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The Covering Radius of Randomly Distributed Points on a Manifold

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We derive fundamental asymptotic results for the expected covering radius $\rho(X_N)$ for N points that are randomly and independently distributed with respect to surface measure on a sphere as well as on a class of smooth manifolds. For the unit sphere $\mathbb{S}^d \subset \mathbb{R}^{d+1}$, we obtain the precise asymptotic that $\mathbb{E}\rho(X_N)[N/\log N]^{1/d}$ has limit $[(d+1)\upsilon_{d+1}/\upsilon_d]^{1/d}$ as $N \to \infty$, where υ_d is the volume of the d-dimensional unit ball. This proves a recent conjecture of Brauchart et al. as well as extends a result previously known only for the circle. Likewise, we obtain precise asymptotics for the expected covering radius of N points randomly distributed on a d-dimensional ball, a d-dimensional cube, as well as on a three-dimensional polyhedron (where the points are independently distributed with respect to volume measure). More generally, we deduce upper and lower bounds for the expected covering radius of N points that are randomly and independently distributed on a compact metric measure space, provided the measure satisfies certain regularity assumptions.

1 Introduction and Notation

The purpose of this paper is to obtain asymptotic results for the expected value of the covering radius of N points $X_N = \{x_1, x_2, \dots, x_N\}$ that are randomly and independently distributed with respect to a given measure μ over a metric space $(\mathcal{X}, m)$. By the *covering*

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radius $\rho(X_N, \mathcal{X})$ (also known as the *mesh norm* or *fill radius*) of the set X_N with respect to $\mathcal{X}$, we mean the radius of the largest neighborhood centered at a point of $\mathcal{X}$ that contains no points of X_N; more precisely,

$$\rho\left(X_N, \mathcal{X}\right) := \sup_{y\in\mathcal{X}} \inf_{j} m\left(y, x_j\right).$$

Our focus is on the limiting behavior as $N \to \infty$ of the expected value $\mathbb{E}\rho(X_N, \mathcal{X})$.

The covering radius of a discrete point set is an important characteristic that arises in a variety of contexts. For example, it plays an essential role in determining the accuracy of various numerical approximation schemes such as those involving radial basis techniques [8, 12]. Another area where the covering radius arises is in "1-bit sensing", that is, the problem of approximating an unknown vector (signal) $x \in K$ from knowledge of m numbers $\operatorname{sign}\langle x, \theta_j\rangle$, $j = 1, \ldots, m$, where the vectors θ_j are selected independently and randomly on a sphere; see discussion after Corollary 2.9 for details.

With regard to asymptotics for the expected value of the covering radius, of particular interest is the case where $\mathcal{X}$ is the unit sphere $\mathbb{S}^d$ in $\mathbb{R}^{d+1}$, and the metric is Euclidean distance in $\mathbb{R}^{d+1}$. Bourgain et al. [2] study local statistics of certain spherical point configurations derived from normalized sums of squares of integers. Their investigation focuses on whether such configurations exhibit features of randomness, and for this purpose, they study various local statistics, including the covering radius of random points on $\mathbb{S}^d$. They prove that this radius is bounded from above by $N^{-1/d+o(1)}$ as $N \to \infty$.

For $d = 1$, that is, the unit circle, it is shown in [6], by using order statistics, that for N points independently and randomly distributed with respect to arclength on the circle,

$$\lim_{N\to\infty} \mathbb{E}\rho\left(X_N, \mathbb{S}^1\right)\left(\frac{N}{\log N}\right) = \pi.$$

Up to now, there has been no extension of this result to higher-dimensional spheres where the order statistics approach is more elusive. Based on a heuristic argument and numerical experiments, Brauchart et al. [4] have conjectured that the appropriate extension of the circle case is the following

$$\lim_{N\to\infty} \mathbb{E}\rho\left(X_N, \mathbb{S}^d\right) \cdot \left(\frac{N}{\log N}\right)^{1/d} = \left(\frac{(d+1)\,\upsilon_{d+1}}{\upsilon_d}\right)^{1/d} = \left(2\sqrt{\pi}\frac{\Gamma\left(\frac{d+2}{2}\right)}{\Gamma\left(\frac{d+1}{2}\right)}\right)^{1/d}, \tag{1.1}$$

where $\upsilon_d := \frac{\pi^{d/2}}{\Gamma(1+d/2)}$ is the volume of a d-dimensional unit ball in $\mathbb{R}^d$, and the points of X_N are randomly and independently distributed with respect to surface measure on $\mathbb{S}^d$ (more precisely, d-dimensional Hausdorff measure $\mathcal{H}_d$). Their conjecture is also consistent with a result of Maehara [10], who obtained probabilistic estimates for the size of random caps that cover the sphere $\mathbb{S}^2$. He showed that with asymptotic probability one, random caps with radii that are a constant factor larger than the expected radii will cover the sphere, whereas this asymptotic probability becomes zero when the random caps all have radii that are a factor smaller. However, his results fall short of providing a sharp asymptotic for the expected covering radius (in addition, his methods do not readily generalize to other smooth manifolds). As discussed in Section 3, our results for the sphere cannot be directly derived from Maehara's; however, his results are a direct consequence of our Corollary 3.3.

The main goal of this article is to provide a proof of (1.1) and its various generalizations.

We remark that for any compact metric space $(\mathcal{X}, m)$ with $\mathcal{X}$ having finite d-dimensional Hausdorff measure, there exists a positive constant C such that for any $Y_N = \{y_1, \ldots, y_N\} \subset \mathcal{X}$, there holds

$$\rho_N := \rho\left(Y_N, \mathcal{X}\right) \geq \frac{C}{N^{1/d}}, \quad N \geq 1. \tag{1.2}$$

Indeed, a lemma of Frostman [11, Theorem 8.17] implies the existence of a finite positive measure μ on $\mathcal{X}$ for which $\mu(B(x,r)) \leq (2r)^d$ for all $x \in \mathcal{X}$ and all $0 < r \leq \operatorname{diam}(\mathcal{X})$, where $B(x,r)$ denotes the closed ball centered at x having radius r. Consequently,

$$0 < \mu\left(\mathcal{X}\right) \leq \sum_{i=1}^{N} \mu\left(B\left(y_i, \rho_N\right)\right) \leq N\left(2\rho_N\right)^d$$

that verifies (1.2). Thus, as also remarked in [2] and made more explicit by (1.1), randomly distributed points have relatively good covering properties, differing from the optimal order by a multiplicative factor of $(\log N)^{1/d}$.

The outline of this paper is as follows. In Section 2, we state our probabilistic and expected covering radius estimates for general compact metric spaces, where the points are randomly distributed with respect to a measure satisfying certain regularity conditions. Results for compact subsets of Euclidean space are given in Section 3, including sharp asymptotic results for randomly distributed points with respect to Hausdorff measure on rectifiable curves, smooth surfaces, bodies with smooth boundaries, d-dimensional cubes, and three-dimensional polyhedra. The proofs of our stated results are provided in Section 5 utilizing properties established in Section 4 for a commonly arising probability function.

We conclude this section with a listing of some notational conventions and terminology that is utilized throughout the paper.

(1) We denote by $B(x,r)$ a closed ball in the metric space $(\mathcal{X},m)$; more precisely, $B(x,r):=\{y\in\mathcal{X}: m(y,x)\leqslant r\}$. For d-dimensional balls in Euclidean space, we write $B_d(x,r)$.

(2) For a positive finite Borel measure μ supported on a set $\mathcal{X}$, we say that a point x is *randomly distributed over $\mathcal{X}$ with respect to μ*, if it is distributed with respect to the probability measure $\mu/\mu(\mathcal{X})$; that is, for any Borel set K, it holds that $\mathbb{P}[x\in K]=\mu(K)/\mu(\mathcal{X})$.

(3) For a positive integer $s\leqslant d$, we denote by $\mathcal{H}_s$ the s-dimensional Hausdorff measure on the Euclidean space $\mathbb{R}^d$ with the Euclidean metric, normalized by $\mathcal{H}_s([0,1]^s)=1$. Thus, $\mathcal{H}_s(E)=\frac{\pi^{s/2}}{2^s\Gamma(1+s/2)}\mathcal{H}^s(E)$, where $\mathcal{H}^s$ is the Hausdorff measure defined in [7].

(4) If K is a subset of the Euclidean space $\mathbb{R}^d$, we always equip it with the Euclidean metric $m(x,y)=|x-y|$.

(5) The symbols $c_1,c_2,\ldots$, and $C_1,C_2,\ldots$ shall denote positive constants that may differ from one inequality to another. These constants never depend on N.

2 Main Theorems for Metric Spaces

Throughout this section, we assume that $(\mathcal{X},m)$ is a metric space, μ is a finite positive Borel measure supported on $\mathcal{X}$, and $X_N=\{x_1,\ldots,x_N\}$ is a set of N points, independently and randomly distributed over $\mathcal{X}$ with respect to μ. Our theorems provide estimates for the probability and expected values of the covering radius $\rho(X_N,\mathcal{X})$ when the measure μ satisfies certain regularity conditions described by a function Φ.

Theorem 2.1. Suppose Φ is a continuous positive strictly increasing function on $(0,\infty)$ satisfying $\Phi(r)\to 0$ as $r\to 0^+$. If there exists a positive number r_0 such that $\mu(B(x,r))\geqslant\Phi(r)$ holds for all $x\in\mathcal{X}$ and every $r<r_0$, then there exist positive constants c_1, c_2, c_3, and α_0 such that for any $\alpha>\alpha_0$, we have

$$\mathbb{P}\left[\rho\left(X_N,\mathcal{X}\right)\geqslant c_1\Phi^{-1}\left(\frac{\alpha\log N}{N}\right)\right]\leqslant c_2N^{1-c_3\alpha}. \tag{2.1}$$

If, in addition, Φ satisfies $\Phi(r)\leqslant r^\sigma$ for all small r and some positive number σ, then there exist positive constants c_1,c_2 such that

$$\mathbb{E}\rho\left(X_N,\mathcal{X}\right)\leqslant c_1\Phi^{-1}\left(c_2\frac{\log N}{N}\right). \tag{2.2}$$

□

A lower bound for the expected covering radius is given in our next result.

Theorem 2.2. Let Φ be a continuous positive strictly increasing function on $(0, \infty)$ satisfying $\Phi(r) \to 0$ as $r \to 0^+$ and the strict doubling property; that is, for some constants $C_1, C_2 > 1$ and any small r, it holds that $C_1\Phi(r) \leqslant \Phi(2r) \leqslant C_2\Phi(r)$. Suppose further that there exists a subset $\mathcal{X}_1 \subset \mathcal{X}$ with the following properties:

(i) $\mu(\mathcal{X}_1) > 0$;

(ii) there exist positive numbers r_0 and c such that for any $x \in \mathcal{X}_1$ and every $r < r_0$ the function $t \mapsto \mu(B(x, t))$ is continuous at $t = r$ and the regularity condition $c\Phi(r) \leqslant \mu(B(x, r)) \leqslant \Phi(r)$ holds.

Then there exist positive constants c_1 , c_2, and c_3 such that

$$\mathbb{P}\left[\rho\left(X_N, \mathcal{X}\right) \geqslant c_1\Phi^{-1}\left(\frac{c_2 \log N - c_3 \log\log N}{N}\right)\right] = 1 - o(1), \quad N \to \infty. \tag{2.3}$$

Consequently, there exist positive constants c_1 and c_2 such that

$$\mathbb{E}\rho\left(X_N, \mathcal{X}\right) \geqslant c_1\Phi^{-1}\left(c_2\frac{\log N}{N}\right). \tag{2.4}$$

□

Combining Theorems 2.1 and 2.2, we deduce the following.

Corollary 2.3. Assume the function Φ is continuous positive, strictly increasing, strictly doubling, and that there exist positive numbers r_0 and σ such that $\Phi(r) \leqslant r^\sigma$ for every $r < r_0$. If, for some positive constants c, C, any $x \in \mathcal{X}$ and every $r < r_0$, the function $t \mapsto \mu(B(x, t))$ is continuous at $t = r$ and

$$c\Phi\left(r\right) \leqslant \mu\left(B\left(x, r\right)\right) \leqslant C\Phi\left(r\right), \tag{2.5}$$

then there exist positive constants c_1, c_2, c_3, c_4 such that for any $\varepsilon > 0$, there is a number $N(\varepsilon)$ such that for any $N > N(\varepsilon)$, we have

$$\mathbb{P}\left[c_1\Phi^{-1}\left(c_2\frac{\log N}{N}\right) \leqslant \rho\left(X_N, \mathcal{X}\right) \leqslant c_3\Phi^{-1}\left(c_4\frac{\log N}{N}\right)\right] > 1 - \varepsilon. \tag{2.6}$$

Moreover, there exist positive constants C_1, C_2, C_3, C_4 such that

$$C_1\Phi^{-1}\left(C_2\frac{\log N}{N}\right) \leqslant \mathbb{E}\rho\left(X_N, \mathcal{X}\right) \leqslant C_3\Phi^{-1}\left(C_4\frac{\log N}{N}\right). \tag{2.7}$$

□

For recent estimates similar to (2.6) and (2.7) for the spherical cap discrepancy of random points on the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$, see [1, Theorems 9 and 10].

An important class of sets in $\mathbb{R}^d$ to which Corollary 2.3 applies is described in Definition 2.4.

Definition 2.4. We call a set $\mathcal{X} \subset \mathbb{R}^d$ *s-regular* if condition (2.5) holds for $\mu = \mathcal{H}_s$ and $\Phi(r) = r^s$; that is, for some positive constants r_0, c, and C there holds

$$cr^s \leqslant \mathcal{H}_s\left(B_d(x,r) \cap \mathcal{X}\right) \leqslant Cr^s \quad \text{for any } x \in \mathcal{X} \text{ and every } r < r_0. \tag{2.8}$$

□

Remark 2.5. Examples of sets in Euclidean space for which Corollary 2.3 holds include the unit cube $[0,1]^d$, a rectifiable curve $\Gamma \subset \mathbb{R}^d$, the unit sphere $\mathbb{S}^{d-1} \subset \mathbb{R}^d$, or any s-regular set $\mathcal{X} \subset \mathbb{R}^d$. Furthermore, the results of Corollary 2.3 hold not only for $\Phi(r) = r^s$, but also for more general regularity functions, such as $\Phi(r) = r^\alpha \log^\beta(1/r)$, with $\alpha > 0$ and $\beta \geqslant 0$.

In particular, Corollary 2.3 applies for the "middle 1/3" Cantor set $\mathcal{C}$ in [0, 1] with $\mathrm{d}\mu = \mathbb{1}_{\mathcal{C}} \mathrm{d}\mathcal{H}_{\log 2/\log 3}$. We remark that for μ-a.e. point $x \in \mathcal{C}$, we have

$$\liminf_{r\to 0^+} \frac{\mu\left(B_1(x,r) \cap \mathcal{C}\right)}{r^{\log 2/\log 3}} \neq \limsup_{r\to 0^+} \frac{\mu\left(B_1(x,r) \cap \mathcal{C}\right)}{r^{\log 2/\log 3}};$$

that is, at μ-a.e. point x of $\mathcal{C}$ the density of μ at x does not exist, which essentially precludes obtaining a sharp asymptotic for $\mathbb{E}\rho(X_N, \mathcal{C})$. However, Corollary 2.3 provides the two-sided estimate

$$c_1 \left(\frac{\log N}{N}\right)^{\log 3/\log 2} \leqslant \mathbb{E}\rho\left(X_N, \mathcal{C}\right) \leqslant c_2 \left(\frac{\log N}{N}\right)^{\log 3/\log 2}.$$

□

Remark 2.6. The condition in Theorem 2.1 that $\mu(B(x,r)) \geqslant \Phi(r)$ for every $x \in \mathcal{X}$ is essential. Indeed, if we consider the set $\mathcal{X} = [0,1] \cup \{2\}$ with μ Lebesgue measure, then $\mu(B_1(x,r)) \geqslant r$ for all $r < 1$ and $x \in \mathcal{X} \setminus \{2\}$. However, we have $\mathbb{P}[\rho(X_N, \mathcal{X}) \geqslant 1] = 1$, and so $\mathbb{E}\rho(X_N, \mathcal{X}) \geqslant 1$. The reason that inequality (2.2) fails in this case is that for the point $x = 2$ we have $\mu(B_1(x,r)) = 0$ for small values of r. However, Theorem 2.1 does apply if $\mu = m_{[0,1]} + \alpha\delta_2$, where $m_{[0,1]}$ is Lebesgue measure on $[0,1]$, δ_2 is the unit point mass at $x = 2$, and $\alpha > 0$. In this case, we obtain

$$\mathbb{E}\rho\left(X_N, \mathcal{X}\right) \leqslant C(\alpha) \cdot \frac{\log N}{N}.$$

In fact, repeating the proofs from Sections 5.5 and 5.6 (with $K_1 = [0,1]$), we obtain

$$\lim_{N\to\infty} \mathbb{E}\rho\left(X_N, \mathcal{X}\right) \cdot \frac{N}{\log N} = \frac{1+\alpha}{2} \quad \text{for any } \alpha > 0.$$

□

The above shown results have immediate consequences for ε-nets. Since different definitions of an "ε-net" occur in the literature, the terminologies that we use are made precise in what follows.

Definition 2.7. A subset A of a metric space $(\mathcal{X}, m)$ is called an *ε-net* (or *ε-covering*) if, for any point $y \in \mathcal{X}$, there exists a point $x \in A$ such that $m(x, y) \leqslant \varepsilon$. Equivalently, A is an ε-net if $\rho(A, \mathcal{X}) \leqslant \varepsilon$. □

Definition 2.8. A subset A of a metric space $(\mathcal{X}, m)$ with a positive Borel measure μ is called a *measure ε-net* if any ball $B(y, r)$ with $\mu(B(y, r)) \geqslant \varepsilon$ intersects A. □

We remind the reader that on $\mathbb{S}^d$ with μ surface area measure $\mathcal{H}_d$, the minimal ε-net has cardinality $c\varepsilon^{-d}$ (for the proof see, for example, [15, Lemma 5.2]), whereas the minimal measure ε-net has cardinality $c\varepsilon^{-1}$.

Corollary 2.9. If Φ and μ are as in the first part of Theorem 2.1, then there exists a positive constant c_1 such that for any positive number α there is a positive constant C_α for which

$$\mathbb{P}\left[X_N \text{ is an } \varepsilon\text{-net}\right] \geqslant 1 - N^{-\alpha}, \quad \text{for } \varepsilon = c_1 \Phi^{-1}\left(C_\alpha \frac{\log N}{N}\right).$$

Furthermore, if the function Φ is doubling, and the measure μ satisfies the condition (2.5), then for any positive number α, there exists a positive constant C_α such that

$$\mathbb{P}\left[X_N \text{ is a measure } \varepsilon\text{-net}\right] \geqslant 1 - N^{-\alpha}, \quad \text{for } \varepsilon = C_\alpha \frac{\log N}{N}.$$ □

By way of illustration, suppose, for the sake of simplicity, that $\Phi(r) = Cr^d$ for some positive constant C and $\varepsilon = [(\log N)/N]^{1/d}$, which implies that N is of the order $\varepsilon^{-d} \log(1/\varepsilon)$. Then, from the first part of Corollary 2.9, if we take $C_1 \varepsilon^{-d} \log(1/\varepsilon)$ random points, we get an ε-net (ε-covering) with high probability.

The cardinality of an ε-covering of a set $K \subset \mathbb{S}^d$ plays an important role in "1-bit compressed sensing". The estimates for the number m of random vectors $\{\theta_j\}_{j=1}^m$, essential to approximate an unknown signal $x \in K$ from knowledge of m "bits" $\operatorname{sign}\langle x, \theta_j \rangle$, involve finding an ε-covering of the set K with $\log(N(K, \varepsilon)) \leqslant C\varepsilon^{-2} w(K)$, where $N(K, \varepsilon)$ is the cardinality of the covering, and w is the so-called "mean width" of K. As can be seen from our results, for many sets K a random set of $C\varepsilon^{-d} \log(1/\varepsilon)$ points satisfies this condition with high probability. For further discussion, see [13, 14].

3 Expected Covering Radii for Subsets of Euclidean Space

In some cases, we are able to "glue" upper and lower estimates together to obtain sharp asymptotic results. For this purpose, we state the following definitions.

Definition 3.1. Let s be a positive integer, $s \leqslant d$. Suppose K is a compact s-dimensional set in $\mathbb{R}^d$ with the Euclidean metric.

We call K an *asymptotically flat s-regular* set if for any $x \in K$ it holds that

$$r^{-s}\mathcal{H}_s\left(B_d\left(x, r\right) \cap K\right) \rightrightarrows \upsilon_s \quad \text{as } r \to 0^+, \tag{3.1}$$

where the convergence is uniform in x, and υ_s is the volume of the s-dimensional unit ball $B_s(0, 1)$.

We call K a *quasi-nice s-regular* set if

(i) K is countably s-rectifiable; that is, K is of the form $\bigcup_{j=1}^{\infty} f_j(E_j) \cup G$, where $\mathcal{H}_s(G) = 0$ and where each f_j is a Lipschitz function from a bounded subset E_j of $\mathbb{R}^s$ to $\mathbb{R}^d$;

(ii) There exist positive numbers c, C, r_0 such that for any $x \in K$ and any $r < r_0$, the s-regularity condition holds: $cr^s \leqslant \mathcal{H}_s(B_d(x, r) \cap K) \leqslant Cr^s$;

(iii) There is a finite set $T \subset K$ such that for any $r < r_0$ and $y \in K \setminus \bigcup_{x_t \in T} B_d(x_t, r)$, it holds that $\mathcal{H}_s(B_d(y, r) \cap K) \geqslant \upsilon_s r^s$. □

We remark that the appearance of the constant υ_s in the above definitions is quite natural. Indeed, if K is a countably s-rectifiable compact set and $0 < \mathcal{H}_s(K) < \infty$, then for $\mathcal{H}_s$-almost every point $x \in K$, the following holds: $r^{-s}\mathcal{H}_s(B_d(x, r) \cap K) \to \upsilon_s$ as $r \to 0^+$. For the details, see the Theorem 17.6 in [11] or Theorem 3.33 in [7]. Thus, if any uniform limit in (3.1) exists, then it must equal υ_s.

For asymptotically flat s-regular and quasi-nice s-regular sets, we deduce the following precise asymptotics for the expected covering radius as well as its moments.

Theorem 3.2. Suppose $K \subset \mathbb{R}^d$ is an asymptotically flat s-regular or a quasi-nice s-regular set for integer $s \leqslant d$. Then for $X_N = \{x_1, \ldots, x_N\}$, a set of N independently and randomly distributed points over K with respect to the measure μ given by $\mathrm{d}\mu := \mathbb{1}_K \cdot \mathrm{d}\mathcal{H}_s/\mathcal{H}_s(K)$, and any $p \geq 1$,

$$\lim_{N\to\infty} \mathbb{E}[\rho\left(X_N, K\right)^p] \cdot \left(\frac{N}{\log N}\right)^{p/s} = \left(\frac{\mathcal{H}_s\left(K\right)}{\upsilon_s}\right)^{p/s}. \tag{3.2}$$

□

Important examples of asymptotically flat s-regular sets are given in the following result, which includes the verification of the conjecture of Brauchart et al. [3] for the expected covering radius of randomly distributed points on the unit sphere.

Corollary 3.3. Suppose K is a closed $C^{(1,1)}$ s-dimensional embedded submanifold of $\mathbb{R}^d$; that is, $0 < \mathcal{H}_s(K) < \infty$ and, for any embedding φ, all its first partial derivatives exist and are uniformly Lipschitz. Then K is an asymptotically flat s-regular manifold, and thus for N points independently and randomly distributed over K with respect to $\mathrm{d}\mu = \mathbb{1}_K \cdot \mathrm{d}\mathcal{H}_s/\mathcal{H}_s(K)$, equation (3.2) holds.

In particular, if $K = \mathbb{S}^d$ is a unit sphere in $\mathbb{R}^{d+1}$ and $p \geq 1$, then

$$\lim_{N\to\infty} \mathbb{E}[\rho\left(X_N, \mathbb{S}^d\right)^p] \cdot \left(\frac{N}{\log N}\right)^{p/d} = \left(\frac{(d+1)\,\upsilon_{d+1}}{\upsilon_d}\right)^{p/d} = \left(2\sqrt{\pi}\frac{\Gamma\left(\frac{d+2}{2}\right)}{\Gamma\left(\frac{d+1}{2}\right)}\right)^{p/d}. \tag{3.3}$$

Thus relation (1.1) holds. □

As a consequence of the corollary, we shall deduce in Section 5 the result of Maehara mentioned in the Introduction.

Corollary 3.4 (Maehara [10]). Suppose $X_N = \{x_1, \ldots, x_N\}$ is a set of N points, independently and randomly distributed over the unit sphere $\mathbb{S}^d$ with respect to the measure μ given by $\mathrm{d}\mu = \mathbb{1}_{\mathbb{S}^d} \cdot \mathrm{d}\mathcal{H}_d/\mathcal{H}_s(\mathbb{S}^d)$ and set

$$Z_N := \rho\left(X_N, \mathbb{S}^d\right) \cdot \left(\frac{\upsilon_d}{(d+1)\,\upsilon_{d+1}} \cdot \frac{N}{\log N}\right)^{1/d}.$$

Then Z_N converges in probability to 1 as $N \to \infty$; that is, for each $\epsilon > 0$,

$$\lim_{N\to\infty} \mathbb{P}\left(|Z_N - 1| \geq \epsilon\right) = 0. \tag{3.4}$$

□

Remark 3.5. We remark that our results for $\mathbb{S}^d$ do not directly (i.e., by means of basic measure theory) follow from (3.4). Maehara's result implies that the bounded sequence

$$p_N(t) := \mathbb{P}\left(Z_N \geqslant t\right) \to \mathbb{1}_{[0,1]}(t) \quad \text{for a.e. } t > 0;$$

however, since the range of t is $[0, \infty)$, the constant function 1 is not integrable, and we cannot apply the Lebesgue dominated convergence theorem to get $\mathbb{E}Z_N = \int_0^\infty p_N(t)\,\mathrm{d}t \to 1$. □

Corollary 3.6 gives an example of a quasi-nice 1-regular set.

Corollary 3.6. Suppose γ is a rectifiable curve in $\mathbb{R}^d$ (i.e., $0<\mathcal{H}_1(\gamma)<\infty$ and γ is a continuous injection of a closed interval of $\mathbb{R}$). If X_N denotes a set of N points independently and randomly distributed over γ with respect to $\mathrm{d}\mu := \mathbb{1}_\gamma \cdot \mathrm{d}\mathcal{H}_1/\mathcal{H}_1(\gamma)$, then γ is a quasi-nice 1-regular set, and for any $p\geqslant 1$

$$\lim_{N\to\infty} \mathbb{E}[\rho\,(X_N,\gamma)^p]\cdot\left(\frac{N}{\log N}\right)^p = \left(\frac{\mathcal{H}_1\,(\gamma)}{2}\right)^p. \tag{3.5}$$

□

Next we deal with the following problem: suppose $A\subset\mathbb{R}^d$ is a d-dimensional set, but the condition

$$\mathcal{H}_d\,(A\cap B_d\,(x,r))\geqslant \upsilon_d r^d$$

fails for a certain subset of points $x\in A$ and the limit (3.1) in Definition 3.1 is not uniform. Such situations arise for sets with boundary, which include the unit ball $B_d(0,1)$ and the unit cube $[0,1]^d$. The case of the ball is included in Theorem 3.7, whereas the case of the cube is studied in Theorem 3.9.

Theorem 3.7. Let $d\geqslant 2$ and $K\subset\mathbb{R}^d$ a set that satisfies the following conditions.

(i) K is compact and $0<\mathcal{H}_d(K)<\infty$;
(ii) $K=\operatorname{clos}(K_0)$, where K_0 is an open set in $\mathbb{R}^d$ with $\partial K_0=\partial K$;
(iii) The boundary ∂K of K is a C^2 smooth $(d-1)$-dimensional embedded submanifold of $\mathbb{R}^d$.

Let $X_N=\{x_1,\dots,x_N\}$ be a set of N points, independently and randomly distributed over K with respect to $\mathrm{d}\mu=\mathbb{1}_K\cdot \mathrm{d}\mathcal{H}_d/\mathcal{H}_d(K)$. Then for any $p\geqslant 1$

$$\lim_{N\to\infty} \mathbb{E}[\rho\,(X_N,K)^p]\cdot\left(\frac{N}{\log N}\right)^{p/d} = \left(\frac{2\,(d-1)}{d}\cdot\frac{\mathcal{H}_d\,(K)}{\upsilon_d}\right)^{p/d}. \tag{3.6}$$

In particular, for the unit ball,

$$\lim_{N\to\infty} \mathbb{E}[\rho\,(X_N,B_d\,(0,1))^p]\cdot\left(\frac{N}{\log N}\right)^{p/d} = \left(\frac{2\,(d-1)}{d}\right)^{p/d}. \tag{3.7}$$

□

Remark 3.8. We see that in the case $d=2$ we have $2(d-1)/d=1$, and so the constant on the right-hand side of (3.6) coincides with the constant for smooth closed manifolds, see (3.2). However, when $d>2$, we have $2(d-1)/d>1$; thus this constant becomes bigger than for smooth closed manifolds. □

Propositions 3.9 and 3.10 deal with cases when the boundary of the set is not smooth. For the sake of simplicity, we formulate them for the unit cube $[0,1]^d$ and a polyhedron in $\mathbb{R}^3$. However, the proof can be applied to other examples, such as cylinders.

Proposition 3.9. Suppose $d \geqslant 2$ and $[0,1]^d$ is the d-dimensional unit cube. Let $d\mu = \mathbb{1}_{[0,1]^d} \cdot d\mathcal{H}_d$. If $X_N = \{x_1, \ldots, x_N\}$ is a set of N points, independently and randomly distributed over $[0,1]^d$ with respect to μ, then for any $p \geqslant 1$

$$\lim_{N\to\infty} \mathbb{E}[\rho\left(X_N, [0,1]^d\right)^p] \cdot \left(\frac{N}{\log N}\right)^{p/d} = \left(\frac{2^{d-1}}{d\upsilon_d}\right)^{p/d}. \tag{3.8}$$

□

Proposition 3.10. Suppose P is a polyhedron in $\mathbb{R}^3$ of volume $V(P)$. Let $X_N = \{x_1, \ldots, x_N\}$ be a set of N points, independently and randomly distributed over P with respect to $d\mu = \mathbb{1}_P \cdot d\mathcal{H}_3 / V(P)$. If θ is the smallest angle at which two faces of P intersect, then for any $p \geqslant 1$

$$\lim_{N\to\infty} \mathbb{E}[\rho\left(X_N, P\right)^p] \cdot \left(\frac{N}{\log N}\right)^{p/3} = \left(\frac{2\pi V(P)}{3\theta\upsilon_3}\right)^{p/3} = \left(\frac{V(P)}{2\theta}\right)^{p/3}, \quad \text{if } \theta \leqslant \frac{\pi}{2}; \tag{3.9}$$

$$\lim_{N\to\infty} \mathbb{E}[\rho\left(X_N, P\right)^p] \cdot \left(\frac{N}{\log N}\right)^{p/3} = \left(\frac{V(P)}{\pi}\right)^{p/3}, \quad \text{if } \theta \geqslant \frac{\pi}{2}. \tag{3.10}$$

□

In the theorems up to now, we dealt with measures μ on sets $\mathcal{X}$ satisfying for all $x \in \mathcal{X}$ the condition $cr^s \leqslant \mu(B(x,r) \cap \mathcal{X}) \leqslant Cr^s$ (i.e., the regularity function Φ was the same for all points of $\mathcal{X}$); only the values of best constants c, C differed for points x deep inside $\mathcal{X}$ from those near the boundary. We now give an example of a measure for which the regularity function parameter s depends upon the distance to the boundary.

Proposition 3.11. Consider the interval $[-1,1]$ and the measure μ given by $d\mu = \frac{dx}{\pi\sqrt{1-x^2}}$. Let $X_N = \{x_1, \ldots, x_N\}$ be a set of N points, independently and randomly distributed over $[-1,1]$ with respect to μ. Define

$$\hat{\rho}\left(X_N, [0,1]\right) := \sup_{y \in [1-\frac{1}{N^a}, 1]} \inf_j |y - x_j|, \quad \tilde{\rho}\left(X_N, [0,1]\right) := \sup_{y \in [-1+\frac{1}{N^a}, 1-\frac{1}{N^a}]} \inf_j |y - x_j|.$$

(i) If $a = 2$, then there exist positive constants c_1 and c_2 such that

$$\frac{c_1}{N^2} \leqslant \mathbb{E}\hat{\rho}\left(X_N, [0,1]\right) \leqslant \frac{c_2}{N^2}. \tag{3.11}$$

(ii) If $0 < a < 2$, then there exist positive constants c_1 and c_2 such that

$$\frac{c_1 \log N}{N^{1+\frac{a}{2}}} \leqslant \mathbb{E}\hat{\rho}\left(X_N, [0,1]\right) \leqslant \frac{c_2 \log N}{N^{1+\frac{a}{2}}}. \tag{3.12}$$

(iii) For any $a > 0$, there exist positive constants c_1 and c_2 such that

$$\frac{c_1 \log N}{N} \leqslant \mathbb{E}\tilde{\rho}\left(X_N, [0,1]\right) \leqslant \frac{c_2 \log N}{N}. \tag{3.13}$$

□

Observe that if we stay away from the endpoints ± 1, the measure μ acts as the Lebesgue measure, and thus the order of the expectation of the covering radius is $(\log N)/N$. However, when we are close to the points ± 1 (where "close" depends on N), the measure μ acts somewhat like the Hausdorff measure $\mathcal{H}_{1/2}$, and we get a different order for the covering radius.

4 An Auxiliary Function

The proofs of the results stated in Sections 2 and 3 rely heavily on the properties of the following function. For three positive numbers N, n, m, with m and N being integers and $m \leqslant n \leqslant N$, set

$$f(N,n,m) := \sum_{k=1}^{m} (-1)^{k+1} \binom{m}{k} \left(1 - \frac{k}{n}\right)^N. \tag{4.1}$$

The useful fact about the function $f(N,n,m)$ is the following.

Lemma 4.1. Suppose $X_N = \{x_1, \ldots, x_N\}$ is a set of N points independently and randomly distributed on a set $\mathcal{X}$ with respect to a Borel probability measure μ. Let $B_1, \ldots, B_m$ be disjoint subsets of $\mathcal{X}$ each of μ-measure $1/n$. Then

$$\mathbb{P}\left(\exists k\colon B_k \cap X_N = \emptyset\right) = f(N,n,m). \tag{4.2}$$

□

Proof. We use well-known formula that, for any m events $A_1, \ldots, A_m$,

$$\mathbb{P}\left(\bigcup_{k=1}^{m} A_j\right) = \sum_{k=1}^{m} (-1)^{k+1} \sum_{(j_1,\ldots,j_k)} \mathbb{P}\left(A_{j_1} \cap A_{j_2} \cap \cdots \cap A_{j_k}\right), \tag{4.3}$$

where the integers $j_1, \ldots, j_k$ are distinct.

Let the event A_i occur if the set B_i does not intersect X_N. Then for any k-tuple $(j_1, \ldots, j_k)$, the event $A_{j_1} \cap \cdots \cap A_{j_k}$ occurs if the points $x_1, \ldots, x_N$ are in the complement of the union $B_{j_1} \cup \cdots \cup B_{j_k}$; that is, $x_1, \ldots, x_N$ are in a set of measure $1 - k/n$. We see that for any k-tuple the probability of this event is equal to $(1 - k/n)^N$. Moreover, there are exactly $\binom{m}{k}$ such k-tuples. Therefore,

$$\sum_{(j_1,\ldots,j_k)} \mathbb{P}\left(A_{j_1} \cap \cdots \cap A_{j_k}\right) = \binom{m}{k} \left(1 - \frac{k}{n}\right)^N,$$

and (4.2) follows from (4.3). ■

For the lower bounds in Theorems 2.2 and 3.2, we will need the following estimate on the function $f(N, n, m)$.

Lemma 4.2. For any three numbers $0 < m \leqslant n \leqslant N$, such that m and N are integers,

$$f(N, n, m) \geqslant 1 - \left[1 - \left(1 - \frac{1}{n}\right)^N\right]^m - \frac{N}{n^2} \cdot \frac{m(m-1)}{2} \cdot \left(1 - \frac{1}{n}\right)^{2(N-1)} \cdot \left[1 + \left(1 - \frac{1}{n}\right)^{N-1}\right]^{m-2}. \tag{4.4}$$

□

Proof. Note first that for $k \geqslant 1$ and $0 \leqslant x \leqslant 1$, we have

$$1 - kx \leqslant (1-x)^k \leqslant 1 - kx + \frac{k(k-1)}{2} x^2.$$

Thus, for $x = 1/n$, we get

$$\left(1 - \frac{1}{n}\right)^k - \frac{k(k-1)}{2} \frac{1}{n^2} \leqslant 1 - \frac{k}{n} \leqslant \left(1 - \frac{1}{n}\right)^k.$$

Suppose $(1 - \frac{1}{n})^k \geqslant \frac{k(k-1)}{2} \frac{1}{n^2}$. Using the inequality

$$a^N - (a-b)^N = b \cdot \left(a^{N-1} + (a-b)\, a^{N-2} + \cdots + (a-b)^{N-1}\right) \leqslant N \cdot b \cdot a^{N-1}, \quad \text{if } a > b > 0,$$

we get

$$\left(1 - \frac{k}{n}\right)^N \geqslant \left(\left(1 - \frac{1}{n}\right)^k - \frac{k(k-1)}{2} \frac{1}{n^2}\right)^N \geqslant \left(1 - \frac{1}{n}\right)^{kN} - N \cdot \frac{k(k-1)}{2} \frac{1}{n^2} \cdot \left(1 - \frac{1}{n}\right)^{k(N-1)}. \tag{4.5}$$

Suppose now that $(1-\frac{1}{n})^k < \frac{k(k-1)}{2}\frac{1}{n^2}$. Then

$$\left(1-\frac{1}{n}\right)^{kN} - N\cdot\frac{k(k-1)}{2}\frac{1}{n^2}\cdot\left(1-\frac{1}{n}\right)^{k(N-1)}$$
$$=\left(1-\frac{1}{n}\right)^{k(N-1)}\left(\left(1-\frac{1}{n}\right)^{k} - N\frac{k(k-1)}{2}\frac{1}{n^2}\right) < 0,$$

so as in inequality (4.5) for $k \leqslant n$,

$$\left(1-\frac{k}{n}\right)^{N} \geqslant \left(1-\frac{1}{n}\right)^{kN} - N\cdot\frac{k(k-1)}{2}\frac{1}{n^2}\cdot\left(1-\frac{1}{n}\right)^{k(N-1)}$$

also holds. Therefore,

$$f(N,n,m) = \sum\nolimits_{k\text{ odd},\, k\leqslant m}\binom{m}{k}\left(1-\frac{k}{n}\right)^{N} - \sum\nolimits_{k\text{ even},\, k\leqslant m}\binom{m}{k}\left(1-\frac{k}{n}\right)^{N}$$
$$\geqslant \sum\nolimits_{k\text{ odd}}\binom{m}{k}\left[\left(1-\frac{1}{n}\right)^{kN} - N\cdot\frac{k(k-1)}{2}\frac{1}{n^2}\cdot\left(1-\frac{1}{n}\right)^{k(N-1)}\right]$$
$$-\sum\nolimits_{k\text{ even}}\binom{m}{k}\left(1-\frac{1}{n}\right)^{kN}$$
$$\geqslant \sum\nolimits_{k=1}^{m}(-1)^{k+1}\binom{m}{k}\left(1-\frac{1}{n}\right)^{kN} - \frac{N}{n^2}\sum\nolimits_{k=0}^{m}\binom{m}{k}\frac{k(k-1)}{2}\cdot\left(1-\frac{1}{n}\right)^{k(N-1)}. \tag{4.6}$$

The first sum in (4.6) is equal to $1-(1-(1-\frac{1}{n})^N)^m$. To calculate the second sum, we note that

$$\frac{m(m-1)}{2}x^2(1+x)^{m-2} = \frac{1}{2}x^2\left((1+x)^m\right)'' = \sum\nolimits_{k=0}^{m}\binom{m}{k}\cdot\frac{k(k-1)}{2}x^k.$$

Thus, for $x=(1-\frac{1}{n})^{N-1}$, we get

$$\sum\nolimits_{k=0}^{m}\binom{m}{k}\frac{k(k-1)}{2}\cdot\left(1-\frac{1}{n}\right)^{k(N-1)} = \frac{m(m-1)}{2}\left(1-\frac{1}{n}\right)^{2(N-1)}\cdot\left(1+\left(1-\frac{1}{n}\right)^{N-1}\right)^{m-2}.$$

Combining the above estimates, we obtain (4.4). ■

With the help of (4.4), we can deduce some asymptotic properties of $f(N,n,m)$ as $N\to\infty$.

Lemma 4.3. Let N be a positive integer and n, m be numbers satisfying $1 \leqslant m \leqslant n \leqslant N$. Further, let κ_n denote constants depending on n such that $0 < c_1 \leqslant \kappa_n \leqslant c_2$ for all n.

(i) If $m = \lfloor \kappa_n n \rfloor$ and $c_2 \leqslant 1$, then there exists a number α such that for $n = \frac{N}{\log N - \alpha \log\log N}$, we have $f(N, n, m) \to 1$ as $N \to \infty$.

(ii) If $d > 1$ and $m = \lfloor \kappa_n n^{\frac{d-1}{d}} \rfloor$, then there exists a number α such that for $n = \frac{N}{\frac{d-1}{d}\log N - \alpha \log\log N}$, we have $f(N, n, m) \to 1$ as $N \to \infty$.

(iii) If $d > 1$ and $m = \lfloor \kappa_n n^{\frac{1}{d}} \rfloor$, then there exists a number α such that for $n = \frac{N}{\frac{1}{d}\log N - \alpha \log\log N}$, we have $f(N, n, m) \to 1$ as $N \to \infty$. □

Proof. We prove only part (i) since the proofs of the second and third parts are similar.

In what follows, to simplify the displays, we omit the symbol for the integer part. If a_N and b_N are two sequences of positive numbers, we write $a_N \sim b_N$ to mean $a_N/b_N \to 1$ as $N \to \infty$.

For our choice of n in part (i), we have

$$\left(1 - \frac{1}{n}\right)^N \sim \exp\left(-\frac{N}{n}\right) \sim \frac{(\log N)^\alpha}{N}.$$

Thus,

$$\left(1 - \left(1 - \frac{1}{n}\right)^N\right)^{\kappa_n n} \sim \left(1 - \frac{(\log N)^\alpha}{N}\right)^{\frac{\kappa_n N}{\log N - \alpha \log\log N}} \sim \exp\left(-\frac{\kappa_n (\log N)^\alpha}{\log N - \alpha \log\log N}\right).$$

If $\alpha > 1$, then the last expression tends to zero. Moreover,

$$\frac{N}{n^2} \cdot \frac{m(m-1)}{2}\left(1 - \frac{1}{n}\right)^{2(N-1)} \cdot \left(1 + \left(1 - \frac{1}{n}\right)^{N-1}\right)^{m-2}$$
$$\sim \frac{\kappa_n^2}{2} \cdot \frac{(\log N)^{2\alpha}}{N} \cdot \left(1 + \frac{(\log N)^\alpha}{N}\right)^{\frac{\kappa_n N}{\log N - \alpha \log\log N}}$$
$$\sim \frac{\kappa_n^2}{2} \cdot \frac{(\log N)^{2\alpha}}{N} \cdot \exp\left(\kappa_n (\log N)^{\alpha - 1}\right).$$

For $\alpha = 3/2$ (actually, any $0 < \alpha < 2$ will work), the last expression is comparable to

$$\frac{(\log N)^3}{N} \exp\left(\kappa_n (\log N)^{\frac{1}{2}}\right),$$

which tends to zero as N tends to infinity. Thus from (4.4), we deduce that

$$\liminf_{N\to\infty} f(N, n, m) \geqslant 1.$$

However, since $f(N, n, m)$ is equal to a certain probability, we have that $f(N, n, m) \leqslant 1$, and so $\lim_{N\to\infty} f(N, n, m) = 1$. ■

5 Proofs

5.1 Preliminary objects

Fix a compact set $\mathcal{X}_0$ with a metric m. For any large positive number n, let $\mathcal{E}_n(\mathcal{X}_0)$ be a maximal set of points such that for any $y, z \in \mathcal{E}_n(\mathcal{X}_0)$, we have $m(y, z) \geqslant 1/n$. Then for any $x \in \mathcal{X}_0$, there exists a point $y \in \mathcal{E}_n(\mathcal{X}_0)$ such that $m(x, y) \leqslant 1/n$ (otherwise, we can add x to $\mathcal{E}_n(\mathcal{X}_0)$, which contradicts its maximality).

5.2 Proof of the Theorem 2.1

Recall that $(\mathcal{X}, m)$ is a metric space, μ is a finite positive measure supported on $\mathcal{X}$, and $B(x, r)$ denotes a closed ball (in the metric m) with center $x \in \mathcal{X}$ and radius r. Put $\mathcal{E}_n := \mathcal{E}_n(\mathcal{X})$ and note that

$$\mu(\mathcal{X}) \geqslant \sum_{x\in\mathcal{E}_n} \mu\left(B\left(x, \frac{1}{3n}\right)\right) \geqslant \text{card}(\mathcal{E}_n)\, \Phi\,(1/(3n))\,. \tag{5.1}$$

Suppose now that $X_N = \{x_1, \dots, x_N\}$ is a set of N random points, independently distributed over $\mathcal{X}$ with respect to the measure μ. We denote its covering radius by

$$\rho(X_N) := \rho(X_N, \mathcal{X})\,.$$

Suppose $\rho(X_N) > \frac{2}{n}$. Then there exists a point $y \in \mathcal{X}$ such that $X_N \cap B(y, \frac{2}{n}) = \emptyset$. Choose a point $x \in \mathcal{E}_n$ such that $m(x, y) < \frac{1}{n}$. Then $B(x, \frac{1}{n}) \subset B(y, \frac{2}{n})$, and so the ball $B(x, \frac{1}{n})$ (and thus $B(x, \frac{1}{3n})$) does not intersect X_N. Therefore,

$$\begin{aligned}\mathbb{P}\left(\rho(X_N) \geqslant \frac{2}{n}\right) &\leqslant \mathbb{P}\,(\exists x \in \mathcal{E}_n \colon B\,(x, 1/(3n)) \cap X_N = \emptyset) \\ &\leqslant \text{card}(\mathcal{E}_n) \cdot \left(1 - \frac{\Phi\left(\frac{1}{3n}\right)}{\mu(\mathcal{X})}\right)^N. \end{aligned}\tag{5.2}$$

We now choose n to be such that $\frac{1}{3n} = \Phi^{-1}(\frac{\alpha \log N}{N})$. There exists such an n since Φ is continuous and $\Phi(r) \to 0$ as $r \to 0^+$. Then utilizing the upper bound for card$(\mathcal{E}_n)$ from

(5.1), we deduce that for some $C>0$, we have

$$\mathbb{P}\left[\rho\left(X_N\right)\geqslant\frac{2}{n}\right]\leqslant C\frac{N}{\log N}\cdot N^{-C\alpha},$$

which concludes the proof of the estimate (2.1).

To establish the estimate (2.2), note that since for small values of r we have $\Phi(r)\leqslant r^{\sigma}$, it follows that for small r and $D=\frac{1}{\sigma}$ we have $\Phi^{-1}(r)\geqslant r^{D}$. Choose α so large that $N^{1-C\alpha}=o(N^{-D})$ as $N\to\infty$. Then

$$\mathbb{E}\rho\left(X_N\right)\leqslant\frac{2}{n}+C\operatorname{diam}\left(\mathcal{X}\right)\cdot o\left(N^{-D}\right)=6\Phi^{-1}\left(\frac{\alpha\log N}{N}\right)+o\left(N^{-D}\right).$$

Finally, since $\Phi^{-1}(\frac{\alpha\log N}{N})\geqslant\Phi^{-1}(N^{-1})\geqslant N^{-D}$, inequality (2.2) follows. ∎

5.3 Proof of the Theorem 2.2

Let $\mathcal{E}_n:=\mathcal{E}_n(\mathcal{X}_1)$, where $\mathcal{X}_1$ is as in the hypothesis. Note that

$$0<\mu\left(\mathcal{X}_1\right)\leqslant\sum\nolimits_{x\in\mathcal{E}_n}\mu\left(B\left(x,\frac{1}{n}\right)\right)\leqslant\operatorname{card}\left(\mathcal{E}_n\right)\Phi\left(\frac{1}{n}\right). \tag{5.3}$$

An estimate as in (5.1) together with the doubling property of Φ implies that

$$\mu\left(\mathcal{X}\right)\geqslant c\cdot\operatorname{card}\left(\mathcal{E}_n\right)\Phi\left(\frac{1}{3n}\right)\geqslant\tilde{c}\cdot\operatorname{card}\left(\mathcal{E}_n\right)\Phi\left(\frac{1}{n}\right).$$

Thus, $\tau_n:=\operatorname{card}(\mathcal{E}_n)\cdot\Phi(1/n)$ satisfies $0<c_1<\tau_n<c_2$ for some constants c_1 and c_2 independent of n. Clearly, if a ball $B(x,\frac{1}{3n})$ does not intersect X_N, then $\rho(X_N)=\rho(X_N,\mathcal{X})\geqslant\frac{1}{3n}$. Thus

$$\mathbb{P}\left(\rho\left(X_N\right)\geqslant\frac{1}{3n}\right)\geqslant\mathbb{P}\left(\exists x\in\mathcal{E}_n\colon\ B\left(x,1/\left(3n\right)\right)\cap X_N=\emptyset\right).$$

Note that the balls $B(x,\frac{1}{3n})$ are disjoint for $x\in\mathcal{E}_n$, and their μ-measure is comparable to $t:=\Phi(\frac{1}{n})$.

Next we claim that for every $x\in\mathcal{E}_n$ there exists a constant $c_x\leqslant 1$ such that the balls $B(x,c_x\frac{1}{3n})$ have the same measure $c_0\Phi(\frac{1}{n})=c_0t$, and moreover, that the uniform estimate $c_x>c>0$ holds for some constant c. To see this, take two points $x_1,x_2\in\mathcal{X}_1$ and assume that the balls $B_i:=B(x_i,r)$, $i=1,2$ are disjoint. Suppose $\mu(B_1)<\mu(B_2)$. Define the function $\varphi(s):=\mu(B(x_2,s\cdot r))$. The strict doubling property of Φ implies

$$\mu\left(B_1\right)\geqslant c\Phi\left(r\right)\geqslant c\cdot C_1^k\Phi\left(r/2^k\right).$$

Choose k such that $c \cdot C_1^k \geqslant 1$. Then

$$\mu(B_1) \geqslant \Phi\left(r/2^k\right) \geqslant \varphi\left(2^{-k}\right).$$

Thus, $\varphi(2^{-k}) \leqslant \mu(B_1) < \mu(B_2) = \varphi(1)$. Since $x_i \in \mathcal{X}_1$ and n is large, we can use continuity of φ to see that there exists a constant c_{x_2} such that $\mu(B(x_2, c_{x_2} r)) = \mu(B(x_1, r))$. Note that $c_{x_2} \geqslant 2^{-k} =: c_0$, where k depends only on the constants c, C_1 from Theorem 2.2 and not on x_1, x_2, or r. Applying this procedure to all balls $B(x, 1/(3n))$, $x \in \mathcal{E}_n$, and using the fact that $\mathrm{card}(\mathcal{E}_n) = \tau_n/t$, we obtain

$$\begin{aligned}\mathbb{P}\left(\rho(X_N) \geqslant \frac{c_0}{3}\Phi^{-1}(t)\right) &\geqslant \mathbb{P}\left(\text{one of } \frac{\tau_n}{t} \text{ disjoint balls of measure } c_0 t \text{ is disjoint from } X_N\right)\\ &= f\left(N, \frac{1}{c_0 t}, \frac{\tau_n}{t}\right) = f\left(N, \frac{1}{c_0 t}, \frac{\kappa_n}{c_0 t}\right),\end{aligned} \tag{5.4}$$

where $\kappa_n := c_0 \tau_n$ and f is given in (4.1). If necessary, we can decrease the size of c_0 so that $\kappa_n \leqslant 1$ for n large. As we have seen in Lemma 4.3(i), there exists a number α such that if

$$\frac{1}{c_0 t} = \frac{N}{\log N - \alpha \log\log N},$$

then $f(N, \frac{1}{c_0 t}, \frac{\kappa_n}{c_0 t}) \to 1$ as $N \to \infty$. Thus, for any sufficiently large number N, we have

$$\mathbb{P}\left(\rho(X_N) \geqslant \frac{c_0}{3}\Phi^{-1}\left(\frac{\log N - \alpha \log\log N}{c_0 N}\right)\right) \geqslant 1 - o(1), \quad N \to \infty,$$

which is the desired inequality (2.3).

Moreover, for large values of N, we have $\log N - \alpha \log\log N \geqslant \frac{1}{2}\log N$, thus

$$\mathbb{E}\rho(X_N) \geqslant c_1 \Phi^{-1}\left(c_2 \frac{\log N}{N}\right),$$

which proves inequality (2.4). ■

5.4 Estimates from above for asymptotically flat sets

Let K be an asymptotically flat s-regular subset of $\mathbb{R}^d$ and put

$$\rho(X_N) = \rho(X_N, K), \quad \varepsilon_N := \frac{1}{\log N}.$$

In order to deduce sharp asymptotic results, we first improve our estimates from above by considering a better net of points. For each $N > 4$, let $\mathcal{E}_{n/\varepsilon_N} := \mathcal{E}_{n/\varepsilon_N}(K)$. From estimates similar to (5.1) and (5.3), we see that $\mathrm{card}(\mathcal{E}_n)$ is comparable to $(n/\varepsilon_N)^s$ independently of N.

Suppose $\rho(X_N) > \frac{1}{n}$. Then, since K is compact, for some $y \in K$, we have $B_d(y, \frac{1}{n}) \cap X_N = \emptyset$, and thus there exists a point $x \in \mathcal{E}_{n/\varepsilon_N}$ such that $B_d(x, \frac{1-\varepsilon_N}{n}) \cap X_N = \emptyset$. We fix a number δ, $0 < \delta < 1$, and take n so large that

$$\mathcal{H}_s\left(B_d\left(x, \frac{1-\varepsilon_N}{n}\right) \cap K\right) \geqslant (1-\delta)\, \upsilon_s \frac{(1-\varepsilon_N)^s}{n^s} \geqslant (1-\delta)\, \upsilon_s \frac{1 - s\varepsilon_N}{n^s}.$$

As in (5.2)

$$\mathbb{P}\left(\rho\left(X_N\right) > \frac{1}{n}\right) \leqslant C\left(\frac{n}{\varepsilon_N}\right)^s \left(1 - \frac{1}{\mathcal{H}_s\left(K\right)}\left(1-\delta\right) \upsilon_s \frac{1 - s\varepsilon_N}{n^s}\right)^N. \tag{5.5}$$

Fix a number $A > 0$ and choose

$$n_1 := \left(\frac{(1-\delta)\, \upsilon_s}{\mathcal{H}_s\left(K\right)} \frac{N}{\log N + A \log\log N}\right)^{1/s}.$$

Then with $n = n_1$ in (5.5), we get for all N large,

$$\mathbb{P}\left(\rho\left(X_N\right) > \frac{1}{n_1}\right) \leqslant C \cdot N\left(\log N\right)^{s-1}\ \mathrm{e}^{-(1-s/\log N)(\log N + A\log\log N)}. \tag{5.6}$$

Recall that C does not depend on N. Thus if A and N are sufficiently large, it follows that

$$\mathbb{P}\left(\rho\left(X_N\right) > \frac{1}{n_1}\right) \leqslant \frac{1}{\log N}. \tag{5.7}$$

Furthermore, if we plug $n = n_2 := (\frac{N}{B\log N})^{1/s}$ in (5.5), we get for sufficiently large B

$$\mathbb{P}\left(\rho\left(X_N\right) > \frac{1}{n_2}\right) \leqslant N^{-p/s-1}. \tag{5.8}$$

With $\mathrm{d}\mu = \mathbb{1}_K \mathrm{d}\mathcal{H}_s/\mathcal{H}_s(K)$, we make use of the formula

$$\begin{aligned}
\mathbb{E}[\rho(X_N)^p] &= \int_{K^N} \rho(X_N)^p \mathrm{d}\mu(x_1)\ldots\mathrm{d}\mu(x_N) = \int_{\rho(X_N)\leqslant 1/n_1} \rho(X_N)^p \mathrm{d}\mu(x_1)\ldots\mathrm{d}\mu(x_N)\\
&\quad + \int_{1/n_1 < \rho(X_N) \leqslant 1/n_2} \rho(X_N)^p \mathrm{d}\mu(x_1)\ldots\mathrm{d}\mu(x_N)\\
&\quad + \int_{\rho(X_N) > 1/n_2} \rho(X_N)^p \mathrm{d}\mu(x_1)\ldots\mathrm{d}\mu(x_N)\\
&\leqslant \frac{1}{n_1^p} + \frac{1}{n_2^p} \cdot \mathbb{P}\left(\rho(X_N) > \frac{1}{n_1}\right) + (\mathrm{diam}(K))^p \cdot \mathbb{P}\left(\rho(X_N) > \frac{1}{n_2}\right).
\end{aligned} \tag{5.9}$$

From (5.7), (5.8), and the definitions of n_1 and n_2, we obtain

$$\mathbb{E}[\rho(X_N)^p] \leqslant \left(\frac{\log N + A\log\log N}{N}\right)^{p/s} \cdot \left(\frac{\mathcal{H}_s(K)}{\upsilon_s}\right)^{p/s} \cdot (1-\delta)^{-p/s} + C\left(\frac{\log N}{N}\right)^{p/s} \frac{1}{\log N} + CN^{-p/s-1}. \tag{5.10}$$

Therefore, for any δ with $0<\delta<1$,

$$\limsup_{N\to\infty} \mathbb{E}[\rho\,(X_N)^p] \cdot \left(\frac{N}{\log N}\right)^{p/s} \leqslant (1-\delta)^{-p/s} \cdot \left(\frac{\mathcal{H}_s\,(K)}{\upsilon_s}\right)^{p/s},$$

and consequently,

$$\limsup_{N\to\infty} \mathbb{E}[\rho\,(X_N)^p] \cdot \left(\frac{N}{\log N}\right)^{p/s} \leqslant \left(\frac{\mathcal{H}_s\,(K)}{\upsilon_s}\right)^{p/s}. \tag{5.11}$$

5.5 Estimate from above for quasi-nice sets

Let K be a quasi-nice s-regular subset of $\mathbb{R}^d$, and again set $\varepsilon_N := 1/\log N$ and $\mathcal{E}_{n/\varepsilon_N} := \mathcal{E}_{n/\varepsilon_N}(K)$, where $n/\varepsilon_N \to \infty$ as $N\to\infty$. Since the set T from part (iii) of Definition 3.1 is finite, the regularity condition (ii) implies

$$\mathcal{H}_s\left(\bigcup_{x\in T} B_d\,(x,r)\right) \leqslant C\cdot \text{card}\,(T)\cdot r^s = C_1 r^s, \quad 0<r<r_0.$$

Suppose $y_1,\dots,y_k \in \mathcal{E}_{n/\varepsilon_N} \cap \bigcup_{x\in T} B_d(x,\frac{1-\varepsilon_N}{n})$. Then the balls $B_d(y_j,\frac{\varepsilon_N}{3n})$ are disjoint and $B_d(y_j,\frac{\varepsilon_N}{3n}) \subset \bigcup_{x\in T} B_d(x,\frac{1+\varepsilon_N}{n})$ for $j=1,\dots,k$. The chain of inequalities

$$C_1\left(\frac{1+\varepsilon_N}{n}\right)^s \geqslant \mathcal{H}_s\left(\bigcup_{x\in T} B_d\left(x,\frac{1+\varepsilon_N}{n}\right)\right) \geqslant \sum\nolimits_{j=1}^k \mathcal{H}_s\left(B_d\left(y_j,\frac{\varepsilon_N}{3n}\right)\right) \geqslant c\cdot k\cdot\left(\frac{\varepsilon_N}{n}\right)^s$$

implies that $k\leqslant C_2/\varepsilon_N^s$, and C_2 does not depend on N. Further, if $y\in \mathcal{E}_{n/\varepsilon_N} \setminus \bigcup_{x\in T} B_d(x,\frac{1-\varepsilon_N}{n})$, then $\mathcal{H}_s(B_d(y,\frac{1-\varepsilon_N}{n})) \geqslant \upsilon_s(\frac{1-\varepsilon_N}{n})^s$.

As we have seen in (5.5), $\mathbb{P}(\rho(X_N)>1/n)$ is bounded from above by the probability that for some $y\in\mathcal{E}_{n/\varepsilon_N}$, we have $B_d(y,\frac{1-\varepsilon_N}{n})\cap X_N=\emptyset$. Taking into account that $\text{card}(\mathcal{E}_{n/\varepsilon_N}) \leqslant C_3(n/\varepsilon_N)^s$, we obtain

$$\mathbb{P}\left(\rho(X_N)>\frac{1}{n}\right) \leqslant \mathbb{P}\left(\text{one of} \leqslant \frac{C_2}{\varepsilon_N^s} \text{ balls of measure} \geqslant \frac{c_1}{n^s} \text{ is disjoint from } X_N \text{ or}\right.$$
$$\left.\text{one of} \leqslant C_3\left(\frac{n}{\varepsilon_N}\right)^s \text{ balls of measure} \geqslant \frac{\upsilon_s(1-\varepsilon_N)^s}{n^s} \text{ is disjoint from } X_N\right).$$

This last probability is bounded from above by

$$\frac{C_2}{\varepsilon_N^s}\left(1-\frac{c_1}{n^s}\right)^N + C_4\left(\frac{n}{\varepsilon_N}\right)^s\left(1-\frac{1}{\mathcal{H}_s(K)}\frac{\upsilon_s(1-\varepsilon_N)^s}{n^s}\right)^N.$$

As in the preceding proof, if

$$n_1 = \left(\frac{\upsilon_s}{\mathcal{H}_s(K)}\frac{N}{\log N + A\log\log N}\right)^{1/s},$$

then, for N large,

$$C_4\left(\frac{n_1}{\varepsilon_N}\right)^s\cdot\left(1-\frac{1}{\mathcal{H}_s(K)}\frac{\upsilon_s(1-\varepsilon_N)^s}{n_1^s}\right)^N \leqslant \frac{C_5}{\log N}.$$

Furthermore, note that if C_6 is sufficiently large, then

$$\frac{C_2}{\varepsilon_N^s}\left(1-\frac{c_1}{n_1^s}\right)^N \leqslant C_6(\log N)^s N^{-c_2}, \quad N\to\infty.$$

Repeating estimates (5.9) and (5.10), we obtain

$$\limsup_{N\to\infty}\mathbb{E}[\rho(X_N)^p]\cdot\left(\frac{N}{\log N}\right)^{p/s} \leqslant \left(\frac{\mathcal{H}_s(K)}{\upsilon_s}\right)^{p/s}. \tag{5.12}$$

Note that (5.12) holds whether or not K is countably s-rectifiable; it requires only that properties (ii) and (iii) of Definition 3.1 hold.

5.6 Estimate from below for quasi-nice sets

For the proof of Theorem 3.2, it remains in view of inequalities (5.11) and (5.12), to establish

$$\liminf_{N\to\infty}\mathbb{E}[\rho(X_N)^p]\cdot\left(\frac{N}{\log N}\right)^{p/s} \geqslant \left(\frac{\mathcal{H}_s(K)}{\upsilon_s}\right)^{p/s} \tag{5.13}$$

for asymptotically flat and quasi-nice s-regular set K. Since by the Hölder inequality, we have

$$\liminf_{N\to\infty}\mathbb{E}[\rho(X_N)^p]\cdot\left[\frac{N}{\log N}\right]^{p/s} \geqslant \left(\liminf_{N\to\infty}\mathbb{E}\rho(N_N)\cdot\left[\frac{N}{\log N}\right]^{1/s}\right)^p,$$

it is enough to prove (5.13) for $p=1$. If K is a quasi-nice s-regular set, then K is countably s-rectifiable (s is an integer) and $0<\mathcal{H}_s(K)<\infty$; thus as previously remarked, the following holds for $\mathcal{H}_s$-almost every point $x \in K$:

$$r^{-s} \cdot \mathcal{H}_s\left(B_d(x,r) \cap K\right) \to \upsilon_s, \quad r \to 0^+.$$

Fix a number δ with $0<\delta<1$ and a countable unbounded set $\mathcal{N}$. For $n \in \mathcal{N}$ define $r_n := 1/n$ and $q_n := (\frac{1-\delta}{1+\delta})^{1/s} \cdot 1/n$. By Egoroff's theorem, there exists a set $K_1 = K_1(\delta) \subset K$ with $\mathcal{H}_s(K_1) > \frac{1}{2}\mathcal{H}_s(K)$ on which the above limit is uniform for radii r equal to r_n and q_n. That is,

$$r^{-s}\mathcal{H}_s\left(B_d(x,r) \cap K\right) \rightrightarrows \upsilon_s, \quad r = r_n \quad \text{or} \quad r = q_n, \quad n \in \mathcal{N}, \quad n \to \infty. \tag{5.14}$$

This means that there exists a large number $n(\delta)$, such that for any $n > n(\delta)$ we have, for every $x \in K_1$,

$$(1-\delta)\,\upsilon_s r_n^s \leqslant \mathcal{H}_s\left(B_d(x,r_n) \cap K\right) \leqslant (1+\delta)\,\upsilon_s r_n^s, \tag{5.15}$$

$$(1-\delta)\,\upsilon_s q_n^s \leqslant \mathcal{H}_s\left(B_d(x,q_n) \cap K\right) \leqslant (1+\delta)\,\upsilon_s q_n^s = (1-\delta)\,\upsilon_s r_n^s. \tag{5.16}$$

Recalling the notation of Section 5.1, we set $\mathcal{E}_{n/2} := \mathcal{E}_{n/2}(K_1)$. Then, as in the preceding sections, there exist positive constants c_1 and c_2 (independent of n) such that $c_1 n^s \leqslant \operatorname{card}(\mathcal{E}_{n/2}) \leqslant c_2 n^s$ where, for the lower bound, we use

$$0 < \mathcal{H}_s(K_1) \leqslant \mathcal{H}_s\left(\bigcup_{x \in \mathcal{E}_{n/2}} \left(B_d(x, 2/n) \cap K\right)\right) \leqslant C \cdot \operatorname{card}\left(\mathcal{E}_{n/2}\right)(2/n)^s.$$

Thus, $\tau_n := \operatorname{card}(\mathcal{E}_{n/2})/n^s$ satisfies $0 < c_1 \leqslant \tau_n \leqslant c_2$. Clearly, if for some $x \in \mathcal{E}_{n/2}$ the ball $B_d(x, \frac{1}{n})$ is disjoint from X_N, then $\rho(X_N) \geqslant \frac{1}{n}$. Thus, for a given $\delta > 0$ and sufficiently large n, we have a family $\{B_d(x, 1/n) \cap K \colon x \in \mathcal{E}_{n/2}(K_1)\}$ of $\tau_n n^s$ balls (relative to K) with disjoint interiors of radius $1/n$ and $\mathcal{H}_s$-measure between $(1-\delta)\upsilon_s/n^s$ and $(1+\delta)\upsilon_s/n^s$. For a fixed $x \in \mathcal{E}_{n/2}(K_1)$, define $\varphi(t) := \mathcal{H}_s(B(x, t/n) \cap K)$. Then $\varphi(1) \geqslant (1-\delta)\upsilon_s/n^s$. On the other hand, inequalities (5.16) imply

$$\varphi\left(\left(\frac{1-\delta}{1+\delta}\right)^{1/s}\right) \leqslant (1-\delta)\,\upsilon_s/n^s.$$

Thus, there is a number $c_x = c_{x,n}$, with $c_x \geqslant (\frac{1-\delta}{1+\delta})^{1/s}$, such that $\varphi(c_x) = (1-\delta)\upsilon_s/n^s$. That is, there exists a new family $\{B_d(x, c_x/n) \cap K \colon x \in \mathcal{E}_{n/2}(K_1)\}$, with $c_x \geqslant (\frac{1-\delta}{1+\delta})^{1/s}$, and the sets $B_d(x, c_x/n) \cap K$ all have the same $\mathcal{H}_s$ measure, namely $(1-\delta)\upsilon_s/n^s$.

As in (5.4), it follows that

$$\mathbb{P}\left(\rho\left(X_N\right)\geqslant\left(\frac{1-\delta}{1+\delta}\right)^{1/s}\frac{1}{n}\right)\geqslant f\left(N,\frac{\mathcal{H}_s\left(K\right)n^s}{\left(1-\delta\right)\upsilon_s},\tau_n n^s\right)$$
$$=f\left(N,\frac{\mathcal{H}_s\left(K\right)n^s}{\left(1-\delta\right)\upsilon_s},\kappa_n\cdot\frac{\mathcal{H}_s\left(K\right)n^s}{\left(1-\delta\right)\upsilon_s}\right),\tag{5.17}$$

where

$$\kappa_n:=\tau_n\cdot\frac{\left(1-\delta\right)\upsilon_s}{\mathcal{H}_s\left(K\right)}.$$

It is easily seen that

$$\mathcal{H}_s\left(K\right)\geqslant\tau_n n^s\cdot\frac{\left(1-\delta\right)\upsilon_s}{n^s}=\mathcal{H}_s\left(K\right)\kappa_n;$$

thus $\kappa_n\leqslant 1$. Part (i) of Lemma 4.3, therefore, implies that the sequence in (5.17) tends to 1 as $N\to\infty$ if (for suitable α), we have

$$\frac{\left(1-\delta\right)\upsilon_s}{\mathcal{H}_s\left(K\right)n^s}=\frac{\log N-\alpha\log\log N}{N},$$

which is equivalent to

$$n:=\left[\frac{\left(1-\delta\right)\upsilon_s}{\mathcal{H}_s\left(K\right)}\cdot\frac{N}{\log N-\alpha\log\log N}\right]^{1/s}.\tag{5.18}$$

We take N so large that n exceeds $n(\delta)$, which ensures that the inequalities (5.15)–(5.16) hold. From (5.17), we obtain

$$\mathbb{E}\rho\left(X_N\right)\geqslant\left(\frac{1-\delta}{1+\delta}\right)^{1/s}\frac{1}{n}\cdot f\left(N,\frac{\mathcal{H}_s\left(K\right)n^s}{\left(1-\delta\right)\upsilon_s},\tau_n n^s\right).$$

Using the definition of n in (5.18), we get

$$\mathbb{E}\rho(X_N)\cdot\left[\frac{N}{\log N}\right]^{1/s}\geqslant\left[\frac{N}{\log N}\right]^{1/s}\cdot\left(\frac{1-\delta}{1+\delta}\right)^{1/s}\cdot\left[\frac{\mathcal{H}_s(K)}{(1-\delta)\upsilon_s}\cdot\frac{\log N-\alpha\log\log N}{N}\right]^{1/s}$$
$$\cdot f\left(N,\frac{\mathcal{H}_s(K)n^s}{(1-\delta)\upsilon_s},\tau_n n^s\right),\tag{5.19}$$

and passing to the lim inf as $N\to\infty$ yields

$$\liminf_{N\to\infty}\mathbb{E}\rho\left(X_N\right)\cdot\left(\frac{N}{\log N}\right)^{1/s}\geqslant\left(\frac{1}{1+\delta}\right)^{1/s}\cdot\left[\frac{\mathcal{H}_s\left(K\right)}{\upsilon_s}\right]^{1/s}.$$

Recalling that δ can be taken arbitrarily small, we obtain (5.13) for quasi-nice sets. For asymptotically flat sets, the same (but even simpler) argument applies.

5.7 Proof of Corollary 3.4

Recall that

$$Z_N = \rho\left(X_N, \mathbb{S}^d\right) \cdot \left(\frac{\upsilon_d}{(d+1)\,\upsilon_{d+1}} \cdot \frac{N}{\log N}\right)^{1/d}.$$

Corollary 3.3 implies that $\mathbb{E}Z_N \to 1$ and $\mathbb{E}[Z_N^2] \to 1$; thus $\mathbb{E}[(Z_N-1)^2] = \mathbb{E}[Z_N^2] - 2\mathbb{E}Z_N + 1 \to 0$. The Chebyshev inequality then implies

$$\mathbb{P}\left(|Z_N - 1| > \varepsilon\right) \leqslant \frac{\mathbb{E}[(Z_N-1)^2]}{\varepsilon^2} \to 0,$$

which completes the proof. ■

5.8 Proof of the Corollaries 3.3 and 3.6

It is well known that a closed $C^{(1,1)}$ manifold is an asymptotically flat set, and a rectifiable curve is a quasi-nice 1-regular set. For the first fact, we refer the reader to a textbook on Riemannian geometry, for instance, [5, Chapters 5–10]. The second fact can be deduced from [7, Section 3.2]. ■

5.9 Proof of the Theorem 3.7: estimate from above

The proof of the theorem is similar to the proof for asymptotically flat sets. However, we need to take into account that the limit (3.1) is not equal to υ_d for points on the boundary. We use properties (ii) and (iii) of K to obtain

$$r^{-d}\mathcal{H}_d\left(B_d(x,r) \cap K\right) \rightrightarrows \tfrac{1}{2}\upsilon_d, \quad r \to 0, \quad x \in \partial K; \tag{5.20}$$

$$x \in K, \quad \operatorname{dist}(x, \partial K) > r \Rightarrow \mathcal{H}_d\left(B_d(x,r) \cap K\right) = \mathcal{H}_d\left(B_d(x,r)\right) = \upsilon_d r^d; \tag{5.21}$$

$$\forall \delta > 0\ \exists r(\delta) > 0 \colon \forall r < r(\delta), \forall x \in K \colon \mathcal{H}_d\left(B_d(x,r) \cap K\right) \geqslant \left(\frac{1}{2} - \delta\right)\upsilon_d r^d. \tag{5.22}$$

For the details, we refer the reader to Lee [9, Chapter 5]. For large N, set $\mathcal{E}_{n/\varepsilon_N} := \mathcal{E}_{n/\varepsilon_N}(K)$ and $\varepsilon_N := 1/\log N$, where $n(N)$ is a sequence such that $n \asymp (N/\log N)^{1/d}$. We now fix a number δ with $0 < \delta < 1/2$. Note that if $x \in \mathcal{E}_{n/\varepsilon_N}$ and $\operatorname{dist}(x, \partial K) > (1-\varepsilon_N)/n$, then

$$\mathcal{H}_d\left(B_d\left(x, (1-\varepsilon_N)/n\right) \cap K\right) = \upsilon_d\left((1-\varepsilon_N)/n\right)^d;$$

if $x \in \mathcal{E}_{n/\varepsilon_N}$ and $\operatorname{dist}(x, \partial K) \leqslant (1-\varepsilon_N)/n$ then, for large enough n,

$$\mathcal{H}_d\left(B_d\left(x, (1-\varepsilon_N)/n\right) \cap K\right) \geqslant \left(\tfrac{1}{2} - \delta\right)\upsilon_d\left((1-\varepsilon_N)/n\right)^d.$$

On considering disjoint balls (relative to K) of radius $\varepsilon_N/(3n)$ and using that

$$\mathcal{H}_d\left(\left\{x\colon \operatorname{dist}(x,\partial K)\leqslant \left(1-\tfrac{2}{3}\varepsilon_N\right)/n\right\}\right)\leqslant C_1/n,$$

we deduce, as in (5.1), that

$$\operatorname{card}\left\{x\in\mathcal{E}_{n/\varepsilon_N}\colon \operatorname{dist}(x,\partial K)\leqslant\frac{1-\varepsilon_N}{n}\right\}\leqslant C_2\frac{n^{d-1}}{\varepsilon_N^d}.$$

Therefore, for large enough n, we get

$$\begin{aligned}\mathbb{P}\left(\rho(X_N)>\frac{1}{n}\right)&\leqslant\mathbb{P}\left(\exists x\in\mathcal{E}_{n/\varepsilon_N}\colon B_d\left(x,\frac{1-\varepsilon_N}{n}\right)\cap K\cap X_N=\emptyset\right)\\&\leqslant C_2\frac{n^{d-1}}{\varepsilon_N^d}\left(1-\frac{(1/2-\delta)\upsilon_d}{\mathcal{H}_d(K)}\left(\frac{1-\varepsilon_N}{n}\right)^d\right)^N\\&\quad+C_3\frac{n^d}{\varepsilon_N^d}\left(1-\frac{\upsilon_d}{\mathcal{H}_d(K)}\left(\frac{1-\varepsilon_N}{n}\right)^d\right)^N.\end{aligned}\tag{5.23}$$

Repeating the estimates (5.7)–(5.11) with

$$n_1:=\left(\frac{(1/2-\delta)\,\upsilon_d}{\mathcal{H}_d(K)}\cdot\frac{N}{\frac{d-1}{d}\log N+A\log\log N}\right)^{1/d},$$

and

$$n_2:=\left(\frac{N}{B\log N}\right)^{1/d},$$

where A and B are sufficiently large, we obtain, after letting $\delta\to 0^+$, the estimate

$$\limsup_{N\to\infty}\mathbb{E}[\rho(X_N)^p]\left(\frac{N}{\log N}\right)^{p/d}\leqslant\left(\frac{2(d-1)}{d}\cdot\frac{\mathcal{H}_d(K)}{\upsilon_d}\right)^{p/d}.$$

5.10 **Proof of the Theorem 3.7: estimate from below**

We repeat the proof from the Section 5.6, but now we will place our net $\mathcal{E}$ only on the boundary ∂K. Namely, put $\mathcal{E}_{n/2}:=\mathcal{E}_{n/2}(\partial K)$. Since ∂K is a smooth $d-1$-dimensional submanifold, we see that $\operatorname{card}(\mathcal{E}_{n/2})=\tau_n n^{d-1}$ with $0<c_1<\tau_n<c_2$. Moreover, from (5.20), we obtain as in (5.14) that

$$r^{-d}\mathcal{H}_d(B_d(x,1/n)\cap K)\rightrightarrows\tfrac{1}{2}\upsilon_d/n^d,\quad r=r_n\text{ or }r=q_n,\ \ n\to\infty,$$

uniformly for $x\in\mathcal{E}_{n/2}$.

The remainder of the proof just involves repeating the estimates (5.17)–(5.19), using part (ii) of Lemma 4.3. ■

5.11 Estimate from above for the cube $[0, 1]^d$

The proof is similar to the case of the bodies with smooth boundary. The only change we need to make is to the formula (5.20). Namely, if a point x lies on a $(d-k)$-dimensional edge of the cube, then $\mathcal{H}_d(B_d(x,r)\cap[0,1]^d)\asymp 2^{-k}\upsilon_d r^d$. Moreover, $\mathcal{H}_d(B_d(x,r)\cap[0,1]^d)=2^{-k}\upsilon_d r^d$ for points x on the $(d-k)$-dimensional edge that are at distance larger than r from all $(d-k-1)$-dimensional edges. Thus, if we consider a set $\mathcal{E}_{n/\varepsilon_N}:=\mathcal{E}_{n/\varepsilon_N}([0,1]^d)$, we have for any $k=0,\dots,d$ at most $C_k n^{d-k}/\varepsilon_N^d$ points $x\in\mathcal{E}_{n/\varepsilon_N}$ with $\mathcal{H}_d(B_d(x,(1-\varepsilon_N)/n)\cap[0,1]^d)\geqslant 2^{-k}\upsilon_d((1-\varepsilon_N)/n)^d$. In particular, if $k=d$, we have only finitely many such points $x\in\mathcal{E}_{n/\varepsilon_N}$; and if $k=d-1$, we have no more than Cn/ε_N^d such points. We now repeat the estimates (5.7)–(5.11) and (5.23) with

$$n_1 := \left(2^{-(d-1)}\cdot d\cdot\upsilon_d\cdot\frac{N}{\log N + A\log\log N}\right)^{1/d}.$$

5.12 Estimate from below for the cube $[0, 1]^d$

The proof is almost identical to the proof in Section 5.10; the only difference is that now we take $\mathcal{E}_{n/2}:=\mathcal{E}_{n/2}(L)$, where L is a one-dimensional edge of the cube $[0,1]^d$. To complete the analysis, we appeal to part (iii) of Lemma 4.3.

5.13 Estimates for a polyhedron in $\mathbb{R}^3$

The estimates here are the same as for the unit cube $[0,1]^d$. The only difference is that, for points $x\in L$, where L is the edge where two faces intersect at angle θ, we have, if x is far enough from the vertices of P:

$$\mathcal{H}_3\left(B\left(x,r\right)\cap P\right)=\frac{\theta}{2\pi}\cdot\upsilon_3\cdot r^3.$$

Consequently, for $k=0,1,2,3$, we have at most $a_k n^{3-k}/\varepsilon_N^3$ points $x\in\mathcal{E}_{n/\varepsilon_N}(P)$ with $\mathcal{H}_3(B_3(x,(1-\varepsilon_N)/n)\cap P)\geqslant c_k\upsilon_3((1-\varepsilon_N)/n)^3$, where $a_0=1$, $a_1=1/2$, and $a_2=\theta/(2\pi)$. In the case $\theta\leqslant\pi/2$, one needs to choose

$$n_1 := \left(\frac{2\theta}{V\left(P\right)}\cdot\frac{N}{\log N + A\log\log N}\right)^{1/3},$$

and in the case $\theta \geqslant \pi/2$, one needs to choose

$$n_1 := \left(\frac{\pi}{V(P)} \cdot \frac{N}{\log N + A \log\log N} \right)^{1/3}.$$

For the estimate from above, consider $\mathcal{E}_{n/2}(L)$ and repeat the estimates for the cube. ■

5.14 Estimates for $d\mu = \frac{dx}{\sqrt{1-x^2}}$

We remind the reader that $\hat{\rho}(X_N) = \hat{\rho}(X_N, [0,1]) = \sup_{y \in [1-\frac{1}{N^a}, 1]} \inf_j |y - x_j|$, where x_j, $j = 1, \ldots, N$, are randomly and independently distributed over $[0, 1]$ with respect to μ.

5.14.1 *Case $a = 2$*

Suppose that an interval $I_\alpha := [1 - \frac{\alpha}{N^2}, 1]$ is disjoint from X_N for some $\alpha > 1$. Then we get

$$\hat{\rho}(X_N) \geqslant \frac{\alpha - 1}{N^2}.$$

We note that if $\alpha < C_1 \log^2(N)$, and N is sufficiently large, then

$$\mu(I_\alpha) \leqslant C_2 \frac{\sqrt{\alpha}}{N}.$$

Therefore, if α is some number greater than 1,

$$\mathbb{P}\left(\hat{\rho} \geqslant \frac{\alpha - 1}{N^2} \right) \geqslant \left(1 - C_2 \frac{\sqrt{\alpha}}{N} \right)^N \geqslant C_3.$$

Consequently,

$$\mathbb{E}\hat{\rho} \geqslant \frac{C_4}{N^2},$$

where $C_4 = C_3(\alpha - 1)$.

For the estimate from above, note that $\mu(I_\alpha) \geqslant \sqrt{\alpha}/(\sqrt{2}\pi N)$. Assuming $\hat{\rho}(X_N) \geqslant \frac{\alpha}{N^2}$, we get that the distance from 1 to any x_j exceeds α/N^2, and thus the interval $[1 - \frac{\alpha}{N^2}, 1]$ is disjoint from X_N. The probability of this event is less than

$$\left(1 - C_5 \frac{\sqrt{\alpha}}{N} \right)^N \leqslant e^{-C_5 \sqrt{\alpha}}.$$

Thus, for any α, $1 < \alpha < N^2$, it follows that

$$\mathbb{P}\left(\hat{\rho}(X_N) \geqslant \frac{\alpha}{N^2} \right) \leqslant e^{-C_5 \sqrt{\alpha}}.$$

In particular, for sufficiently large C_6, we have

$$\mathbb{P}\left(\hat{\rho}\left(X_N\right) \geqslant \frac{C_6 \log^2(N)}{N^2}\right) \leqslant N^{-3}.$$

Therefore,

$$\mathbb{E}\hat{\rho}\left(X_N\right) \leqslant \frac{1}{N^2} + \sum_{\alpha=1}^{C_6 \log^2(N)} \frac{\alpha+1}{N^2} \mathrm{e}^{-C_5\sqrt{\alpha}} + N^{-3}.$$

It is easy to see that the latter expression is bounded by C_7/N^2, which completes the proof for this case.

5.14.2 *Case* $0 < a < 2$

We again note that, if α is a number and $I = [\alpha, \alpha+\varepsilon] \subset [1 - \frac{1}{N^a}, 1]$ is an interval of length ε, then

$$\mu(I) = \int_\alpha^{\alpha+\varepsilon} \frac{\mathrm{d}t}{\pi\sqrt{1-t^2}} \geqslant \frac{1}{\pi}\frac{\varepsilon}{\sqrt{1-\alpha^2}} \geqslant \frac{1}{\pi}\frac{\varepsilon}{\sqrt{1-\left(1-\frac{1}{N^a}\right)^2}} \geqslant C_1 \varepsilon N^{\frac{a}{2}}.$$

Now consider n intervals of length $\frac{1}{nN^a}$ (and thus having μ-measure μ greater than $\frac{C_1}{nN^{\frac{a}{2}}}$) inside $[1-\frac{1}{N^a}, 1]$. As we have seen before, if $\hat{\rho}(X_N) > \frac{2}{nN^a}$, then for some $y \in [1-\frac{1}{N^a}, 1]$, the interval of length $\frac{2}{nN^a}$ centered at y is disjoint from X_N; thus one of the fixed intervals of length $\frac{1}{nN^a}$ is disjoint from X_N. Consequently,

$$\mathbb{P}\left(\hat{\rho}\left(X_N\right) \geqslant \frac{2}{nN^a}\right) \leqslant n\left(1 - \frac{C_1}{nN^{\frac{a}{2}}}\right)^N.$$

With

$$n := \frac{N^{1-\frac{a}{2}}}{A \log N},$$

where A large enough, we get

$$\mathbb{P}\left(\hat{\rho}\left(X_N\right) \geqslant \frac{2}{nN^a}\right) \leqslant n\left(1 - \frac{C_1}{nN^{\frac{a}{2}}}\right)^N \leqslant N^{-3}.$$

Therefore,

$$\mathbb{E}\hat{\rho} \leqslant C\frac{\log N}{N^{1+\frac{a}{2}}} + N^{-3},$$

which finishes the estimate from above.

For the estimate from below, we note that if $I=[\alpha,\alpha+\varepsilon]\subset[1-\frac{1}{N^a},1-\frac{1}{2N^a}]$, then

$$\mu\left(I\right)\leqslant C_2\frac{\varepsilon}{\sqrt{1-\left(1-\frac{1}{2N^a}\right)^2}}\leqslant 2C_2\varepsilon N^{\frac{a}{2}}.$$

Take n intervals in $[1-\frac{1}{N^a},1-\frac{1}{2N^a}]$ of length comparable to $\frac{1}{nN^a}$ and having equal μ-measures $C_3\frac{1}{nN^{\frac{a}{2}}}$ (note that if we are allowed to take such intervals near 1, then the best measure we can get is $\frac{1}{\sqrt{nN^a}}$). If one of them is disjoint from X_N, then $\hat{\rho}(X_N)\geqslant\frac{C_4}{nN^a}$. Thus,

$$\mathbb{P}\left(\hat{\rho}\left(X_N\right)\geqslant\frac{C_4}{nN^a}\right)\geqslant f\left(N,nN^{\frac{a}{2}}/C_3,n\right).$$

It is easy to see that if we take

$$n:=\frac{N^{1-\frac{a}{2}}}{A\log N-B\log\log N}$$

for suitable A and B, then the latter expression tends to one. Recall that $0<a<2$. Therefore, for large values of N, we have

$$\mathbb{P}\left(\hat{\rho}\left(X_N\right)\geqslant C_4\frac{\log N}{N^{1+\frac{a}{2}}}\right)\geqslant\frac{1}{2},$$

which completes the proof for this case.

5.14.3 *The estimate for $\tilde{\rho}$*

For the estimate from above simply note that for any interval I, we have $\mu(I)\geqslant|I|$. For the estimate from below, take the interval $[-\frac{1}{2},\frac{1}{2}]$. For any interval $I\subset[-\frac{1}{2},\frac{1}{2}]$, we have $\mu(I)\leqslant C|I|$, and thus the estimation from below runs as usual. ■

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References

[1] Aistleitner, C., J. Brauchart, and J. Dick. "Point sets on the sphere $\mathbb{S}^2$ with small spherical cap discrepancy." *Discrete & Computational Geometry* 48, no. 4 (2012): 990–1024.

[2] Bourgain, J., P. Sarnak, and Z. Rudnick. "Local statistics of lattice points on the sphere," (2012): preprint arXiv:1204.0134.

[3] Brauchart, J. S., J. Dick, E. B. Saff, I. H. Sloan, Y. G. Wang, and R. S. Womersley. "Covering of spheres by spherical caps and worst-case error for equal weight cubature in Sobolev

spaces." *The Australian Journal of Mathematical Analysis and Applications* 431, no. 2 (2015): 782–811.

[4] Brauchart, J. S., J. Dick, E. B. Saff, I. H. Sloan, Y. G. Wang, and R. S. Womersley. "Covering, separation, and hole radii for random point sets on the sphere," in preparation.

[5] Burago, D., Y. Burago, and S. Ivanov. *A Course in Metric Geometry*. Providence, RI: American Mathematical Society, 2001.

[6] David, H. A. and H. N. Nagaraja. *Order Statistics. Wiley Series in Probability and Statistics*, 3rd ed. Hoboken, NJ: Wiley-Interscience (John Wiley & Sons), 2003.

[7] Falconer, K. J. *The Geometry of Fractal Sets*. Cambridge Tracts in Mathematics 85. Cambridge: Cambridge University Press, 1986, xiv+162 pp.

[8] Fuselier, E. J. and G. B. Wright. "Stability and error estimates for vector field interpolation and decomposition on the sphere with RBFs." *SIAM Journal on Numerical Analysis* 47, no. 5 (2009): 3213–39.

[9] Lee, J. *Introduction to Smooth Manifolds*, 2nd ed. Graduate Texts in Mathematics 218. New York: Springer, 2013, xvi+708 pp.

[10] Maehara, H. "A threshold for the size of random caps to cover a sphere." *Annals of the Institute of Statistical Mathematics* 40, no. 4 (1988): 665–70.

[11] Mattila, P. *Geometry of Sets and Measures in Euclidean Spaces. Fractals and Rectifiability*. Cambridge Studies in Advanced Mathematics 44. Cambridge: Cambridge University Press, 1995, xii+343 pp.

[12] Mhashar, H. N., F. J. Narcowich, J. Prestin, and J. D. Ward. "L^p Bernstein estimates and approximation by spherical basis functions." *Mathematics of Computation* 79, no. 271 (2010): 1647–49.

[13] Plan, Y. and R. Vershynin. "One-bit compressed sensing by linear programming," (2011): preprint arXiv:1109.4299.

[14] Plan, Y. and R. Vershynin. "Dimension reduction by random hyperplane tessellations," (2011): preprint arXiv:1111.4452.

[15] Vershynin, R. "Introduction to the non-asymptotic analysis of random matrices. *Compressed Sensing*, 210–68. Cambridge: Cambridge University Press, 2012.