Angular Overconvergence for Rational Functions Converging Geometrically on $[0, +\infty)$

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Dedicated to the memory of our teacher Prof. J. L. Walsh

§1. Introduction.

The classical results of Bernstein, Walsh, Gončar, and others concerning the overconvergence of rational functions are roughly of the following type (cf. [19]): It is assumed that

(i) $f(z)$ is defined (finite) on some compact set E in the complex plane $\mathbb{C}$;

(ii) $\{r_n(z)\}_{n=1}^{\infty}$ is a sequence of rational functions of respective degrees n which converge geometrically to f on E, i.e.,

$$\overline{\lim_{n\to\infty}}\{\|f-r_n\|_{L_\infty(E)}\}^{1/n} < 1;$$

and

(iii) the set of poles of the sequence $\{r_n(z)\}_{n=1}^{\infty}$ has no accumulation points on E.

It is then concluded that

(iv) the sequence $\{r_n(z)\}_{n=1}^{\infty}$ converges geometrically to an analytic extension of f on some open set in the plane containing E.

The aim of the present paper is to investigate the phenomenon of overconvergence in the case where E is a closed line segment $[a,b]$ and the hypothesis (iii) above

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is weakened to allow accumulation points of poles at the endpoints of E, i.e., assumption (iii) is replaced by

(iii)' the set of poles of the sequence $\{r_n(z)\}_{n=1}^{\infty}$ has no accumulation points on the open subinterval (a,b) of E = [a,b].

Of course with the hypotheses (i), (ii), and (iii)', we must modify conclusion (iv) to read

(iv)' the sequence $\{r_n(z)\}_{n=1}^{\infty}$ converges geometrically to an analytic extension of f on some open set in the plane containing (a,b).

For the precise statements of such results on "angular overconvergence" it is sufficient to take $E = [0, +\infty)$, because any interval [a,b] can be mapped onto $[0, +\infty)$ by means of a bilinear transformation, and such bilinear transformations preserve rational functions of degree n. For example, one of the results which we prove asserts that if rational functions $r_n(z)$ of respective degrees n converge geometrically on $E = [0, +\infty)$, and the poles of the $r_n(z)$ lie outside an infinite sector of the form

$$\{z \in \mathbb{C}: |\arg z| < \phi_1\}, \quad 0 < \phi_1 \leq \pi,$$

then the $r_n(z)$ converge geometrically on some smaller infinite sector

$$\{z \in \mathbb{C}: |\arg z| < \phi_2\}, \quad 0 < \phi_2 < \phi_1.$$

It is important to note that a number of results have appeared in the literature ([8], [10], [11]) which give classes of functions f and examples of approximating rational functions $r_n(z)$ for which condition (ii) above is satisfied on $E=[0, +\infty)$. Furthermore, for some special sequences of approximating rational functions, the existence of pole-free open sets (in the plane) containing $(0, +\infty)$ follows from the results in [18], [12], [13], among others. Hence the main results of this paper, which we state in Section 2, have immediate applications. These applications will be discussed primarily in Section 3.

§2. Statements of Main Results.

We now introduce the necessary notation and state our main results. Their proofs will be given in [14].

For an arbitrary set A in the complex plane $\mathbb{C}$, we denote by $\|\cdot\|_A$ the sup norm on A, i.e.,

$$\|f\|_A := \sup \{|f(z)| : z \in A\}.$$

We use the symbol π_n to denote the set of all complex polynomials in the variable z having degree at most n, and let $\pi_{n,n}$ denote the set of all complex rational functions $r_n(z)$ of the form

$$r_n(z) = \frac{p_n(z)}{q_n(z)}, \text{ where } p_n \in \pi_n,\ q_n \in \pi_n,\ q_n \not\equiv 0.$$

The first three results which we state concern pole-free regions whose boundaries are tangent to the ray $E=[0,+\infty)$ at $x=+\infty$. It is convenient in this regard to introduce the set $\mathcal{H}$ which consists of all real nonnegative continuous functions h on $[0,+\infty)$ such that for x large, $h(x)>0$, and $h'(x)$ exists, is nonnegative, and satisfies

$$(2.1)\qquad \lim_{x\to+\infty} h'(x) = 0.$$

Corresponding to each $h \in \mathcal{H}$ we define generically the set $E_s(h)$, $0 \le s \le 1$, in the complex plane by

$$(2.2)\qquad E_s(h) := \{z = x+iy : x \ge 0 \text{ and } |y| \le sh(x)\}.$$

Notice that, by condition (2.1), the boundary of each set $E_s(h)$ defined in (2.2) makes an angle of zero with the positive real axis at $x=+\infty$.

Our first result is the following:

Theorem 2.1. *Assume that for a function f, defined and finite on $[0,+\infty)$, there exists a sequence of rational functions $\{r_n\}_{n=1}^{\infty}$, with $r_n \in \pi_{n,n}$ for all $n \ge 1$, and a real number $q>1$ such that*

(2.3) $\overline{\lim_{n\to\infty}} \{\|f-r_n\|_{[0,+\infty)}\}^{1/n} < \frac{1}{q} < 1.$

Assume further that for some function $h \in \mathcal{H}$ the interior of the region $E_1(h)$ (defined in (2.2)) contains no poles of the $r_n(z)$ for all n sufficiently large. Then for every d satisfying the inequality

(2.4) $0 < d < \frac{\sqrt{q}-1}{\sqrt{q}+1} < 1,$

there exists a bounded subset K_d of $E_d(h)$ and an analytic function $F(z)$ on $E_d(h) - K_d$ with $F(x) = f(x)$ for all real x in this set, such that $\{r_n(z)\}_{n=1}^{\infty}$ converges geometrically to $F(z)$ on $E_d(h) - K_d$. Moreover

(2.5) $\overline{\lim_{n\to\infty}}\{\|F-r_n\|_{E_d(h)-K_d}\}^{1/n} < \frac{1}{q}\cdot\left(\frac{1+d}{1-d}\right)^2 < 1.$

The next result shows that in certain cases the conclusion of Theorem 2.1 can hold on the whole set $E_d(h)$, rather than on $E_d(h) - K_d$.

Corollary 2.2. Assume that for a continuous function $g(\not\equiv 0)$ on $[0,+\infty)$ there exists a sequence of polynomials $\{p_n\}_{n=1}^{\infty}$, with $p_n \in \pi_n$ for all $n \geq 1$, and a real number $q > 1$ such that

(2.6) $\overline{\lim_{n\to\infty}}\{\|\frac{1}{g} - \frac{1}{p_n}\|_{[0,+\infty)}\}^{1/n} \leq \frac{1}{q} < 1.$

Then, as is known [7, Theorem 3], there exists an entire function $G(z)$ of finite order with $G(x) = g(x)$ for all $x \geq 0$. Next assume that for some function $h \in \mathcal{H}$, with $h(x) > 0$, for all $x > 0$, the interior of the region $E_1(h)$ (defined in (2.2)) contains no zeros of $p_n(z)$ for all n large. If d satisfies (2.4) and if G is nonzero on the vertical segment $\{z=iy : |y| \leq dh(0)\}$, then

$$(2.7)\quad \overline{\lim_{n\to\infty}} \{\|\frac{1}{G} - \frac{1}{p_n}\|_{E_d(h)}\}^{1/n} \le \frac{1}{q}(\frac{1+d}{1-d})^2 < 1.$$

As a concrete application of Corollary 2.2, we first recall from Meinardus and Varga [8] that

$$(2.8)\quad \lim_{n\to\infty}\{\|e^{-x} - \frac{1}{s_n(x)}\|_{[0,+\infty)}\}^{1/n} = \frac{1}{2},$$

where $s_n(z) = \sum_{k=0}^{n} z^k/k!$ denotes the familiar n-th partial sum of e^z. It is further known from Saff and Varga [12] that for

$$(2.9)\quad \hat{h}(x) := 2(x+1)^{1/2},\ x \ge 0,$$

the region

$$(2.10)\quad E_1(\hat{h}) = \{z = x+iy:\ x\ge 0,\ |y| \le 2(x+1)^{1/2}\}$$

contains no zeros of the $s_n(z)$ for all n. Note that $\hat{h} \in \mathcal{H}$, and that with $G(z) = e^z$ (so that G is nonzero at every finite point z), with $p_n \equiv s_n$ for all $n \ge 1$, and with $q = 2$, the hypotheses of Corollary 2.2 are all fulfilled. Thus for any d satisfying $0 < d < (\sqrt{2} - 1)/(\sqrt{2} + 1)$, we have from (2.7) that

$$(2.11)\quad \overline{\lim_{n\to\infty}} \{\|e^{-z} - \frac{1}{s_n(z)}\|_{E_d(\hat{h})}\}^{1/n} \le \frac{1}{2}(\frac{1+d}{1-d})^2 < 1,$$

which is effectively the result of [11, Theorem 4.1]. We remark that for any $d > 0$ the set

$$(2.12)\quad E_d(\hat{h}) = \{z = x+iy:\ x\ge 0,\ |y| \le 2d(x+1)^{1/2}\}$$

is an unbounded parabolic region truncated at the origin.

As a consequence of Corollary 2.2 and of the results in [12], similar overconvergence results in unbounded parabolic regions also hold for each column of the Padé table for e^{-z}, i.e., for the Padé approximants $\{R_{\nu,n}(z)\}_{n=1}^{\infty}$ where the degree, ν, of the numerator is fixed.

Applications of Corollary 2.2 can in fact be made to a certain class of entire functions which contains the above example, and this will be described in the next section.

From Corollary 2.2 it is possible to deduce the following result which concerns geometric convergence on related unbounded sets whose widths grow more slowly at infinity.

Corollary 2.3. *With the hypotheses of Corollary 2.2, assume that* $c(x)$ *is a nonnegative continuous function on* $[0,+\infty)$ *with* $c(x)<h(x)$ *for all* $x>0$, *such that*

$$\lim_{x\to+\infty}\frac{c(x)}{h(x)}=0, \tag{2.13}$$

and let

$$\mathscr{C}:=\{z=x+iy:\ x\geq 0,\ |y|\leq c(x)\}. \tag{2.14}$$

If G *is nonzero on the segment* $\{z=iy:\ |y|\leq c(0)\}$, *then*

$$\overline{\lim_{n\to\infty}}\,\{\|\frac{1}{G}-\frac{1}{p_n}\|_{\mathscr{C}}\}^{1/n}=\overline{\lim_{n\to\infty}}\,\{\|\frac{1}{g}-\frac{1}{p_n}\|_{[0,+\infty)}\}^{1/n}. \tag{2.15}$$

The remaining results concern overconvergence on regions having a positive angle at infinity. In stating them it is convenient to introduce the sets $S(\theta,\mu)$ and $S(\theta)$ defined by

(2.16) $S(\theta,\mu) := \{z : |\arg z| < \theta,\ |z| > \mu\}$,

(2.17) $S(\theta) := \{z : |\arg z| < \theta\}$.

Theorem 2.4. Assume that for a function f, defined and finite on $[0,+\infty)$, there exists a sequence of rational functions $\{r_n\}_{n=1}^{\infty}$, with $r_n \in \pi_{n,n}$ for all $n \geq 1$, and a real number $q > 1$ such that

$$(2.18)\qquad \overline{\lim_{n\to\infty}} \{\|f - r_n\|_{[0,+\infty)}\}^{1/n} < \frac{1}{q} < 1.$$

Assume further that for some θ_o and μ_o, with $0 < \theta_o \leq \pi$, $\mu_o > 0$, the region $S(\theta_o, \mu_o)$ (defined in (2.16)) contains no poles of the $r_n(z)$ for all n large. Then for every θ satisfying the inequality

$$(2.19)\qquad 0 < \theta < 4\tan^{-1}\left\{\left(\frac{\sqrt{q}-1}{\sqrt{q}+1}\right)\cdot \tan\left(\frac{\theta_o}{4}\right)\right\},$$

there exists a $\mu = \mu(\theta) > 0$ and an analytic function $F(z)$ on the closure $\overline{S}(\theta,\mu)$ with $F(x) = f(x)$ for all real x in this set, such that $\{r_n(z)\}_{n=1}^{\infty}$ converges geometrically to $F(z)$ on $\overline{S}(\theta,\mu)$. Moreover

$$(2.20)\qquad \overline{\lim_{n\to\infty}}\{\|F - r_n\|_{\overline{S}(\theta,\mu)}\}^{1/n} < \frac{1}{q}\cdot\left\{\frac{\sin[\frac{1}{4}(\theta_o+\theta)]}{\sin[\frac{1}{4}(\theta_o-\theta)]}\right\}^2 < 1.$$

It is interesting to note that while Theorem 2.1 cannot be deduced from Theorem 2.4, the former result can be considered as a limiting case of the latter. Indeed, for the situation of Theorem 2.1, we regard θ_o and θ as functions of x which tend to zero as $x \to +\infty$; specifically, we define θ_o and θ by the equations

$$\tan\theta_o = \frac{h(x)}{x}, \quad \tan\theta = \frac{dh(x)}{x}.$$

Then, on writing (2.19) in the equivalent form

$$\frac{\tan(\theta/4)}{\tan(\theta_o/4)} < \frac{\sqrt{q}-1}{\sqrt{q}+1},$$

and taking the limit as $x \to +\infty$, we derive the condition

$$\lim_{x\to+\infty} \frac{\tan(\theta/4)}{\tan(\theta_o/4)} = \lim_{x\to+\infty} \frac{\tan\theta}{\tan\theta_o} = d < \frac{\sqrt{q}-1}{\sqrt{q}+1},$$

which is the same as inequality (2.4) of Theorem 2.1.

Using Theorem 2.4 we can deduce the following analogs of Corollaries 2.2 and 2.3:

Corollary 2.5. *Let the functions g, G, and the sequence of polynomials $\{p_n\}_{n=1}^{\infty}$ be as in Corollary 2.2 (so that, in particular, inequality (2.6) holds). Assume further that no zeros of p_n lie in the infinite sector $S(\theta_o)$ (defined in (2.17)), $0 < \theta_o \le \pi$, for all n sufficiently large, and that $g(0) \ne 0$. If θ satisfies (2.19), then on the closure $\overline{S}(\theta)$,*

$$\text{(2.21)} \quad \overline{\lim_{n\to\infty}} \left\{\left\|\frac{1}{G} - \frac{1}{p_n}\right\|_{\overline{S}(\theta)}\right\}^{1/n} \le \frac{1}{q}\left\{\frac{\sin[\frac{1}{4}(\theta_o+\theta)]}{\sin[\frac{1}{4}(\theta_o-\theta)]}\right\}^2 < 1.$$

Corollary 2.6. *Let the functions g, G, and the sequence of polynomials $\{p_n\}_{n=1}^{\infty}$ be as in Corollary 2.2 (so that, in particular, inequality (2.6) holds). Assume that no zeros of p_n lie in $S(\theta_o)$, $0 < \theta_o \le \pi$, for all n sufficiently large, and that $g(0) \ne 0$. Then for any nonnegative continuous function $c(x)$ on $[0, +\infty)$ such that $c(x) = o(x)$ as $x \to +\infty$ and such that (i) $c(0) = 0$ if $\theta_o = \pi/2$, (ii) $c(x) < x\tan(\theta_o)$ for $x > 0$ if $0 < \theta_o < \pi/2$, we have*

$$(2.22)\quad \overline{\lim_{n\to\infty}}\left\{\left\|\frac{1}{G}-\frac{1}{p_n}\right\|_{\mathbb{C}}\right\}^{1/n} = \overline{\lim_{n\to\infty}}\left\{\left\|\frac{1}{g}-\frac{1}{p_n}\right\|_{[0,+\infty)}\right\}^{1/n},$$

where the region $\mathbb{C}$ *is defined as in* (2.14).

If, in Corollary 2.5, we weaken the hypothesis by replacing the reciprocals of polynomials, $1/p_n$, by arbitrary rational functions $r_n \in \pi_{n,n}$ whose poles omit a full sector, then we obtain the following less specific conclusion:

Theorem 2.7. *Assume that for a function* f, *defined and finite on* $[0,+\infty)$, *there exists a sequence of rational functions* $\{r_n\}_{n=1}^{\infty}$, *with* $r_n \in \pi_{n,n}$ *for all* $n \geq 1$, *and a real number* $q > 1$ *such that inequality* (2.18) *holds. Suppose further that the infinite sector* $S(\theta_o)$ *(defined in* (2.17)), $0 < \theta_o \leq \pi$, *contains no poles of the* $r_n(z)$ *for all* n *large. Then there exists a* θ, $0 < \theta < \theta_o$, *and a function* $F(z)$ *analytic on the sector* $S(\theta)$, *continuous on* $\overline{S}(\theta)$, *with* $F(x) = f(x)$ *for all* $x \geq 0$, *such that* $\{r_n(z)\}_{n=1}^{\infty}$ *converges geometrically to* $F(z)$ *on* $\overline{S}(\theta)$.

Theorem 2.7 has an important application to the problem (raised at the International Conference on Approximation Theory, Maryland, 1970) of finding a sequence of rational functions which converges geometrically to e^{-z} in an infinite sector. It is well-known that the sequence $1/s_n(z)$, $s_n(z) = \sum_{o}^{n} z^k/k!$, does not have this property because no infinite sector is devoid of zeros of $s_n(z)$ for all n large (cf. [3] or [15]). However, it is shown by the authors in [11] and [13], that certain sequences of Padé approximants of e^{-z} converge geometrically on $[0,+\infty)$

to e^{-x}, and furthermore have all their poles outside some infinite sector $\{z: |\arg z| < \theta_0\}$. Hence, by Theorem 2.7, such a sequence must converge geometrically to e^{-z} on some infinite sector $\{z: |\arg z| < \theta\}$, $0 < \theta < \theta_0$. The precise details of this application shall be reserved for a later occasion.

The last result of this section concerns rational functions which converge faster than geometrically on $[0, +\infty)$, i.e.,

$$(2.23) \quad \lim_{n\to\infty} \{\|f-r_n\|_{[0,+\infty)}\}^{1/n} = 0.$$

Corollary 2.8. *If in Theorem 2.7, the assumption of inequality (2.18) is replaced by (2.23), then the sequence* $\{r_n(z)\}_{n=1}^{\infty}$ *converges faster than geometrically on every closed sector* $\overline{S}(\theta)$, $0 < \theta < \theta_0$, i.e.,

$$(2.24) \quad \lim_{n\to\infty} \{\|F-r_n\|_{\overline{S}(\theta)}\}^{1/n} = 0.$$

§3. Some Applications.

In order to apply results such as Corollaries 2.2 and 2.3 we first need conditions on the entire function $G(z)$ which insure that there exists a sequence of polynomials p_n, with $p_n \in \pi_n$ for all $n \geq 1$, such that

$$\text{(3.1)} \quad \overline{\lim_{n\to\infty}} \{ \| \frac{1}{G} - \frac{1}{p_n} \|_{[0,+\infty)} \}^{1/n} < 1.$$

Second, we need a specific result, like that of (2.9), which asserts that for an appropriate function $h \in \mathcal{H}$, the interior of the region $E_1(h)$ defined in (2.2) is free of zeros of the polynomials p_n in (3.1) for all n large. Results of both these types are already known for the case where the p_n are the n-th partial sums of the Maclaurin expansion for G. In order to state these results we remind the reader of some standard terminology.

If $g(z) = \sum_{k=0}^{\infty} a_k z^k$ is an entire function, we let $M_g(r) := \max\{|g(z)| : |z| = r\}$ denote its maximum modulus function, and let $\rho = \rho_g$ denote the order of g (for non-constant g), i.e., (cf. [2, p. 8], [16, p. 34])

$$\text{(3.2)} \quad \rho = \overline{\lim_{r\to+\infty}} \frac{\ln \ln M_g(r))}{\ln r}.$$

Furthermore, an entire function $g(z)$ of order ρ, $0 < \rho < \infty$, is said to be of perfectly regular growth (cf. [16, p. 44]) if there exists a real $B > 0$ such that

$$\text{(3.3)} \quad 0 < B = \lim_{r\to+\infty} \frac{\ln M_g(r)}{r^\rho}.$$

We remark that if a non-constant entire function g satisfies a linear differential equation with rational function coefficients, then g is necessarily of perfectly regular growth (cf. [16, p. 108]).

We now state a result which gives sufficient conditions for geometric convergence on $[0, +\infty)$.

Theorem 3.1 (Meinardus and Varga [8]). *Let* $g(z) = \sum_{k=0}^{\infty} a_k z^k$ *be an entire function of perfectly regular growth* (ρ, B) *with real nonnegative coefficients* a_k. *Then*

$$(3.4) \quad \lim_{n\to\infty}\{\|\frac{1}{g} - \frac{1}{s_n}\|_{[0,+\infty)}\}^{1/n} = \frac{1}{2^{1/\rho}} < 1,$$

where $s_n(z) = \sum_{k=0}^{n} a_k z^k$ *denotes the* n-th *partial sum of the Maclaurin expansion for* g.

Concerning zero-free regions for the partial sums s_n we state a previously unpublished result from one of the authors' theses [18]. For related published results see [17].

Theorem 3.2. *Let* S *denote the set of all entire functions* $g(z) = \sum_{k=0}^{\infty} a_k z^k$ *for which*

(i) $a_o > 0$ *and* $a_k \geq 0$ *for all* $k \geq 1$;

(ii) *if* $a_m = 0$, *then* $a_{m+2j} = 0$ *for every* $j \geq 1$;

(iii) *if* $K := \{k: a_k > 0$ *and* $a_{k+2} > 0\}$ *is non-empty, then*

$$(3.5) \quad \inf_{k\in K} \left\{\frac{a_k}{(k+1)(k+2)a_{k+2}}\right\} > 0.$$

Then, for $g \in S$, *there exists a nondecreasing continuous function* h_g *defined on* $[0, +\infty)$ *with* $h_g(0) > 0$, *such that* $g(z)$ *and all its partial sums* $s_n(z) = \sum_{k=0}^{n} a_k z^k$, $n \geq 1$, *have no zeros in*

$$(3.6) \quad \{z = x+iy: x \geq 0 \text{ and } |y| \leq h_g(x)\}.$$

Moreover, for each $g \in S$, the order ρ_g of g satisfies $0 \leq \rho_g \leq 1$.

We remark that the set S of Theorem 3.2 contains many familiar elements. For example, $u(z) = e^z$, $v(z) = \cosh(\sqrt{z}) = \sum_0^\infty z^k/(2k)!$, the modified Bessel functions $J_n(iz)/(iz)^n$ for any $n \geq 0$, and the hypergeometric function ${}_1F_1(c;d;z)$ with $c > 0$, $d > 0$, are easily seen to be elements of S.

If $\mathcal{W}_g$ denotes the nonempty (from Theorem 3.2) collection of all positive nondecreasing continuous functions h_g on $[0, +\infty)$ for which $g(z)$ and all its partial sums $s_n(z)$, $n \geq 1$, have no zeros in the region defined by (3.6), then we define the (maximal) width function $H_g(x)$ by

(3.7) $H_g(x) := \sup\{h_g(x) : h_g \in \mathcal{W}_g\}$, for each $x \geq 0$.

The function $H_g(x)$ so defined is clearly nondecreasing on $[0, +\infty)$, and $g(z)$ and all its partial sums $s_n(z)$ have no zeros in the interior of the region defined by

(3.8) $\{z = x + iy : x \geq 0 \text{ and } |y| \leq H_g(x)\}$.

More precisely, it is easily seen that either $H_g(x) = +\infty$ for all x sufficiently large, or that H_g is a step function, i.e., there exists a denumerable set of points $\{z_j = x_j + iy_j\}_{j=1}^\infty$, with $0 \leq x_j < x_{j+1}$ and $0 \leq y_j \leq y_{j+1}$ for all j, such that

$$H_g(x_j) = y_j \, ; \; H_g(x) = y_{j+1} \, , \text{ for } x_j < x \leq x_{j+1} ; \; j \geq 1 .$$

Here, each z_j is a zero of g or one of its partial sums s_n. Moreover, if g is of order $\rho_g > 0$, then a result of

Carlson [3] states that no proper sector, with vertex at the origin, can be devoid of zeros of the partial sums s_n, for all n large. Consequently, when $\rho_g > 0$, $H_g(x)$ is not only finite for all finite $x \geq 0$, but also $\lim_{j\to\infty} y_j/x_j = 0$.

The next corollary provides lower bounds for $H_g(x)$ for particular elements in $\mathcal{S}$.

Corollary 3.3. *Let* $g(z) = \sum_0^\infty a_k z^k$ *be an entire function such that* $a_k > 0$ *for all* k *and such that*

$$(3.9) \quad \inf_{k\geq 1} \{\frac{a_k}{k^2 a_{k+1}}\} > 0.$$

Then $g \in \mathcal{S}$ *and its associated width function* H_g of (3.7) *satisfies, for some constant* $c > 0$,

$$(3.10) \quad H_g(x) \geq cx^{1/2}, \text{ for all } x \geq 0.$$

Proof. It is trivial to verify that $g(z) = \sum_0^\infty a_k z^k \in \mathcal{S}$. Furthermore, it follows from the hypotheses above that the entire function f defined by $f(z) := \sum_0^\infty a_k z^{2k}$ is also in $\mathcal{S}$. Thus, from Theorem 3.2, we can associate with f a continuous nondecreasing function h_f defined on $[0, +\infty)$, with $h_f(0) > 0$, such that f and all its partial sums $S_n(z)$ have no zeros in

$$\mathfrak{F} := \{z = x+iy:\ x \geq 0 \text{ and } |y| \leq h_f(x)\}.$$

But if $s_n(z)$ denotes the n-th partial sum of g(z), then $s_n(z^2) = S_{2n}(z)$ for all $n = 1, 2, \cdots$, which allows us to relate the corresponding zeros of the partial sums of g with those of f. Thus, defining

$$\mathfrak{G} := \{z^2 : z \in \mathfrak{F}\},$$

then g and all its partial sums s_n have no zeros in $\mathcal{G}$. Now, since $h_f(0) > 0$ and h_f is nondecreasing on $[0, +\infty)$, then evidently

$$\mathcal{G} \supset \{z^2 : z = x+iy,\ x \geq 0 \text{ and } |y| \leq h_f(0)\}.$$

Thus, if H_g is the associated width function for g, the above inclusion implies that

$$H_g(t) \geq 2h_f(0)(t+h_f^2(0))^{1/2} \geq 2h_f(0)t^{1/2}, \text{ for all } t \geq 0,$$

which is the desired result of (3.10). ∎

As previously noted, $u(z) = e^z$, of order $\rho_u = 1$, and $v(z) = \cosh(\sqrt{z})$, of order $\rho_v = 1/2$, are elements of the set S, and furthermore each is of perfectly regular growth. Moreover, for $u(z) = e^z$, the authors' result of (2.9) implies that

$$H_u(x) \geq 2(x+1)^{1/2}, \text{ for all } x \geq 0.$$

Also, applying Corollary 3.3 to $v(z) = \cosh(\sqrt{z})$ gives

$$H_v(x) \geq cx^{1/2}, \text{ for all } x \geq 0.$$

However, we believe that this last inequality can be improved. In fact, we conjecture more generally that, for any element $g \in S$ of perfectly regular growth, its associated width function satisfies

$$H_g(x) \geq cx^{(2-\rho_g)/2}, \text{ for all } x \geq 0.$$

As a consequence of Theorems 3.1 and 3.2, which apply to both e^z and $\cosh(\sqrt{z})$, we have the following application of Corollary 2.2.

Corollary 3.4. For any $g \in S$ of order $\rho > 0$ which is of perfectly regular growth, let H_g be its associated non-

decreasing width function of (3.7), *and let* $h \in \mathcal{H}$ *be any positive function for which* $h(x) \le H_g(x)$ *for all* $x \ge 0$. *Then for*

$$0 < d < (2^{1/2\rho}-1)/(2^{1/2\rho}+1),$$

we have

$$(3.11)\quad \overline{\lim_{n\to\infty}} \{\|\frac{1}{g} - \frac{1}{s_n}\|_{E_d(h)}\}^{1/n} \le \frac{1}{2^{1/\rho}} \left(\frac{1+d}{1-d}\right)^2 < 1,$$

where the region $E_d(h)$ *is defined as in* (2.2), *and* $s_n(z)$ *denotes the* n-th *partial sum of* $g(z)$.

Proof. Because $g \in \mathcal{S}$ implies that the Maclaurin coefficients of g are all nonnegative, and because g is assumed to be of perfectly regular growth, then the conclusion (3.4) of Theorem 3.1 is valid. Next, by definition of $H_g(x)$ and the fact that $h(x) \le H_g(x)$ for all $x \ge 0$, it follows that g and all its partial sums s_n have no zeros in the interior of the region $E_1(h)$. Consequently, applying Corollary 2.2, with $q = 2^{1/\rho}$, gives the desired result of (3.11). ∎

We remark that the existence of a function $h \in \mathcal{H}$ satisfying the conditions of Corollary 3.4 is obvious. As a simple example, take h_g of Theorem 3.2 and set $h(x) \equiv h_g(0)$.

Concerning rational approximation to entire functions of order $\rho = 0$, it is shown in [7, Thm. 7] and in [4, Thm. 2] that if g is an entire function of order zero and satisfies certain growth and coefficient restrictions, then

$$(3.12)\quad \lim_{n\to\infty} \{\inf_{p\in\pi_n} \|\frac{1}{g} - \frac{1}{p}\|_{[0,+\infty)}\}^{1/n} = 0.$$

As an illustration of how our techniques apply to such situations, we present

Proposition 3.5. *Let* $g(z) = \sum_{k=0}^{\infty} z^k/a^{k^2}$, *where* $a \geq 2$, *and let* $s_n(z) = \sum_{k=0}^{n} z^k/a^{k^2}$. *Then, on every closed sector* $\overline{S}(\theta)$ *(defined in* (2.17)) *with* $0 < \theta < \pi$, *we have*

$$(3.13) \quad \lim_{n\to\infty} \{\|\frac{1}{g} - \frac{1}{s_n}\|_{\overline{S}(\theta)}\}^{1/n^2} = \frac{1}{\sqrt{a}} .$$

Of course, for the functions of Proposition 3.5, we see that the conclusion of (3.13) is far stronger, and implies the result of (3.12) as a special case.

The proof of Proposition 3.5 will be given in [14].

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