ISSN: 1889-3066

Web site: jja.ujaen.es

Jaen J. Approx. 5(2) (2013), 179–208

Jaen Journal on Approximation

Determining singularities using rows of Padé-orthogonal approximants†

Nattapong Bosuwan, Guillermo López Lagomasino and Edward B. Saff

Communicated by
F. Marcellán

Received
June 2, 2013
Accepted
September 12, 2013

Abstract

Starting from the orthogonal polynomial expansion of a function F corresponding to a finite positive Borel measure with infinite compact support, we study the asymptotic behavior of certain associated rational functions (Padé-orthogonal approximants). We obtain both direct and inverse results relating the convergence of the poles of the approximants and the singularities of F. Thereby, we obtain analogues of theorems by E. Fabry, R. de Montessus de Ballore, V. I. Buslaev, and S. P. Suetin.

Keywords: Padé approximants, orthogonal polynomials, Fabry's theorem, Montessus de Ballore's theorem.

MSC: Primary 30E10, 41A27; Secondary 41A21.

§1. Introduction

Let E be an infinite compact subset of the complex plane $\mathbb{C}$ such that $\overline{\mathbb{C}} \setminus E$ is simply connected. There exists a unique exterior conformal representation Φ from $\overline{\mathbb{C}} \setminus E$ onto

†Results of this article constitute part of N.B.'s Ph.D. dissertation under the mentorship of E.B.S. at Vanderbilt University, G.L.L. was supported by Ministerio de Economía y Competitividad under grant MTM2012-36372-C03-01, E.B.S. was supported, in part, by the U.S. National Science Foundation under grants DMS-0808093 and DMS-1109266.

$\overline{\mathbb{C}} \setminus \{w : |w| \le 1\}$ satisfying $\Phi(\infty) = \infty$ and $\Phi'(\infty) > 0$. We assume that E is such that the inverse function $\Psi = \Phi^{-1}$ can be extended continuously to $\overline{\mathbb{C}} \setminus \{w : |w| < 1\}$ (the closure of a bounded Jordan region and a finite interval satisfy the above conditions). Unless otherwise stated, E will be as described above.

Let μ be a finite positive Borel measure with infinite support $\mathrm{supp}(\mu)$ contained in E. We write $\mu \in \mathcal{M}(E)$ and define the associated inner product,

$$\langle g, h\rangle_\mu := \int g(\zeta)\overline{h(\zeta)}d\mu(\zeta), \quad g, h \in L_2(\mu).$$

Let

$$p_n(z) := \kappa_n z^n + \cdots, \quad \kappa_n > 0, \quad n = 0, 1, \ldots,$$

be the orthonormal polynomial of degree n with respect to μ having positive leading coefficient; that is, $\langle p_n, p_m\rangle_\mu = \delta_{n,m}$. Denote by $\mathcal{H}(E)$ the space of all functions holomorphic in some neighborhood of E.

Definition 1.1. Let $F \in \mathcal{H}(E)$, $\mu \in \mathcal{M}(E)$ and a pair of nonnegative integers (n, m) be given. A rational function $[n/m]_F^\mu := P_{n,m}^\mu / Q_{n,m}^\mu$ is called an (n, m) *Padé-orthogonal approximant* of F with respect to μ if $P_{n,m}^\mu$ and $Q_{n,m}^\mu$ are polynomials satisfying

$$\deg(P_{n,m}^\mu) \le n, \quad \deg(Q_{n,m}^\mu) \le m, \quad Q_{n,m}^\mu \not\equiv 0, \tag{1.1}$$

$$\langle Q_{n,m}^\mu F - P_{n,m}^\mu, p_j\rangle_\mu = 0, \quad \text{for } j = 0, 1, \ldots, n+m. \tag{1.2}$$

Since $Q_{n,m}^\mu \not\equiv 0$, we normalize it to have leading coefficient equal to 1.

When $E = \{z \in \mathbb{C} : |z| \le 1\}$ and $d\mu = d\theta/2\pi$ on the boundary of E, then $p_n(z) = z^n$ and the Padé-orthogonal approximants reduce to the classical Padé approximants. In this case, we write $P_{n,m}, Q_{n,m}$ and $[n/m]_F$, respectively.

The study of the convergence properties of row sequences of Padé approximants (when m is fixed and $n \to \infty$) has a long history beginning with the classical results of J. Hadamard [11], R. de Montessus de Ballore [12], and E. Fabry [5]. These results have attracted considerable attention motivating different extensions and generalizations to other approximation schemes using rational functions in which the degree of the denominator remains bounded as $n \to \infty$ (see, for example, [2, 4, 7, 8, 9, 10, 14, 18, 19, 20, 21, 22]). For the case of measures supported on the real line and the unit circle, some results in this direction are contained in [2, 3, 18, 19]. However, up to the present there are no results of this nature for measures supported on general compact subsets E of the complex plane. The object of this paper is to fill this gap.

The general theory covers direct and inverse type results. In direct results one starts with a function for which the analytic properties and location of singularities in a certain domain is known, and using this information one draws conclusions about the asymptotic behavior of the approximants and their poles. In the inverse direction, the information is given in terms of the asymptotic behavior of the poles of the approximating functions from which the analyticity and location of the singularities of the function must be deduced. We give results in both directions.

For any $\rho > 1$, set

$$\Gamma_\rho := \{z \in \mathbb{C} : |\Phi(z)| = \rho\} \qquad \text{and} \qquad \gamma_\rho := \{w \in \mathbb{C} : |w| = \rho\}.$$

Denote by D_ρ the bounded connected component of the complement of Γ_ρ and by $\mathbb{B}(a, \rho)$ the open disk centered at $a \in \mathbb{C}$ of radius ρ. We call Γ_ρ and D_ρ a *level curve* and a *canonical domain* (with respect to E), respectively. We denote by $\rho_0(F)$ the index ρ (> 1) of the largest canonical domain D_ρ to which F can be extended as a holomorphic function, and by $\rho_m(F)$ the index ρ of the largest canonical domain D_ρ to which F can be extended as a meromorphic function with at most m poles (counting multiplicities).

Let $\mu \in \mathcal{M}(E)$ be such that

$$\lim_{n\to\infty} |p_n(z)|^{1/n} = |\Phi(z)|, \tag{1.3}$$

uniformly inside $\mathbb{C} \setminus E$. Such measures are called *regular* (cf. [17]). Here and in what follows, the phrase "uniformly inside a domain" means "uniformly on each compact subset of the domain". The Fourier coefficient of F with respect to p_n is given by

$$F_n := \langle F, p_n \rangle_\mu = \int F(z)\overline{p_n(z)}d\mu(z).$$

As in the case of Taylor series (see, for example, [17, Theorem 6.6.1]), it is easy to show that

$$\rho_0(F) = \left(\limsup_{n\to\infty} |F_n|^{1/n}\right)^{-1}.$$

Additionally, the series $\sum_{n=0}^\infty F_n p_n(z)$ converges to $F(z)$ uniformly inside $D_{\rho_0(F)}$ and diverges pointwise for all $z \in \mathbb{C} \setminus \overline{D_{\rho_0(F)}}$. Therefore, if (1.3) holds, then

$$Q^\mu_{n,m}(z)F(z) - P^\mu_{n,m}(z) = \sum_{k=n+m+1}^{\infty} \langle Q^\mu_{n,m}F, p_k\rangle_\mu \, p_k(z) \tag{1.4}$$

for all $z \in D_{\rho_0(F)}$ and $P^{\mu}_{n,m} = \sum_{k=0}^{n} \langle Q^{\mu}_{n,m} F, p_k \rangle_\mu \, p_k$ is uniquely determined by $Q^{\mu}_{n,m}$.

In contrast with classical Padé approximants, the rational function $[n/m]^{\mu}_F$ may not be unique as the following example shows.

Example 1.2. Let $E = [-1, 1]$, $d\mu = dx/\sqrt{1-x^2}$ and

$$F(x) = \frac{37}{x-3} + \sum_{k=0}^{4} c_k p_k(x),$$

where the p_k are normalized Chebyshev polynomials, and

$$c_0 := 37, \quad c_1 := 6(-271\sqrt{\pi} + 192\sqrt{2\pi}), \quad c_2 := -\sqrt{2} + 315\sqrt{\pi} - 222\sqrt{2\pi},$$

$$c_3 := 3513\sqrt{\pi} - 2484\sqrt{2\pi}, \quad c_4 := \sqrt{2} + 10674\sqrt{\pi} - 7548\sqrt{2\pi}.$$

Using the program Mathematica it is easy to check that both $Q^{\mu}_{1,2}(x) := x$ and $Q^{\mu}_{1,2}(x) = (x-3)^2$ satisfy $\langle Q^{\mu}_{1,2} F, p_k \rangle_\mu = 0, k = 2, 3$. These denominators $Q^{\mu}_{1,2}$ give us

$$[1/2]^{\mu}_F(x) = \frac{4756\sqrt{\pi} - 3363\sqrt{2\pi} - 36\sqrt{2\pi}x + 144x}{4\sqrt{\pi}x}$$

and

$$[1/2]^{\mu}_F(x) = \frac{1404 - 28536\sqrt{\pi} + 19827\sqrt{2\pi} - 864x + 90364\sqrt{\pi}x - 63681\sqrt{2\pi}x}{4\sqrt{\pi}(x-3)^2},$$

respectively, which are clearly distinct.

It is easy to see, however, that the condition

$$\Delta_{n,m}(F,\mu) := \begin{vmatrix} \langle F, p_{n+1} \rangle_\mu & \langle zF, p_{n+1} \rangle_\mu & \cdots & \langle z^{m-1}F, p_{n+1} \rangle_\mu \\ \vdots & \vdots & \vdots & \vdots \\ \langle F, p_{n+m} \rangle_\mu & \langle zF, p_{n+m} \rangle_\mu & \cdots & \langle z^{m-1}F, p_{n+m} \rangle_\mu \end{vmatrix} \neq 0$$

and the condition that every solution of (1.1)-(1.2) has $\deg Q^{\mu}_{n,m} = m$ are equivalent. In turn, they imply the uniqueness of $[n/m]^{\mu}_F$.

An outline of the paper is as follows. In Section 2, we state our main results and comment on their connection with classical and recent developments of the theory. Theorem 2.1 is a direct result whereas the rest of the theorems are of inverse type. Section 3 is devoted to the proof of Theorem 2.11, the rest is proved in Section 4.

§2. Main results

We will make the following assumptions on the asymptotic behavior of the sequence of orthonormal polynomials with respect to a given measure $\mu \in \mathcal{M}(E)$. We write $\mu \in \mathcal{R}(E)$ when the corresponding sequence of orthonormal polynomials has ratio asymptotics; that is,

$$\lim_{n\to\infty} \frac{p_n(z)}{p_{n+1}(z)} = \frac{1}{\Phi(z)}. \tag{2.1}$$

We say that Szegő or strong asymptotics takes place, and write $\mu \in \mathcal{S}(E)$, if

$$\lim_{n\to\infty} \frac{p_n(z)}{c_n \Phi^n(z)} = S(z) \quad \text{and} \quad \lim_{n\to\infty} \frac{c_n}{c_{n+1}} = 1. \tag{2.2}$$

The first limit in (2.2) and the one in (2.1) are assumed to hold uniformly inside $\overline{\mathbb{C}} \setminus E$, the c_n's are positive constants, and $S(z)$ is some holomorphic and non-vanishing function on $\overline{\mathbb{C}} \setminus E$. Obviously, (2.2) $\Rightarrow$ (2.1) $\Rightarrow$ (1.3).

Our first result is of direct type.

Theorem 2.1. *Suppose $F \in \mathcal{H}(E)$ has poles of total multiplicity exactly m in $D_{\rho_m(F)}$ at the (not necessarily distinct) points $\lambda_1, \ldots, \lambda_m$ and let $\mu \in \mathcal{R}(E)$. Then $[n/m]_F^\mu$ is uniquely determined for all sufficiently large n and the sequence converges uniformly to F inside $D_{\rho_m(F)} \setminus \{\lambda_1, \ldots, \lambda_m\}$ as $n \to \infty$. Moreover, for any compact subset K of $D_{\rho_m(F)} \setminus \{\lambda_1, \ldots, \lambda_m\}$,*

$$\limsup_{n\to\infty} \|F - [n/m]_F^\mu\|_K^{1/n} \leq \frac{\max\{|\Phi(z)| : z \in K\}}{\rho_m(F)}, \tag{2.3}$$

where $\|\cdot\|_K$ denotes the sup-norm on K and, if $K \subset E$, then $\max\{|\Phi(z)| : z \in K\}$ is replaced by 1. Additionally,

$$\limsup_{n\to\infty} \|Q_{n,m}^\mu - Q_m\|^{1/n} \leq \frac{\max\{|\Phi(\lambda_j)| : j = 1, \ldots, m\}}{\rho_m(F)} < 1, \tag{2.4}$$

where $\|\cdot\|$ denotes (for example) the coefficient norm in the space of polynomials of degree m and $Q_m(z) = \prod_{k=1}^m (z - \lambda_k)$.

Remark 2.2. When $K = E$, the rate of convergence in (2.3) cannot be improved; that is,

$$\limsup_{n\to\infty} \|F - [n/m]_F^\mu\|_E^{1/n} = \limsup_{n\to\infty} \sigma_{n,m}^{1/n} = \frac{1}{\rho_m(F)}, \tag{2.5}$$

where

$$\sigma_{n,m} := \inf_r \|F - r\|_E,$$

and the infimum is taken over the class of all rational functions of type (n, m),

$$r(z) = \frac{a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0}{b_m z^m + b_{m-1} z^{m-1} + \cdots + b_0}.$$

We refer the reader to [7, 13] for more information on the second equality in (2.5).

In the context of classical Padé approximation, Theorem 2.1 is known as the Montessus de Ballore theorem (see [12]). In [18, Theorem 1], S. P. Suetin proves an analogous result for measures supported on a bounded interval of the real line and states without proof that a similar theorem may be obtained for measures supported on a continuum of the complex plane whose sequence of orthonormal polynomials and their associated second type functions have strong asymptotic behavior. The assumptions of our Theorem 2.1 are substantially weaker.

In the inverse direction we have the following.

Theorem 2.3. *Let $F \in \mathcal{H}(E)$ and $\mu \in \mathcal{S}(E)$. If*

$$\lim_{n\to\infty} \frac{F_n}{F_{n+1}} = \tau,$$

then $\Psi(\tau)$ is a singularity of F and $\rho_0(F) = |\tau|$.

For expansions in Taylor series and classical Padé approximation, this result reduces to Fabry's theorem (see [5]).

If $E = \overline{\mathbb{B}}$, where $\mathbb{B} = \mathbb{B}(0, 1)$, and the measure μ supported on $\mathbb{T}$, the unit circle, satisfies the Szegő condition

$$\int_0^{2\pi} \log w(\theta) d\theta > -\infty, \tag{2.6}$$

(where $d\mu(\theta) = w(\theta)d\theta/2\pi + d\mu_s(\theta)$ is the Radon-Nikodym decomposition of μ), it is well known that the orthonormal polynomials $\varphi_n(z) = \kappa_n z^n + \cdots$ with respect to μ satisfy the Szegő asymptotics (2.2) (with $c_n = 1$). In particular, this allows us to use Theorem 2.3 to locate the first singularity of the reciprocal of the interior Szegő function

$$S_{\text{int}}(z) := \exp\left(\frac{1}{4\pi} \int_0^{2\pi} \log w(\theta) \frac{e^{i\theta} + z}{e^{i\theta} - z} d\theta\right), \quad z \in \mathbb{B},$$

in terms of the *Verblunsky (or Schur) coefficients* α_n ($\alpha_n := -\overline{\Phi_n(0)}$). It is well known that the Szegő condition (2.6) also implies that

$$\lim_{n\to\infty} \kappa_n = \kappa := \exp\left\{-\frac{1}{4\pi}\int_0^{2\pi} \log w(\theta) d\theta\right\}$$

and

$$\frac{1}{S_{\text{int}}(z)} = \frac{1}{\kappa}\sum_{k=0}^{\infty} \overline{\varphi_k(0)}\varphi_k(z)$$

uniformly inside $\mathbb{B}$ (see [6, p. 19-20]). By Theorem 2.3, we immediately obtain the following.

Corollary 2.4. *Let μ satisfy* (2.6) *and assume that $1/S_{\text{int}} \in \mathcal{H}(\overline{\mathbb{B}})$. Suppose that*

$$\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = \lambda.$$

Then λ is a singularity of $1/S_{\text{int}}$ and $1/S_{\text{int}}$ is holomorphic in $\mathbb{B}(0, |\lambda|)$.

This result complements [1, Theorem 2] where, under stronger assumptions, it is shown that λ is a simple pole and $1/S_{\text{int}}$ has no other singularity in a neighborhood of $\overline{\mathbb{B}(0, |\lambda|)}$.

Using the definition of $Q^{\mu}_{n,1}$ it is easy to verify that whenever $F_{n+1} \neq 0$, we have

$$Q^{\mu}_{n,1}(z) = z - \frac{\langle zF, p_{n+1}\rangle_\mu}{F_{n+1}}.$$

The next result enables one to locate the first singularity of F using the zeros of $Q^{\mu}_{n,1}$.

Theorem 2.5. *Let $F \in \mathcal{H}(E)$ and $\mu \in \mathcal{S}(E)$. If*

$$\lim_{n\to\infty} \frac{\langle zF, p_n\rangle_\mu}{F_n} = \lambda,$$

then λ is a singularity of F and $\rho_0(F) = |\Phi(\lambda)|$.

The proofs of Theorems 2.3 and 2.5 are reduced to Fabry's theorem by using the following result.

Theorem 2.6. *Let $F \in \mathcal{H}(E)$ and $\mu \in \mathcal{S}(E)$. Define $f(w) := F(\Psi(w))$ and denote the Laurent series of f about 0 by $\sum_{k=-\infty}^{\infty} f_k w^k$. Then, the following limits are equivalent:*

(a) $\lim_{n\to\infty} F_n/F_{n+1} = \tau$,

(b) $\lim_{n\to\infty} \langle zF, p_n\rangle_\mu / F_n = \lambda$,

(c) $\lim_{n\to\infty} f_n/f_{n+1} = \tau$,

where τ and λ are finite and related by the formula $\Phi(\lambda) = \tau$.

Theorem 2.5 admits the following extension to general row sequences.

Theorem 2.7. *Let $F \in \mathcal{H}(E)$ and $\mu \in \mathcal{S}(E)$. If for all n sufficiently large, $[n/m]_F^\mu$ has precisely m finite poles $\lambda_{n,1}, \ldots, \lambda_{n,m}$ and*

$$\lim_{n\to\infty} \lambda_{n,j} = \lambda_j, \quad j = 1, 2, \ldots, m$$

($\lambda_1, \ldots, \lambda_m$ are not necessarily distinct), then

(i) *F is holomorphic in $D_{\rho_{\min}}$, where $\rho_{\min} := \min_{1\le j\le m} |\Phi(\lambda_j)|$;*

(ii) *$\rho_{m-1}(F) = \max_{1\le j\le m} |\Phi(\lambda_j)|$;*

(iii) *$\lambda_1, \ldots, \lambda_m$ are singularities of F; those lying in $D_{\rho_{m-1}(F)}$ are poles (counting multiplicities), and F has no other poles in $D_{\rho_{m-1}(F)}$.*

For classical Padé approximants, this theorem was proved by S. P. Suetin in [21] (see also [20]). In [2, Theorem 1], V. I. Buslaev provides an analogue for measures supported on a bounded interval of the real line. Buslaev reduces the proof of his result to Suetin's statement through an extension of Poincaré's theorem on difference equations (see Lemmas 4.1-4.2 below). We will follow this approach by proving the next result.

Theorem 2.8. *Let $F \in \mathcal{H}(E)$ and $\mu \in \mathcal{S}(E)$. Define $f(w) := F(\Psi(w))$ and denote the Laurent series of f about 0 by $\sum_{k=-\infty}^{\infty} f_k w^k$ and the regular part of f by $\hat{f}(w) := \sum_{k=0}^{\infty} f_k w^k$. For each fixed $m \ge 1$, the following conditions are equivalent:*

(a) *The poles of $[n/m]_{\hat{f}}$ have finite limits $\tau_1, \ldots, \tau_m$, as $n \to \infty$.*

(b) *The poles of $[n/m]_F^\mu$ have finite limits $\lambda_1, \ldots, \lambda_m$, as $n \to \infty$.*

Under appropriate enumeration of the sub-indices, the values λ_j and τ_j, $j = 1, \ldots, m$, are related by the formula $\Phi(\lambda_j) = \tau_j$ for all $j = 1, \ldots, m$.

§3. Proof of Theorem 2.1

The second type functions $s_n(z)$ defined by

$$s_n(z) := \int \frac{\overline{p_n(\zeta)}}{z-\zeta} d\mu(\zeta), \quad z \in \overline{\mathbb{C}} \setminus \mathrm{supp}(\mu),$$

play a major role in the proofs that follow.

Lemma 3.1. *If $\mu \in \mathcal{R}(E)$, then*

$$\lim_{n\to\infty} p_n(z)s_n(z) = \frac{\Phi'(z)}{\Phi(z)},$$

uniformly inside $\overline{\mathbb{C}} \setminus E$. Consequently, for any compact set $K \subset \mathbb{C} \setminus E$, there exists n_0 such that $s_n(z) \neq 0$ for all $z \in K$ and $n \geq n_0$.

Proof. From orthogonality, we get

$$p_n(z)s_n(z) = \int \frac{|p_n(\zeta)|^2}{z-\zeta} d\mu(\zeta), \quad z \notin \mathrm{supp}(\mu).$$

Since p_n is of norm 1 in $L_2(\mu)$, the sequence $\left(\int |p_n(\zeta)|^2/(z-\zeta)d\mu(\zeta)\right)_{n\geq 0}$ forms a normal family in $\overline{\mathbb{C}} \setminus E$. Consequently, the limit stated follows from pointwise convergence in a neighborhood of infinity. Now, for all z sufficiently large, since $\mu \in \mathcal{R}(E)$, from [15, Theorem 1.8] it follows that[1]

$$\begin{aligned} \lim_{n\to\infty} \int \frac{|p_n(\zeta)|^2}{z-\zeta} d\mu(\zeta) &= \lim_{n\to\infty} \sum_{k=0}^{\infty} \frac{1}{z^{k+1}} \int \zeta^k |p_n(\zeta)|^2 d\mu(\zeta) = \sum_{k=0}^{\infty} \frac{1}{z^{k+1}} \frac{1}{2\pi} \int_{\mathbb{T}} \Psi(w)^k \frac{dw}{wi} \\ &= \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{1}{w(z-\Psi(w))} dw = \frac{1}{2\pi i} \int_{\Psi(\mathbb{T})} \frac{\Phi'(\zeta)}{\Phi(\zeta)(z-\zeta)} d\zeta = \frac{\Phi'(z)}{\Phi(z)}. \end{aligned}$$

Since the function on the right-hand side never vanishes in $\mathbb{C} \setminus E$, the rest of the statements follow at once. ■

Proof of Theorem 2.1. For $l = 0, 1, \ldots$, from (2.1) it follows that

$$\lim_{n\to\infty} \frac{p_n(z)}{p_{n+l}(z)} = \frac{1}{\Phi(z)^l}, \quad l = 0, 1, \ldots, \tag{3.1}$$

[1] We note that in [15, Theorem 1.8] it is assumed that E is a compact set bounded by a Jordan curve. However, as pointed out to us by the author, the result remains valid if E verifies the conditions imposed in this paper.

uniformly inside $\mathbb{C} \setminus E$ and by (3.1) and Lemma 3.1

$$\lim_{n\to\infty} \frac{s_{n+l}(z)}{s_n(z)} = \lim_{n\to\infty} \frac{p_n(z)}{p_{n+l}(z)} \frac{p_{n+l}(z)s_{n+l}(z)}{p_n(z)s_n(z)} = \frac{1}{\Phi(z)^l}, \tag{3.2}$$

uniformly inside $\mathbb{C} \setminus E$. From (3.1) and (3.2) we obtain

$$\lim_{n\to\infty} |p_n(z)|^{1/n} = |\Phi(z)| \qquad \text{and} \qquad \lim_{n\to\infty} |s_n(z)|^{1/n} = \frac{1}{|\Phi(z)|}, \tag{3.3}$$

uniformly inside $\mathbb{C} \setminus E$.

By the definition of Padé-orthogonal approximant and the first relation in (3.3), we have (see (1.4))

$$Q^{\mu}_{n,m}(z)F(z) - P^{\mu}_{n,m}(z) = \sum_{k=n+m+1}^{\infty} a_{k,n} p_k(z), \quad z \in D_{\rho_0(F)}. \tag{3.4}$$

Using Cauchy's integral formula and Fubini's theorem, we obtain, for $k \geq n+1$,

$$a_{k,n} := \langle Q^{\mu}_{n,m}F, p_k\rangle_{\mu} = \int \frac{1}{2\pi i} \int_{\Gamma_{\rho_1}} \frac{Q^{\mu}_{n,m}(t)F(t)}{t-z} dt\, \overline{p_k(z)}\, d\mu(z) \tag{3.5}$$

$$= \frac{1}{2\pi i} \int_{\Gamma_{\rho_1}} Q^{\mu}_{n,m}(t)F(t) \int \frac{\overline{p_k(z)}}{t-z} d\mu(z) dt = \frac{1}{2\pi i} \int_{\Gamma_{\rho_1}} Q^{\mu}_{n,m}(t)F(t)s_k(t)dt,$$

where $1 < \rho_1 < \rho_0(F)$. Let $\{\alpha_1, \ldots, \alpha_\gamma\}$ be the set of distinct poles of F in $D_{\rho_m(F)}$ and m_k the multiplicity of α_k so that

$$Q(z) := \prod_{j=1}^{m} (z - \lambda_j) = \prod_{k=1}^{\gamma} (z - \alpha_k)^{m_k}, \quad m := \sum_{k=1}^{\gamma} m_k.$$

Multiplying (3.4) by Q and expanding $QQ^{\mu}_{n,m}F - QP^{\mu}_{n,m} \in H(D_{\rho_m(F)})$ in terms of the orthonormal system $\{p_\nu\}_{\nu=0}^{\infty}$, for $z \in D_{\rho_m(F)}$ we obtain

$$Q(z)Q^{\mu}_{n,m}(z)F(z) - Q(z)P^{\mu}_{n,m}(z) = \sum_{k=n+m+1}^{\infty} a_{k,n} Q(z) p_k(z) = \sum_{\nu=0}^{\infty} b_{\nu,n} p_\nu(z), \tag{3.6}$$

where

$$b_{\nu,n} := \sum_{k=n+m+1}^{\infty} a_{k,n} \langle Qp_k, p_\nu \rangle_\mu, \quad \nu = 0, 1, \ldots .$$

First of all, we will estimate $|a_{k,n}|$ in terms of $|\tau_{k,n}|$, where

$$\tau_{k,n} := \frac{1}{2\pi i} \int_{\Gamma_{\rho_2}} Q^\mu_{n,m}(t) F(t) s_k(t) dt, \quad \rho_{m-1}(F) < \rho_2 < \rho_m(F), \quad k = 0, 1 \ldots . \tag{3.7}$$

Since $\rho_2 > \rho_1$ the integral in (3.7) allows a better upper bound than the last integral in (3.5). For each $k \geq 0$, the function $Q^\mu_{n,m} F s_k$ is meromorphic on $\overline{D_{\rho_2}} \setminus D_{\rho_1} = \{z \in \mathbb{C} : \rho_1 \leq |\Phi(z)| \leq \rho_2\}$ and has poles at $\alpha_1, \ldots, \alpha_\gamma$ with multiplicities at most $m_1, \ldots, m_\gamma$, respectively. Applying Cauchy's residue theorem we obtain

$$\frac{1}{2\pi i} \int_{\Gamma_{\rho_2}} Q^\mu_{n,m}(t) F(t) s_k(t) dt - \frac{1}{2\pi i} \int_{\Gamma_{\rho_1}} Q^\mu_{n,m}(t) F(t) s_k(t) dt \tag{3.8}$$

$$= \sum_{j=1}^{\gamma} \operatorname{res}(Q^\mu_{n,m} F s_k, \alpha_j),$$

for $k \geq 0$. The limit formula for the residue gives

$$\operatorname{res}(Q^\mu_{n,m} F s_k, \alpha_j) = \frac{1}{(m_j - 1)!} \lim_{z \to \alpha_j} ((z - \alpha_j)^{m_j} Q^\mu_{n,m}(z) F(z) s_k(z))^{(m_j - 1)}.$$

Since $s_n(z) \neq 0$ for all sufficiently large n and $z \in \mathbb{C} \setminus E$ (see Lemma 3.1), Leibniz' formula allows us to write

$$((z - \alpha_j)^{m_j} Q^\mu_{n,m}(z) F(z) s_k(z))^{(m_j - 1)} = \left((z - \alpha_j)^{m_j} Q^\mu_{n,m}(z) F(z) s_n(z) \frac{s_k(z)}{s_n(z)} \right)^{(m_j - 1)}$$

$$= \sum_{p=0}^{m_j - 1} \binom{m_j - 1}{p} ((z - \alpha_j)^{m_j} Q^\mu_{n,m}(z) F(z) s_n(z))^{(m_j - 1 - p)} \left(\frac{s_k(z)}{s_n(z)} \right)^{(p)} .$$

For $j = 1, \ldots, \gamma$ and $p = 0, \ldots, m_j - 1$, set

$$\beta_n(j, p) := \frac{1}{(m_j - 1)!} \binom{m_j - 1}{p} \lim_{z \to \alpha_j} ((z - \alpha_j)^{m_j} Q^\mu_{n,m}(z) F(z) s_n(z))^{(m_j - 1 - p)},$$

(notice that the $\beta_n(j,p)$ do not depend on k). So, we can rewrite (3.8) as

$$a_{k,n} = \tau_{k,n} - \sum_{j=1}^{\gamma} \left(\sum_{p=0}^{m_j-1} \beta_n(j,p) \left(\frac{s_k}{s_n}\right)^{(p)} (\alpha_j) \right), \quad n \geq n_0 \quad \text{and} \quad k = 0,1,\ldots. \tag{3.9}$$

Since $a_{k,n} = 0$, for $k = n+1, n+2, \ldots, n+m$, it follows from (3.9) that

$$\sum_{j=1}^{\gamma} \sum_{p=0}^{m_j-1} \beta_n(j,p) \left(\frac{s_k}{s_n}\right)^{(p)} (\alpha_j) = \tau_{k,n}, \quad k = n+1,\ldots,n+m. \tag{3.10}$$

We view (3.10) as a system of m equations on the m unknowns $\beta_n(j,p)$. If we show that

$$\Lambda_n := \left| \begin{matrix} \left(\frac{s_{n+1}}{s_n}\right)(\alpha_j) & \left(\frac{s_{n+1}}{s_n}\right)'(\alpha_j) & \cdots & \left(\frac{s_{n+1}}{s_n}\right)^{(m_j-1)}(\alpha_j) \\ \left(\frac{s_{n+2}}{s_n}\right)(\alpha_j) & \left(\frac{s_{n+2}}{s_n}\right)'(\alpha_j) & \cdots & \left(\frac{s_{n+2}}{s_n}\right)^{(m_j-1)}(\alpha_j) \\ \vdots & \vdots & \vdots & \vdots \\ \left(\frac{s_{n+m}}{s_n}\right)(\alpha_j) & \left(\frac{s_{n+m}}{s_n}\right)'(\alpha_j) & \cdots & \left(\frac{s_{n+m}}{s_n}\right)^{(m_j-1)}(\alpha_j) \end{matrix} \right|_{j=1,\ldots,\gamma} \neq 0$$

(this expression represents the determinant of order m in which the indicated group of columns is written out successively for $j = 1,\ldots,\gamma$), then we can express $\beta_n(j,p)$ in terms of $(s_k/s_n)^{(p)}(\alpha_j)$ and $\tau_{k,n}$, for $k = n+1,\ldots n+m$. In fact,

$$\lim_{n\to\infty} \Lambda_n = \Lambda := \left| \begin{matrix} R(\alpha_j) & R'(\alpha_j) & \cdots & R^{(m_j-1)}(\alpha_j) \\ R^2(\alpha_j) & (R^2)'(\alpha_j) & \cdots & (R^2)^{(m_j-1)}(\alpha_j) \\ \vdots & \vdots & \vdots & \vdots \\ R^m(\alpha_j) & (R^m)'(\alpha_j) & \cdots & (R^m)^{(m_j-1)}(\alpha_j) \end{matrix} \right|_{j=1,\ldots,\gamma}$$
$$= \prod_{j=1}^{\gamma} (m_j-1)!!(-\Phi'(\alpha_j))^{m_j(m_j-1)/2} \Phi(\alpha_j)^{-m_j^2} \prod_{1\leq i<j\leq\gamma} \left(\frac{1}{\Phi(\alpha_j)} - \frac{1}{\Phi(\alpha_i)}\right)^{m_i m_j},$$

where $R(z) = 1/\Phi(z)$ and $n!! = 0!1!\cdots n!$ (use, for example, [16, Theorem 1] for the last equality). Hence, $\Lambda \neq 0$ and, for all sufficiently large n, $|\Lambda_n| \geq c_1 > 0$, where the number c_1 does not depend on n (from now on, we will denote some constants that do not depend on n by $c_2, c_3, \ldots$ and we will consider only n large enough so that $|\Lambda_n| \geq c_1 > 0$).

Applying Cramer's rule to (3.10), we have

$$\beta_n(j,p) = \frac{\Lambda_n(j,p)}{\Lambda_n} = \frac{1}{\Lambda_n}\sum_{s=1}^{m} \tau_{n+s,n} C_n(s,q), \tag{3.11}$$

where $\Lambda_n(j,p)$ is the determinant obtained from Λ_n by replacing the column with index $q = (\sum_{l=0}^{j-1} m_l) + p + 1$ (where $m_0 := 0$) with the column $[\tau_{n+1,n} \ \ldots \ \tau_{n+m,n}]^T$ and $C_n(s,q)$ is the determinant of the $(s,q)^{\text{th}}$ cofactor matrix of $\Lambda_n(j,p)$. Substituting $\beta_n(j,p)$ in (3.9) with the expression in (3.11), we obtain

$$a_{k,n} = \tau_{k,n} - \frac{1}{\Lambda_n}\sum_{j=1}^{\gamma}\sum_{p=0}^{m_j-1}\sum_{s=1}^{m} \tau_{n+s,n} C_n(s,q) \left(\frac{s_k}{s_n}\right)^{(p)} (\alpha_j), \quad k \geq n+m+1. \tag{3.12}$$

Let $\delta > 0$ and $\epsilon > 0$ be sufficiently small so that $\rho_0(F) - 2\delta > 1$,

$$\{z \in \mathbb{C} : |z - \alpha_j| = \epsilon\} \subset \{z \in \mathbb{C} : |\Phi(z)| \geq \rho_0(F) - \delta\}.$$

For $k = 0, 1 \ldots$ and $p = 0, \ldots, m_j - 1$, we have

$$\left(\frac{s_k}{s_n}\right)^{(p)} (\alpha_j) = \frac{p!}{2\pi i}\int_{|z-\alpha_j|=\epsilon} \frac{s_k(z)}{s_n(z)(z-\alpha_j)^{p+1}} dz. \tag{3.13}$$

Applying (3.2) and (3.13), we can easily check that for $p = 0, \ldots, m_j - 1, j = 1, \ldots, \gamma$, and $k = n+1, \ldots, n+m$,

$$\left|\left(\frac{s_k}{s_n}\right)^{(p)} (\alpha_j)\right| \leq c_2, \tag{3.14}$$

for $n \geq n_1$, and if $k \geq n+m+1$,

$$\left|\left(\frac{s_k}{s_n}\right)^{(p)} (\alpha_j)\right| \leq \frac{c_3}{(\rho_0(F) - 2\delta)^{k-n}}, \tag{3.15}$$

for $n \geq n_2$. The inequality (3.14) implies that

$$|C_n(s,q)| \leq (m-1)! c_2^{m-1} = c_4, \quad s, q = 1, \ldots, m, \tag{3.16}$$

for $n \geq n_3$. Combining (3.12), (3.14), (3.15), (3.16) and $|\Lambda_n| \geq c_1 > 0$, we see that for $n \geq n_4$

$$|a_{k,n}| \leq |\tau_{k,n}| + \frac{mc_4c_3}{c_1} \frac{1}{(\rho_0(F) - 2\delta)^{k-n}} \sum_{s=1}^{m} |\tau_{n+s,n}| \tag{3.17}$$

$$\leq |\tau_{k,n}| + \frac{c_5}{(\rho_0(F) - 2\delta)^{k-n}} \sum_{s=1}^{m} |\tau_{n+s,n}|, \quad k \geq n+m+1.$$

Now, we estimate $|b_{\nu,n}|$ in terms of $|\tau_{k,n}|$. By the Cauchy-Schwarz inequality and the orthonormality of p_ν, for $k, \nu = 0, 1, \ldots$ we have

$$|\langle Qp_k, p_\nu\rangle_\mu|^2 \leq \langle Qp_k, Qp_k\rangle_\mu \langle p_\nu, p_\nu\rangle_\mu \leq \max_{z\in E} |Q(z)|^2 = c_6. \tag{3.18}$$

By (3.17), (3.18) and the fact that $\sum_{k=n+m+1}^{\infty}(\rho_0(F)-2\delta)^{n-k} < \infty$, we obtain, for n sufficiently large and for all $\nu \geq 0$,

$$|b_{\nu,n}| \leq \sum_{k=n+m+1}^{\infty} |a_{k,n}||\langle Qp_k, p_\nu\rangle| \leq \sqrt{c_6} \sum_{k=n+m+1}^{\infty} |a_{k,n}| \tag{3.19}$$

$$\leq \sqrt{c_6} \left(\sum_{k=n+m+1}^{\infty} |\tau_{k,n}| + c_5 \sum_{k=n+m+1}^{\infty} \frac{1}{(\rho_0(F) - 2\delta)^{k-n}} \sum_{s=1}^{m} |\tau_{n+s,n}| \right) \leq c_7 \sum_{k=n+1}^{\infty} |\tau_{k,n}|.$$

Let K be a compact subset of $D_{\rho_m(F)}$ and $\sigma > 1$ be such that $K \subset \overline{D_\sigma} \subset D_{\rho_m(F)}$. Choose $\delta > 0$ sufficiently small so that

$$\rho_2 := \rho_m(F) - \delta > \rho_{m-1}(F), \quad \rho_0(F) - 2\delta > 1 \quad \text{and} \quad \frac{\sigma+\delta}{\rho_2 - \delta} < 1. \tag{3.20}$$

We write (3.6) in the form

$$|Q(z)Q^\mu_{n,m}(z)F(z) - Q(z)P^\mu_{n,m}(z)| \leq \sum_{\nu=0}^{n+m} |b_{\nu,n}||p_\nu(z)| + \sum_{\nu=n+m+1}^{\infty} |b_{\nu,n}||p_\nu(z)|. \tag{3.21}$$

Define

$$A^1_n(z) := \frac{\sum_{\nu=0}^{n+m} |b_{\nu,n}||p_\nu(z)|}{|Q(z)Q^\mu_{n,m}(z)|} \quad \text{and} \quad A^2_n(z) := \frac{\sum_{\nu=n+m+1}^{\infty} |b_{\nu,n}||p_\nu(z)|}{|Q(z)Q^\mu_{n,m}(z)|},$$

and let $Q^{\mu}_{n,m}(z) := \prod_{j=1}^{m_n}(z-\lambda_{n,j})$. Then (3.21) implies

$$\left|F(z) - \frac{P^{\mu}_{n,m}(z)}{Q^{\mu}_{n,m}(z)}\right| \leq A^1_n(z) + A^2_n(z),$$

for all $z \in \hat{D}_\sigma := \overline{D}_\sigma \setminus (\cup_{n=0}^{\infty}\{\lambda_{n,1},\ldots,\lambda_{n,m_n}\} \cup \{\lambda_1,\ldots,\lambda_m\})$.

Let us bound $A^1_n(z)$ from above. We will first estimate $|\tau_{k,n}/Q^{\mu}_{n,m}(z)|$ for $z \in \hat{D}_\sigma$ and for $k \geq n+1$. By definition of $\tau_{k,n}$,

$$\frac{\tau_{k,n}}{Q^{\mu}_{n,m}(z)} = \frac{1}{2\pi i}\int_{\Gamma_{\rho_2}} s_k(t)F(t)\frac{Q^{\mu}_{n,m}(t)}{Q^{\mu}_{n,m}(z)}dt, \quad k \geq n+1. \tag{3.22}$$

For n sufficiently large,

$$|s_k(t)| \leq \frac{c_8}{(\rho_2-\delta)^k}, \quad k \geq n+1.$$

Define

$$Q^{\mu}_{n,m,\rho_2}(t) = \prod_{\lambda_{n,j}\in D_{\rho_2}}(t-\lambda_{n,j}).$$

It is easy to see that

$$\left|\frac{t-\zeta}{z-\zeta}\right| \leq c_9,$$

for all $t \in \Gamma_{\rho_2}, z \in \hat{D}_\sigma$ and $\zeta \in \mathbb{C} \setminus D_{\rho_2}$ (according to (3.20), $\rho_2 > \sigma$). Then,

$$\left|\frac{Q^{\mu}_{n,m}(t)}{Q^{\mu}_{n,m}(z)}\right| \leq {c_9}^m \left|\frac{Q^{\mu}_{n,m,\rho_2}(t)}{Q^{\mu}_{n,m,\rho_2}(z)}\right| \leq \frac{c_{10}}{|Q^{\mu}_{n,m,\rho_2}(z)|}, \quad z \in \hat{D}_\sigma, \quad t \in \Gamma_{\rho_2}.$$

By (3.22), we obtain

$$\left|\frac{\tau_{k,n}}{Q^{\mu}_{n,m}(z)}\right| \leq \frac{c_{11}}{|Q^{\mu}_{n,m,\rho_2}(z)|(\rho_2-\delta)^k}, \quad z \in \hat{D}_\sigma, \quad k \geq n+1, \quad n \geq n_5,$$

which implies

$$\left|\frac{b_{\nu,n}}{Q^{\mu}_{n,m}(z)}\right| \leq \frac{c_{12}}{|Q^{\mu}_{n,m,\rho_2}(z)|(\rho_2-\delta)^n}, \quad z \in \hat{D}_\sigma, \quad n \geq n_6. \tag{3.23}$$

Applying (3.3) and the maximum modulus principle, we have

$$|p_\nu(z)| \leq c_{13}(\sigma+\delta)^\nu, \quad z \in \overline{D_\sigma}, \quad \nu \geq 0. \tag{3.24}$$

Using (3.23) and (3.24), we obtain

$$A_n^1(z) = \frac{1}{|Q(z)|}\sum_{\nu=0}^{n+m}\frac{|b_{\nu,n}||p_\nu(z)|}{|Q^\mu_{n,m}(z)|} \leq \frac{(n+m+1)c_{12}c_{13}(\sigma+\delta)^{n+m}}{|Q(z)Q^\mu_{n,m,\rho_2}(z)|(\rho_2-\delta)^n}, \quad z \in \hat{D}_\sigma.$$

Choose $\theta > 0$ such that $(\sigma+\delta)/(\rho_2-\delta) < \theta < 1$. Then, for n sufficiently large,

$$A_n^1(z) \leq \frac{c_{14}\theta^n}{|Q(z)Q^\mu_{n,m,\rho_2}(z)|}, \quad z \in \hat{D}_\sigma. \tag{3.25}$$

Next, we bound $A_n^2(z)$. Since $\deg(QP^\mu_{n,m}) \leq n+m$, by a computation similar to (3.5), we obtain

$$b_{\nu,n} = \langle QQ^\mu_{n,m}F, p_\nu\rangle_\mu = \frac{1}{2\pi i}\int_{\Gamma_{\rho_2}} Q(t)Q^\mu_{n,m}(t)F(t)s_\nu(t)dt, \quad \nu \geq n+m+1. \tag{3.26}$$

As before, from (3.3) and (3.26), we have

$$\frac{|b_{\nu,n}|}{|Q(z)Q^\mu_{n,m}(z)|} \leq \frac{c_{15}}{|Q(z)Q^\mu_{n,m,\rho_2}(z)|(\rho_2-\delta)^\nu}, \quad z \in \hat{D}_\sigma, \quad \nu \geq n+m+1, \tag{3.27}$$

for $n \geq n_7$. Using (3.24) and (3.27), for n sufficiently large, we obtain

$$A_n^2(z) \leq \frac{c_{16}(\sigma+\delta)^n}{|Q(z)Q^\mu_{n,m,\rho_2}(z)|(\rho_1-\delta)^n} < \frac{c_{17}\theta^n}{|Q(z)Q^\mu_{n,m,\rho_2}(z)|}, \quad z \in \hat{D}_\sigma. \tag{3.28}$$

Combining (3.25) and (3.28), for n sufficiently large, we have

$$\left|F(z) - \frac{P^\mu_{n,m}(z)}{Q^\mu_{n,m}(z)}\right| \leq \frac{c_{18}\theta^n}{|Q(z)Q_{n,m,\rho_2}(z)|}, \quad z \in \hat{D}_\sigma.$$

Let $T_n(z) := Q(z)Q_{n,m,\rho_2}(z)$. Then, $T_n(z)$ is a monic polynomial of degree at most $2m$. Let $\varepsilon > 0$. Clearly,

$$e_n := \left\{z \in \hat{D}_\sigma : \left|F(z) - \frac{P^\mu_{n,m}(z)}{Q^\mu_{n,m}(z)}\right| \geq \varepsilon\right\} \subset \left\{z \in \hat{D}_\sigma : |Q(z)Q_{n,m,\rho_2}(z)| \leq \frac{c_{18}\theta^n}{\varepsilon}\right\} =: E_n.$$

The logarithmic capacity is a monotonic set function and satisfies

$$\operatorname{cap}\{z \in \mathbb{C} : |z^n + a_{n-1}z^{n-1} + \ldots + a_0| \le \rho^n\} = \rho, \quad \rho > 0.$$

Hence, we find that for n sufficiently large

$$\operatorname{cap} e_n \le \operatorname{cap} E_n \le \left(\frac{1}{\varepsilon} c_{18}\theta^n\right)^{1/\deg T_n} \le \left(\frac{1}{\varepsilon} c_{18}\theta^n\right)^{1/2m} \le c_{19}\theta^{n/2m}.$$

This means that $\operatorname{cap}\left\{z \in \overline{D}_\sigma : \left|F(z) - P^\mu_{n,m}(z)/Q^\mu_{n,m}(z)\right| \ge \varepsilon\right\} = \operatorname{cap} e_n \to 0$, as $n \to \infty$. This proves that $[n/m]^\mu_F$ converges in capacity to F on each compact subset of $D_{\rho_m(F)}$, as $n \to \infty$. On the other hand, the number of poles of $[n/m]^\mu_F$ in $D_{\rho_m(F)}$ does not exceed m. Applying [7, Lemma 1] it follows that $[n/m]^\mu_F$ converges to F uniformly inside $D_{\rho_m(F)} \setminus \{\lambda_1, \ldots, \lambda_m\}$, as $n \to \infty$. In addition, it follows that each pole of F in $D_{\rho_m(F)}$ attracts as many zeros of $Q^\mu_{n,m}$ as its order. Therefore, $\deg Q^\mu_{n,m} = m$ for all sufficiently large n which in turn implies that $[n/m]^\mu_F$ is uniquely determined for such n. We have obtained (2.3) and (2.4) except for the rate of convergence exhibited in those relations.

To prove (2.3), let K be a compact subset of $D_{\rho_m(F)} \setminus \{\lambda_1, \ldots \lambda_m\}$. Take σ to be the smallest positive number ≥ 1 such that $K \subset \overline{D_\sigma} \subset D_{\rho_m(F)}$, and choose an arbitrarily small number $\delta > 0$ such that ρ_2 satisfies (3.20). Note that what we proved above implies that

$$\max_{z \in D_{\rho_m(F)}} |Q^\mu_{n,m}(z)| \le c_{20}.$$

From (3.3), for $n \ge n_8$,

$$|b_{\nu,n}| = \left|\frac{1}{2\pi i}\int_{\Gamma_{\rho_2}} Q(t)Q^\mu_{n,m}(t)F(t)s_\nu(t)dt\right| \le \frac{c_{21}}{(\rho_2 - \delta)^\nu}, \quad \nu \ge n + m + 1,$$

$$|\tau_{k,n}| = \left|\frac{1}{2\pi i}\int_{\Gamma_{\rho_2}} Q^\mu_{n,m}(t)F(t)s_k(t)dt\right| \le \frac{c_{22}}{(\rho_2 - \delta)^k}, \quad k \ge n + 1. \tag{3.29}$$

Then, by (3.19) and (3.29), for $n \ge n_9$,

$$|b_{\nu,n}| \le c_7 \sum_{k=n+1}^{\infty} |\tau_{k,n}| \le \frac{c_{23}}{(\rho_2 - \delta)^n}, \quad 0 \le \nu \le n + m.$$

Using (3.24), we can prove that for $z \in K$ and for $n \geq n_{10}$,

$$|Q(z)Q^{\mu}_{n,m}(z)F(z) - Q(z)P^{\mu}_{n,m}(z)| \leq \sum_{\nu=0}^{\infty} |b_{\nu,n}||p_\nu(z)| \leq c_{24} \left(\left(\frac{\sigma+\delta}{\rho_2-\delta} \right) + \delta \right)^n. \tag{3.30}$$

Consequently, for $n \geq n_{10}$ we have

$$\left| F(z) - \frac{P^{\mu}_{n,m}(z)}{Q^{\mu}_{n,m}(z)} \right| \leq \frac{c_{25}}{|Q(z)Q^{\mu}_{n,m}(z)|} \left(\left(\frac{\sigma+\delta}{\rho_2-\delta} \right) + \delta \right)^n, \quad z \in K.$$

Since for n sufficiently large, the zeros of $Q^{\mu}_{n,m}(z)$ are distant from K, it follows that

$$\limsup_{n\to\infty} \|F - [n/m]^{\mu}_F\|_K^{1/n} \leq \left(\frac{\sigma+\delta}{\rho_2-\delta} \right) + \delta.$$

Letting $\delta \to 0^+$ and $\rho_2 \to \rho_m(F)$, we obtain (2.3).

Finally, we prove (2.4). We first need to show that for $k = 1, \ldots, \gamma$,

$$\limsup_{n\to\infty} |(Q^{\mu}_{n,m})^{(j)}(\alpha_k)|^{1/n} \leq \frac{|\Phi(\alpha_k)|}{\rho_m(F)}, \quad j = 0, \ldots, m_k - 1. \tag{3.31}$$

Let $\varepsilon > 0$ be sufficiently small so that $\overline{\mathbb{B}(\alpha_k, \varepsilon)} \subset D_{\rho_m(F)}$ for all $k = 1, \ldots, \gamma$ and the disks $\overline{\mathbb{B}(\alpha_k, \varepsilon)}$, $k = 1, \ldots, \gamma$, are pairwise disjoint. As a consequence of (3.30), we have

$$\limsup_{n\to\infty} \left\| (z-\alpha_k)^{m_k} F Q^{\mu}_{n,m} - (z-\alpha_k)^{m_k} P^{\mu}_{n,m} \right\|^{1/n}_{\overline{\mathbb{B}(\alpha_k,\varepsilon)}} \leq \frac{\|\Phi\|_{\overline{\mathbb{B}(\alpha_k,\varepsilon)}}}{\rho_m(F)},$$

so by Cauchy's integral formula for the derivative, we obtain

$$\limsup_{n\to\infty} \left\| \left[(z-\alpha_k)^{m_k} F Q^{\mu}_{n,m} - (z-\alpha_k)^{m_k} P^{\mu}_{n,m} \right]^{(j)} \right\|^{1/n}_{\overline{\mathbb{B}(\alpha_k,\varepsilon)}} \leq \frac{\|\Phi\|_{\overline{\mathbb{B}(\alpha_k,\varepsilon)}}}{\rho_m(F)}, \tag{3.32}$$

for all $j \geq 0$. Since $\varepsilon > 0$ can be taken arbitrarily small, this implies that

$$\limsup_{n\to\infty} \left| L_k Q^{\mu}_{n,m}(\alpha_k) \right|^{1/n} \leq \frac{|\Phi(\alpha_k)|}{\rho_m(F)},$$

where $L_k := \lim_{z\to\alpha_k} (z-\alpha_k)^{m_k} F(z) \neq 0$ (because F has a pole of order m_k at α_k). Therefore,

$$\limsup_{n\to\infty} \left| Q^{\mu}_{n,m}(\alpha_k) \right|^{1/n} \leq \frac{|\Phi(\alpha_k)|}{\rho_m(F)}.$$

Proceeding by induction, let $r \leq m_k - 1$ and assume that

$$\limsup_{n\to\infty} \left|(Q^{\mu}_{n,m})^{(j)}(\alpha_k)\right|^{1/n} \leq \frac{|\Phi(\alpha_k)|}{\rho_m(F)}, \quad j = 0, \dots, r-1. \tag{3.33}$$

Let us show that the above inequality also holds for $j = r$. Using (3.32), since $r < m_k$, we obtain

$$\limsup_{n\to\infty} \left|[(z-\alpha_k)^{m_k} F Q^{\mu}_{n,m}]^{(r)}(\alpha_k)\right|^{1/n} \leq \frac{|\Phi(\alpha_k)|}{\rho_m(F)}. \tag{3.34}$$

By the Leibniz formula, we have

$$\left[(z-\alpha_k)^{m_k} F Q^{\mu}_{n,m}\right]^{(r)}(\alpha_k) = \sum_{l=0}^{r} \binom{r}{l} [(z-\alpha_k)^{m_k} F]^{(l)}(\alpha_k)(Q^{\mu}_{n,m})^{(r-l)}(\alpha_k).$$

Therefore, by (3.33), (3.34), and the fact that $L_k \neq 0$, it follows that

$$\limsup_{n\to\infty} \left|(Q^{\mu}_{n,m})^{(r)}(\alpha_k)\right|^{1/n} \leq \frac{|\Phi(\alpha_k)|}{\rho_m(F)}$$

which completes the induction and the proof of (3.31).

Let $\{q_{k,s}\}_{k=1,\dots,\gamma,\, s=0,\dots,m_k-1}$ be a system of polynomials such that $\deg q_{k,s} \leq m-1$ for all k, s and

$$q^{(i)}_{k,s}(\alpha_j) = \delta_{j,k}\delta_{i,s}, \quad 1 \leq j \leq \gamma, \quad 0 \leq i \leq m_j - 1.$$

It is not difficult to check that $q_{k,s}$ exists (using for example [16, Theorem 1]). Then,

$$Q^{\mu}_{n,m}(z) = \sum_{k=1}^{\gamma} \sum_{s=0}^{m_k-1} (Q^{\mu}_{n,m})^{(s)}(\alpha_k) q_{k,s}(z) + Q_m(z).$$

This formula combined with (3.31) implies

$$\limsup_{n\to\infty} \left\|Q^{\mu}_{n,m} - Q_m\right\|^{1/n} \leq \frac{\max_{k=1,\dots,\gamma} |\Phi(\alpha_k)|}{\rho_m(F)}.$$

∎

§4. Proofs of inverse type results

We begin stating two lemmas due to V. I. Buslaev (see [2, Theorems 5-6]). These results constitute the basic tools for proving our inverse type results. We make use of the following notation. Let $f(w) = \sum_{k=-\infty}^{\infty} f_k w^k$ be a Laurent series. We denote the regular part of $f(w)$ by $\hat{f}(w) := \sum_{k=0}^{\infty} f_k w^k$. If $\hat{f}(w)$ is holomorphic at 0, we denote by $R_m(\hat{f})$ the radius of the largest disk centered at the origin to which $\hat{f}(w)$ can be extended as a meromorphic function with at most m poles (counting multiplicities). Define the annulus

$$T_{\delta,m}(f) := \{w \in \mathbb{C} : e^{-\delta} R_0(\hat{f}) \le |w| \le e^{\delta} R_{m-1}(\hat{f})\},$$

where $m \in \mathbb{N}$ and $\delta \ge 0$. We will use $[\cdot]_n$ to denote the coefficient of w^n in the Laurent series expansion around 0 of the function in the square brackets. Set

$$U := \overline{\mathbb{C}} \setminus \overline{\mathbb{B}}.$$

Lemma 4.1 (Buslaev [2]). *Let $m \in \mathbb{N}, \delta > 0$ and let $f(w) = \sum_{n=-\infty}^{\infty} f_n w^n$ be a Laurent series such that*

$$0 < R_0(\hat{f}) \le R_{m-1}(\hat{f}) < \infty \quad \text{and} \quad \varlimsup_{n\to\infty} |f_{-n}|^{1/n} \le R_0(\hat{f}).$$

Assume further that

$$\lim_{n\to\infty} [f\alpha_n \eta_{n,j}]_n R_{m-1}^n(\hat{f}) e^{\delta n} = 0, \quad j = 0, \ldots, m-1, \tag{4.1}$$

where the functions $\alpha_n, \eta_{n,j} \in H(T_{\delta,m}(f))$ have the limits

$$\alpha(w) := \lim_{n\to\infty} \alpha_n(w) \not\equiv 0, \quad \eta_j(w) := \lim_{n\to\infty} \eta_{n,j}(w) = \eta^j(w), \quad j = 0, \ldots, m-1,$$

uniformly in $T_{\delta,m}(f)$, $\eta(w)$ is a univalent function in $T_{\delta,m}(f)$ and $\alpha(w)$ has at most m zeros in the annulus $T_{0,m}(f)$. Then the function $\alpha(w)$ has precisely m zeros $\tau_1, \ldots, \tau_m$ in $T_{0,m}(f)$ and $\lim_{n\to\infty} \tau_{n,j} = \tau_j$, where the $\tau_{n,j}, j = 1, \ldots, m$, are poles of the classical approximants $[n/m]_{\hat{f}}(w)$. Moreover, for any functions $K_{n,1}, \ldots, K_{n,m}, L_{n,1}, \ldots, L_{n,m} \in H(T_{\nu,m}(f))$, $\nu > 0$, that converge to $K_1, \ldots, K_m$, $L_1, \ldots, L_m$ uniformly on $T_{\nu,m}(f)$,

$$\lim_{n\to\infty} \frac{\det([fK_{n,i}L_{n,j}]_n)_{i,j=1,\ldots,m}}{\det(f_{n-i-j})_{i,j=0,\ldots,m-1}} = \frac{\det(K_r(\tau_s))_{s,r=1,\ldots,m} \det(L_r(\tau_s))_{s,r=1,\ldots,m}}{W^2(\tau_1, \ldots, \tau_m)}, \tag{4.2}$$

where $W(\tau_1, \ldots, \tau_m) = \det(\tau_s^{r-1})_{s,r=1,\ldots,m}$ is the Vandermonde determinant of the numbers $\tau_1, \ldots, \tau_m$ (for multiple zeros, the right-hand side of (4.2) is defined by continuity). In particular, for any

$k_1, \dots, k_m, q_1, \dots, q_m \in \mathbb{Z}$, the limits

$$\lim_{n\to\infty} \frac{\det(f_{n-k_i-q_j})_{i,j=1,\dots,m}}{\det(f_{n-i-j})_{i,j=0,\dots,m-1}} = \frac{\det(\tau_s^{k_r})_{s,r=1,\dots,m} \det(\tau_s^{q_r})_{s,r=1,\dots,m}}{W^2(\tau_1,\dots,\tau_m)}$$

exist.

The assumptions $R_{m-1}(\hat{f}) < \infty$ and (4.1) in Lemma 2 can be replaced by the following: the functions $\alpha_n(w)$ and $w^{-j}\eta_{n,j}(w)$ are holomorphic in the set $\overline{\mathbb{C}} \setminus \mathbb{B}(0, e^{-\delta}R_0(\hat{f}))$, and

$$[f\alpha_n\eta_{n,j}]_n = 0, \quad j = 0,\dots,m-1, \quad n \geq n_0.$$

Hence, we also have:

Lemma 4.2 (Buslaev [2])**.** *Let $m \in \mathbb{N}$, $\sigma > 1$ and $f(w) = \sum_{n=-\infty}^{\infty} f_n w^n$ be a holomorphic function in the annulus $\{1 < |w| < \sigma\}$. Assume further that*

$$[f\alpha_n\eta_{n,j}]_n = 0, \quad j = 0,\dots,m-1, \quad n \geq n_0,$$

hold, where $\alpha_n(w)$ and $w^{-j}\eta_{n,j}(w)$ are holomorphic functions in U, the limits

$$\alpha(w) := \lim_{n\to\infty} \alpha_n(w) \not\equiv 0, \quad \eta_j(w) := \lim_{n\to\infty} \eta_{n,j}(w) = \eta^j(w), \quad j = 0,\dots,m-1,$$

exist uniformly inside $U \setminus \{\infty\}$, the function $\alpha(w)$ has at most m zeros in $U \setminus \{\infty\}$ and $\eta(w)$ is a univalent function in U such that $\eta(\infty) = \infty$. Then, only one of the following assertions takes place:

(i) *$\hat{f}(w)$ is a rational function with at most $m-1$ poles;*

(ii) *$\alpha(w)$ has precisely m zeros, $\tau_1,\dots,\tau_m$ in $U \setminus \{\infty\}$, these zeros are singularities of $f(w)$, with an appropriate ordering, $|\tau_1| = R_0(\hat{f}),\dots,|\tau_m| = R_{m-1}(\hat{f})$, and the limits $\lim_{n\to\infty} \tau_{n,j} = \tau_j$ exist, where the $\tau_{n,j}, j = 1,\dots,m$, are the poles of the classical Padé approximants $[n/m]_{\hat{f}}(w)$.*

Define

$$h_n(w) := c_n w^{n+1} s_n(\Psi(w))\Psi'(w).$$

Lemma 4.3. *Let $F \in \mathcal{H}(E)$. Define $f(w) := F(\Psi(w))$. The functions $h_n(w)$ are holomorphic in U, $F_n = [fh_n]_n/c_n$ and $\langle zF, p_n\rangle_\mu = [\Psi f h_n]_n/c_n$. If $\mu \in \mathcal{S}(E)$, then the sequence $h_n(w)$ converges to some non-vanishing function $h(w)$ uniformly inside U.*

Proof. Clearly, $h_n(w)$ is holomorphic in U. Let $\epsilon > 0$ be a small number so that $\Gamma_{1+\varepsilon}$ is in the domain of holomorphy of $F(z)$. By Fubini's theorem and Cauchy's integral formula, we have

$$F_n = \int F(z)\overline{p_n(z)}d\mu(z) = \int \left(\frac{1}{2\pi i}\int_{\Gamma_{1+\varepsilon}} \frac{F(\zeta)}{\zeta - z}d\zeta\right)\overline{p_n(z)}d\mu(z)$$

$$= \frac{1}{2\pi i}\int_{\Gamma_{1+\varepsilon}} F(\zeta)\int \frac{\overline{p_n(z)}}{\zeta - z}d\mu(z)d\zeta = \frac{1}{2\pi i}\int_{\Gamma_{1+\varepsilon}} F(\zeta)s_n(\zeta)d\zeta$$

$$= \frac{1}{2\pi i}\int_{\gamma_{1+\varepsilon}} f(w)s_n(\Psi(w))\Psi'(w)dw = \frac{1}{c_n}\frac{1}{2\pi i}\int_{\gamma_{1+\varepsilon}} \frac{f(w)h_n(w)}{w^{n+1}}dw = \frac{1}{c_n}[fh_n]_n.$$

The other formula is obtained similarly.

If $\mu \in \mathcal{S}(E)$ then $\mu \in \mathcal{R}(E)$ and using Lemma 3.1, we have

$$h(w) := \lim_{n\to\infty} h_n(w) = \lim_{n\to\infty} c_n w^{n+1} s_n(\Psi(w))\Psi'(w)$$

$$= w\Psi'(w)\lim_{n\to\infty}\frac{c_n w^n}{p_n(\Psi(w))}\lim_{n\to\infty} p_n(\Psi(w))s_n(\Psi(w)) = \frac{1}{S(\Psi(w))},$$

uniformly inside U. ∎

Proof of Theorem 2.6. Using Lemma 4.2 for $m = 1$, we will prove that (a) or (b) imply (c). Let

$$\tau_n := \frac{F_n}{F_{n+1}}, \quad \lambda_n := \frac{\langle zF, p_n\rangle_\mu}{F_n}.$$

For $w \in U$, we define $\eta_{n,0}(w) \equiv 1$,

$$\alpha_{n,1}(w) := \frac{c_n}{c_{n+1}}\frac{\tau_n h_{n+1}(w)}{w} - h_n(w), \quad \alpha_{n,2}(w) := \frac{h_{n+1}(w)(\lambda_{n+1} - \Psi(w))}{w}.$$

The functions $\alpha_{n,1}(w)$ and $\alpha_{n,2}(w)$ are holomorphic in U. By Lemma 4.3, for $\varepsilon > 0$ sufficiently small so that $f(w)$ is holomorphic in a neighborhood of $\gamma_{1+\varepsilon}$,

$$[f\alpha_{n,1}]_n = \frac{c_n}{c_{n+1}}\frac{\tau_n}{2\pi i}\int_{\gamma_{1+\varepsilon}} \frac{f(w)h_{n+1}(w)}{w^{n+2}}dw - \frac{1}{2\pi i}\int_{\gamma_{1+\varepsilon}} \frac{f(w)h_n(w)}{w^{n+1}}dw$$

$$= \frac{c_n}{c_{n+1}} \frac{F_n}{F_{n+1}} [fh_{n+1}]_{n+1} - [fh_n]_n = 0$$

and

$$[f\alpha_{n,2}]_n = \frac{\lambda_{n+1}}{2\pi i} \int_{\gamma_{1+\varepsilon}} \frac{f(w)h_{n+1}(w)}{w^{n+2}} dw - \frac{1}{2\pi i} \int_{\gamma_{1+\varepsilon}} \frac{\Psi(w)f(w)h_{n+1}(w)}{w^{n+2}} dw$$

$$= \frac{c_{n+1}}{c_{n+1}} \frac{\langle zF, p_{n+1}\rangle_\mu}{F_{n+1}} [fh_{n+1}]_{n+1} - [\Psi fh_{n+1}]_{n+1} = 0.$$

If (a) holds, then

$$\alpha_1(w) := \lim_{n\to\infty} \alpha_{n,1}(w) = h(w)\left(\frac{\tau}{w} - 1\right), \quad \text{uniformly inside } U,$$

and if (b) holds, then

$$\alpha_2(w) := \lim_{n\to\infty} \alpha_{n,2}(w) = \frac{h(w)(\lambda - \Psi(w))}{w}, \quad \text{uniformly inside } U.$$

Since $h(w)$ is never zero on U, each function $\alpha_j(w)$, $j = 1, 2$, has at most one zero in U (which is τ). By Lemma 4.3, $\alpha_1(\infty) = -h(\infty) \neq 0$ and $\alpha_2(\infty) = -\mathrm{cap}(E)h(\infty) \neq 0$. Moreover, if $f_n = 0$ for $n \geq n_0$, then $F_n = [fh_n]_n = 0$ (recall that $h_n(w)$ is analytic at ∞). Therefore, by (ii) in Lemma 4.2, $|\tau| > 1$ and $\lim_{n\to\infty} f_n/f_{n+1} = \tau$.

Now, using Lemma 4.1 for $m = 1$, we prove that (c) implies claims (a) and (b). Assume that $\lim_{n\to\infty} f_n/f_{n+1} = \tau$. Set $\eta_{n,0}(w) \equiv 1$,

$$\tau_n := \frac{f_n}{f_{n+1}} \quad \text{and} \quad \alpha_n(w) := \frac{\tau_n}{w} - 1, \quad z \in U.$$

Therefore,

$$[f\alpha_n]_n = \tau_n f_{n+1} - f_n = 0,$$

$$\alpha(w) := \lim_{n\to\infty} \alpha_n(w) = \frac{\tau}{w} - 1, \quad \text{uniformly inside } U,$$

$\alpha(\infty) = -1$, and $\alpha(w)$ has at most one zero in U. Applying (4.2) in Lemma 4.1, if we select $K_{n,1}(w) = h_n(w)$ and $L_{n,1}(w) = 1$, we have

$$\lim_{n\to\infty} \frac{[fh_n]_n}{f_n} = h(\tau),$$

and, if we select $K_{n,1}(w) = \Psi(w)h_n(w)$ and $L_{n,1}(w) = 1$, we have

$$\lim_{n\to\infty} \frac{[\Psi f h_n]_n}{f_n} = \Psi(\tau)h(\tau).$$

Since $h(w)$ vanishes nowhere in the domain U,

$$\lim_{n\to\infty} \frac{F_n}{F_{n+1}} = \lim_{n\to\infty} \frac{c_{n+1}}{c_n} \frac{[fh_n]_n}{[fh_{n+1}]_{n+1}} = \lim_{n\to\infty} \frac{c_{n+1}}{c_n} \frac{[fh_n]_n}{f_n} \frac{f_n}{f_{n+1}} \frac{f_{n+1}}{[fh_{n+1}]_{n+1}} = \tau$$

and

$$\lim_{n\to\infty} \frac{\langle zF, p_n\rangle_\mu}{F_n} = \lim_{n\to\infty} \frac{c_n}{c_n} \frac{[\Psi f h_n]_n}{[fh_n]_n} = \lim_{n\to\infty} \frac{[\Psi f h_n]_n}{f_n} \frac{f_n}{[fh_n]_n} = \Psi(\tau) = \lambda.$$

The proof is complete. ■

Proof of Theorem 2.8. First of all, we prove that (b) implies (a) using Lemma 4.2. We assume that the zeros of $Q^\mu_{n,m}(z)$ have limits $\lambda_1, \ldots, \lambda_m$, as $n \to \infty$. For $w \in U$, we define

$$\alpha_n(w) := w^{-m} h(w) Q^\mu_{n,m}(\Psi(w)),$$

$$\eta_{n,j}(w) := \frac{c_{n+m-j} w^{n+m+1} s_{n+m-j}(\Psi(w))\Psi'(w)}{h(w)}, \quad j = 0, \ldots, m-1.$$

The functions $\alpha_n(w)$ and $w^{-j}\eta_{n,j}(w) = h_{n+m-j}(w)/h(w)$, $j = 1, \ldots, m-1$, are holomorphic in U and

$$\alpha(w) := \lim_{n\to\infty} \alpha_n(w) = w^{-m} h(w) \prod_{j=1}^{m} (\Psi(w) - \lambda_j),$$

$$\eta_j(w) := \lim_{n\to\infty} \eta_{n,j}(w) = w^j, \quad j = 0, 1, \ldots, m-1,$$

uniformly inside $U \setminus \{\infty\}$. Since $h(w)$ is never zero in U, $\alpha(w)$ has at most m zeros in $U \setminus \{\infty\}$. By Cauchy's integral formula, Fubini's theorem and the definition of $Q^\mu_{n,m}$, we have, for $\epsilon > 0$ sufficiently small so that $F(z)$ is analytic on $D_{1+\varepsilon}$ and for $j = 0, \ldots, m-1$,

$$[f\alpha_n\eta_{n,j}]_n = \frac{c_n}{2\pi i} \int_{\gamma_{1+\varepsilon}} F(\Psi(w)) Q^\mu_{n,m}(\Psi(w)) s_{n+m-j}(\Psi(w))\Psi'(w) dw$$

$$= \frac{c_n}{2\pi i} \int_{\Gamma_{1+\varepsilon}} F(t) Q^\mu_{n,m}(t) s_{n+m-j}(t) dt = \frac{c_n}{2\pi i} \int_{\Gamma_{1+\varepsilon}} F(t) Q^\mu_{n,m}(t) \int \frac{\overline{p_{n+m-j}(z)}}{t-z} d\mu(z) dt$$

$$= c_n \int \frac{1}{2\pi i} \int_{\Gamma_{1+\varepsilon}} \frac{F(t)Q^{\mu}_{n,m}(t)}{t-z} dt\overline{p_{n+m-j}(z)}d\mu(z) = c_n \int F(z)Q^{\mu}_{n,m}(z)\overline{p_{n+m-j}(z)}d\mu(z) = 0.$$

Therefore, the assumptions of Lemma 4.2 are satisfied. If the regular part of $f(w)$ is a rational function with at most $m-1$ poles, then $F(z)$ is a rational function with at most $m-1$ poles which implies that $\Delta_{n,m}(F,\mu) = 0$ for n sufficiently large. This is impossible, because $\deg(Q^{\mu}_{n,m}) = m$, for n sufficiently large. Therefore, by Lemma 4.2, $\alpha(w)$ has precisely m zeros $\tau_1, \ldots, \tau_m$ in $U \setminus \{\infty\}$ and the limits of the poles of the classical Padé approximants $[n/m]_{\hat{f}}(w)$ are $\tau_1, \ldots, \tau_m$, as $n \to \infty$.

Now, we prove that (a) implies (b) using Lemma 4.1. Assume that the poles of $[n/m]_{\hat{f}}(w)$ have limits $\tau_1, \ldots, \tau_m$, as $n \to \infty$. We assume further that $Q_{n,m}(w)$ is monic.

Define, for $w \in U$,

$$\tilde{\alpha}_n(w) := w^{-m} Q_{n,m}(w),$$
$$\tilde{\eta}_{n,\nu}(w) := w^{\nu}, \quad \nu = 0, \ldots, m-1.$$

Then,

$$\tilde{\alpha}(w) := \lim_{n\to\infty} \tilde{\alpha}_n(z) = w^{-m} \prod_{j=1}^{m} (w - \tau_j),$$
$$\tilde{\eta}_{\nu}(w) = w^{\nu}, \quad \nu = 0, \ldots, m-1,$$

uniformly inside $U \setminus \{\infty\}$. By the definition of $Q_{n,m}(z)$, it follows that, for $\epsilon > 0$ sufficiently small so that $f(w)$ is holomorphic on $\gamma_{1+\varepsilon}$ and for n sufficiently large,

$$[f\tilde{\alpha}_n\tilde{\eta}_{n,\nu}]_n = [\hat{f}\tilde{\alpha}_n\tilde{\eta}_{n,\nu}]_n = \frac{1}{2\pi i} \int_{\gamma_{1+\varepsilon}} \frac{\hat{f}(w)Q_{n,m}(w)}{w^{m-\nu+n+1}} dw = 0, \quad \nu = 0, \ldots, m-1.$$

We can easily check the rest of the conditions required in Lemma 4.1 for $\tilde{\alpha}_n(w)$ and $\tilde{\eta}_{n,\nu}(w)$, so we can apply the equality (4.2) in Lemma 4.1.

Next, set

$$\tilde{Q}_{n,m}(z) := \begin{vmatrix} c_{n+1}\langle F, p_{n+1}\rangle_\mu & c_{n+1}\langle zF, p_{n+1}\rangle_\mu & \cdots & c_{n+1}\langle z^m F, p_{n+1}\rangle_\mu \\ \vdots & \vdots & \cdots & \vdots \\ c_{n+m}\langle F, p_{n+m}\rangle_\mu & c_{n+m}\langle zF, p_{n+m}\rangle_\mu & \cdots & c_{n+m}\langle z^m F, p_{n+m}\rangle_\mu \\ 1 & z & \cdots & z^m \end{vmatrix}. \tag{4.3}$$

Note that the polynomials $\tilde{Q}_{n,m}(z)$ satisfy

$$\langle \tilde{Q}_{n,m}F, p_\nu\rangle_\mu = 0, \quad \nu = n+1, \ldots, n+m,$$

and if we show that $\Delta_{n,m}(F,\mu) \neq 0$ (the coefficient of $\tilde{Q}_{n,m}(z)/\prod_{j=1}^{m} c_{n+j}$), which will be verified at the end of this proof, then $Q^{\mu}_{n,m}(z)$ is unique and

$$Q^{\mu}_{n,m}(z) = \frac{\tilde{Q}_{n,m}(z)}{\Delta_{n,m}(F,\mu)\prod_{j=1}^{m} c_{n+j}}.$$

Using Cauchy's integral formula and Fubini's theorem, for $\varepsilon > 0$ sufficiently small so that $F(z)$ is holomorphic on $D_{1+\varepsilon}$, for $j = 1,\ldots,m+1$ and $\nu = 1,\ldots,m$, we have

$$c_{n+\nu}\langle z^{j-1}F, p_{n+\nu}\rangle_{\mu} = c_{n+\nu}\int \frac{1}{2\pi i}\int_{\Gamma_{1+\varepsilon}} \frac{\zeta^{j-1}F(\zeta)}{\zeta - z}d\zeta\overline{p_{n+\nu}(z)}d\mu(z)$$

$$= \frac{c_{n+\nu}}{2\pi i}\int_{\Gamma_{1+\varepsilon}} \zeta^{j-1}F(\zeta)\int \frac{\overline{p_{n+\nu}(z)}}{\zeta - z}d\mu(z)d\zeta = \frac{c_{n+\nu}}{2\pi i}\int_{\Gamma_{1+\varepsilon}} \zeta^{j-1}F(\zeta)s_{n+\nu}(\zeta)d\zeta$$

$$= \frac{c_{n+\nu}}{2\pi i}\int_{\gamma_{1+\varepsilon}} \Psi^{j-1}(w)f(w)s_{n+\nu}(\Psi(w))\Psi'(w)dw = [f(w)w^{-\nu}h_{n+\nu}(w)\Psi^{j-1}(w)]_n.$$

Computing the determinant in (4.3) by expanding along the last row and applying the previous formula, we obtain

$$\tilde{Q}_{n,m}(z) = \sum_{k=0}^{m}(-1)^{m+k}z^k \det([fK_{n,t}L_{n,r}]_n)_{t=1,\ldots,m,\ r=1,\ldots,k,k+2,\ldots,m+1}, \tag{4.4}$$

where

$$K_{n,t}(w) := w^{-t}h_{n+t}(w), \quad t = 1,\ldots,m,$$
$$L_{n,r}(w) := \Psi^{r-1}(w), \quad r = 1,\ldots,m+1.$$

Moreover, all the functions $K_{n,t}(w)$ and $L_{n,r}(w)$ are holomorphic in $U \setminus \{\infty\}$ and

$$K_t(w) := \lim_{n\to\infty} K_{n,t}(w) = w^{-t}h(w), \quad t = 1,\ldots,m,$$

$$L_r(w) := \Psi^{r-1}(w), \quad r = 1,\ldots,m+1,$$

uniformly inside $U \setminus \{\infty\}$. By Lemma 4.1 and (4.4), we have $\tau_1,\ldots,\tau_m \in U$ and

$$\lim_{n\to\infty} \frac{\tilde{Q}_{n,m}(z)}{\det(f_{n-i-j})_{i,j=0,1,\ldots,m-1}} = \lim_{n\to\infty}\sum_{k=0}^{m}(-1)^{m+k}z^k\frac{\det([fK_{n,t}L_{n,r}]_n)_{t=1,\ldots,m,\ r=1,\ldots,k,k+2,\ldots,m+1}}{\det(f_{n-i-j})_{i,j=0,1,\ldots,m-1}}$$

$$= \sum_{k=0}^{m}(-1)^{m+k} z^k \frac{\det(K_r(\tau_t))_{t,r=1,\ldots,m} \det(L_r(\tau_t))_{t=1,\ldots,m,\ r=1,\ldots,k,k+2,\ldots,m+1}}{W^2(\tau_1,\tau_2,\ldots,\tau_m)}$$

$$= \frac{\det(K_r(\tau_t))_{r,t=1,2,\ldots,m}}{W^2(\tau_1,\tau_2,\ldots,\tau_m)} \begin{vmatrix} 1 & \Psi(\tau_1) & \cdots & \Psi^m(\tau_1) \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \Psi(\tau_m) & \cdots & \Psi^m(\tau_m) \\ 1 & z & \cdots & z^m \end{vmatrix}$$

$$= (-1)^{(m)(m-1)/2} \frac{\prod_{i=1}^m h(\tau_i)}{\prod_{i=1}^m \tau_i^m} \prod_{1\le i<j\le m} \left(\frac{\Psi(\tau_j)-\Psi(\tau_i)}{\tau_j-\tau_i}\right) z^m + \ldots,$$

where $W(\tau_1,\tau_2,\ldots,\tau_m) = \det(\tau_t^{r-1})_{t,r=1,\ldots,m}$ is the Vandermonde determinant of the numbers $\tau_1,\ldots,\tau_m$. Since the degree of the polynomial in the last expression is m, the degree of $\tilde{Q}_{n,m}(z)$ is m for all n sufficiently large. Thus $\Delta_{n,m}(F,\mu) \neq 0$ and

$$Q^{\mu}_{n,m}(z) = \frac{\tilde{Q}_{n,m}(z)}{\Delta_{n,m}(F,\mu)\prod_{j=1}^m c_{n+j}}.$$

Moreover, the zeros of the polynomial in the second last equality are $\lambda_1,\ldots,\lambda_m$, so the zeros of $\tilde{Q}_{n,m}(z)$ (and $Q^{\mu}_{n,m}(z)$) converge to $\lambda_1,\ldots,\lambda_m$, as $n\to\infty$. ∎

Acknowledgements

We thank Brian Simanek for very useful discussions and references.

References

[1] Barrios Rolanía D., G. López Lagomasino and E. B. Saff (2001)
Asymptotics of orthogonal polynomials inside the unit circle and Szegő-Padé approximants, J. Comput. Appl. Math. 133(1-2), 171–181.

[2] Buslaev V. I. (2009)
An analogue of the Fabry theorem for generalized Padé approximants, Sb. Math. 200(7), 981–1050.

[3] Cacoq J. and G. López Lagomasino (2012)
Convergence of row sequences of simultaneous Fourier-Padé approximation, Jaen J. Approx. 4, 101–120.

[4] Cacoq J., B. de la Calle Ysern and G. López Lagomasino (2013)
Direct and inverse results on row sequences of Hermite-Padé approximants, Constr. Approx. 38(1), 133–160.

[5] Fabry E. (1896)
Sur les points singuliers d'une fonction donnée par son développement en série et l'impossibilité du prolongement analytique dans des cas très généraux, Ann. Sci. École Norm. Sup. (3) 13, 367–399.

[6] Geronimus L. Ya. (1961)
Orthogonal Polynomials: Estimates, Asymptotic Formulas, and Series of Polynomials Orthogonal on the Unit Circle and on an Interval, Consultants Bureau, New York.

[7] Gonchar A. A. (1977)
On the convergence of generalized Padé approximants of meromorphic functions, Math. USSR Sb. 27(1975), 503–514.

[8] Gonchar A. A. (1981)
Poles of the rows of Padé table and meromorphic continuation of functions, Mat. Sb. N. Ser. 115, 590–613.

[9] Graves-Morris P. R. and E. B. Saff (1984),
A de Montessus theorem for vector-valued rational interpolants, in: Rational Approximation and Interpolation, Lecture Notes in Math. 1105, P. R. Graves-Morris, E. B. Saff and R. S. Varga (eds.), Springer, Berlin, pp. 227–242.

[10] Gutknecht M. H. and E. B. Saff (1986)
A de Montessus-type theorem for CF approximation, J. Comput. Appl. Math. 16(2), 251–254.

[11] Hadamard J. (1892)
Essai sur l'étude des fonctions données par leur développement de Taylor, Journ. de Math. (4) VIII, 101–186.

[12] de Montessus de Ballore R. (1902)
Sur les fractions continues algébriques, S. M. F. Bull. 30, 28–36.

[13] Saff E. B. (1971)
Regions of meromorphy determined by the degree of best rational approximation, Proc. Amer. Math. Soc. 29, 30–38.

[14] Saff E. B. (1972)
An extension of Montessus de Ballore's theorem on the convergence of interpolating rational functions, J. Approximation Theory 6, 63–67.

[15] Simanek B. (2013)
Ratio asymptotics, Hessenberg matrices, and weak asymptotic measures, Int. Math. Res. Notices (doi: 10.1093/imrn/rnt187, published on-line Sept. 6).

[16] Sobczyk G. (2002)
Generalized Vandermonde determinants and applications, Aportaciones Mat. Serie Comun. 30, 203–213.

[17] Stahl H. and V. Totik (1992),
General Orthogonal Polynomials, Encyclopedia of Mathematics and its Applications Vol. 43, Cambridge University Press, Cambridge.

[18] Suetin S. P. (1978)
On the convergence of rational approximations to polynomial expansions in domains of meromorphy of a given function, Math USSR Sb. 34, 367–381.

[19] Suetin S. P. (1980)
On de Montessus de Ballore's theorem for nonlinear Padé approximants of orthogonal expansions and Faber series, Sov. Math. Dokl. 22, 274–277.

[20] Suetin S. P. (1984)
On poles of the mth row of a Padé table, Math. USSR Sb. 48, 493–497.

[21] Suetin S. P. (1985)
On an inverse problem for the mth row of the Padé table, Math. USSR Sb. 52, 231–244.

[22] Suetin S. P. (2002)
Padé approximants and efficient analytic continuation of power series, Russ. Math. Surv. 57(1), 43–141.

Nattapong Bosuwan, Edward B. Saff,
Center for Constructive Approximation,
Department of Mathematics,
Vanderbilt University,
1326 Stevenson Center,
37240 Nashville, TN, USA.
nattapong.bosuwan@vanderbilt.edu
edward.b.saff@vanderbilt.edu

Guillermo López Lagomasino,
Departamento de Matemáticas,
Universidad Carlos III de Madrid,
Avda. de la Universidad 30,
28911 Leganés, Spain.
lago@math.uc3m.es