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ORTHOGONAL POLYNOMIALS FOR AREA-TYPE MEASURES AND IMAGE RECOVERY*

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Abstract. Let G be a finite union of disjoint and bounded Jordan domains in the complex plane, let $\mathcal{K}$ be a compact subset of G, and consider the set $G^\star$ obtained from G by removing $\mathcal{K}$; i.e., $G^\star := G \setminus \mathcal{K}$. We refer to G as an archipelago and $G^\star$ as an archipelago with lakes. Denote by $\{p_n(G,z)\}_{n=0}^\infty$ and $\{p_n(G^\star,z)\}_{n=0}^\infty$ the sequences of the Bergman polynomials associated with G and $G^\star$, respectively, that is, the orthonormal polynomials with respect to the area measure on G and $G^\star$. The purpose of the paper is to show that $p_n(G,z)$ and $p_n(G^\star,z)$ have comparable asymptotic properties, thereby demonstrating that the asymptotic properties of the Bergman polynomials for $G^\star$ are determined by the boundary of G. As a consequence we can analyze certain asymptotic properties of $p_n(G^\star,z)$ by using the corresponding results for $p_n(G,z)$, which were obtained in a recent work by B. Gustafsson, M. Putinar, and two of the present authors. The results lead to a reconstruction algorithm for recovering the shape of an archipelago with lakes from a partial set of its complex moments.

Key words. Bergman space, orthogonal polynomials, image recovery, Christoffel functions, reproducing kernel

AMS subject classifications. Primary, 65E05, 30E05, 42C05; Secondary, 41A10, 94A08, 32A36

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1. Introduction. Let $G := \cup_{j=1}^m G_j$ be a finite union of bounded Jordan domains G_j, $j = 1, \ldots, m$, in the complex plane $\mathbf{C}$, with pairwise disjoint closures, let $\mathcal{K}$ be a compact subset of G, and consider the set $G^\star$ obtained from G by removing $\mathcal{K}$, i.e., $G^\star := G \setminus \mathcal{K}$. Set $\Gamma_j := \partial G_j$ for the respective boundaries and let $\Gamma := \cup_{j=1}^m \Gamma_j$ denote the boundary of G. For later use we introduce also the (unbounded) complement Ω of $\overline{G}$ with respect to $\overline{\mathbf{C}}$, i.e., $\Omega := \overline{\mathbf{C}} \setminus \overline{G}$; see Figure 1. Note that $\Gamma = \partial G = \partial\Omega$. We call G an *archipelago* and $G^\star$ an *archipelago with lakes*.

Let $\{p_n(G,z)\}_{n=0}^\infty$ denote the sequence of *Bergman polynomials* associated with G. This is defined as the unique sequence of polynomials

$$p_n(G,z) = \gamma_n(G)z^n + \cdots, \quad \gamma_n(G) > 0, \quad n = 0, 1, 2, \ldots,$$

that are orthonormal with respect to the inner product

$$\langle f, g\rangle_G := \int_G f(z)\overline{g(z)}dA(z), \tag{1.1}$$

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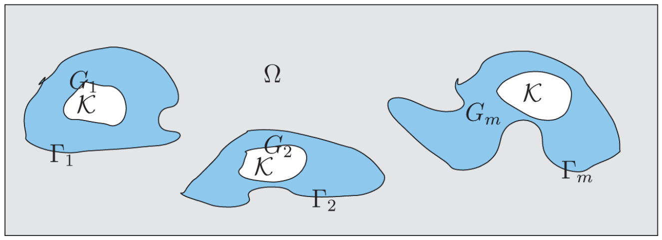

Fig. 1.

where dA stands for the differential of the area measure. We use $L^2(G)$ to denote the associated Lebesgue space with norm $\|f\|_{L^2(G)} := \langle f, f\rangle_G^{1/2}$.

The corresponding monic polynomials $p_n(G,z)/\gamma_n(G)$ can be equivalently defined by the extremal property

$$\left\|\frac{1}{\gamma_n(G)}p_n(G,\cdot)\right\|_{L^2(G)} := \min_{z^n+\cdots}\|z^n+\cdots\|_{L^2(G)}.$$

Thus,

$$\frac{1}{\gamma_n(G)} = \min_{z^n+\cdots}\|z^n+\cdots\|_{L^2(G)}. \tag{1.2}$$

A related extremal problem leads to the sequence $\{\lambda_n(G,z)\}_{n=1}^{\infty}$ of the so-called Christoffel functions associated with the area measure on G. These are defined, for any $z \in \mathbf{C}$, by

$$\lambda_n(G,z) := \inf\{\|P\|^2_{L^2(G)},\, P \in \mathbb{P}_n \text{ with } P(z)=1\}, \tag{1.3}$$

where $\mathbb{P}_n$ stands for the space of complex polynomials of degree up to n. Using the Cauchy–Schwarz inequality it is easy to verify (see, e.g., [17, section 3]) that

$$\frac{1}{\lambda_n(G,z)} = \sum_{k=0}^{n}|p_k(G,z)|^2, \quad z \in \mathbf{C}. \tag{1.4}$$

Clearly, $\lambda_n(G,z)$ is the inverse of the diagonal of the kernel polynomial

$$K_n^G(z,\zeta) := \sum_{k=0}^{n}\overline{p_k(G,\zeta)}p_k(G,z). \tag{1.5}$$

We use $L^2_a(G)$ to denote the *Bergman space* associated with G and the inner product (1.1), i.e.,

$$L^2_a(G) := \left\{f \text{ analytic in } G \text{ and } \|f\|_{L^2(G)} < \infty\right\},$$

and note that $L^2_a(G)$ is a Hilbert space that possesses a reproducing kernel, which we denote by $K_G(z,\zeta)$. That is, $K_G(z,\zeta)$ is the unique function $K_G(z,\zeta): G\times G \to \mathbf{C}$ such that $K_G(\cdot,\zeta) \in L^2_a(G)$, for all $\zeta \in G$, with the reproducing property

$$f(\zeta) = \langle f, K_G(\cdot,\zeta)\rangle_G \quad \forall\, f \in L^2_a(G). \tag{1.6}$$

In particular, for any $z \in G$,

$$K_G(z,z) = \|K_G(\cdot,z)\|^2_{L^2(G)} > 0, \tag{1.7}$$

which, in view of the reproducing property and the Cauchy–Schwarz inequality, yields the characterization

$$\frac{1}{K_G(z,z)} = \inf\{\|f\|^2_{L^2(G)},\ f \in L^2_a(G) \text{ with } f(z) = 1\}; \tag{1.8}$$

cf. (1.3)–(1.5). Furthermore, due to the same property and the completeness of polynomials in $L^2_a(G)$ (see, e.g., [7, Lemma 3.3]), the kernel $K_G(z,\zeta)$ is given, for any $\zeta \in G$, in terms of the Bergman polynomials by

$$K_G(z,\zeta) = \sum_{n=0}^{\infty} \overline{p_n(G,\zeta)}\, p_n(G,z), \tag{1.9}$$

locally uniformly with respect to $z \in G$.

Consider now the Bergman spaces $L^2_a(G_j)$, $j = 1, 2, \ldots, m$, associated with the components G_j,

$$L^2_a(G_j) := \left\{ f \text{ analytic in } G_j \text{ and } \|f\|_{L^2(G_j)} < \infty \right\},$$

and let $K_{G_j}(z,\zeta)$ denote their respective reproducing kernels. Then it is straightforward to verify using the uniqueness property of $K_G(\cdot,\zeta)$ the following relation:

$$K_G(z,\zeta) = \begin{cases} K_{G_j}(z,\zeta) & \text{if } z,\zeta \in G_j,\ j = 1,\ldots,m, \\ 0 & \text{if } z \in G_j,\ \zeta \in G_k,\ j \neq k. \end{cases} \tag{1.10}$$

This relation leads to expressing $K_G(z,\zeta)$ in terms of conformal mappings $\varphi_j : G_j \to \mathbb{D}$, $j = 1, 2, \ldots, m$. This is so because, as is well-known (see, e.g., [5, p. 33]), for $z, \zeta \in G_j$,

$$K_{G_j}(z,\zeta) = \frac{\varphi_j'(z)\overline{\varphi_j'(\zeta)}}{\pi\left[1 - \varphi_j(z)\overline{\varphi_j(\zeta)}\right]^2}.$$

For $G^\star := G \setminus \mathcal{K}$, we likewise define $\langle f, g\rangle_{G^\star}$, the norm $\|f\|_{L^2(G^\star)}$, the Bergman space $L^2_a(G^\star)$ along with its reproducing kernel $K_{G^\star}(z,\zeta) : G^\star \times G^\star \to \mathbf{C}$, and associated orthonormal polynomials

$$p_n(G^\star, z) = \gamma_n(G^\star) z^n + \cdots, \quad \gamma_n(G^\star) > 0, \quad n = 0, 1, 2, \ldots,$$

as well as the associated Christoffel functions $\lambda_n^\star(G,z)$ and polynomial kernel functions $K_n^{G^\star}(z,\zeta)$. It is important to note, however, that the analogue of (1.9) with G replaced by $G^\star$ does not hold because the polynomials $\{p_n(G^\star,z)\}_{n=0}^{\infty}$ are not complete in $L^2_a(G^\star)$.

Since $G^\star \subset G$, it is readily verified that the following two comparison principles hold:

$$\lambda_n(G^\star, z) \le \lambda_n(G, z), \quad z \in \mathbf{C}, \tag{1.11}$$

and

$$K_G(z,z) \leq K_{G^\star}(z,z), \quad z \in G^\star. \tag{1.12}$$

The paper is organized as follows. In the next three sections we prove that holes inside the domains have little influence on the external asymptotics (a fact anticipated in [10, section 3]). Then, in section 5, we use this to modify the recent domain recovery algorithm from [7] to the case when one has no a priori knowledge about the holes. Another modification allows us to recover even the holes. We devote the last section to some comments on issues of stability of our algorithm.

2. Bergman polynomials on full domains versus domains with holes. The following theorem shows that in many respect Bergman polynomials on G and on $G^\star$ behave similarly.

THEOREM 2.1. *If G is a union of a finite family of bounded Jordan domains lying a positive distance apart and $G^\star = G \setminus \mathcal{K}$, where $\mathcal{K} \subset G$ is compact, then, as $n \to \infty$,*

(a) $\gamma_n(G^\star)/\gamma_n(G) \to 1$,

(b) $\|p_n(G^\star,\cdot) - p_n(G,\cdot)\|_{L^2(G)} \to 0$,

(c) $\lambda_n(G^\star,z)/\lambda_n(G,z) \to 1$ *uniformly on compact subsets of* $\overline{\mathbf{C}} \setminus \overline{G}$,

(d) $p_n(G^\star,z)/p_n(G,z) \to 1$ *uniformly on compact subsets of* $\overline{\mathbf{C}} \setminus \mathrm{Con}(G)$.

Here $\mathrm{Con}(G)$ *denotes the convex hull of* G.

Since outside $\overline{G}$ both $\lambda_n(G^\star,z)$ and $\lambda_n(G,z)$ tend to zero locally uniformly (see (2.10) below), while inside G both quantities tend to a positive finite limit (see the next lemma), part (c) of Theorem 2.1 is particularly useful in domain reconstruction (see section 5), because it tells us that, in the algorithm considered, for reconstructing the outer boundary Γ one does not need to know in advance whether there are holes inside G.

The proof of Theorem 2.1 is based on the following.

LEMMA 2.2. *We have*

$$\sum_{n=0}^{\infty} |p_n(G^\star,z)|^2 < \infty \tag{2.1}$$

uniformly on compact subsets of G. In particular, $p_n(G^\star,z) \to 0$ uniformly on compact subsets of G.

Proof. Let V be a compact subset of G. Choose a system $\sigma \subset G^\star$ of closed broken lines separating V from ∂G (meaning each $V \cap G_j$ is separated from each ∂G_j), and choose $r > 0$ such that the disk $D_r(z)$ of radius r about z lies in $G^\star$ for all $z \in \sigma$. For any $N > 1$ and fixed $z \in \sigma$ we obtain from the subharmonicity in t of

$$|P_N(t)|^2 := \left| \sum_{n=0}^{N} \overline{p_n(G^\star,z)} p_n(G^\star,t) \right|^2$$

the estimate

$$\begin{aligned}\left(\sum_{n=0}^{N} |p_n(G^\star,z)|^2 \right)^2 &= |P_N(z)|^2 \leq \frac{1}{r^2\pi} \int_{D_r(z)} |P_N(t)|^2 dA(t) \\ &\leq \frac{1}{r^2\pi} \int_{G^\star} |P_N(t)|^2 dA(t) = \frac{1}{r^2\pi} \sum_{n=0}^{N} |p_n(G^\star,z)|^2.\end{aligned}$$

Thus,

$$\sum_{n=0}^{N} |p_n(G^\star, z)|^2 \le \frac{1}{r^2\pi} \tag{2.2}$$

on σ; hence, again by subharmonicity, the same is true inside σ (i.e., in every bounded component of $\mathbf{C}\setminus\sigma$). For $N\to\infty$ we get

$$\sum_{n=0}^{\infty} |p_n(G^\star, z)|^2 \le \frac{1}{r^2\pi} \tag{2.3}$$

on and inside σ, but we still need to prove the uniform convergence on V of the series on the left-hand side.

Let σ_1 be another family of closed broken lines lying inside σ separating V and σ. If δ is the distance of σ and σ_1, then for any N and any choice $|\varepsilon_n| = 1$ we have, by Cauchy's formula for the derivative of an analytic function for $z, w \in \sigma_1$,

$$\left|\sum_{n=0}^{N} \varepsilon_n p_n(G^\star, z) p_n'(G^\star, w)\right| \le \frac{L}{2\pi\delta^2}\max_{t\in\sigma}\left|\sum_{n=0}^{N} \varepsilon_n p_n(G^\star, z) p_n(G^\star, t)\right|$$

$$\le \frac{L}{2\pi\delta^2}\max_{t\in\sigma}\left(\sum_{n=0}^{N}|p_n(G^\star,z)|^2\right)^{1/2}\left(\sum_{n=0}^{N}|p_n(G^\star,t)|^2\right)^{1/2} \le \frac{L}{2\delta^2}\frac{1}{r^2\pi^2},$$

where L is the length of σ. So for $w = z$ an appropriate choice of the ε_n's gives

$$\sum_{n=0}^{N} |p_n(G^\star, z)||p_n'(G^\star, z)| \le \frac{L}{2\delta^2}\frac{1}{r^2\pi^2}$$

for all $z \in \sigma_1$. But then, if ds is arc-length on σ_1, we obtain on σ_1

$$\frac{d}{ds}\sum_{n=0}^{N}|p_n(G^\star,\cdot)|^2\Bigg|_{\cdot\, =\, z} \le 2\sum_{n=0}^{N}|p_n(G^\star,z)||p_n'(G^\star,z)| \le \frac{L}{\delta^2}\frac{1}{r^2\pi^2},$$

which shows that on σ_1 the family

$$\left\{\sum_{n=0}^{N}|p_n(G^\star,z)|^2\right\}_{N=0}^{\infty}$$

is uniformly equicontinuous. Since it converges pointwise to a finite limit (see (2.3)), we can conclude that the convergence in (2.3) is uniform on σ_1 and hence (by subharmonicity) also on V (which lies inside σ_1). $\square$

Proof of Theorem 2.1. In view of (1.2) we have

$$\begin{aligned}\frac{1}{\gamma_n(G)^2} &\le \int_G \frac{|p_n(G^\star,z)|^2}{\gamma_n(G^\star)^2}dA(z) = \int_{G^\star} + \int_{\mathcal{K}} \\ &\le \frac{1}{\gamma_n(G^\star)^2} + \frac{\varepsilon_n^2|\mathcal{K}|}{\gamma_n(G^\star)^2} = \frac{1+\varepsilon_n^2|\mathcal{K}|}{\gamma_n(G^\star)^2},\end{aligned} \tag{2.4}$$

where

$$\varepsilon_n := \|p_n(G^\star, \cdot)\|_{\mathcal{K}} \to 0 \tag{2.5}$$

by Lemma 2.2. (Here and below we use $|\mathcal{K}|$ to denote the area measure of $\mathcal{K}$.) On the other hand, (1.11) gives that $\gamma_n(G^\star) \geq \gamma_n(G)$, which together with the preceding inequality shows

$$1 \leq \frac{\gamma_n(G^\star)^2}{\gamma_n(G)^2} \leq 1 + \varepsilon_n^2 |\mathcal{K}|, \tag{2.6}$$

and this proves (a).

Next we apply a standard parallelogram argument:

$$\int_{G^\star} \left| \frac{1}{2}\left(\frac{p_n(G,\cdot)}{\gamma_n(G)} - \frac{p_n(G^\star,\cdot)}{\gamma_n(G^\star)} \right) \right|^2 dA + \int_{G^\star} \left| \frac{1}{2}\left(\frac{p_n(G,\cdot)}{\gamma_n(G)} + \frac{p_n(G^\star,\cdot)}{\gamma_n(G^\star)} \right) \right|^2 dA$$

$$= \frac{1}{2}\int_{G^\star} \left| \frac{p_n(G,\cdot)}{\gamma_n(G)} \right|^2 dA + \frac{1}{2}\int_{G^\star} \left| \frac{p_n(G^\star,\cdot)}{\gamma_n(G^\star)} \right|^2 dA.$$

By (1.2) the second term on the left is $\geq 1/\gamma_n(G^\star)^2$, the second term on the right is $1/(2\gamma_n(G^\star)^2)$, and, according to (2.4), the first term on the right is

$$\leq \frac{1}{2}\int_{G} \left| \frac{p_n(G,\cdot)}{\gamma_n(G)} \right|^2 dA = \frac{1}{2\gamma_n(G)^2} \leq \frac{1+\varepsilon_n^2|\mathcal{K}|}{2\gamma_n(G^\star)^2}.$$

Therefore, we can conclude

$$\int_{G^\star} \left| \frac{p_n(G,\cdot)}{\gamma_n(G)} - \frac{p_n(G^\star,\cdot)}{\gamma_n(G^\star)} \right|^2 dA \leq \frac{2\varepsilon_n^2|\mathcal{K}|}{\gamma_n(G^\star)^2},$$

and since (2.6) implies

$$\left| 1 - \frac{\gamma_n(G^\star)}{\gamma_n(G)} \right| \leq \varepsilon_n^2 |\mathcal{K}|,$$

we arrive at

$$\int_{G^\star} |p_n(G,\cdot) - p_n(G^\star,\cdot)|^2 \, dA = O(\varepsilon_n^2), \tag{2.7}$$

as $n \to \infty$. It is easy to see that the norms on $G^\star$ and G for functions in $L_a^2(G)$ are equivalent; indeed, if $f \in L_a^2(G)$ and Γ_0 is the union of m Jordan curves lying in $G^\star$ and containing $\mathcal{K}$ in its interior, then

$$\|f\|^2_{L^2(G^\star)} \leq \|f\|^2_{L^2(G)} = \|f\|^2_{L^2(G^\star)} + \|f\|^2_{L^2(\mathcal{K})}$$

and, by subharmonicity,

$$\|f\|^2_{L^2(\mathcal{K})} \leq |\mathcal{K}| \max_{z\in\mathcal{K}} |f(z)|^2 \leq |\mathcal{K}| \max_{z\in\Gamma_0} |f(z)|^2 \leq \frac{|\mathcal{K}|}{R^2\pi} \|f\|^2_{L^2(G^\star)},$$

where $R := \operatorname{dist}(\Gamma_0, \partial G^\star)$. Hence part (b) follows from (2.7).

To prove (c), let z lie in $\overline{\mathbf{C}} \setminus \overline{G}$. For an $\varepsilon > 0$ select an M such that

$$\sum_{j=M}^{\infty} |p_j(G^\star, t)|^2 \leq \varepsilon, \qquad t \in \mathcal{K} \tag{2.8}$$

(see Lemma 2.2). For the polynomial

$$P_n(t) := \frac{\sum_{j=M}^{n} \overline{p_j(G^\star, z)} p_j(G^\star, t)}{\sum_{j=M}^{n} |p_j(G^\star, z)|^2}, \qquad n > M,$$

we have $P_n(z) = 1$ and

$$\int_{G^\star} |P_n(t)|^2 dA(t) = \frac{1}{\sum_{j=M}^{n} |p_j(G^\star, z)|^2}.$$

For its square integral over $\mathcal{K}$ we have by Hölder's inequality

$$\int_{\mathcal{K}} |P_n(t)|^2 dA(t) \leq \int_{\mathcal{K}} \frac{\sum_{j=M}^{n} |p_j(G^\star, t)|^2}{\sum_{j=M}^{n} |p_j(G^\star, z)|^2} dA(t) \leq \frac{|\mathcal{K}|\varepsilon}{\sum_{j=M}^{n} |p_j(G^\star, z)|^2}.$$

If we add together these last two integrals we obtain

$$\lambda_n(G, z) \leq \frac{1 + |\mathcal{K}|\varepsilon}{\sum_{j=M}^{n} |p_j(G^\star, z)|^2}. \tag{2.9}$$

On the other hand, it is easy to see that outside $\overline{G}$ we always have

$$\sum_{j=0}^{n} |p_j(G^\star, z)|^2 \to \infty \tag{2.10}$$

as $n \to \infty$, and actually this convergence to infinity is uniform on compact subsets of $\Omega := \overline{\mathbf{C}} \setminus \overline{G}$. Indeed, if $\{F_n\}$ denotes a sequence of Fekete polynomials associated with $\overline{G}$, then it is known (see, e.g., [12, Chapter III, Theorems 1.8, 1.9]) that

$$\|F_n\|_{\overline{G}}^{1/n} \to \operatorname{cap}(\overline{G}) = \operatorname{cap}(\Gamma), \quad n \to \infty, \tag{2.11}$$

where $\operatorname{cap}(\overline{G})$ denotes the *logarithmic capacity* of $\overline{G}$. At the same time

$$|F_n(z)|^{1/n} \to \operatorname{cap}(\overline{G}) \exp\left(g_\Omega(z, \infty)\right), \quad n \to \infty, \tag{2.12}$$

uniformly on compact subsets of $\overline{\mathbf{C}} \setminus \overline{G}$, where $g_\Omega(z, \infty)$ denotes the Green function of Ω with pole at infinity. Thus,

$$\lambda_n(G^\star, z) \leq \int_{G^\star} \left| \frac{F_n(t)}{F_n(z)} \right|^2 dA(t) \to 0, \quad n \to \infty, \tag{2.13}$$

uniformly on compact subsets of Ω. (Note that $g_\Omega(z, \infty)$ has positive lower bound there.) Since $1/\lambda_n(G^\star, z)$ is the left-hand side of (2.10), the relation (2.10) follows.

Combining (2.9) and (2.10) we can write

$$\lambda_n(G^\star, z) \leq \lambda_n(G, z) \leq \frac{1 + |\mathcal{K}|\varepsilon}{\sum_{j=M}^{n} |p_j(G^\star, z)|^2} = (1 + o(1)) \frac{1 + |\mathcal{K}|\varepsilon}{\sum_{j=0}^{n} |p_j(G^\star, z)|^2}$$

$$= (1 + o(1))(1 + |\mathcal{K}|\varepsilon)\lambda_n(G^\star, z), \tag{2.14}$$

and since this relation is uniform on compact subsets of Ω, part (c) follows since $\varepsilon > 0$ was arbitrary.

Finally, we prove part (d). Notice first of all that for $i, j \le n$ the expression $(z^i t^j - z^j t^i)/(z-t)$ is a polynomial in t of degree smaller than n, and therefore the same is true of

$$\frac{p_n(G,z)p_n(G^\star,t) - p_n(G,t)p_n(G^\star,z)}{z-t},$$

so this expression is orthogonal to $p_n(G,t)$ on G with respect to area measure. Hence,

$$\int_G \frac{p_n(G,z)p_n(G^\star,t)\overline{p_n(G,t)}}{z-t}dA(t) = \int_G \frac{p_n(G,t)p_n(G^\star,z)\overline{p_n(G,t)}}{z-t}dA(t),$$

and then division gives

$$\frac{p_n(G^\star,z)}{p_n(G,z)} - 1 = \frac{\int_G \frac{(p_n(G^\star,t)-p_n(G,t))\overline{p_n(G,t)}}{z-t}dA(t)}{\int_G \frac{|p_n(G,t)|^2}{z-t}dA(t)}. \tag{2.15}$$

Let now z be outside the convex hull of G and let z_0 be the closest point in the convex hull to z. Then G lies in the half-plane $\{t \mid \Re\{(z-t)/(z-z_0)\} \ge 1\}$, so for $t \in G$

$$\Re\frac{z-z_0}{z-t} = \frac{\Re\{(z-t)/(z-z_0)\}}{(|z-t|/|z-z_0|)^2} \ge \frac{|z-z_0|^2}{|z-t|^2} \ge \frac{|z-z_0|^2}{(|z-z_0| + \operatorname{diam}(G))^2}.$$

This gives the following bound for the modulus of the denominator in (2.15):

$$\begin{aligned}\left|\int_G \frac{|p_n(G,t)|^2}{z-t}dA(t)\right| &\ge \frac{1}{|z-z_0|}\Re\int_G \frac{z-z_0}{z-t}|p_n(G,t)|^2 dA(t)\\ &\ge \frac{|z-z_0|}{(|z-z_0|+\operatorname{diam}(G))^2}\int_G |p_n(G,t)|^2 dA(t)\\ &= \frac{|z-z_0|}{(|z-z_0|+\operatorname{diam}(G))^2}.\end{aligned}$$

On the other hand, in the numerator of (2.15) we have $1/|z-t| \le 1/|z-z_0|$, so we obtain from the Cauchy–Schwarz inequality that

$$\begin{aligned}&\left|\int_G \frac{(p_n(G^\star,t)-p_n(G,t))\overline{p_n(G,t)}}{z-t}dA(t)\right|\\ &\qquad\le \frac{1}{|z-z_0|}\left(\int_G |p_n(G^\star,t)-p_n(G,t)|^2 dA(t)\right)^{1/2}.\end{aligned}$$

Collecting these estimates we can see that

$$\left|\frac{p_n(G^\star,z)}{p_n(G,z)} - 1\right| \le \frac{(|z-z_0|+\operatorname{diam}(G))^2}{|z-z_0|^2}\|p_n(G^\star,\cdot) - p_n(G,\cdot)\|_{L^2(G)}.$$

Now invoking part (b), we can see that the left-hand side is uniformly small on compact subsets of $\overline{\mathbf{C}} \setminus \operatorname{Con}(G)$ since for $\operatorname{dist}(z,G) \ge \delta$ we have

$$\frac{|z-z_0|+\operatorname{diam}(G)}{|z-z_0|} \le \frac{\delta + \operatorname{diam}(G)}{\delta}.$$

This proves (d).[1] □

3. Smooth outer boundary. Next, we make Theorem 2.1 more precise when the boundary Γ of G is $C(p,\alpha)$-smooth, by which we mean that, for $j = 1,\dots,m$, if γ_j is the arc-length parametrization of Γ_j, then γ_j is p-times differentiable, and its pth derivative belongs to the Lip α.

Let $\|\cdot\|_{\overline{G}}$ denote the supremum norm on the closure $\overline{G}$ of G.

THEOREM 3.1. *If each of the boundary curves Γ_j is $C(p,\alpha)$-smooth for some $p \in \{1,2,\dots\}$ and $0 < \alpha < 1$, then*

(a) $\gamma_n(G^\star)/\gamma_n(G) = 1 + O(n^{-2p+2-2\alpha})$,

(b) $\|p_n(G^\star,\cdot) - p_n(G,\cdot)\|_{\overline{G}} = O(n^{-p+2-\alpha})$,

(c) $\lambda_n(G^\star,z)/\lambda_n(G,z) = 1 + O(n^{-2p+3-2\alpha})$, *uniformly on compact subsets of* $\overline{\mathbf{C}} \setminus \overline{G}$,

(d) $p_n(G^\star,z)/p_n(G,z) = 1 + O(n^{-p+1-\alpha})$, *uniformly on compact subsets of* $\overline{\mathbf{C}} \setminus \mathrm{Con}(G)$.

If each Γ_j is analytic, then (a)–(d) *are true with $O(q^n)$ on the right-hand sides for some $0 < q < 1$.*

Note that now in (b) we have the supremum norm, so $p_n(G^\star,z) - p_n(G,z) \to 0$ uniformly on $\overline{G}$ if $p > 1$. Note also that nothing like (d) is possible in the convex hull of G since $p_n(G,\cdot)$ may have zeros there, which need not be zeros of $p_n(G^\star,\cdot)$.

As background for the proof of Theorem 3.1, we shall first define m special holes (lakes) whose union contains $\mathcal{K}$. For this purpose, let φ_j map G_j conformally onto the unit disk $\mathbb{D}$, and select an $0 < r < 1$ such that each of the holes $\mathcal{K}_j := \mathcal{K} \cap G_j$ is mapped by φ_j into the disk $\mathbb{D}_r := \{w : |w| < r\}$. Let $\widetilde{\mathbb{D}} := \{w : r < |w| < 1\}$ and define $\widetilde{G}_j := \varphi_j^{-1}(\widetilde{\mathbb{D}})$, $\widetilde{G} := \cup_{j=1}^m \widetilde{G}_j$. Thus, the special holes $\widetilde{\mathcal{K}}_j := G_j \setminus \widetilde{G}_j$ we are considering are the preimages of the closed disk $\overline{\mathbb{D}}_r$ under φ_j. Clearly, the above construction leads to the inclusions

$$\widetilde{G} \subset G^\star \subset G. \tag{3.1}$$

We shall need to work with functions in the Bergman space $L_a^2(G)$ but with the inner product

$$\langle f, g\rangle_{\widetilde{G}} := \int_{\widetilde{G}} f(z)\overline{g(z)}dA(z), \tag{3.2}$$

and corresponding norm $\|\cdot\|_{\widetilde{G}}$. Let $L_a^{2\#}(G)$ denote the space of functions in $L_a^2(G)$ endowed with the inner product (3.2). It is easy to see that $L_a^{2\#}(G)$ is again a Hilbert space, but note that it is different from $L_a^2(\widetilde{G})$. (The definition of the norm on the two spaces is the same, but the latter space contains also functions that may not be analytically continued throughout G, while the former space contains only analytic functions in G.) In fact, in $L_a^{2\#}(G)$, the polynomials $\{p_n(\widetilde{G},\cdot)\}_{n=0}^\infty$ form a *complete* orthonormal system (they also form an orthonormal system in $L_a^2(\widetilde{G})$, which, however, is not complete). Consequently, the reproducing kernel of $L_a^{2\#}(G)$ is

$$K^\#(z,\zeta) = \sum_{k=0}^\infty \overline{p_k(\widetilde{G},\zeta)}p_k(\widetilde{G},z). \tag{3.3}$$

[1] The analysis used in the proof of part (d) was also found independently by B. Simanek (see [13, Lemma 2.1 and Theorem 2.2]).

Note that by Lemma 2.2 (with $G^\star$ replaced by $\widetilde{G}$) the series on the right-hand side converges uniformly on compact subsets of $G \times G$.

Analogously, we define the Hilbert space $L_a^{2\#}(\mathbb{D})$ consisting of functions in $L_a^2(\mathbb{D})$, but with inner product

$$\langle f, g\rangle_{\widetilde{\mathbb{D}}} := \int_{\widetilde{\mathbb{D}}} f(w)\overline{g(w)}dA(w). \tag{3.4}$$

The following lemma provides a representation for the reproducing kernel $K^\#(z,\zeta)$ in terms of the reproducing kernel for the space $L_a^{2\#}(\mathbb{D})$.

Lemma 3.2. *Let $J(w,\omega)$ denote the reproducing kernel for $L_a^{2\#}(\mathbb{D})$. Then,*

$$K^\#(z,\zeta) = \begin{cases} \overline{\varphi_j'(\zeta)}\varphi_j'(z)J(\varphi_j(z),\varphi_j(\zeta)) & \text{if } z,\zeta \in G_j,\ j=1,\dots,m, \\ 0 & \text{if } z \in G_j,\ \zeta \in G_k,\ j \neq k. \end{cases} \tag{3.5}$$

Furthermore,

$$J(w,\omega) = \sum_{\nu=0}^{\infty} \frac{r^{2\nu}}{\pi(1 - r^{2\nu}w\overline{\omega})^2}, \quad w,\omega \in \mathbb{D}, \tag{3.6}$$

and consequently, for $z,\zeta \in G_j$,

$$K^\#(z,\zeta) = \overline{\varphi_j'(\zeta)}\varphi_j'(z)\sum_{\nu=0}^{\infty} \frac{r^{2\nu}}{\pi[1 - r^{2\nu}\overline{\varphi_j(\zeta)}\varphi_j(z)]^2}. \tag{3.7}$$

Proof. As with (1.10) it suffices to verify (3.5) for $z,\zeta \in G_j$, $j=1,\dots,m$. In fact, for $z,\zeta \in G_j$ the relation in (3.5) is quite standard; see, e.g., [3, section 1.3, Theorem 3]. To derive this relation, observe that since the Jacobian of the mapping $w = \varphi_j(z)$ is $|\varphi_j'(z)|^2$, we have

$$\int_{\widetilde{G}_j} |F(\varphi_j(z))|^2|\varphi_j'(z)|^2 dA(z) = \int_{\widetilde{\mathbb{D}}} |F(w)|^2 dA(w)$$

for any $F \in L_a^{2\#}(\mathbb{D})$. Hence, the mapping $F \to F(\varphi_j)\varphi_j'$ is an isometry from $L_a^{2\#}(\mathbb{D})$ into $L_a^{2\#}(G_j) := \{f\chi_{G_j} : f \in L_a^{2,\#}(G)\}$. This mapping is actually onto $L_a^{2\#}(G_j)$, with inverse $f \to f(\varphi_j^{-1})(\varphi_j^{-1})'$.

Next, from the reproducing property of $J(w,\omega)$, it follows that for $\omega \in \mathbb{D}$,

$$F(\omega) = \int_{\widetilde{\mathbb{D}}} F(w)\overline{J(w,\omega)}dA(w), \quad F \in L_a^{2\#}(\mathbb{D}).$$

If we make the change of variable $w = \varphi_j(z)$, $\omega = \varphi_j(\zeta)$, this takes the form

$$F(\varphi_j(\zeta)) = \int_{\widetilde{G}_j} F(\varphi_j(z))\overline{J(\varphi_j(z),\varphi_j(\zeta))}|\varphi_j'(z)|^2 dA(z), \quad \zeta \in G_j,$$

which, after multiplication by $\varphi_j'(\zeta)$, gives for $f(\zeta) := F(\varphi_j(\zeta))\varphi_j'(\zeta)$ that

$$f(\zeta) = \int_{\widetilde{G}_j} f(z)\overline{\overline{\varphi_j'(\zeta)}\varphi_j'(z)J(\varphi_j(z),\varphi_j(\zeta))}dA(z), \quad \zeta \in G_j. \tag{3.8}$$

Thus $\overline{\varphi_j'(\zeta)}\varphi_j'(z)J(\varphi_j(z),\varphi_j(\zeta))$ is the reproducing kernel for the space $L_a^{2\#}(G_j)$, which establishes (3.5).

To obtain the formula for $J(w, \omega)$, we note that the polynomials

$$\left(\frac{\pi}{n+1}\left(1-r^{2n+2}\right)\right)^{-1/2} w^n, \quad n = 0, 1, \ldots,$$

form a complete orthonormal system in the space $L_a^{2\#}(\mathbb{D})$. Therefore, we obtain the following representation:

$$\begin{aligned} J(w,\omega) &= \sum_{n=0}^{\infty}\left(\frac{\pi}{n+1}\left(1-r^{2n+2}\right)\right)^{-1} w^n\overline{\omega}^n = \sum_{n=0}^{\infty}\frac{n+1}{\pi}\sum_{\nu=0}^{\infty} r^{2\nu}r^{2n\nu}w^n\overline{\omega}^n \\ &= \sum_{\nu=0}^{\infty} r^{2\nu}\sum_{n=0}^{\infty}\frac{n+1}{\pi}r^{2n\nu}w^n\overline{\omega}^n = \sum_{\nu=0}^{\infty}\frac{r^{2\nu}}{\pi(1-r^{2\nu}w\overline{\omega})^2}, \end{aligned}$$

and the result (3.7) follows from (3.5). ▯

Proof of Theorem 3.1. With the above preparations we now turn to the proof of part (a) in Theorem 3.1. First, we need a good polynomial approximation of the kernel $K^{\#}(\cdot,\zeta)$ on $\overline{G}$, for fixed $\zeta \in V$, where V is a compact subset of G_j. By the Kellogg–Warschawskii theorem (see, e.g., [9, Theorem 3.6]), our assumption $\Gamma_j \in C(p,\alpha)$ implies that φ_j belongs to the class $C^{p+\alpha}$ on Γ_j. Thus, $\varphi_j' \in C^{p-1+\alpha}$ on Γ_j and (3.7) shows that the kernel $K^{\#}(\cdot,\zeta)$ is a $C^{p-1+\alpha}$-smooth function on Γ_j and the smoothness is uniform when ζ lies in a compact subset V of G_j. Consequently (see, e.g., [16, p. 34]), there are polynomials $P_{\nu,j,\zeta}(z)$ of degree ν such that for $\zeta \in V$

$$\sup_{z\in\Gamma_j}|K^{\#}(z,\zeta) - P_{\nu,j,\zeta}(z)| \le C(\Gamma_j, V)\frac{1}{\nu^{p-1+\alpha}}, \quad \nu \in \mathbb{N},\ j = 1,\ldots,m,$$

where $C(\Gamma_j, V)$ here and below denotes a positive constant, not necessarily the same at each appearance, that depends on Γ_j and V but is independent of ν. Therefore, the maximum modulus principle gives

$$\sup_{z\in\overline{G}_j}|K^{\#}(z,\zeta) - P_{\nu,j,\zeta}(z)| \le C(\Gamma_j, V)\frac{1}{\nu^{p-1+\alpha}}, \quad \zeta \in V. \tag{3.9}$$

Note that this provides a good approximation to $K^{\#}(z,\zeta)$ only for $z \in \overline{G}_j$. However, $K^{\#}(z,\zeta)$ is also defined for $z \in \overline{G}_k$, $k \neq j$. Actually, as we have seen in (3.5), for such values $K^{\#}(z,\zeta) = 0$. Therefore, in order to obtain a good approximation to $K^{\#}(z,\zeta)$ for all $z \in \overline{G}$, we have to modify the polynomials $\{P_{\nu,j,\zeta}(z)\}$. To this end, we note that since (3.9) implies that the $\{P_{\nu,j,\zeta}(z)\}$ are bounded uniformly for $z \in \overline{G}_j$, $\zeta \in V$ and $\nu \ge 1$, the Bernstein–Walsh lemma [18, p. 77] implies that there is a constant $\tau > 0$ such that

$$|P_{\nu,j,\zeta}(z)| \le C(\Gamma, V)\tau^{\nu}, \quad z \in \overline{G}. \tag{3.10}$$

Consider next the characteristic function

$$\chi_{\overline{G}_j}(z) := \begin{cases} 1 & \text{if } z \in \overline{G}_j, \\ 0 & \text{if } z \in \overline{G}_k,\ k \neq j. \end{cases} \tag{3.11}$$

Since $\chi_{\overline{G}_j}$ has an analytic continuation to an open set containing $\overline{G}$, it is known from the theory of polynomial approximation (cf. [18, p. 75]) that there exist polynomials $H_{n/2,j}(z)$ of degree at most $n/2$ such that

$$\sup_{z\in\overline{G}}|\chi_{\overline{G}_j}(z) - H_{n/2,j}(z)| \le C(\Gamma, V)\eta^n \tag{3.12}$$

for some $0 < \eta < 1$.

For some small $\epsilon > 0$ we set

$$Q_{n,j,\zeta}(z) := P_{\epsilon n,j,\zeta}(z) H_{n/2,j}(z).$$

This is a polynomial in z of degree at most $\epsilon n + (n/2) < n$, and (3.11)–(3.12), in conjunction with (3.9)–(3.10), yield for large n

$$\sup_{z \in \overline{G}_j} |K^{\#}(z,\zeta) - Q_{n,j,\zeta}(z)| \leq C(\Gamma_j, V) \frac{1}{(\epsilon n)^{p-1+\alpha}} + C(\Gamma, V) \tau^{\epsilon n} \eta^n$$

and

$$\sup_{z \in \overline{G} \setminus \overline{G}_j} |K^{\#}(z,\zeta) - Q_{n,j,\zeta}(z)| \leq C(\Gamma, V) \tau^{\epsilon n} \eta^n, \quad \zeta \in V \subset G_j.$$

Thus, if we fix $\epsilon > 0$ so small that $\tau^{\epsilon} \eta < 1$ is satisfied, we obtain for large enough n

$$\sup_{z \in \overline{G}} |K^{\#}(z,\zeta) - Q_{n,j,\zeta}(z)| \leq C(\Gamma, V) \frac{1}{n^{p-1+\alpha}}. \tag{3.13}$$

This is our desired estimate.

Since $Q_{n,j,\zeta}(z)$ is of degree smaller than n, using the reproducing property of the kernel $K^{\#}(z,\zeta)$ and the orthonormality of $p_n(\widetilde{G}, z)$ with respect to the inner product (3.2), we conclude that

$$\begin{aligned} p_n(\widetilde{G}, \zeta) &= \langle p_n(\widetilde{G}, \cdot), K^{\#}(\cdot, \zeta) \rangle_{\widetilde{G}} \\ &= \langle p_n(\widetilde{G}, \cdot), K^{\#}(\cdot, \zeta) - Q_{n,j,\zeta} \rangle_{\widetilde{G}}. \end{aligned}$$

Therefore, from the Cauchy–Schwarz inequality and (3.13), we obtain the following uniform estimate for $\zeta \in V$:

$$|p_n(\widetilde{G}, \zeta)| \leq C(\Gamma, V) \frac{1}{n^{p-1+\alpha}},$$

where we recall that V is a compact subset of G_j. Since this is true for any $j = 1, \ldots, m$, we have shown that

$$|p_n(\widetilde{G}, \zeta)| \leq C(\Gamma, V) \frac{1}{n^{p-1+\alpha}}, \quad \zeta \in V, \tag{3.14}$$

where now V is any compact subset of G.

Consequently, with $V = \widetilde{\mathcal{K}} := \cup_{j=1}^{m} \widetilde{\mathcal{K}}_j$ in (3.14), and $G^{\star}$ and $\mathcal{K}$ replaced by $\widetilde{G}$ and $\widetilde{\mathcal{K}}$ in (2.4) and (2.5), from (2.6) we get

$$\frac{\gamma_n(\widetilde{G})}{\gamma_n(G)} = 1 + O\left(\frac{1}{n^{2(p-1+\alpha)}} \right), \tag{3.15}$$

which in view of the fact

$$\gamma_n(G) \leq \gamma_n(G^{\star}) \leq \gamma_n(\widetilde{G})$$

yields part (a) of the theorem.

To prove part (b), notice that (3.15) is (2.6) with $\varepsilon_n = O(n^{-p+1-\alpha})$, and so the argument leading from (2.6) to (2.7) yields

$$\|p_n(G,\cdot) - p_n(G^\star,\cdot)\|_{L^2(G^\star)} = O\left(\frac{1}{n^{p-1+\alpha}}\right). \tag{3.16}$$

The L^2-estimate in (3.16) holds also over G since, as was previously remarked, the two norms $\|\cdot\|_{L^2(G)}$ and $\|\cdot\|_{L^2(G^\star)}$ are equivalent in $L^2_a(G)$. The uniform estimate in part (b) then follows from the L^2-estimate by using the inequality

$$\|Q_n\|_{\overline{G}} \le C(\Gamma) n \|Q_n\|_{L^2(G)},$$

which is valid for all polynomials Q_n of degree at most $n \in \mathbb{N}$, where the constant $C(\Gamma)$ depends on Γ only; see [16, p. 38].

In proving part (c) we may assume $p + \alpha > 3/2$ (see Theorem 2.1(c)). It follows from (3.14) that

$$\sum_{k=n}^{\infty} |p_k(\widetilde{G}, z)|^2 = O(n^{-2p+3-2\alpha})$$

uniformly on compact subsets of G, i.e., (2.8) holds (for $\widetilde{G}$ in place of $G^\star$) with $\varepsilon = O(n^{-2p+3-2\alpha})$. Copying the proof leading from (2.8) to (2.14) with this ε we get

$$\lambda_n(\widetilde{G}, z) \le \lambda_n(G, z) = (1 + O(n^{-2p+3-2\alpha}))\lambda_n(\widetilde{G}, z)$$

(indeed, by that proof the $o(1)$ in (2.14) is exponentially small). In view of $\widetilde{G} \subset G^\star \subset G$ this then implies

$$\begin{aligned}\lambda_n(G^\star, z) \le \lambda_n(G, z) &= (1 + O(n^{-2p+3-2\alpha}))\lambda_n(\widetilde{G}, z)\\ &\le (1 + O(n^{-2p+3-2\alpha}))\lambda_n(G^\star, z),\end{aligned}$$

which is part (c) in the theorem.

Part (d) follows at once from the L^2-estimate in (3.16), by working as in the proof of (d) in Theorem 2.1.

Regarding the case when all the curves Γ_j are analytic, we have that the conformal maps φ_j are analytic on $\overline{G}_j$, and then so is the kernel $K^\#(z,\zeta)$ for $z \in \overline{G}$, and all fixed $\zeta \in \widetilde{G}$. More precisely, if V is a compact subset of $\widetilde{G}$, then there is an open set $\overline{G} \subset U$ such that for $\zeta \in V$ the kernel $K(z,\zeta)$ is analytic for $z \in U$. Then, from the proof of the classical polynomial approximation theorem for analytic functions mentioned previously, together with the formula for $K^\#(z,\zeta)$, it follows that there is a $0 < q < 1$ and a constant C independent of $\zeta \in V$, such that in place of (3.9) we have

$$\sup_{z \in \tilde{G}_j} |K^\#(z,\zeta) - P_{n-1,j,\zeta}(z)| \le Cq^n, \quad \zeta \in V. \tag{3.17}$$

Thus, instead of (3.14), we obtain

$$\begin{aligned}\overline{|p_n(\widetilde{G}, \zeta)|} &= \left| \int_{\widetilde{G}} K^\#(z,\zeta) \overline{p_n(\widetilde{G}, z)}\, dA(z) \right|\\ &= \left| \int_{\widetilde{G}} (K^\#(z,\zeta) - P_{n-1,j,\zeta}(z)) \overline{p_n(\widetilde{G}, z)}\, dA(z) \right| \le C|G|^{1/2} q^n,\end{aligned}$$

so the ε_n in (2.5) is $O(q^n)$, and then the proofs of (a)–(d) above give the same statements with error $O(q^n)$ (for a possibly different $0 < q < 1$). ∎

Remark 3.1. Our theorems thus far have emphasized the similar asymptotic behavior of the Bergman orthogonal polynomials for an archipelago without lakes and the Bergman polynomials for an archipelago with lakes. Differences appear, however, when one considers the asymptotic behaviors of the zeros of the two sequences of polynomials. A future paper will be devoted to this topic.

4. Asymptotics behavior. Since area measure on the archipelago G belongs to the class Reg of measures (cf. [14]), it readily follows from Theorem 2.1 that so does area measure on $G^\star$. In particular,

$$\lim_{n\to\infty} \gamma_n(G^\star)^{1/n} = \frac{1}{\operatorname{cap}(\Gamma)}. \tag{4.1}$$

In order to describe the nth root asymptotic behavior for the Bergman polynomials $p_n(G^\star, z)$ in Ω, we need the Green function $g_\Omega(z,\infty)$ of Ω with pole at infinity. We recall that $g_\Omega(z,\infty)$ is harmonic in $\Omega \setminus \{\infty\}$, vanishes on the boundary Γ of G, and near ∞ satisfies

$$g_\Omega(z,\infty) = \log|z| + \log\frac{1}{\operatorname{cap}(\Gamma)} + O\left(\frac{1}{|z|}\right), \quad |z| \to \infty. \tag{4.2}$$

Our next result corresponds to Proposition 4.1 of [7] and follows in a similar manner.

PROPOSITION 4.1. *The following assertions hold:*

(a) *For every $z \in \overline{\mathbf{C}} \setminus \operatorname{Con}(G)$ and for any $z \in \operatorname{Con}(G) \setminus \overline{G}$ not a limit point of zeros of the $p_n(G^\star, \cdot)$'s, we have*

$$\lim_{n\to\infty} |p_n(G^\star, z)|^{1/n} = \exp\{g_\Omega(z,\infty)\}. \tag{4.3}$$

The convergence is uniform on compact subsets of $\overline{\mathbf{C}} \setminus \operatorname{Con}(G)$.

(b) *There holds*

$$\limsup_{n\to\infty} |p_n(G^\star, z)|^{1/n} = \exp\{g_\Omega(z,\infty)\}, \ z \in \overline{\Omega}, \tag{4.4}$$

locally uniformly in Ω.

For our next result we assume that all the boundary curves Γ_j are analytic. Its proof is a simple consequence of Theorem 4.1 of [7] in conjunction with Theorem 3.1 above.

PROPOSITION 4.2. *Assume that every curve Γ_j, $j = 1, \dots, m$, constituting Γ is analytic. Then there exist positive constants $C_1(\Gamma, \mathcal{K})$ and $C_2(\Gamma, \mathcal{K})$ such that*

$$C_1(\Gamma,\mathcal{K}) \le \sqrt{\frac{n+1}{\pi}} \frac{1}{\gamma_n(G^\star)\operatorname{cap}(\Gamma)^{n+1}} \le C_2(\Gamma,\mathcal{K}), \quad n \in \mathbb{N}. \tag{4.5}$$

As the following example emphasizes, we cannot expect that the limit of the sequence in (4.5) exists when $m \ge 2$.

Example 4.1 (see [7, Remark 7.1]). Consider the m-component lemniscate $G := \{z : |z^m - 1| < r^m\}$, $m \ge 2$, $0 < r < 1$, for which $\operatorname{cap}(\Gamma) = r$. Then, the sequence

$$\sqrt{\frac{n+1}{\pi}} \frac{1}{\gamma_n(G)\operatorname{cap}(\Gamma)^{n+1}}, \quad n \in \mathbb{N},$$

has exactly m limit points:

$$r^{m-1}, r^{m-2}, \ldots, r, 1.$$

Combining the result of Theorem 3.1 with that of Theorem 4.4 of [7], we arrive at estimates for the Bergman polynomials $\{p(G^\star, z\}$ in the exterior domain Ω, where we use $\operatorname{dist}(z, E)$ to denote the (Euclidean) distance of z from a set E.

THEOREM 4.3. *With G as in Proposition 4.2, the following hold:*

(a) *There exists a positive constant C such that*

$$|p_n(G^\star, z)| \leq \frac{C}{\operatorname{dist}(z, \Gamma)} \sqrt{n} \exp\{n g_\Omega(z, \infty)\}, \quad z \notin \overline{G}. \tag{4.6}$$

(b) *For every $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that*

$$|p_n(G^\star, z)| \geq C_\varepsilon \sqrt{n} \exp\{n g_\Omega(z, \infty)\}, \quad \operatorname{dist}(z, \operatorname{Con}(G)) \geq \varepsilon. \tag{4.7}$$

5. Reconstruction algorithm from moments. The present section contains the description and analysis of a reconstruction algorithm for the archipelago with lakes $G^\star$ for the case when the lakes are themselves finite unions of disjoint Jordan regions. The algorithm is motivated by the "reconstruction from moments" algorithm of [7, section 5] and the estimates established in the previous sections. In [7] the functional $\lambda_n^{1/2}(G, z)$ was used as the main reconstruction tool for recovering the shape of the archipelago G using area complex moment measurements. Here we describe how to recover from $\lambda_n^{1/2}(G^\star, z)$ both the shape of G and its lakes.

Assume that the following set of area complex moments is available:

$$\mu_{ij}^\star := \int_{G^\star} z^i \overline{z}^j \, dA(z), \quad i, j = 0, 1, \ldots, n.$$

(For a discussion of how these moments are related to the real moments

$$\tau_{mn}^\star := \int_{G^\star} x^m y^n dxdy$$

that arise in geometric tomography from measurements of the Radon transform, see [7] and [11].)

Before describing our algorithm, we remark that several other techniques exist for shape recovery from complex moments. For example, Elad, Milanfar, and Golub [4] and Beckermann, Golub, and Labahn [2] analyze a method based on solving a generalized Hankel eigenvalue problem to recover the vertices of a planar polygon. This method differs from our algorithm in that it involves only the analytic moments $\mu_{i,0}$ and produces a polygonal region approximation, which seems not so appropriate for the recovery of several pairwise disjoint nonpolygonal regions with lakes. In Gustafsson et al. [6] a reconstruction method is presented that is based on the exponential transform. This approach is particularly suited for quadrature domains, but as illustrated in their paper may yield nonsmooth approximations to regions with smooth boundaries (such as an ellipse) and, for regions with corners, may display distortions near the corners. In neither of these methods is there a discussion of the recovery of finitely many disjoint domains with lakes.

More detailed comparisons with these and other recovering algorithms will be investigated in a future paper. (See section 6 for a discussion related to the stability of our algorithm.)

Our algorithm consists of two phases.

Reconstruction Algorithm.

`Phase A: Recovery of` G.

I. Use the Arnoldi Gram–Schmidt (GS) process described below to compute $p_0(G^\star, z)$, $p_1(G^\star, z), \dots, p_n(G^\star, z)$, from the given set of moments $\mu^\star_{i,j}$ of $G^\star$, $i, j = 0, 1, \dots, n$.

II. Plot the zeros of $p_n(G^\star, z)$.

III. Form $\lambda_n^{1/2}(G^\star, z)$.

IV. Plot the level curves of the function $\lambda_n^{1/2}(G^\star, x + iy)$ on a suitable rectangular frame for (x, y) that surrounds the plotted zero set.[2] The outermost level curves will provide an approximation to the boundary of G. Denote by $\widehat{G}$ the region(s) bounded by this approximation.

`Phase B: Recovery of` $\mathcal{K}$.

I. Use the approximation $\widehat{G}$ of G to calculate the moments

$$\widehat{\mu}_{i,j} := \int_{\widehat{G}} z^i \overline{z}^j \, dA(z), \quad i, j = 0, 1, \dots, n.$$

II. Compute the approximate moments $\mu'_{i,j}$ for the lakes $\mathcal{K}$ by taking the difference $\widehat{\mu}_{i,j} - \mu^\star_{i,j}$

III. Repeat steps I–IV of Phase A with data $\mu'_{i,j}$ in the place of $\mu^\star_{i,j}$ to produce an approximation $\widehat{\mathcal{K}}$ to $\mathcal{K}$.

Step I of Phase B is computationally demanding but can be carried out by approximating the outermost level curves by polygonal curves which will facilitate the computation of the area moments of $\widehat{G}$. This aspect of the algorithm will be explored in a future paper. Here, we shall illustrate our method by using the moments of G instead of $\widehat{G}$.

We recall that the GS process (mentioned in step I) converts, in an iterative fashion, a set of linearly independent functions in some inner product space into a set of orthonormal polynomials $\{p_0, p_1, \dots, p_{n-1}, p_n\}$. By the *Arnoldi* GS we mean the application of the GS process in the following way: At the k-step, where the orthonormal polynomial p_k is to be constructed, we use the polynomials $\{p_0, p_1, \dots, p_{k-1}, zp_{k-1}\}$ as input of the process. We refer to [15, section 7.4] for a discussion regarding the stability properties of the Arnoldi GS. In particular, we note that the Arnoldi GS does not suffer from the severe ill-conditioning associated with the conventional GS as reported, for instance, by theoretical and numerical evidence in [8].

Remark 5.1. A well-known result of Fejér asserts that the zeros of orthogonal polynomials with respect to a compactly supported measure are contained in the convex hull of the support of the measure. Thus the frames chosen in Phases A and B should at least contain such zeros. However, adjustments to the size of such frames may be required, as may be indicated by the appearance of level lines for $\lambda_n^{1/2}$ that are not closed (see Figure 5).

The following theorem contains estimates for the asymptotic behavior of $\lambda_n^{1/2}(G^\star, z)$, thus providing the theoretical support of the reconstruction algorithm given above.

Theorem 5.1. *Under the general assumption that* Γ *consists of a finite union of Jordan curves we have the following:*

(a) *There exists a positive constant* C *such that*

$$\lambda_n^{1/2}(G^\star, z) \geq C \operatorname{dist}(z, \Gamma), \quad z \in G. \tag{5.1}$$

[2]See Remark 5.1.

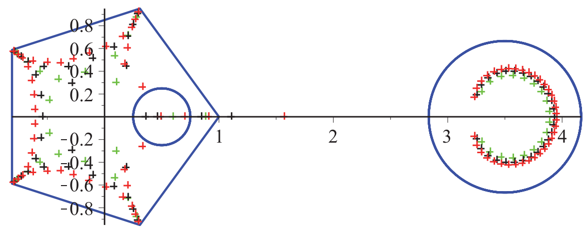

FIG. 2. *Zeros of the polynomials* $p_n(G^\star, z)$ *of Example* 5.1 *for* $n = 40, 60,$ *and* 80.

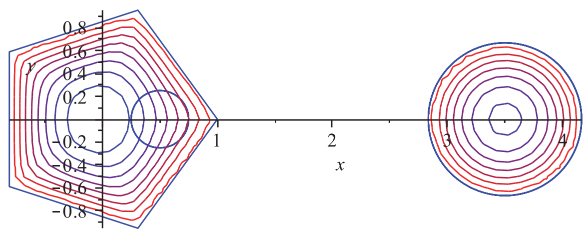

FIG. 3. *Phase A: Level curves of* $\lambda_{80}^{1/2}(G^\star, x+iy)$, *on* $\{(x,y) : -2 \le x \le 5, -2 \le y \le 2\}$, *with* $G^\star$ *as in Example* 5.1.

(b) *For every compact subset* B *of* Ω, *there exists a positive constant* $C(B)$ *such that*

$$\lambda_n^{1/2}(G^\star, z) \le C(B) \exp\{-n g_\Omega(z, \infty)\}, \quad z \in B. \tag{5.2}$$

The estimate in (5.1) is immediate from (2.2), while (5.2) follows from (2.11) and (2.12).

Example 5.1. Recovery for the archipelago $G = G_1 \cup G_2$, with G_1 denoting the canonical pentagon with vertices at the fifth roots of unity, $G_2 = \{z : |z - 7/2| < 2/3\}$, and lake $\mathcal{K}$ the closed disc centered at $1/2$ with radius $1/4$. The boundaries of the archipelago $G^\star := G \setminus \mathcal{K}$ are depicted in Figure 2.

In view of Remark 5.1, the zeros of the polynomial $p_n(G^\star, z)$ will give an indication of the position of G in the complex plane. Accordingly, in Figure 2 we show the zeros for $n = 40, 60$, and 80. This should be compared with Figure 8 in [7], which depicts zeros of $p_n(G, z)$.

In Figures 3 and 4 we show the application of the two phases of the algorithm on a frame that was suggested by the position of the zeros in Figure 2. In order to emphasize the importance of the information about zeros, we depict in Figure 5 the application of Phase A, with an arbitrarily chosen frame.

Regarding the use of the square root $\lambda_n^{1/2}$ rather than λ_n itself, as indicated in (5.1), the former quantity decays linearly to zero with the distance to the boundary $\Gamma = \partial G$, while the latter has a more rapid decay which will affect the omission (due to negligibility) of level curves that are closer to Γ. This can be seen by comparing Figure 6 with the more accurate Figure 3, where the Maple routine `contourplot` was used to generate the level curves.

Example 5.2. Recovery for the archipelago of the three disks $G_1 = \{z : |z + 1| < 1/2\}$, $G_2 = \{z : |z - 2| < 1\}$, and $G_3 = \{z : |z - 2i| < 1/2\}$ and lake $\mathcal{K} := \cup_{j=1}^3 \mathcal{K}_j$, where $\mathcal{K}_j$ are the closed disks $\mathcal{K}_1 = \{z : |z + 1| \le 1/3\}$, $\mathcal{K}_2 = \{z : |z - 2| \le 1/3\}$, and $\mathcal{K}_3 = \{z : |z - 2i| \le 1/4\}$.

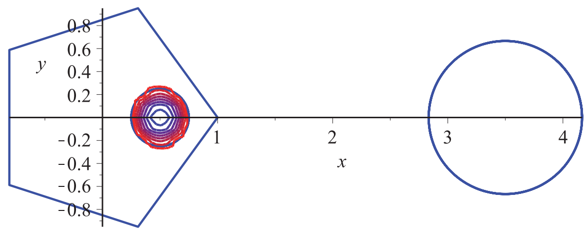

FIG. 4. *Phase B: Level curves of* $\lambda_{80}^{1/2}(\widehat{\mathcal{K}}, x+iy)$*, on* $\{(x,y) : -2 \le x \le 5, -2 \le y \le 2\}$*, with* $G^\star$ *as in Example* 5.1.

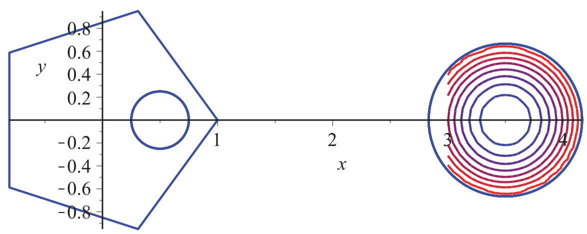

FIG. 5. *Phase A: Level curves of* $\lambda_{80}^{1/2}(G^\star, x+iy)$*, for the inappropriately frame* $\{(x,y) : 3 \le x \le 6, -2 \le y \le 2\}$*, with* $G^\star$ *as in Example* 5.1.

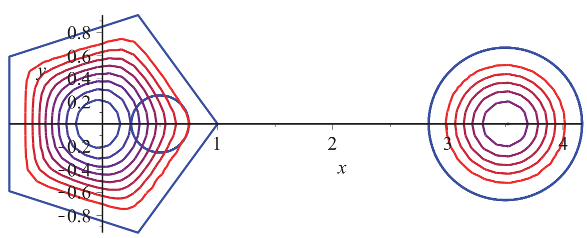

FIG. 6. *Phase A: Level curves of* $\lambda_{80}(G^\star, x+iy)$*, on* $\{(x,y) : -2 \le x \le 5, -2 \le y \le 2\}$*, with* G *as in Example* 5.1.

In Figure 7 we show the zeros $p_n(G^\star, z)$, for $n = 80, 90$, and 100. This should be compared with Figure 13 in [7], which depicts zeros of $p_n(G, z)$. In Figures 8 and 9 we show the application of the two phases of the algorithm on a frame that was suggested by the position of zeros in Figure 7.

All the computations were carried out on a MacBook Pro 2.4-GHz Intel Core i7 using Maple 16.

6. Comments on stability. The examples presented in the preceding section utilized exact measurements for the moments. Here we comment briefly on the effect of noise corruption in the measurements. Ill-conditioning is known to be an inherent problem in mappings that take moments to the support of the generating measure (see, e.g., Beckermann, Golub, and Labahn [2]). A detailed analysis of this issue for the recovery algorithm presented in the preceding section is far from trivial and will be left for a future investigation. However, since the matter is clearly of great practical importance, we provide below some illustrations of the sensitivity of our method to the presence of white noise with mean zero and with several different standard deviations.

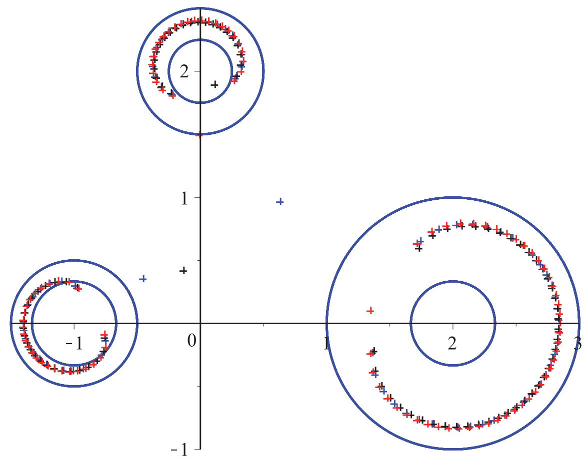

FIG. 7. *Zeros of the polynomials* $p_n(G^\star, z)$ *of Example* 5.2 *for* $n = 80, 90,$ *and* 100.

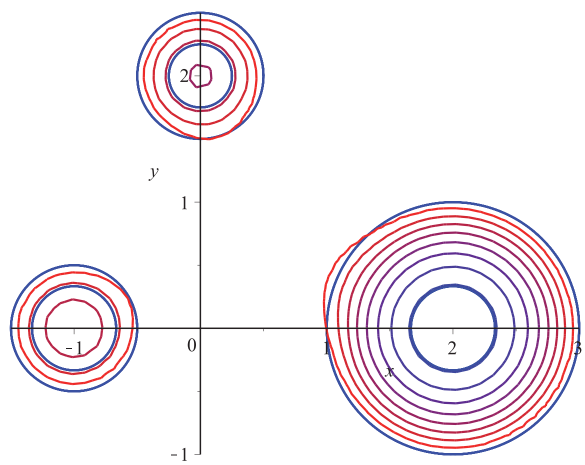

FIG. 8. *Phase A: Level curves of* $\lambda_{100}^{1/2}(G^\star, x+iy)$*, on* $\{(x,y) : -3 \le x \le 4, -2 \le y \le 3\}$*, with* $G^\star$ *as in Example* 5.2.

Our examples are only for Phase A of the recovery. The first case we consider is the union of the regular pentagon and disk (without lakes), which are now both contained in the unit disk.

Example 6.1. Recovery from noisy data of the archipelago $G = G_1 \cup G_2$, with G_1 denoting the canonical pentagon with vertices inscribed on the circle centered at the origin and radius $1/4$ and $G_2 = \{z : |z - 0.7| < 1/6\}$.

The Gaussian noise is added in a relative sense; i.e., we replace the exact moments $\mu_{i,j}$ by $\mu'_{i,j} := \mu_{i,j}(1 + X_{i,j})$, where $X_{i,j}$ is generated by a Gaussian with mean $\mu = 0$ and standard deviation σ, with σ taking the values 10^{-k}, for $k = 2, 4, 6, \ldots, 12$.

For each fixed σ, the recovery algorithm was repeated 10 times for the perturbed moments $\mu'_{i,j}$ with i and j running from 0 up to 20. The computations were carried out with 32-digit accuracy in Maple 16, using the `RandomVariable` tool with parameter `Normal(mu,sigma)` in the `Statistics` package, which is suitable for generating Gaussian white noise. What we observed was that the Arnoldi GS part of the

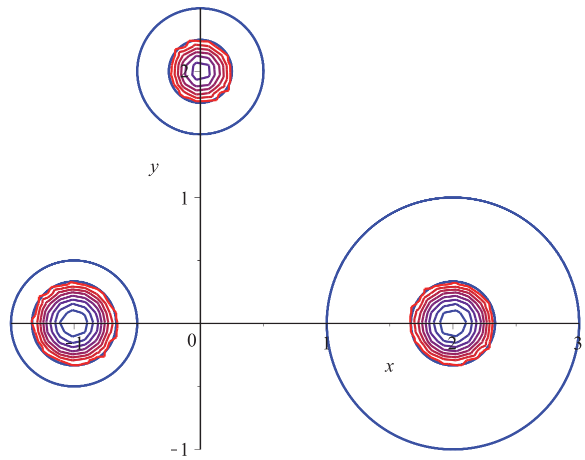

FIG. 9. *Phase B: Level curves of* $\lambda_{100}^{1/2}(\widehat{\mathcal{K}}, x+iy)$, *on* $\{(x,y) : -3 \le x \le 4, -2 \le y \le 3\}$, *with* $G^\star$ *as in Example* 5.2.

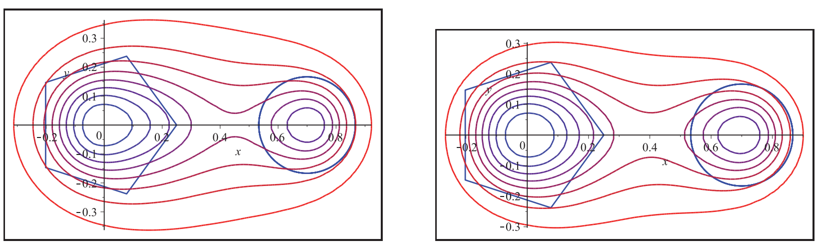

FIG. 10. *Level curves of* $\lambda_7(G, x+iy)$, *on* $\{(x,y) : -1.2 \le x \le 1.0, -0.5 \le y \le 0.5\}$, $\sigma = 10^{-4}$ *(left) and no noise (right), with* G *as in Example* 6.1.

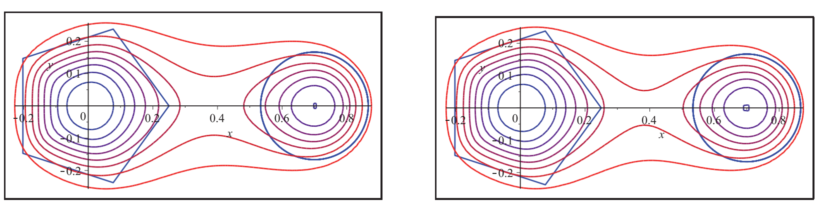

FIG. 11. *Level curves of* $\lambda_{11}(G, x+iy)$, *on* $\{(x,y) : -1.2 \le x \le 1.0, -0.3 \le y \le 0.3\}$, $\sigma = 10^{-8}$ *(left) and no noise (right), with* G *as in Example* 6.1.

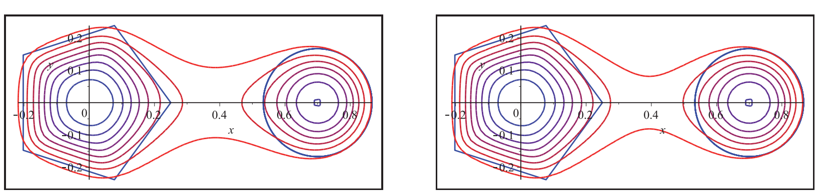

FIG. 12. *Level curves of* $\lambda_{16}(G, x+iy)$, *on* $\{(x,y) : -1.2 \le x \le 1.0, -0.3 \le y \le 0.3\}$, $\sigma = 10^{-12}$ *(left) and no noise (right), with* G *as in Example* 6.1.

algorithm for the generation of orthogonal polynomials breaks down on average for a certain polynomial degree N_b as listed in Table 1, yielding no approximation to the archipelago. (This breakdown occurs because the perturbed moments $\mu'_{i,j}$ fail to be

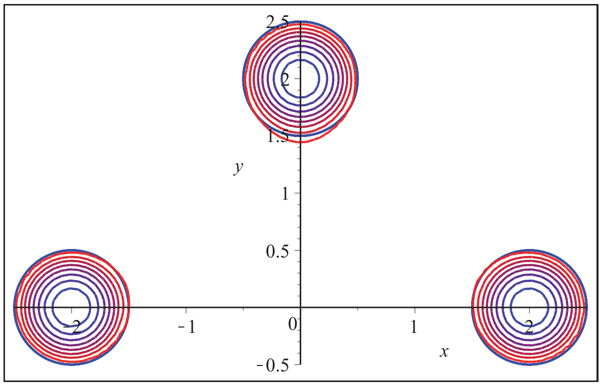

FIG. 13. *Level curves of* $\lambda_{75}(G, x+iy)$*, on* $\{(x,y) : -2.6 \le x \le 3.2, -1.2 \le y \le 2.5\}$*, generated from moments perturbed by unit point masses at* -1 *and* I*, with* G *as in Example* 6.2.

TABLE 1
Median polynomial degree N_b *of Arnoldi GS breakdown for standard deviation* σ.

σ	N_b
10^{-2}	5
10^{-4}	8
10^{-6}	10
10^{-8}	12
10^{-10}	15
10^{-12}	16

part of a measure-defining infinite sequence of complex numbers; see the two criteria in [1, Theorem 2.1].) However, when the algorithm is repeated with noisy data $\mu'_{i,j}$ with i and j up to $N_b - 1$, it yields results that are only modestly distorted from the results using exact moments up to $N_b - 1$. The situation is illustrated in Figures 10, 11, and 12, where the left-hand graphs are typical of those produced from noisy data (as defined in the caption) and should be compared with the right-hand figure computed by the algorithm with exact moments. Notice that all graphs display a concentration of level lines on the two bodies, with the remaining curves approximating the level lines for the Green function with pole at infinity associated with the complement of the union of the two bodies.

To summarize, our very preliminary examples suggest that the crucial issue with regard to unstructured noisy data is the breakdown in the computation of the orthogonal polynomial sequence. Whenever such a sequence can be generated, our algorithm yields useful approximations to the generating shapes. How accurate these approximations are for a given number of moments is yet another area for future investigation.

One advantage of our recovery scheme not to be found, for example, in the generalized Hankel eigenvalue approach based on Davis's theorem (cf. [4]) is its lack of sensitivity to structured perturbations of the form

$$\mu''_{i,j} := \mu_{i,j} + \gamma_{i,j},$$

where the $\gamma_{i,j}$'s are moments arising from a compact set of logarithmic capacity zero, or from a set of positive capacity lying in the polynomial convex hull of the archipelago.

For example, if $\gamma_{i,j} := \gamma$ is any fixed positive constant, which corresponds to a point mass of γ at $z = 1$, or any countable number of such point masses, then the recovery algorithm yields results essentially identical to those obtained with exact measurements of the moments. As a graphical illustration of such a structured perturbation we present the next example.

Example 6.2. Recovery from a structured perturbation of the moments for the archipelago of the three disks $G_1 = \{z : |z + 2| < 1/2\}$, $G_2 = \{z : |z - 2| < 1/2\}$, and $G_3 = \{z : |z - 2i| < 1/2\}$.

In Figure 13, the exact moments $\mu_{i,j}$ are perturbed by

$$\gamma_{i,j} := (-1)^{i+j} + I^i(-I)^j, \quad I = \sqrt{-1},$$

which corresponds to adding the moments of point measures at $z = -1$ and $z = I$. No breakdown now occurs in the recovery algorithm, enabling us to compute orthonormal polynomials of large degree, resulting in an accurate approximation of the archipelago as illustrated in Figure 13.

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