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A COMMENT ON THE CONVEXITY CONDITION

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1. *Introduction*

The study of the topological property of convexity has been shown to be of interest in many areas of mathematics including functional analysis ([2], pp. 24-25) and univalent function theory ([1], pp. 12-13). Classically a subset D of the complex plane C is said to be *convex* if for all $a \in (0, 1)$

(1.1) $$d_1, d_2 \in D \text{ implies } ad_2 + (1-a)d_1 \in D.$$

A natural question that arises is the following: What are the geometrical properties of a subset $D \subseteq C$ which satisfies the condition (1.1) for a *single fixed* $a \in (0, 1)$ or, more generally, for a fixed complex a? The surprising answer is that if a is real and $0<a<1$, then the interior of D is convex. Furthermore, if $a \in C-[0, 1]$ and D is open, then $D=C$ or $D=\phi$.

The formal definitions, statements and proofs of these facts are presented in Section 2 in the more general setting of a subset D of a topological vector space over the real or complex field of scalars. Some further consequences are stated in Section 3.

2. *Definitions and Main Results*

Let V be a topological vector space over the real or complex field of scalars F, and let $f : F \times V \times V \to V$ and $g : F^3 \to F$ be the continuous functions defined by

$$f(a, \mathbf{v}_1, \mathbf{v}_2) = a\mathbf{v}_2 + (1-a)\mathbf{v}_1$$

and

$$g(a, a_1, a_2) = aa_2 + (1-a)a_1.$$

We also define the continuous binary operations $f_a : V^2 \to V$ and $g : F^2 \to F$ by $f_a(v_1, v_2) = f(a, v_1, v_2)$ and $g_a(a_1, a_2) = g(a, a_1, a_2)$.

DEFINITION 2.1. A subset D of V (or F) is said to be G-convex with respect to a scalar $a \in F$ if D is closed under the binary operation f_a (or g_a). Further, D is G-convex with respect to a subset A of F if D is G-convex with respect to each $a \in A$.

Thus a set D is classically convex if and only if it is G-convex with respect to the unit interval $[0, 1]$. G-convexity of a set D in V is most easily studied through the associated set $A(D)$ defined by

$$A(D) = \{a \in F : D \text{ is } G\text{-convex with respect to } a\}.$$

Note that for any set D we have $0, 1 \in A(D)$. Our aim is to prove the following main result:

THEOREM 2.1. *If D is G-convex with respect to a fixed scalar $a \in F$, $0 < a < 1$, then the interior of D is convex, i.e.*

$$A(D) \cap (0, 1) \neq \phi \Rightarrow [0, 1] \subseteq A(D^\circ), \tag{2.1}$$

where D° denotes the interior of D.

In order to prove Theorem 2.1 we first investigate $A(D)$ for an arbitrary subset D of V. If $b \in F$ and $v \in V$ define

$$bD + v = \{bd + v : d \in D\}.$$

Also define

$$[A(D)]^2 = \{ab : a, b \in A(D)\}.$$

LEMMA 2.1. *If $v \in V$ and b is a non-zero scalar in F, then $A(D) = A(bD + v)$ for any subset D of V.*

Proof. Fix $a \in A(D)$. Then for any $bd_1, bd_2 \in bD$ we have $f_a(bd_1 + v, bd_2 + v) = bf_a(d_1, d_2) + v$ which belongs to $bD + v$ so that $a \in A(bD + v)$. Therefore $A(D) \subseteq A(bD + v)$. The reverse inclusion follows by replacing D by $bD + v$, b by b^{-1}, and v by $-b^{-1}v$, i.e. $A(bD + v) \subseteq A(b^{-1}(bD + v) - b^{-1}v) = A(D)$. □

LEMMA 2.2. (i) $A(D) = 1 - A(D)$,

(ii) $[A(D)]^2 = A(D)$,

(iii) $A(A(D)) = A(D)$.

Proof. (i) Note that $f(a, \mathrm{d}_1, \mathrm{d}_2)=f(1-a, \mathrm{d}_2, \mathrm{d}_1)$.

(ii) Since $1\in A(D)$, we have $A(D)\subseteq[A(D)]^2$. To establish the reverse inclusion, let $a, b\in A(D)$ and observe that

$f_{ab}(\mathrm{d}_1, \mathrm{d}_2)=f_a(\mathrm{d}_1, f_b(\mathrm{d}_1, \mathrm{d}_2))\in D$ for all $\mathrm{d}_1, \mathrm{d}_2\in D$.

(iii) If $a\in A(A(D))$, then $a=g_a(0, 1)\in A(D)$ so that $A(A(D))\subseteq A(D)$. On the other hand, let $a\in A(D)$. To show that $a\in A(A(D))$, we must prove that $g_a(a_1, a_2)\in A(D)$ for all a_1 $a_2\in A(D)$. To this end let $\mathrm{d}_0, \mathrm{d}_1\in D$ and set $\mathrm{d}_2=\mathrm{d}_1-\mathrm{d}_0$. Then $0, \mathrm{d}_2\in D-\mathrm{d}_0$ and so $f_{g_a(a_1, a_2)}(0, \mathrm{d}_2)=g_a(a_1, a_2)\mathrm{d}_2=f_a(a_1\mathrm{d}_2, a_2\mathrm{d}_2)\in D-\mathrm{d}_0$ by Lemma 2.1. Thus $f_{g_a(a_1, a_2)}(\mathrm{d}_0, \mathrm{d}_1)=f_{g_a(a_1, a_2)}(0, \mathrm{d}_2)+\mathrm{d}_0\in D$. □

LEMMA 2.3. *If $A(D)\cap(0, 1)\neq\phi$, then $[0, 1]\subseteq A(D)^*$, the closure of $A(D)$.*

Proof. We shall show that if $t\in(0, 1)$, then t is a limit point of $A(D)$. Choose $a\in A(D)\cap(0, 1)$ and a sequence of positive integers $k_1, k_2, \ldots$ so that

$$s_1\equiv a^{k_1}<t\leq a^{k_1-1}\equiv r_1$$

$$s_2\equiv s_1+a^{k_2}(r_1-s_1)<t\leq s_1+a^{k_2-1}(r_1-s_1)\equiv r_2$$

$$\cdots \quad \cdots \quad \cdots$$

and in general

$$s_n\equiv s_{n-1}+a^{k_n}(r_{n-1}-s_{n-1})<t\leq s_{n-1}+a^{k_n-1}(r_{n-1}-s_{n-1})\equiv r_n.$$

By Lemma 2.2, s_1, $r_1\in A(D)$. Also $r_1-s_1\leq 1-a$. By induction r_n, $s_n\in A(D)$ and $r_n-s_n\leq(1-a)^n$. Since $s_n<t\leq r_n$ for all n, the assertion follows. □

LEMMA 2.4. *If $\mathrm{d}_1, \mathrm{d}_2\in D^\circ$, the interior of D, then there is an open neighborhood W of the zero vector $0\in V$ such that $f_a(\mathrm{d}_1, \mathrm{d}_2)+W\subseteq D^\circ$ for all $a\in A(D)$.*

Proof. If $\mathrm{d}_1, \mathrm{d}_2\in D^\circ$, then $W=(D^\circ-\mathrm{d}_1)\cap(D^\circ-\mathrm{d}_2)$ is an open neighborhood of the zero vector. Further if $\mathrm{w}\in W$, then $\mathrm{d}_1+\mathrm{w}\in D$ and $\mathrm{d}_2+\mathrm{w}\in D$ so that

$f_a(\mathrm{d}_1, \mathrm{d}_2)+\mathrm{w}=f_a(\mathrm{d}_1+\mathrm{w}, \mathrm{d}_2+\mathrm{w})\in D$ whenever $a\in A(D)$. Hence for all $a\in A(D)$, $f_a(\mathrm{d}_1, \mathrm{d}_2)+W\subseteq D$. However since $f_a(\mathrm{d}_1, \mathrm{d}_2)+W$ is open, it follows that $f_a(\mathrm{d}_1, \mathrm{d}_2)+W\subseteq D^\circ$. □

Proof of Theorem 2.1. Let $\mathbf{d}_1$, $\mathbf{d}_2$ be fixed but arbitrary vectors in D°. Define the continuous function $h : F \to V$ by $h(a) = f_a(\mathbf{d}_1, \mathbf{d}_2) = a\mathbf{d}_2 + (1-a)\mathbf{d}_1$. We shall show that $h(a) \in D^\circ$ for all $a \in [0, 1]$.

By Lemma 2.4, let W be an open neighborhood of the zero vector such that $h(a) + W \subseteq D^\circ$ for all $a \in A(D)$. Then $h(0) + W = \mathbf{d}_1 + W \subseteq D^\circ$. Further, for all $a \in A(D)$,

$$h^{-1}(W + \mathbf{d}_1) + a \subseteq h^{-1}(D^\circ). \tag{2.2}$$

To see this let $u = b + a \in h^{-1}(W + \mathbf{d}_1) + a$. Then $h(u) = h(b+a) = f_{b+a}(\mathbf{d}_1, \mathbf{d}_2) = f_a(\mathbf{d}_1, \mathbf{d}_2) + f_b(\mathbf{d}_1, \mathbf{d}_2) - \mathbf{d}_1 = h(a) + h(b) - \mathbf{d}_1 \in h(a) + W \subseteq D^\circ$, since $h(b) \in W + \mathbf{d}_1$. Thus $u \in h^{-1}(D^\circ)$.

By the continuity of h there is a positive r such that $(-r, r) \subseteq h^{-1}(w + \mathbf{d}_1)$ since $h^{-1}(W + \mathbf{d}_1)$ is an open set containing positive $0 \in F$. Therefore, for each $a \in A(D) \cap [0, 1]$, $(a - r, a + r) \subseteq h^{-1}(D^\circ)$ by (2.2), and by Lemma 2.3, $[0, 1] \subseteq h^{-1}(D^\circ)$, i.e. $h(a) \in D^\circ$ for all $a \in [0, 1]$.

Since $\mathbf{d}_1$, $\mathbf{d}_2 \in D^\circ$ were chosen arbitrarily, D° must be convex. □

COROLLARY 2.1. *Fix $0 < a < 1$. An open subset D of a topological vector space V over the real or complex field of scalars is G-convex with respect to a if and only if D is convex.*

We now consider V a topological vector space over the complex field of scalars C and examine the property of G-convexity of a subset $D \subseteq V$ with respect to an arbitrary $a \in C - [0, 1]$.

THEOREM 2.2. *If D is G-convex with respect to a fixed scalar $a \notin [0, 1]$, then the interior of D is either empty or equal to V,* i.e.

$$A(D) \cap (C - [0, 1]) \neq \phi \Rightarrow D^\circ = \phi \text{ or } D^\circ = D = V \tag{2.3}$$

where D° denotes the interior of D.

Proof. Assume $D^\circ \neq \phi$ and, without loss of generality by Lemma 2.1, $0 \in D^\circ$. Choose $a \in A(D) \cap (C - [0, 1])$. Suppose that $D^\circ \neq V$ and let $\mathbf{v}_0 \in V - D^\circ$. Let r be the least upper bound of the set

$$T = \{r' : s\mathbf{v}_0 \in D^\circ \text{ for } 0 \leqslant |s| \leqslant r'\}.$$

Then $0 < r \leqslant 1$ by the fact that $0 \in D^\circ$, $\mathbf{v}_0 \notin D^\circ$, and scalar multiplication is continuous. Now write $a = pe^{i\alpha}$ and $1 - a = qe^{i\beta}$ for $0 \leqslant \alpha, \beta \leqslant 2\pi$. Since $a \notin [0, 1]$ the triangle inequality implies that $p + q = 1 + \delta$ for some positive δ.

If $0<\epsilon\leqslant r$, then $r-\epsilon$ belongs to T so that for any real γ it follows that $(r-\epsilon)e^{i(\gamma-\alpha)}\mathbf{v}_0$ and $(r-\epsilon)e^{i(\gamma-\beta)}\mathbf{v}_0$ belong D°. Thus,

$$(r-\epsilon)(1+\delta)e^{i\gamma}\mathbf{v}_0=a(r-\epsilon)e^{i(\gamma-\alpha)}\mathbf{v}_0+(1-a)(r-\epsilon)e^{i(\gamma-\beta)}\mathbf{v}_0\in D^\circ,$$

which implies $s\mathbf{v}_0\in D^\circ$ for all $0\leqslant|s|<r(1+\delta)$. This contradicts the choice of r showing that $D^\circ=V$. □

Corollary 2.2. *Fix $a\in C-[0, 1]$. A non-empty open subset D of a topological vector space V over C is G-convex with respect to a if and only if $D=V$.*

3. *Consequences*

We continue our investigation of $A(D)$ by presenting some elementary observations based upon the results in Section 2. As before we denote the interior of D by D°, and the closure of D by D^*.

Notice that if D is closed in V, then $A(D)$ is closed in F. For if $\{a_n\}$ is a net in $A(D)$ converging to a in $A(D)^*$, then the continuity of f implies that for $\mathbf{d}_1, \mathbf{d}_2\in D$, $\{f(a_n, \mathbf{d}_1, \mathbf{d}_2)\}$ is a net in D converging to $f(a, \mathbf{d}_1, \mathbf{d}_2)$, an element of D since $D^*=D$. Hence if D is closed, then $A(D)=A(D)^*=A(D^*)$. The general situation where D is arbitrary is given in

Lemma 3.1. $A(D)\subseteq A(D)^*\subseteq A(D^*)$.

Proof. Let $a\in A(D)$ and $\mathbf{c}, \mathbf{d}\in D^*$. Choose nets $\{\mathbf{c}_n\}$ and $\{\mathbf{d}_n\}$ in D converging to $\mathbf{c}$ and $\mathbf{d}$, respectively. Then $\{f(a, \mathbf{c}_n, \mathbf{d}_n)\}$ is a net in D converging to $f(a, \mathbf{c}, \mathbf{d})\in D^*$. Thus $a\in A(D^*)$ and so $A(D)\subseteq A(D^*)$. Since $A(D^*)$ is closed, $A(D)\subseteq A(D)^*\subseteq A(D^*)$. □

Analogously we observe

Lemma 3.2. $A(D)^\circ\subseteq A(D)\subseteq A(D^\circ)$.

Proof. Let $a\in A(D)$ and $\mathbf{d}_1, \mathbf{d}_2\in D^\circ$. Choose W as in Lemma 2.4, i.e. $f_a(\mathbf{d}_1, \mathbf{d}_2)+W\subseteq D^\circ$. Since $0\in W$ we conclude that $f_a(\mathbf{d}_1, \mathbf{d}_2)\in D^\circ$ whence $a\in A(D^\circ)$. □

Lemma 3.3. *If $S\subseteq A(D)$ and $S\subseteq T\subseteq S^*$, then $T\subseteq A(D^*)$.*

Proof. Note that $T\subseteq S^*\subseteq A(D)^*\subseteq A(D^*)$. □

Theorem 3.1. *If D is G-convex with respect to a scalar a, $0<a<1$, then D^* is convex.*

Proof. By Lemma 2.3 and Lemma 3.1, $[0, 1] \subseteq A(D)^* \subseteq A(D^*)$. □

Theorem 2.1 and Theorem 3.1 indicate that if D is G-convex with respect to a fixed scalar a, $0<a<1$, then D must be "squeezed" between two classically convex sets, viz. $D' \subseteq D \subseteq D^*$.

REMARK: Just recently, the authors became aware that the results of this paper generalize a problem on midpoint convexity which appears in the text *Linear Topological Spaces* by J. L. Kelley and I. Namioka. The result stated there is for the space of real numbers.

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