INVERSE BERNSTEIN INEQUALITIES AND MIN-MAX-MIN PROBLEMS ON THE UNIT CIRCLE

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Abstract. We give a short and elementary proof of an inverse Bernstein-type inequality found by S. Khrushchev for the derivative of a polynomial having all its zeros on the unit circle . The inequality is used to show that equally-spaced points solve a min-max-min problem for the logarithmic potential of such polynomials. Using techniques recently developed for polarization (Chebyshev-type) problems, we show that this optimality also holds for a large class of potentials, including the Riesz potentials $1/r^s$ with $s > 0$.

1. Inverse Bernstein-type inequality

Inequalities involving the derivatives of polynomials often occur in approximation theory (see, e.g. [4], [6]). One of the most familiar of these inequalities is due to Bernstein which provides an upper bound for the derivative of a polynomial on the unit circle $\mathbb{T}$ of the complex plane. In [9], S. Khrushchev derived a rather striking inverse Bernstein-type inequality, a slight improvement of which may be stated as follows.

Theorem 1.1. *If*

$$P(z) = \prod_{j=1}^{n} (z - z_j)\,, \qquad z_j \in \mathbb{T}\,,$$

$$z_j = e^{it_j}\,, \qquad 0 \le t_1 < t_2 < \cdots < t_n < 2\pi\,, \quad t_{n+1} := t_1 + 2\pi\,,$$

$$m := \min_{1 \le j \le n} \left(\max_{t \in [t_j, t_{j+1}]} |P(e^{it})| \right), \tag{1.1}$$

then

$$|P'(z)|^2 \ge \left(\frac{n}{2}\right)^2 \left(|P(z)|^2 + (m^2 - |P(z)|^2)_+\right) \ge \left(\frac{nm}{2}\right)^2, \qquad z \in \mathbb{T}\,, \tag{1.2}$$

where $(x)_+ := \max\{x, 0\}$.

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Khrushchev used a potential theoretic method to prove his inequality. Here we offer a simple and short proof based on an elementary zero counting argument.

Proof. Write

$$P(e^{it}) = R(t)e^{i\varphi(t)}\,, \qquad R(t) := |P(e^{it})|\,,$$

where R and φ are differentiable functions on

$$[0, 2\pi) \setminus \{t_1, t_2, \dots, t_n\}\,.$$

Since $P'(e^{it})$ is a continuous function of $t \in \mathbb{R}$, in the rest of the proof we may assume that

$$t \in [0, 2\pi) \setminus \{t_1, t_2, \dots, t_n\}\,.$$

We have

$$P'(e^{it})e^{it}i = R'(t)e^{i\varphi(t)} + iR(t)e^{i\varphi(t)}\varphi'(t)\,,$$

and it follows that

$$|P'(e^{it})|^2 = (R'(t))^2 + R(t)^2(\varphi'(t))^2\,. \tag{1.3}$$

Using the fact that $w = z/(z - z_j)$ maps $\mathbb{T}$ onto the vertical line $\operatorname{Re} w = 1/2$, we have

$$\varphi'(t) = \operatorname{Re}\left(\varphi'(t) - \frac{R'(t)}{R(t)}\,i\right) = \operatorname{Re}\left(\frac{P'(e^{it})e^{it}}{P(e^{it})}\right) = \operatorname{Re}\left(\sum_{j=1}^{n} \frac{e^{it}}{e^{it} - e^{it_j}}\right) = \frac{n}{2}\,.$$

Thus from (1.3) we get

$$|P'(e^{it})|^2 = (R'(t))^2 + \left(\frac{n}{2}\right)^2 R(t)^2\,, \tag{1.4}$$

and so, if $R(t) \geq m$, then (1.2) follows immediately.

Assume now that $R(t) < m$. Observe that Q defined by $Q(t) := R(t)^2 = |P(e^{it})|^2$ is a real trigonometric polynomial of degree n; that is, $Q \in \mathcal{T}_n$. Now let

$$t_0 \in [0, 2\pi) \setminus \{t_1, t_2, \dots, t_n\}$$

be fixed, and let $T \in \mathcal{T}_n$ be defined by

$$T(t) := m^2 \cos^2(n(t-\alpha)/2) = \frac{m^2}{2}\,(1 + \cos(n(t-\alpha)))\,,$$

where $\alpha \in \mathbb{R}$ is chosen so that

$$T(t_0) = Q(t_0) \tag{1.5}$$

and

$$\operatorname{sign}(T'(t_0)) = \operatorname{sign}(Q'(t_0))\,. \tag{1.6}$$

We claim that

$$|Q'(t_0)| \geq |T'(t_0)|\,. \tag{1.7}$$

Indeed, $|Q'(t_0)| < T'(t_0)|$ together with (1.5) and (1.6) would imply that the not identically zero trigonometric polynomial $T - Q \in \mathcal{T}_n$ had at least

$2n+2$ zeros in the period $[0,2\pi)$ (at least two zeros on each of the intervals $[t_1,t_2)$, $[t_2,t_3)$,…, $[t_n,t_{n+1})$, and at least four zeros on the interval (t_j,t_{j+1}) containing t_0) by counting multiplicities, a contradiction. Thus (1.7) holds and implies that

$$|Q'(t_0)| \geq |m^2 n \cos(n(t_0-\alpha)/2)\sin(n(t_0-\alpha)/2)|\,,$$

which, together with (1.5), yields

$$\begin{aligned}|Q'(t_0)|^2 &\geq n^2(m^2\cos^2(n(t_0-\alpha)/2))(m^2\sin^2(n(t_0-\alpha)/2))\\ &= n^2(|Q(t_0)|(m^2-|Q(t_0)|)\,.\end{aligned}$$

Substituting $Q(t_0)=R(t_0)^2$ and $Q'(t_0)=2R(t_0)R'(t_0)$ in the above inequality, we conclude that

$$(R'(t_0))^2 \geq \frac{n^2}{4}(m^2-R(t_0)^2)\,.$$

Finally, combining this last inequality with (1.4) and recalling that $R(t_0)=|P(e^{it_0})|$ yields (1.2). □

A natural question that arises is finding the maximal value $m^*(n)$ of the quantity m given in (1.1) or, equivalently (using the notation of Theorem 1.1), determining

$$m^*(n) := \max_{\omega_n\in\Omega_n}\ \min_{1\leq j\leq n}\ \max_{t\in[t_j,t_{j+1}]}\left|\prod_{k=1}^{n}(e^{it}-z_k)\right|, \qquad z_k=e^{it_k}, \tag{1.8}$$

where Ω_n is the collection of all n-tuples $\omega_n\in[0,2\pi)^n$ of the form

$$\omega_n=(t_1,\dots,t_n), \qquad 0\leq t_1\leq\ldots\leq t_n<2\pi.$$

In Corollary 6.9 of [9], Khrushchev proved that $m^*(n)=2$, the value of m corresponding to $P(z)=z^n-1$ for which equality holds throughout in (1.2). Here we deduce this fact as a simple consequence of Theorem 1.1.

Corollary 1.2. *Let $m^*(n)$ be as in (1.8). Then $m^*(n)=2$ and this maximum is attained only for n distinct equally spaced points $\{z_1,\dots,z_n\}$ on the unit circle.*

In other words, for any monic polynomial of degree n all of whose zeros lie on the unit circle, there must be some sub-arc formed from consecutive zeros on which the modulus of the polynomial is at most 2.

Proof of Corollary 1.2. Assume $m^*(n)>2$. According to Theorem 1.1, $|P_o'(z)|>n$ for all z on $\mathbb{T}$, where P_o is a monic polynomial of degree n for which the maximum value $m^*(n)$ is attained. By the Gauss-Lucas theorem, P_o' has all its zeros in the open unit disk (clearly it can't have any on $\mathbb{T}$). So now consider the $f(z):=P_o'(z)/z^{n-1}$, which is analytic on and outside $\mathbb{T}$, even at infinity where it equals n. Since f does not vanish outside or on $\mathbb{T}$, its modulus must attain its minimum on $\mathbb{T}$. But $|f(z)|>n$ on $\mathbb{T}$, while $f(\infty)=n$, which gives the desired contradiction. Thus $m^*(n)=2$

and the argument above also shows that if this maximum is attained by a polynomial P_o, then $|f(z)| = 2$ for all z on or outside $\mathbb{T}$, which implies that f is constant and so P_o has equally spaced zeros on $\mathbb{T}$. □

Observe that the determination of $m^*(n)$ can alternatively be viewed as a min-max-min problem on the unit circle for the logarithmic potential $\log(1/r)$ with r denoting Euclidean distance between points on $\mathbb{T}$. In the next section we consider such problems for a general class of potentials.

2. Min-max-min problems on $\mathbb{T}$

Let g be a positive, extended real-valued, even function defined on $\mathbb{R}$ that is periodic with period 2π and satisfies $g(0) = \lim_{t\to 0} g(t)$. Further suppose that g is non-increasing and strictly convex on $(0, \pi]$. For $\omega_n = (t_1, \dots, t_n) \in \Omega_n$, we set

$$P_{\omega_n}(t) := \sum_{j=1}^{n} g(t - t_j). \tag{2.1}$$

Here and in the following we assume that t_j is extended so that

$$t_{j+n} = t_j + 2\pi, \qquad (j \in \mathbb{Z});$$

in particular, we have $t_0 = t_n - 2\pi$ and $t_{n+1} = t_1 + 2\pi$. For $\omega_n = (t_1, \dots, t_n) \in \Omega_n$ and $\gamma \in [0, 2\pi)$, let $\omega_n + \gamma$ denote the element in Ω_n corresponding to the set $\{e^{i(t_k+\gamma)}\}_{k=1}^n$. Then $P_{\omega_n+\gamma}(t) = P_{\omega_n}(t - \gamma)$. We further let $\widetilde{\omega}_n := (\widetilde{t}_1, \dots, \widetilde{t}_n)$ denote the equally-spaced configuration given by

$$\widetilde{t}_j := 2(j-1)\pi/n\,, \qquad j = 1, 2, \dots, n\,.$$

By the convexity of g, it follows that

$$\min_{t\in[0,2\pi)} P_{\widetilde{\omega}_n}(t) = P_{\widetilde{\omega}_n}(\pi/n)\,.$$

Motivated by recent articles on polarization of discrete potentials on the unit circle (cf. [1], [2], [7], [8]) we shall prove the following generalization of Corollary 1.2.

Theorem 2.1. *Let g be a positive, extended real-valued, even function defined on $\mathbb{R}$ that is periodic with period 2π and satisfies $g(0) = \lim_{t\to 0} g(t)$. Suppose further that g is non-increasing and strictly convex on $(0, \pi]$. Then we have*

$$\min_{\omega_n\in\Omega_n} \left\{ \max_{1\le j\le n} \left\{ \min_{t\in[t_j,t_{j+1}]} P_{\omega_n}(t) \right\} \right\} = P_{\widetilde{\omega}_n}(\pi/n)\,; \tag{2.2}$$

that is, the solution to the min-max-min problem on $\mathbb{T}$ is given by n distinct equally-spaced points on $\mathbb{T}$ and, moreover, these are the only solutions.

2.1. **Logarithmic and Riesz kernels.** It is straightforward to verify that $g(t) = g_{\log}(t) := \log(1/|e^{it} - 1|) = -\log(2\sin|t/2|)$ satisfies the hypotheses of Theorem 2.1 providing an alternate proof of Corollary 1.2 . Furthermore, for the case (relating to Euclidean distance),

$$g(t) = g_s(t) := |e^{it} - 1|^{-s} = (2\sin|t/2|)^{-s},\ s > 0, \tag{2.3}$$

we obtain the *Riesz s-potential* and it is again easily verified that g_s satisfies the hypotheses of Theorem 2.1. Consequently, with $z_k = e^{it_k}$,

(2.4)

$$\min_{\omega_n \in \Omega_n} \max_{1 \le j \le n} \min_{t \in [t_j, t_{j+1}]} \sum_{k=1}^{n} |e^{it} - z_k|^{-s} = \sum_{k=1}^{n} |e^{i\pi/n} - e^{2k\pi i/n}|^{-s} = M_n^s(\mathbb{T}),$$

where $M_n^s(\mathbb{T})$ is the *Riesz s-polarization constant* for n points on the unit circle (cf. [8]). We remark that for s an even integer, say $s = 2m$, the precise value of $M_n^s(\mathbb{T})$ can be expressed as a polynomial in n; namely, as a consequence of the formulas derived in [5],

$$M_n^{2m}(\mathbb{T}) = \frac{2}{(2\pi)^{2m}} \sum_{k=1}^{m} n^{2k} \zeta(2k) \alpha_{m-k}(2m)(2^{2k} - 1), \quad m \in \mathbb{N}, \tag{2.5}$$

where $\zeta(s)$ is the classical Riemann zeta function and $\alpha_j(s)$ is defined via the power series for $\operatorname{sinc} z = (\sin \pi z)/(\pi z)$:

$$(\operatorname{sinc} z)^{-s} = \sum_{j=0}^{\infty} \alpha_j(s) z^{2j}\,; \quad \alpha_0(s) = 1\,,$$

see Corollary 3 from [8]. In particular,

$$M_n^2(\mathbb{T}) = \frac{n^2}{4}, \quad M_n^4(\mathbb{T}) = \frac{n^2}{24} + \frac{n^4}{48}, \quad M_n^6(\mathbb{T}) = \frac{n^2}{120} + \frac{n^4}{192} + \frac{n^6}{480}.$$

2.2. **Proof of Theorem 2.1.** Theorem 2.1 is a consequence of the following lemma which is the basis of the proof of the polarization theorem established by Hardin, Kendall and Saff in [8]. (In Section 3 we state a slightly stronger version of this polarization result as Theorem 3.1 and present some related results.)

Lemma 2.2. *Let g be as in Theorem 2.1 and suppose $\omega_n = (t_1, \ldots, t_n)$ and $\omega_n' = (t_1', \ldots, t_n')$ are in Ω_n. Then there is some $\ell \in \{0, 1, \ldots, n\}$ and some $\gamma \in [0, 2\pi)$ (where ℓ and γ depend on ω_n and ω_n' but not on g) such that*

$$P_{\omega_n'}(t - \gamma) \le P_{\omega_n}(t)\,, \qquad t \in [t_\ell, t_{\ell+1}], \tag{2.6}$$

and $[t_\ell, t_{\ell+1}] \subset [t_\ell' + \gamma, t_{\ell+1}' + \gamma]$.

The inequality is strict for $t \in (t_\ell, t_{\ell+1})$ unless $t_{j+1} - t_j = t_{j+1}' - t_j'$ for all $j = 1, \ldots, n$.

Sketch of proof. This lemma follows from techniques developed in [8], specifically from Lemmas 5 and 6 in that paper. For the convenience of the reader, we provide here an outline of its proof. First, the convexity of g implies that, for $n = 2$, the inequality

$$P_{(t_1-\Delta,\, t_2+\Delta)}(t) < P_{(t_1,t_2)}(t), \qquad t \in (t_1, t_2), \tag{2.7}$$

holds for sufficiently small $\Delta > 0$ (this observation was also used in [2]). That is, the potential due to two points decreases on an interval when the points are moved symmetrically *away* from the interval. For simplicity, we consider the case that

$$\operatorname{sep}(\omega_n) := \min_j (t_{j+1} - t_j) > 0,$$

(see [8] for the case of coincident points where $\operatorname{sep}(\omega_n) = 0$).

Next, using elementary linear algebra, we find a vector $\boldsymbol{\Delta} = (\Delta_1, \ldots, \Delta_n)$ such that (a) $\Delta_k \geq 0$ for all k, (b) $\Delta_\ell = 0$ for some ℓ and (c) $\boldsymbol{\Delta}$ solves the equations

$$(t'_{j+1} - t'_j) = (t_{j+1} - t_j) - \Delta_{j+1} + 2\Delta_j - \Delta_{j-1}, \qquad (j = 1, \ldots, n) \tag{2.8}$$

where we take $\Delta_0 := \Delta_n$ and $\Delta_{n+1} := \Delta_1$. For $j = 1, \ldots, n$, consider the transformation

$$\tau_{j,\Delta}(\omega_n) := (t_1, \ldots, t_{j-2}, t_{j-1} - \Delta, t_j + \Delta, t_{j+1}, \ldots, t_n).$$

Then (2.8) implies that $\omega''_n := \tau_{1,\Delta_1} \circ \tau_{2,\Delta_2} \circ \cdots \circ \tau_{n,\Delta_n}(\omega_n)$ equals $\omega'_n + \gamma$ for some $\gamma \in [0, 2\pi)$. If $\max_j \Delta_j \leq (1/2)\operatorname{sep}(\omega_n)$ then, since $\Delta_\ell = 0$ and $\Delta_k \geq 0$, we may apply the inequality (2.7) n times to obtain (2.6). Moreover, unless $\Delta_k = 0$ for all k, inequality (2.8) is strict. If $\max_j \Delta_j > (1/2)\operatorname{sep}(\omega_n)$, then we may choose m such that $(1/m)\max_j \Delta_j < (1/2)\operatorname{sep}(\omega_n)$ and then recursively applying $\tau_{(1/m)\boldsymbol{\Delta}}$ to ω_n the number m times, we again obtain (2.6).

Finally, since $\Delta_\ell = 0$ and $\Delta_{\ell-1}, \Delta_{\ell+1} \geq 0$, we have $[t_\ell, t_{\ell+1}] \subset [t''_\ell, t''_{\ell+1}] = [t'_\ell + \gamma, t'_{\ell+1} + \gamma]$. □

Proof of Theorem 2.1. Let $\omega_n \in \Omega_n$ be fixed but arbitrary and recall that $\widetilde{\omega}_n$ denotes an equally spaced configuration. By Lemma 2.2, there is some $\ell \in \{0, 1, \ldots, n\}$ and some $\gamma \in [0, 2\pi)$ such that

$$P_{\widetilde{\omega}_n}(t - \gamma) \leq P_{\omega_n}(t)\,, \qquad t \in [t_\ell, t_{\ell+1}].$$

Hence,

$$\begin{aligned} P_{\widetilde{\omega}_n}(\pi/n) &= \min_{t\in[0,2\pi)} P_{\widetilde{\omega}_n}(t) \leq \min_{t\in[t_\ell,t_{\ell+1}]} P_{\widetilde{\omega}_n}(t-\gamma) \\ &\leq \min_{t\in[t_\ell,t_{\ell+1}]} P_{\omega_n}(t) \leq \max_j \min_{t\in[t_j,t_{j+1}]} P_{\omega_n}(t). \end{aligned}$$

□

2.3. **Derivatives of logarithmic potentials.** We next consider a class of kernels g derived from $g_{\log}$ that were considered in [7]. For an even positive integer m, we define the kernel:

$$g_m(t) := g_{\log}^{(m)}(t) = \frac{d^m}{dt^m} g_{\log}(t).$$

Then, for $t \in [0, 2\pi)$,

$$g_2(t) = \frac{d}{dt}\left(-\frac{1}{2}\cot\left(\frac{t}{2}\right)\right) = \frac{1}{4}\csc^2\left(\frac{t}{2}\right)$$

and hence

$$g_m(t) = \frac{1}{4} f^{(m-2)}(t)\,,$$

where $f(t) := \csc^2(t/2)$. Following [7], we next verify that g_m satisfies the hypotheses of Theorem 2.1. It is well known and elementary to check that

$$\tan t = \sum_{j=1}^{\infty} a_j t^j\,, \qquad t \in (-\pi/2, \pi/2)\,,$$

with each $a_j \geq 0$, $j = 1, 2, \ldots$. Hence, if $h(t) := \tan(t/2)$, then

$$h^{(k)}(t) > 0, \qquad t \in (0, \pi), \qquad k = 0, 1, \ldots\,.$$

Now observe that

$$f(t) = \csc^2\left(\frac{t}{2}\right) = \sec^2 \frac{\pi - t}{2} = 2h'(\pi - t)\,,$$

and hence,

$$(-1)^k f^{(k)}(t) = 2h^{(k+1)}(\pi - t) > 0, \qquad t \in (0, \pi)\,.$$

This implies that if m is even, then $g_m(t) = \frac{1}{4} f^{(m-2)}(t)$ is a positive, decreasing, strictly convex function on $(0, \pi)$. It is also clear that if m is even, then g_m is even since f is even. Thus, $g = g_m$ satisfies the hypotheses of Theorem 2.1.

We remark that, for an even positive integer m, an induction argument implies that

$$g_m(t) = p_m(r^{-2}), \qquad r = 2\sin(t/2),$$

for some polynomial p_m of degree $m/2$. The induction follows from the recursive relation

$$p_{m+2}(x) = (6x^2 - x)p_m'(x) + (4x^3 - x^2)p_m''(x), \tag{2.9}$$

which is easily derived using $(r')^2 = 1 - (r/2)^2$ and $r'' = -r/4$. Thus, g_m can be expressed as a linear combination of Riesz s-potentials with $s = 2, 4, \ldots, m$ with coefficients corresponding to the polynomial p_m. Table 1 displays p_m for $m = 2, 4, 6$, and 8.

For $\omega_n \in \Omega_n$, we let

$$Q_{\omega_n}(t) := \prod_{j=1}^{n} \sin\left|\frac{t - t_j}{2}\right|$$

and set

$$T_n(t) := Q_{\widetilde{\omega}_n}(t) = 2^{1-n} \sin\left|\frac{nt}{2}\right| .$$

Our next two results are consequences of Lemma 2.2 and Theorem 2.1, respectively.

Theorem 2.3. *Let m be a positive even integer and $\omega_n \in \Omega_n$. Then there is some $\gamma \in [0, 2\pi)$ and some $j \in \{1, 2, \dots, n\}$ (with γ and j depending on ω_n) such that*

$$-(\log|Q_{\omega_n}|)^{(m)}(t) \geq -(\log|T_n|)^{(m)}(t-\gamma)\,, \qquad t \in (t_j, t_{j+1})\,.$$

Proof. This is an immediate consequence of Lemma 2.2 with $g = g_m$ and $\omega_n' = \widetilde{\omega}_n$, and so $P_{\omega_n}(t) = -(\log|Q_{\omega_n}|)^{(m)}(t)$ and $P_{\widetilde{\omega}_n}(t) = -(\log|T_n|)^{(m)}(t)$). □

Since g_m satisfies the hypotheses of Theorem 2.1, we obtain the following theorem.

Theorem 2.4. *We have*

$$\min_{\omega_n \in \Omega_n} \left\{ \max_{1 \leq j \leq n} \left\{ \min_{t \in [t_j, t_{j+1}]} -(\log|Q_{\omega_n}|)^{(m)}(t) \right\}\right\} = -(\log|T_n|)^{(m)}(\pi/n)\,,$$

for every even positive integer m.

From (2.9), one can show that the leading coefficient of p_m is $(m-1)!$. A somewhat more detailed computation using (2.5) and (2.9) yields

$$-(\log|T_n|)^{(m)}(\pi/n) = \frac{2}{(2\pi)^m}\zeta(m)(m-1)!(2^m-1). \tag{2.10}$$

Table 1 gives the values $-(\log|T_n|)^{(m)}(\pi/n)$ for $m = 2, 4, 6, 8$, and for $n \in \mathbb{N}$.

m	$p_m(x)$	$-(\log\lvert T_n\rvert)^{(m)}(\pi/n)$
2	x	$n^2/4$
4	$6x^2 - x$	$n^4/8$
6	$120x^3 - 30x^2 + x$	$n^6/4$
8	$5040x^4 - 1680x^3 + 126x^2 - x$	$17n^8/16$

TABLE 1. The polynomials $p_m(x)$ and the values $-(\log|T_n|)^{(m)}(\pi/n)$ from (2.10) (see Theorem 2.4 and Corollary 3.3) for $m = 2, 4, 6, 8$, and for $n \in \mathbb{N}$.

3. Comments on polarization

The main part of the following 'polarization' theorem was proved in [8]. As observed in [7], for each $\omega_n \in \Omega_n$, we may restrict the set over which we search for a minimum to

$$E(\omega_n) := [0, 2\pi) \setminus \bigcup_{j=1}^{n} (t_j - \pi/n, t_j + \pi/n) \pmod{2\pi}.$$

Theorem 3.1. *Let g be as in Theorem 2.1. Then*

$$\max_{\omega_n \in \Omega_n} \left\{ \min_{t \in [0,2\pi)} P_{\omega_n}(t) \right\} = \max_{\omega_n \in \Omega_n} \left\{ \min_{t \in E(\omega_n)} P_{\omega_n}(t) \right\} = P_{\widetilde{\omega}_n}(\pi/n). \tag{3.1}$$

Proof. Let $\omega_n \in \Omega_n$ be arbitrary. The proof follows from Lemma 2.2 and is similar to the proof of Theorem 2.1, except that the roles of $\widetilde{\omega}_n$ and ω_n are switched. By Lemma 2.2, there is some $\ell \in \{0, 1, \ldots, n\}$ and some $\gamma \in [0, 2\pi)$ such that

$$P_{\omega_n}(t - \gamma) \le P_{\widetilde{\omega}_n}(t), \qquad t \in [\widetilde{t}_\ell, \widetilde{t}_{\ell+1}],$$

and $[\widetilde{t}_\ell, \widetilde{t}_{\ell+1}] \subset [t_\ell + \gamma, t_{\ell+1} + \gamma]$. Then

$$t_\ell + \pi/n \le \pi(2\ell - 1)/n - \gamma \le t_{\ell+1} - \pi/n,$$

and so $\pi(2\ell + 1)/n - \gamma \in E(\omega_n)$. We then obtain

$$\begin{aligned} \min_{t \in [0,2\pi)} P_{\omega_n}(t) \le \min_{t \in E(\omega_n)} P_{\omega_n}(t) &\le P_{\omega_n}(\pi(2\ell + 1)/n - \gamma) \\ &\le P_{\widetilde{\omega}_n}(\pi(2\ell + 1)/n) = P_{\widetilde{\omega}_n}(\pi/n), \end{aligned}$$

which completes the proof. □

Theorem 3.2. *Let $\omega_n \in \Omega_n$. Then there is a number $\theta \in [0, 2\pi)$ (depending on ω_n) such that*

$$-(\log |Q_{\omega_n}|)^{(m)}(t) \le -(\log |T_n|)^{(m)}(t - \theta), \qquad t \in (\theta, \theta + 2\pi/n),$$

for every nonnegative even integer m.

Proof. Let m be a nonnegative even integer. We apply Lemma 2.2 with $g = g_m$, $\omega_n' = \omega_n$ and $\omega_n = \widetilde{\omega}_n$ (in which case, $P_{\omega_n}(t) = -(\log |Q_{\omega_n}|)^{(m)}(t)$ and $P_{\widetilde{\omega}_n}(t) = -(\log |T_n|)^{(m)}(t)$) to deduce that there is an $\ell \in \{1, 2, \ldots, n\}$ and a number $\gamma \in [0, 2\pi)$ (depending on ω_n) such that

$$-(\log |Q_{\omega_n}|)^{(m)}(t - \gamma) \le -(\log |T_n|)^{(m)}(t), \qquad t \in [\widetilde{t}_\ell, \widetilde{t}_{\ell+1}),$$

which can be rewritten using $\theta := \widetilde{t}_\ell - \gamma$, $u := t - \gamma$, and the fact that T_n is $2\pi/n$ periodic as

$$-(\log |Q_{\omega_n}|)^{(m)}(u) \le -(\log |T_n|)^{(m)}(u - \theta), \qquad u \in [\theta, \theta + 2\pi/n).$$

□

Corollary 3.3. *We have*

$$\max_{\omega_n \in \Omega_n} \left\{ \min_{t \in [0,2\pi)} -(\log |Q_{\omega_n}|)^{(m)}(t) \right\} = \max_{\omega_n \in \Omega_n} \left\{ \min_{t \in E(\omega_n)} -(\log |Q_{\omega_n}|)^{(m)}(t) \right\}$$
$$= -(\log |T_n|)^{(m)}(\pi/n)$$

for every even integer m.

Proof. This is an immediate corollary of Theorem 3.1. □

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