α-Riesz balayage, we get

$$U_\alpha^{(\beta_{D^c}^\alpha \nu)^*}(x^*) = |x - x_0|^{n-\alpha} U_\alpha^{\beta_{D^c}^\alpha \nu}(x) = |x - x_0|^{n-\alpha} U_\alpha^\nu(x) = U_\alpha^{\nu^*}(x^*),$$

the relation being valid for nearly all $x \in D^c$. Consequently, it also holds for nearly all $x^* \in (\overline{D^c})^*$, because the inversion of a set with $C_\alpha(\cdot) = 0$ has the interior α-Riesz capacity zero as well (see [16, Chapter IV, Section 5, n° 19]). Since $(\beta_{D^c}^\alpha \nu)^*$ is supported by $(\overline{D^c})^*$, identity (10.11) follows.

Applying [16, Lemma 4.3] to ν_{k_ℓ}, $\ell \in \mathbb{N}$, and ν_0 (where ν_{k_ℓ}, $\ell \in \mathbb{N}$, and ν_0 are as above), on account of (10.1) and (10.2) we have

$$\nu_{k_\ell}^* \to \nu_0^* \quad \text{vaguely as } \ell \to \infty. \tag{10.12}$$

Also observe that, according to (10.9) and the fact that $\{\widetilde{\nu_{k_\ell}}\}_{\ell \in \mathbb{N}}$ is strongly fundamental, so is the sequence $\big(\widetilde{\nu_{k_\ell}}\big)^* \in \mathscr{E}_\alpha(\mathbb{R}^n)$, $\ell \in \mathbb{N}$, which in consequence of (10.4), (10.10) and (10.11) can be rewritten in the form

$$\big(\widetilde{\nu_{k_\ell}}\big)^* = \nu_{k_\ell}^* - \big(\beta_{D^c}^\alpha \nu_{k_\ell}\big)^* = \nu_{k_\ell}^* - \beta_{(\overline{D^c})^*}^\alpha \nu_{k_\ell}^*, \quad \ell \in \mathbb{N}. \tag{10.13}$$

The positive and the negative parts of $\big(\widetilde{\nu_{k_\ell}}\big)^*$ are supported by the sets E^* and $(\overline{D^c})^*$, respectively, which are closed in $\mathbb{R}^n$ and nonintersecting; hence, the strong completeness theorem from [23] (see Theorem 2.3 above) can be applied. Therefore, there exists the unique measure $\hat{\nu} = \hat{\nu}^+ - \hat{\nu}^- \in \mathscr{E}_\alpha(\mathbb{R}^n)$, where $\hat{\nu}^+$ and $\hat{\nu}^-$ are supported by E^* and $(\overline{D^c})^*$, respectively, such that

$$\lim_{\ell \to \infty} \big\| \big(\widetilde{\nu_{k_\ell}}\big)^* - \hat{\nu} \big\|_\alpha = 0. \tag{10.14}$$

Furthermore, $\hat{\nu}^+$ and $\hat{\nu}^-$ are the vague limits of the positive and the negative parts of $\bigl(\widetilde{\nu_{k_\ell}}\bigr)^*$, $\ell \in \mathbb{N}$, respectively. When combined with (10.12), (10.13) and the fact that the vague topology is Hausdorff, this implies

$$\hat{\nu}^+ = \nu_0^*. \tag{10.15}$$

In view of (10.14) and the remark in [11, p. 166], one can choose a subsequence of the sequence $\bigl\{\bigl(\widetilde{\nu_{k_\ell}}\bigr)^*\bigr\}_{\ell\in\mathbb{N}}$ (denote it again by the same symbol) so that

$$U_\alpha^{\hat{\nu}}(x) = \lim_{\ell\to\infty} U_\alpha^{(\widetilde{\nu_{k_\ell}})^*}(x) \quad \text{n.e. in } \mathbb{R}^n.$$

On account of (10.13), we thus have $U_\alpha^{\hat{\nu}}(x) = 0$ n.e. in $(\overline{D^c})^*$, and therefore, by (10.15),

$$\hat{\nu}^- = \beta^\alpha_{(\overline{D^c})^*}\hat{\nu}^+ = \beta^\alpha_{(\overline{D^c})^*}\nu_0^*. \tag{10.16}$$

Using the fact that the Kelvin transformation is an involution and applying (10.9), (10.10) and (10.11) again, we conclude from (10.14), (10.15) and (10.16) that

$$\widetilde{\nu_{k_\ell}} \to \nu_0 - \beta^\alpha_{D^c}\nu_0 \quad (\text{as } \ell\to\infty) \quad \text{in } \mathcal{E}_\alpha(\mathbb{R}^n),$$

or equivalently, by the definition of $\widetilde{\nu_{k_\ell}}$,

$$\lim_{\ell\to\infty} \bigl\| (\nu_{k_\ell} - \nu_0) - \beta^\alpha_{D^c}(\nu_{k_\ell} - \nu_0) \bigr\|_\alpha = 0.$$

Repeated application of (2.8) then proves relation (10.3) also in the case $\overline{E} \cap \partial_{\overline{\mathbb{R}^n}} D = \{x_0\}$, where $x_0 \neq \infty_{\mathbb{R}^n}$. This completes Step 2.

Since the sequence $\{\nu_k\}_{k\in\mathbb{N}}$ is strongly fundamental, $\nu_k \to \nu_0$ strongly by (10.3). It has thus been proved that $\{\nu_k\}_{k\in\mathbb{N}}$ converges strongly to any of its vague cluster points. As the α-Green kernel is strictly positive definite, any two cluster points of $\{\nu_k\}_{k\in\mathbb{N}}$, ν_0 and ν_0', have to coincide. Thus, ν_0 is the only vague cluster point of $\{\nu_k\}_{k\in\mathbb{N}}$, and so $\nu_k \to \nu_0$ also vaguely (cf. [2, Chapter I, Section 9, n° 1, cor.]). □

11 α-Green equilibrium measure

Theorem 11.1 *Let $E \subset D$ be relatively closed. Suppose that $\alpha = 2$ and D is regular, or $\overline{E} \cap \partial_{\overline{\mathbb{R}^n}} D$ consists of at most one point. If, moreover, $C_{g^\alpha_D}(E) < \infty$, then there exists an α-Green equilibrium measure $\gamma = \gamma_E$ on E, that is, a one possessing the properties $\gamma \in \mathcal{E}^+_{g^\alpha_D}(E)$ and*[14]

$$E_{g^\alpha_D}(\gamma) = \gamma(E) = C_{g^\alpha_D}(E), \tag{11.1}$$

$$U^\gamma_{g^\alpha_D}(x) \geqslant 1 \quad \textit{n.e. in } E, \tag{11.2}$$

$$U^\gamma_{g^\alpha_D}(x) \leqslant 1 \quad \textit{for all } x \in S^\gamma_D. \tag{11.3}$$

This γ solves the problem of minimizing the energy $E_{g^\alpha_D}(\nu)$ over the convex class Γ_E of all $\nu \in \mathcal{E}_{g^\alpha_D}(D)$ such that $U^\nu_{g^\alpha_D}(x) \geqslant 1$ n.e. in E, and hence it is unique.

[14] See also Remark 11.1.

Proof The theorem needs to be established only in the case where either $\alpha < 2$, or both $\alpha = 2$ and D is non-regular, since otherwise it is a special case of [11, Theorem 4.1] in view of the perfectness of the classical g_D^2-Green kernel.

Also note that, while proving the existence of an α-Green equilibrium measure $\gamma = \gamma_E$ on E possessing the properties $\gamma \in \mathscr{E}^+_{g_D^\alpha}(E)$ and (11.1)–(11.3), one can certainly assume E to be noncompact in D, for if not, then this follows from [11, Theorem 2.5]. Now, consider an exhaustion of E by an increasing sequence of sets $K_k \subset E$, $k \in \mathbb{N}$, compact in D, and let $\gamma_k = \gamma_{K_k}$ be an α-Green equilibrium measure on K_k. Then, by [11, Lemma 2.3.3] and relation (11.1) with $E = K_k$,

$$\lim_{k\to\infty} \|\gamma_k\|^2_{g_D^\alpha} = \lim_{k\to\infty} C_{g_D^\alpha}(K_k) = C_{g_D^\alpha}(E) < \infty. \tag{11.4}$$

Since $\gamma_k \in \Gamma_{K_p}$ for all $k \geqslant p$, which is seen from the monotonicity of $\{K_k\}_{k\in\mathbb{N}}$ and inequality (11.2) with $E = K_k$, Lemma 2.2 yields

$$\|\gamma_k - \gamma_p\|^2_{g_D^\alpha} \leqslant \|\gamma_k\|^2_{g_D^\alpha} - \|\gamma_p\|^2_{g_D^\alpha} \quad \text{for all } \ k \geqslant p.$$

In consequence of the last two relations, $\{\gamma_k\}_{k\in\mathbb{N}} \subset \mathscr{E}^+_{g_D^\alpha}(E)$ is strongly fundamental. In addition, by (11.1) with $E = K_k$,

$$\gamma_k(E) = C_{g_D^\alpha}(K_k) \leqslant C_{g_D^\alpha}(E) < \infty \quad \text{for all } \ k \in \mathbb{N}, \tag{11.5}$$

so that all the assumptions of Theorem 10.1 for $\{\gamma_k\}_{k\in\mathbb{N}}$ are satisfied. Hence, there exists the unique $\gamma \in \mathscr{E}^+_{g_D^\alpha}(E)$ such that $\gamma_k \to \gamma$ both strongly and vaguely.

On account of (11.4), we thus get

$$\|\gamma\|^2_{g_D^\alpha} = \lim_{k\to\infty} \|\gamma_k\|^2_{g_D^\alpha} = C_{g_D^\alpha}(E). \tag{11.6}$$

According to [11] (see the remark on p. 166 therein), the strong convergence of γ_k to γ also yields that there exists a subsequence $\gamma_{k_\ell} = \gamma_{K_{k_\ell}}$, $\ell \in \mathbb{N}$, of $\{\gamma_k\}_{k\in\mathbb{N}}$ such that

$$\lim_{\ell\to\infty} U^{\gamma_{k_\ell}}_{g_D^\alpha}(x) = U^{\gamma}_{g_D^\alpha}(x) \quad \text{n.e. in } \ D,$$

while by (11.2) for $E = K_{k_\ell}$,

$$U^{\gamma_{k_\ell}}_{g_D^\alpha}(x) \geqslant 1 \quad \text{n.e. in } \ K_{k_\ell}.$$

Since the sets K_{k_ℓ}, $\ell \in \mathbb{N}$, increase and $E = \bigcup_{\ell\in\mathbb{N}} K_{k_\ell}$, the last two relations imply (11.2). Here we have used the fact that the α-Green capacity of a countable union of Borel sets with zero α-Green capacity is still zero; see [11].

Fix $x \in S_D^\gamma$. As $\gamma_k \to \gamma$ vaguely, one can choose $x_k \in S_D^{\gamma_k}$ so that $x_k \to x$ as $k \to \infty$. Because of the lower semicontinuity of $U^\mu_{g_D^\alpha}(x)$ on the product space $D \times \mathfrak{M}^+(D)$, where $\mathfrak{M}^+(D)$ is equipped with the vague topology (cf. [11, Lemma 2.2.1]), we get

$$U^\gamma_{g_D^\alpha}(x) \leqslant \liminf_{k\to\infty} U^{\gamma_k}_{g_D^\alpha}(x_k).$$

Since, by (11.3) for $E = K_k$, $U^{\gamma_k}_{g_D^\alpha}(x_k) \leqslant 1$ for all $k \in \mathbb{N}$, inequality (11.3) follows.

In view of the vague convergence of γ_k to γ, we also have

$$\gamma(E) \leqslant \liminf_{k\to\infty} \gamma_k(E),$$

so that $\gamma(E) \leqslant C_{g_D^\alpha}(E)$ by (11.5). When combined with (11.6), this shows that, in order to complete the proof of (11.1), it is left to establish the inequality $\gamma(E) \geqslant C_{g_D^\alpha}(E)$, but it follows at once by integrating (11.3) with respect to γ.

Finally, [11, Lemma 3.2.2] with $t = 1$ yields that such a γ solves the problem of minimizing the energy $E_{g_D^\alpha}(\nu)$ over the convex class Γ_E of all $\nu \in \mathscr{E}_{g_D^\alpha}(D)$ such that $U_{g_D^\alpha}^\nu(x) \geqslant 1$ n.e. in E. Application of Lemma 2.2 then shows that any two α-Green equilibrium measures on E are actually equal. $\square$

Remark 11.1 Let the hypotheses of Theorem 11.1 be satisfied. To exclude the trivial case $\gamma = 0$, assume $C_{g_D^\alpha}(E) > 0$. Then $\lambda_E := C_{g_D^\alpha}(E)^{-1}\gamma$ solves Problem 3.2 for $F = E$, $\sigma = \infty$ and $f = 0$. See Corollaries 6.4 and 6.5 for a more detailed information about the properties of the α-Green potential and the support of λ_E (and, hence, γ). In particular, relations (6.10) and (6.11) are equivalent to

$$U_{g_D^\alpha}^\gamma(x) = 1 \quad \text{n.e. in } E, \qquad U_{g_D^\alpha}^\gamma(x) \leqslant 1 \quad \text{for all } x \in D,$$

respectively, so that γ is an actual equilibrium.

12 Proof of Theorem 5.2

In this section we follow methods developed in [26] (see Theorems 2.2 and 3.1 therein). Under the assumptions of Theorem 5.2, the following auxiliary result holds.

Lemma 12.1 *For any $\sigma \in \mathfrak{C}(F) \cup \{\infty\}$, the metric space*

$$\mathscr{E}_{g_D^\alpha}^\sigma(F,1) := \left\{\mu \in \mathscr{E}_{g_D^\alpha}^+(F,1) : \ \mu \leqslant \sigma\right\}$$

is strongly complete. In more detail, any strong Cauchy sequence $\{\mu_k\}_{k\in\mathbb{N}} \subset \mathscr{E}_{g_D^\alpha}^\sigma(F,1)$ converges both strongly and vaguely to the unique $\mu_0 \in \mathscr{E}_{g_D^\alpha}^\sigma(F,1)$.

Proof Fix a strong Cauchy sequence $\{\mu_k\}_{k\in\mathbb{N}} \subset \mathscr{E}_{g_D^\alpha}^\sigma(F,1)$. According to Theorem 10.1, there exists the unique $\mu_0 \in \mathscr{E}_{g_D^\alpha}^+(F)$ such that

$$\mu_k \to \mu_0 \quad \text{strongly and vaguely.}$$

Actually, $\mu_0 \in \mathscr{E}_{g_D^\alpha}^\sigma(F)$, since $\mathscr{E}_{g_D^\alpha}^\sigma(F)$ is vaguely closed. Hence, it is left to show that

$$\mu_0(F) = 1. \tag{12.1}$$

Assume F to be noncompact, for if not, then (12.1) is evident. Consider an exhaustion of F by an increasing sequence of sets $K_m \subset F$, $m \in \mathbb{N}$, compact in D; then

$$\begin{aligned} 1 = \lim_{k\to\infty} \mu_k(F) \geqslant \mu_0(F) = \lim_{m\to\infty} \mu_0(K_m) &\geqslant \lim_{m\to\infty} \limsup_{k\to\infty} \mu_k(K_m) \\ &= 1 - \lim_{m\to\infty} \liminf_{k\to\infty} \mu_k(F \setminus K_m). \end{aligned}$$

Therefore, identity (12.1) will be established once we prove

$$\lim_{m\to\infty} \liminf_{k\to\infty} \mu_k(F \setminus K_m) = 0. \tag{12.2}$$

Write $K_m^* := \mathrm{C}\ell_D(F \setminus K_m)$. It is seen from Theorem 11.1 that, under the assumptions made, there exists the α-Green equilibrium measure γ_m on K_m^*, and it solves the problem of minimizing $E_{g_D^\alpha}(\nu)$ over the convex cone $\Gamma_{K_m^*}$. Since, by the monotonicity of K_m^*, $m \in \mathbb{N}$, and relation (11.2) for $E = K_m^*$, γ_m belongs to Γ_p for all $p \geqslant m$, Lemma 2.2 yields

$$\|\gamma_m - \gamma_p\|^2_{g_D^\alpha} \leqslant \|\gamma_m\|^2_{g_D^\alpha} - \|\gamma_p\|^2_{g_D^\alpha} \quad \text{for all } \ p \geqslant m.$$

Furthermore, it is clear from (11.1) for $E = K_m^*$ that the sequence $\|\gamma_m\|^2_{g_D^\alpha}$, $m \in \mathbb{N}$, is bounded and nonincreasing, and hence it is fundamental in $\mathbb{R}$. The preceding inequality thus implies that γ_m, $m \in \mathbb{N}$, is strongly fundamental in $\mathscr{E}^+_{g_D^\alpha}(D)$. Since it obviously converges vaguely to zero, zero is also its strong limit due to Theorem 10.1. Hence,

$$\lim_{m\to\infty} \|\gamma_m\|_{g_D^\alpha} = 0.$$

Besides, by (11.2) for $E = K_m^*$,

$$\mu_k(F \setminus K_m) \leqslant \mu_k(K_m^*) \leqslant \left\langle U^{\gamma_m}_{g_D^\alpha}, \mu_k \right\rangle \leqslant \|\gamma_m\|_{g_D^\alpha} \cdot \|\mu_k\|_{g_D^\alpha} \quad \text{for all } \ k, m \in \mathbb{N}.$$

As $\|\mu_k\|_{g_D^\alpha}$, $k \in \mathbb{N}$, is bounded, combining the last two relations yields (12.2). □

Now we are able to complete the proof of Theorem 5.2. In view of (3.8), one can choose $\nu_k \in \mathscr{E}^\sigma_{g_D^\alpha,f}(F,1)$, $k \in \mathbb{N}$, so that

$$\lim_{k\to\infty} G_{g_D^\alpha,f}(\nu_k) = G^\sigma_{g_D^\alpha,f}(F,1) < \infty. \tag{12.3}$$

Based on the convexity of the class $\mathscr{E}^\sigma_{g_D^\alpha,f}(F,1)$ and the pre-Hilbert structure on $\mathscr{E}_{g_D^\alpha}(D)$, with the help of arguments similar to those in the proof of Lemma 4.1 we obtain

$$0 \leqslant \|\nu_k - \nu_p\|^2_{g_D^\alpha} \leqslant -4G^\sigma_{g_D^\alpha,f}(F,1) + 2G_{g_D^\alpha,f}(\nu_k) + 2G_{g_D^\alpha,f}(\nu_p) \quad \text{for all } \ k, p \in \mathbb{N}.$$

Substituting (12.3) into this relation implies that $\{\nu_k\}_{k\in\mathbb{N}}$ is strongly fundamental in the metric space $\mathscr{E}^\sigma_{g_D^\alpha}(F,1)$. By Lemma 12.1, $\{\nu_k\}_{k\in\mathbb{N}}$ therefore converges both strongly and vaguely to the unique $\nu_0 \in \mathscr{E}^\sigma_{g_D^\alpha}(F,1)$. On account of Lemma 4.4, we thus get

$$G_{g_D^\alpha,f}(\nu_0) \leqslant \lim_{k\to\infty} G_{g_D^\alpha,f}(\nu_k) = G^\sigma_{g_D^\alpha,f}(F,1) < \infty. \tag{12.4}$$

Hence, $\nu_0 \in \mathscr{E}^\sigma_{g_D^\alpha,f}(F,1)$ and, consequently, $G_{g_D^\alpha,f}(\nu_0) \geqslant G^\sigma_{g_D^\alpha,f}(F,1)$. Combined with (12.4), this shows that $\nu_0 =: \lambda_F^\sigma$ is the solution to Problem 3.2. □

13 Proof of the assertions formulated in Section 6

13.1 Proof of Theorem 6.1

Fix $\lambda \in \mathscr{E}^\xi_{g_D^\alpha,f}(F,1)$, and first assume that it solves Problem 3.2. Then inequality (6.2) holds for $w_\lambda = L$, where

$$L := \sup\left\{q \in \mathbb{R} : \ W^\lambda_{g_D^\alpha,f}(x) \geqslant q \quad (\xi - \lambda)\text{-a.e. in } \ F\right\}.$$

In turn, (6.2) with $w_\lambda = L$ implies $L < \infty$, since $W^\lambda_{g^\alpha_D,f}(x) < \infty$ holds n.e. in F_0, hence $(\xi-\lambda)$-a.e. in F_0, while $(\xi-\lambda)(F_0) > 0$. Also, $L > -\infty$, for f is bounded from below.

We proceed by establishing (6.3) for $w_\lambda = L$. Having denoted (cf. [9, 19])

$$F^+(w) := \bigl\{x \in F:\ W^\lambda_{g^\alpha_D,f}(x) > w\bigr\} \quad \text{and} \quad F^-(w) := \bigl\{x \in F:\ W^\lambda_{g^\alpha_D,f}(x) < w\bigr\},$$

where $w \in \mathbb{R}$ is arbitrary, we assume on the contrary that (6.3) for $w_\lambda = L$ does not hold. In view of the lower semicontinuity of $W^\lambda_{g^\alpha_D,f}$ on F, then one can choose $w_1 \in (L,\infty)$ so that $\lambda\bigl(F^+(w_1)\bigr) > 0$. At the same time, as $w_1 > L$, relation (6.2) with $w_\lambda = L$ yields

$$(\xi-\lambda)\bigl(F^-(w_1)\bigr) > 0.$$

Therefore, there exist compact sets $K_1 \subset F^+(w_1)$ and $K_2 \subset F^-(w_1)$ such that

$$0 < \lambda(K_1) < (\xi-\lambda)(K_2).$$

Write $\tau := (\xi-\lambda)\big|_{K_2}$; then $\tau \in \mathcal{E}^+_{g^\alpha_D}(K_2)$. Since $\bigl\langle W^\lambda_{g^\alpha_D,f}, \tau\bigr\rangle \leqslant w_1\tau(K_2) < \infty$, we get $\langle f,\tau\rangle < \infty$. Define

$$\theta := \lambda - \lambda\big|_{K_1} + c\tau, \quad \text{where } c := \lambda(K_1)\big/\tau(K_2) \in (0,1).$$

A straightforward verification shows that $\theta(F) = 1$ and $\theta \leqslant \xi$, and so $\theta \in \mathcal{E}^\xi_{g^\alpha_D,f}(F,1)$. On the other hand,

$$\begin{aligned}\bigl\langle W^\lambda_{g^\alpha_D,f}, \theta-\lambda\bigr\rangle &= \bigl\langle W^\lambda_{g^\alpha_D,f} - w_1, \theta-\lambda\bigr\rangle\\ &= -\bigl\langle W^\lambda_{g^\alpha_D,f} - w_1, \lambda\big|_{K_1}\bigr\rangle + c\bigl\langle W^\lambda_{g^\alpha_D,f} - w_1, \tau\bigr\rangle < 0,\end{aligned}$$

which is impossible in view of Lemma 4.3. This proves the necessary part of the theorem.

Next, let λ satisfy both (6.2) and (6.3) for some $w_\lambda \in \mathbb{R}$. Then $\lambda\bigl(F^+(w_\lambda)\bigr) = 0$ and $\bigl(\xi-\lambda\bigr)\bigl(F^-(w_\lambda)\bigr) = 0$. For any $\nu \in \mathcal{E}^\xi_{g^\alpha_D,f}(F,1)$, we therefore obtain

$$\begin{aligned}\bigl\langle W^\lambda_{g^\alpha_D,f}, \nu-\lambda\bigr\rangle &= \bigl\langle W^\lambda_{g^\alpha_D,f} - w_\lambda, \nu-\lambda\bigr\rangle\\ &= \bigl\langle W^\lambda_{g^\alpha_D,f} - w_\lambda, \nu\big|_{F^+(w_\lambda)}\bigr\rangle + \bigl\langle W^\lambda_{g^\alpha_D,f} - w_\lambda, (\nu-\xi)\big|_{F^-(w_\lambda)}\bigr\rangle \geqslant 0.\end{aligned}$$

Application of Lemma 4.3 shows that, indeed, λ is the solution to Problem 3.2. $\square$

13.2 Proof of Corollary 6.1

Under the conditions of the corollary, $W^\lambda_{g^\alpha_D,f}(x) = U^{\lambda+\chi}_{g^\alpha_D}(x) > 0$ for all $x \in D$; hence, the number w_λ from Theorem 6.1 satisfies the relation

$$w_\lambda \in (0,\infty). \tag{13.1}$$

Furthermore, since $\lambda+\chi \in \mathcal{E}^+_{g^\alpha_D}(D)$ is bounded, Corollary 2.1 with $\lambda+\chi$ in place of μ yields

$$U^{\lambda+\chi}_{\hat{g}^\alpha_D}(x) = U^{\lambda+\chi}_\alpha(x) - U^{\beta^\alpha_{D^c}(\lambda+\chi)}_\alpha(x) \quad \text{n.e. in } \mathbb{R}^n.$$

As $U^{\lambda+\chi}_{g^\alpha_D}(x)=U^{\lambda+\chi}_{\hat g^\alpha_D}(x)$ for all $x\in D$, while $\xi-\lambda$ is C-absolutely continuous, inequality (6.2) can be rewritten in the form

$$U^{\lambda+\chi}_{\hat g^\alpha_D}(x)\geqslant w_\lambda>0\quad(\xi-\lambda)\text{-a.e. in } F.$$

We can certainly assume that there is $y_0\in\partial D\cap\mathrm{C}\ell_{\mathbb{R}^n}S^{\xi-\lambda}_D$, for if not, then the corollary is obvious. Since for every $\varepsilon>0$ it holds $(\xi-\lambda)\bigl(B(y_0,\varepsilon)\bigr)>0$, one can choose $x_\varepsilon\in F\cap B(y_0,\varepsilon)$ so that $U^{\lambda+\chi}_{\hat g^\alpha_D}(x_\varepsilon)\geqslant w_\lambda>0$. Therefore,

$$\limsup_{x\to y_0,\ x\in D}U^{\lambda+\chi}_{\hat g^\alpha_D}(x)\geqslant w_\lambda>0.$$

On the other hand, $U^{\lambda+\chi}_{\hat g^\alpha_D}(x)=0$ for all $x\in\operatorname{Int}D^c$, for $I_{D^c}\subset\partial D$. As, by assumption, y_0 is a boundary point of $\operatorname{Int}D^c$ as well, we get

$$\liminf_{x\to y_0,\ x\in D^c}U^{\lambda+\chi}_{\hat g^\alpha_D}(x)=0.$$

Consequently, $U^{\lambda+\chi}_{\hat g^\alpha_D}$ is discontinuous on $\partial D\cap\mathrm{C}\ell_{\mathbb{R}^n}S^{\xi-\lambda}_D$, and Lusin's type theorem for the α-Riesz potentials (see [16, Theorem 3.6]) establishes the corollary. $\square$

13.3 Proof of Corollary 6.2

Fix $\lambda_0\in\mathscr{E}^{\xi}_{g^\alpha_D,f}(F)$. We first assume that it solves Problem 3.2, and let $w'_{\lambda_0}\in\mathbb{R}$ be the number from (6.2) and (6.3) for $f=0$. Note that then $w'_{\lambda_0}>0$ by (13.1), and also that $W^{\lambda_0}_{g^\alpha_D,f}(x)=U^{\lambda_0}_{g^\alpha_D}(x)$ for all $x\in D$. Therefore, using Lemma 2.5, from (6.3) we obtain (6.6), while assertion (2.11) takes the form

$$U^{\lambda_0}_\alpha(x)\leqslant w'_{\lambda_0}+U^{\beta^\alpha_{D^c}\lambda_0}_\alpha(x)\quad\text{for all } x\in\mathbb{R}^n.\tag{13.2}$$

Combining (6.6) with (6.2) for $\lambda=\lambda_0$ and $w_\lambda=w'_{\lambda_0}$ results in (6.5).

Assuming now that both (6.5) and (6.6) hold for some $w'_{\lambda_0}\in(0,\infty)$, we conclude from Theorem 6.1 that λ_0 solves Problem 3.2, as was to be proved.

Finally, let $\alpha<2$ and let λ_0 solve Problem 3.2. To establish (6.7), assume on the contrary that there exists $x_0\in F$ such that $x_0\notin S^{\lambda_0}_D$. Then one can choose $r>0$ so that

$$\overline{B}(x_0,r):=\{x\in\mathbb{R}^n:|x-x_0|\leqslant r\}\subset D\quad\text{and}\quad\overline{B}(x_0,r)\cap S^{\lambda_0}_D=\varnothing.$$

It follows that $(\xi-\lambda_0)\bigl(B(x_0,r)\cap F\bigr)>0$. Therefore, by (6.5),

$$U^{\lambda_0}_\alpha(x_1)=w'_{\lambda_0}+U^{\beta^\alpha_{D^c}\lambda_0}_\alpha(x_1)\quad\text{for some } x_1\in B(x_0,r)\cap F.\tag{13.3}$$

As $U^{\lambda_0}_\alpha(\cdot)$ is α-harmonic in $B(x_0,r)$ and continuous on $\overline{B}(x_0,r)$, while $w'_{\lambda_0}+U^{\beta^\alpha_{D^c}\lambda_0}_\alpha(\cdot)$ is α-superharmonic in $\mathbb{R}^n$, we conclude from (13.2) and (13.3) with the help of [16, Theorem 1.28] that

$$U^{\lambda_0}_\alpha(x)=w'_{\lambda_0}+U^{\beta^\alpha_{D^c}\lambda_0}_\alpha(x)\quad m_n\text{-a.e. in } \mathbb{R}^n.$$

This implies $w'_{\lambda_0}=0$, for $U^{\beta^\alpha_{D^c}\lambda_0}_\alpha(x)=U^{\lambda_0}_\alpha(x)$ holds n.e. in D^c, hence, also m_n-a.e. in D^c. A contradiction. $\square$

13.4 Proof of Theorem 6.2

This theorem is a very particular case of [27, Theorems 7.1, 7.2, 7.3] (see also [24] and Theorems 1, 2 and Proposition 1 therein). □

13.5 Proof of Corollary 6.3

Since (a) follows directly from Theorem 5.3, assume the conditions of assertion (b) to hold. Then the number w_f from Theorem 6.2 is strictly positive. Hence, by (6.9),

$$U_{\hat{g}_D^\alpha}^{\lambda+\chi}(x) \geqslant w_f > 0 \quad \text{n.e. in } F.$$

We can certainly assume that there is $y_0 \in \partial D \cap \mathrm{C}\ell_{\mathbb{R}^n}\breve{F}$, for if not, then (b) is obvious. For every $\varepsilon > 0$, it holds $C_\alpha\big(B(y_0,\varepsilon)\cap\breve{F}\big) > 0$, and therefore one can choose $x_\varepsilon \in B(y_0,\varepsilon)\cap\breve{F}$ so that $U_{\hat{g}_D^\alpha}^{\lambda+\chi}(x_\varepsilon) \geqslant w_f > 0$. This yields

$$\limsup_{x\to y_0,\ x\in D} U_{\hat{g}_D^\alpha}^{\lambda+\chi}(x) \geqslant w_f > 0.$$

Likewise as in Section 13.2, we can thus see that $U_{\hat{g}_D^\alpha}^{\lambda+\chi}$ is discontinuous on $\partial D \cap \mathrm{C}\ell_{\mathbb{R}^n}\breve{F}$, and Lusin's type theorem for the α-Riesz potentials establishes the corollary. □

13.6 Proof of Corollary 6.4

Let $w := w_f$ be the number from Theorem 6.2 for $f = 0$; then $w > 0$. Proof of the statement that $\lambda_F \in \mathscr{E}^+_{g_D^\alpha}(F,1)$ solves Problem 3.2 if and only if both (6.10) and (6.11) hold is based on Theorem 6.2 and runs in a way similar to that in the proof of Corollary 6.2. In particular, if λ_F solves Problem 3.2, then (compare with (13.2))

$$U_\alpha^{\lambda_F}(x) \leqslant w + U_\alpha^{\beta_{D^c}^\alpha \lambda_F}(x) \quad \text{for all } x \in \mathbb{R}^n. \tag{13.4}$$

Integrating (6.10) with respect to λ_F and taking (2.1) into account, we obtain (6.12).

To complete the proof, assume the minimizer λ_F to exist and consider $\nu \in \mathscr{E}^+_{g_D^\alpha}(F,1)$ with the property that $U_{g_D^\alpha}^\nu(x) = c$ n.e. in F, where $c \in \mathbb{R}$. Then, by (2.1) and (6.12),

$$\|\nu\|^2_{g_D^\alpha} = \big\langle U_{g_D^\alpha}^\nu, \nu \big\rangle = c \geqslant w_{g_D^\alpha}(F) = \|\lambda_F\|^2_{g_D^\alpha}$$

and, hence,

$$\begin{aligned}\|\nu-\lambda_F\|^2_{g_D^\alpha} &= \|\nu\|^2_{g_D^\alpha} + \|\lambda_F\|^2_{g_D^\alpha} - 2E_{g_D^\alpha}(\nu,\lambda_F)\\ &= c + w_{g_D^\alpha}(F) - 2\big\langle U_{g_D^\alpha}^\nu, \lambda_F\big\rangle = w_{g_D^\alpha}(F) - c \leqslant 0,\end{aligned}$$

which in view of the strict positive definiteness of g_D^α proves $\nu = \lambda_F$. □

13.7 Proof of Corollary 6.5

Having first assumed $\alpha < 2$, we start by showing that

$$U_{g_D^\alpha}^{\lambda_F}(x) < w \quad \text{for all } x \in D \setminus S_D^{\lambda_F}. \tag{13.5}$$

Suppose to the contrary that (13.5) is not satisfied for some $x_0 \in D \setminus S_D^{\lambda_F}$. Then $U_{g_D^\alpha}^{\lambda_F}(x_0) = w$ in accordance with (6.11), or equivalently

$$U_\alpha^{\lambda_F}(x_0) = w + U_\alpha^{\beta_{D^c}^\alpha \lambda_F}(x_0). \tag{13.6}$$

Choose $\varepsilon > 0$ so that $\overline{B}(x_0,\varepsilon) \subset D \setminus S_D^{\lambda_F}$. Since then $U_\alpha^{\lambda_F}(\cdot)$ is α-harmonic in $B(x_0,\varepsilon)$ and continuous on $\overline{B}(x_0,\varepsilon)$, while $w + U_\alpha^{\beta_{D^c}^\alpha \lambda_F}(\cdot)$ is α-superharmonic in $\mathbb{R}^n$, we conclude from (13.4) and (13.6) with the help of [16, Theorem 1.28] that

$$U_\alpha^{\lambda_F}(x) = w + U_\alpha^{\beta_{D^c}^\alpha \lambda_F}(x) \quad m_n\text{-a.e. in } \mathbb{R}^n.$$

As $U_\alpha^{\beta_{D^c}^\alpha \lambda_F}(x) = U_\alpha^{\lambda_F}(x)$ n.e. in D^c, we thus get $w = 0$. A contradiction.

We next proceed by proving the former identity in (6.14). Let, on the contrary, there exist $x_1 \in \breve{F}$ such that $x_1 \notin S_D^{\lambda_F}$, and let $V \subset D \setminus S_D^{\lambda_F}$ be an open neighborhood of x_1. Then, by (13.5), $U_{g_D^\alpha}^{\lambda_F}(x) < w$ for all $x \in V$. On the other hand, since $V \cap F$ has nonzero capacity, $U_{g_D^\alpha}^{\lambda_F}(x_2) = w$ for some $x_2 \in V$ by (6.10). The contradiction obtained shows that, indeed, $S_D^{\lambda_F} = \breve{F}$. Substituting this identity into (13.5) establishes (6.13) for $\alpha < 2$.

In the rest of the proof, $\alpha = 2$. To verify (6.13), assume, on the contrary, that it does not hold for some x_3 in the domain $D_0 := D \setminus \breve{F}$. According to (6.11), then $U_{g_D^2}^{\lambda_F}(x_3) = w$, which in view of the harmonicity of $U_{g_D^2}^{\lambda_F}$ in D_0 implies, by the maximum principle, that

$$U_{g_D^2}^{\lambda_F}(x) = w \quad \text{for all } x \in D_0.$$

Thus,

$$\lim_{x \to z,\ x \in D_0} U_{g_D^2}^{\lambda_F}(x) = w > 0 \quad \text{for all } z \in \partial D_0.$$

Since $C_\alpha(\partial D \cap \partial D_0) > 0$ in consequence of Corollary 6.3, (b), Lusin's type theorem for the Newtonian potentials shows that the preceding relation is impossible.

In view of (6.10), [16, Theorem 1.13] yields $\lambda_F\big|_{\operatorname{Int} F} = 0$, and so $S_D^{\lambda_F} \subset \partial_D \breve{F}$. Thus, if we prove the converse inclusion, the latter identity in (6.14) follows. Assume, on the contrary, it not to hold; then one can choose a point $y \in \partial_D \breve{F}$ and a neighborhood $V_1 \subset D$ of y so that $V_1 \cap S_D^{\lambda_F} = \varnothing$. As $V_1 \cap F$ has nonzero capacity, we see from (6.10) that there exists $y_1 \in V_1$ such that $U_{g_D^2}^{\lambda_F}(y_1) = w$. Taking (6.11) into account and applying the maximum principle to the harmonic in V_1 function $U_{g_D^2}^{\lambda_F}$, we thus have $U_{g_D^2}^{\lambda_F}(x) = w$ for all $x \in V_1$. This contradicts (6.13), because $V_1 \cap D_0 \neq \varnothing$. $\square$

13.8 Proof of Theorem 6.3

Since $U^{\xi}_{g^{\alpha}_{D}}(x)$ is continuous, so is $U^{\lambda_0}_{g^{\alpha}_{D}}(x)$. Indeed, $U^{\lambda_0}_{g^{\alpha}_{D}}(x) = U^{\xi}_{g^{\alpha}_{D}}(x) - U^{\xi-\lambda_0}_{g^{\alpha}_{D}}(x)$, which implies that $U^{\lambda_0}_{g^{\alpha}_{D}}(x)$ is both lower semicontinuous and upper semicontinuous. Next, since λ_0 solves the non-weighted Problem 3.2 with the constraint ξ, both (6.5) and (6.6) are fulfilled. As $U^{\lambda_0}_{g^{\alpha}_{D}}(\cdot)$ is continuous, equality in (6.5) holds in fact everywhere on $S^{\xi-\lambda_0}_{D}$. This allows us to rewrite (6.5) and (6.6) respectively as

$$U^{\xi-\lambda_0}_{g^{\alpha}_{D}}(x) - U^{\xi}_{g^{\alpha}_{D}}(x) = -w'_{\lambda_0} \quad \text{on } S^{\xi-\lambda_0}_{D},$$
$$U^{\xi-\lambda_0}_{g^{\alpha}_{D}}(x) - U^{\xi}_{g^{\alpha}_{D}}(x) \geqslant -w'_{\lambda_0} \quad \text{on } D,$$

which in the notations accepted in Section 6.3 are equivalent to (6.15) and (6.16). Since $\theta \in \mathscr{E}^{q\xi}_{g^{\alpha}_{D},f}(F,1) \subset \mathscr{E}^{+}_{g^{\alpha}_{D},f}(F,1)$, application of Theorem 6.2 completes the proof. $\square$

14 Examples

In this section, $n = 3$ and $x = (x_1, x_2, x_3)$ is a point in $\mathbb{R}^3$. In the following Examples 14.1–14.3, consider $0 < \alpha \leqslant 2$, $D := B(0,1)$ and $A_2 := D^c$; then A_2 is not α-thin at $\infty_{\mathbb{R}^n}$.

Example 14.1 Write $E := \{x \in B(0,1) : \ 0 \leqslant x_1 < 1,\ x_2 = x_3 = 0\}$. Since $C_\alpha(E) = 0$, Lemma 2.6 yields $C_{g^{\alpha}_{D}}(E) = 0$. Consequently, there exists a neighborhood F of E, closed in D, with $0 < C_{g^{\alpha}_{D}}(F) < \infty$. We can certainly assume that $\partial D \cap \mathrm{C}\ell_{\mathbb{R}^3} F = \{(1,0,0)\}$. Consider an external field f such that $f(x) < \infty$ n.e. in F unless Case II holds. Application of Lemma 4.6, Theorems 5.2 and 5.4 then shows that, in both Cases I and II, Problem 3.1 is (uniquely) solvable for every $\sigma \in \mathscr{A}(F) \cup \{\infty\}$.

Example 14.2 Let $F = D$. Define $\xi := m_3|_F$; then $\xi \in \mathfrak{C}_0(F)$ and has finite α-Riesz energy and thus it is admissible (see Definition 4.1). Consider an external field f such that Case I holds and $f(x) < \infty$ $m_3|_F$-a.e. Hence, by Lemma 4.6, assumptions (3.7) and (3.8) hold and so we can apply Theorems 5.1 and 5.4 to conclude that Problem 3.1 is solvable; that is, no short-circuit between the conductors F and D^c occurs, though they touch each other over the whole sphere $S(0,1)$.

Example 14.3 Let $F = S(x_0, 1/2) \cap D$, where $x_0 = (1/2, 0, 0)$. Consider an external field f such that Case I holds and $f(x) < \infty$ $m_2|_F$-a.e. We further assume $1 < \alpha \leq 2$. Define $\xi := m_2|_F$; then (since $\alpha > 1$) ξ has finite α-Riesz energy and so, as in the previous example, we can apply Theorems 5.1 and 5.4 to obtain the solvability of Problem 3.1.

Example 14.4 Let $\alpha = 2$, $D = \{x \in \mathbb{R}^3 : \ x_1 > 0\}$ and $F = \{x \in D : \ x_1 = 1\}$; then $A_2 = D^c$ is not 2-thin at $\infty_{\mathbb{R}^n}$, while $C_{g^{2}_{D}}(F) = \infty$, for $C_2(F) = \infty$ by [16, Chapter II, Section 3, n° 14]. Let, in addition, Case II with $\zeta \geqslant 0$ hold. Then, by Theorems 5.3 and 5.4, Problem 3.1 is nonsolvable for every $\sigma \in \mathscr{A}(F) \cup \{\infty\}$ such that $\sigma \geqslant \xi_0$, where $\xi_0 \in \mathfrak{C}(F) \setminus \mathfrak{C}_0(F)$ is properly chosen. Thus, for these σ, a short-circuit between F and D^c occurs at $\infty_{\mathbb{R}^n}$. To construct a constraint which would not allow such a short-circuit, consider K_k, $k \in \mathbb{N}$, where $K_k := \{x \in F : \ (k-1)^2 \leqslant x_2^2 + x_3^2 \leqslant k^2\}$ for all $k \geqslant 2$ and $K_1 := \{x \in F : \ x_2^2 + x_3^2 \leqslant 1\}$, and write

$$\xi := \sum_{k \in \mathbb{N}} \frac{m_2|_{K_k}}{k^3}.$$

Then ξ is bounded and admissible, and so we can again use Theorems 5.1 and 5.4 to see that Problem 3.1 for this constraint ξ is solvable.

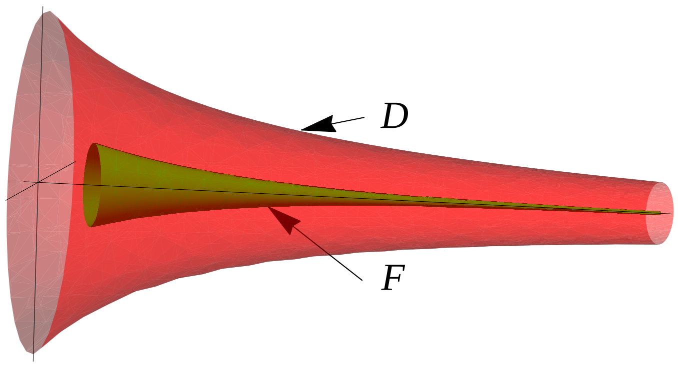

Fig. 14.1 The condenser for Example 14.5.

Example 14.5 Let $\alpha = 2$, Case II with $\zeta \geqslant 0$ hold, and let F and D be defined by

$$F := \{x \in \mathbb{R}^3 : \quad 2 \leqslant x_1 < \infty, \quad x_2^2 + x_3^2 = \rho_1^2(x_1), \quad \text{where } \rho_1(x_1) = \exp(-x_1)\},$$
$$D := \{x \in \mathbb{R}^3 : \quad 1 < x_1 < \infty, \quad x_2^2 + x_3^2 < \rho_2^2(x_1), \quad \text{where } \rho_2(x_1) = x_1^{-1}\}.$$

Then $A_2 = D^c$ is not 2-thin at $\infty_{\mathbb{R}^n}$, while $C_{g_D^2}(F) = \infty$, for $C_2(F) = \infty$ by [21]. Hence, by Theorems 5.3 and 5.4, Problem 3.1 is nonsolvable for every $\sigma \in \mathscr{A}(F) \cup \{\infty\}$ such that $\sigma \geqslant \xi_0$, where $\xi_0 \in \mathfrak{C}(F) \setminus \mathfrak{C}_0(F)$ is properly chosen. However, Problem 3.1 with $\xi := m_2|_F$ is already solvable, which is seen from Theorems 5.1 and 5.4.

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