# POLARIZATION OPTIMALITY OF EQUALLY SPACED POINTS ON THE CIRCLE FOR DISCRETE POTENTIALS

DOUGLAS P. HARDIN, AMOS P. KENDALL, AND EDWARD B. SAFF

ABSTRACT. We prove a conjecture of Ambrus, Ball and Erdélyi that equally spaced points maximize the minimum of discrete potentials on the unit circle whenever the potential is of the form

$$\sum_{k=1}^{n} f(d(z,z_k)),$$

where $f : [0,\pi] \to [0,\infty]$ is non-increasing and convex and $d(z,w)$ denotes the geodesic distance between $z$ and $w$ on the circle.

## 1. INTRODUCTION AND MAIN RESULTS

Let $\mathbb{S}^1 := \{z = x + iy \mid x, y \in \mathbb{R},\ x^2 + y^2 = 1\}$ denote the unit circle in the complex plane $\mathbb{C}$. For $z, w \in \mathbb{S}^1$, we denote by $d(z,w)$ the geodesic (shortest arclength) distance between $z$ and $w$. Let $f : [0,\pi] \to [0,\infty]$ be non-increasing and convex on $(0,\pi]$ with $f(0) = \lim_{\theta\to 0^+} f(\theta)$. It then follows that $f$ is a continuous extended real-valued function on $[0,\pi]$. For a list of $n$ points (not necessarily distinct) $\omega_n = (z_1,\ldots,z_n) \in (\mathbb{S}^1)^n$, we consider the *$f$-potential* of $\omega_n$,

$$U^f(\omega_n; z) := \sum_{k=1}^{n} f(d(z,z_k)) \qquad (z \in \mathbb{S}^1), \tag{1}$$

and the *$f$-polarization* of $\omega_n$,

$$M^f(\omega_n; \mathbb{S}^1) := \min_{z\in\mathbb{S}^1} U^f(\omega_n; z). \tag{2}$$

In this note, we are chiefly concerned with the *$n$-point $f$-polarization* of $\mathbb{S}^1$ (also called the *$n$th $f$-Chebyshev constant* of $\mathbb{S}^1$),

$$M_n^f(\mathbb{S}^1) := \sup_{\omega_n\in(\mathbb{S}^1)^n} M^f(\omega_n; \mathbb{S}^1), \tag{3}$$

which has been the subject of several recent papers (e.g., [1], [2], [5], [6]).

---

*Date*: August 25, 2012.

2000 *Mathematics Subject Classification.* Primary 52A40, 30C15.

*Key words and phrases.* Polarization, Chebyshev constants, roots of unity, potentials, max-min problems.

This research was supported, in part, by the U. S. National Science Foundation under grants DMS-0808093 and DMS-1109266.

In the case (relating to Euclidean distance) when

$$f(\theta) = f_s(\theta) := |e^{i\theta} - 1|^{-s} = (2\sin|\theta/2|)^{-s},\ s > 0, \tag{4}$$

we abbreviate the notation for the above quantities by writing

$$\begin{aligned} U^s(\omega_n; z) &:= \sum_{k=1}^{n} f_s(d(z, z_k)) = \sum_{k=1}^{n} \frac{1}{|z - z_k|^s}, \\ M^s(\omega_n; \mathbb{S}^1) &:= \min_{z\in\mathbb{S}^1} \sum_{k=1}^{n} \frac{1}{|z - z_k|^s}, \\ M_n^s(\mathbb{S}^1) &:= \sup_{\omega_n \in (\mathbb{S}^1)^n} M^s(\omega_n; \mathbb{S}^1). \end{aligned} \tag{5}$$

The main result of this note is the following theorem conjectured by G. Ambrus et al [2]. Its proof is given in the next section.

**Theorem 1.** *Let $f : [0, \pi] \to [0, \infty]$ be non-increasing and convex on $(0, \pi]$ with $f(0) = \lim_{\theta\to 0^+} f(\theta)$. If $\omega_n$ is any configuration of $n$ distinct equally spaced points on $\mathbb{S}^1$, then $M^f(\omega_n; \mathbb{S}^1) = M_n^f(\mathbb{S}^1)$. Moreover, if the convexity condition is replaced by strict convexity, then such configurations are the only ones that achieve this equality.*

Applying this theorem to the case of $f_s$ given in (4) we immediately obtain the following.

**Corollary 2.** *Let $s > 0$ and $\omega_n^* := \{e^{i2\pi k/n} : k = 1, 2, \ldots, n\}$. If $(z_1, \ldots, z_n) \in (\mathbb{S}^1)^n$, then*

$$\min_{z\in\mathbb{S}^1} \sum_{k=1}^{n} \frac{1}{|z - z_k|^s} \leq M^s(\omega_n^*; \mathbb{S}^1) = M_n^s(\mathbb{S}^1), \tag{6}$$

*with equality if and only if $(z_1, \ldots, z_n)$ consists of distinct equally spaced points.*

The following representation of $M^s(\omega_n^*; \mathbb{S}^1)$ in terms of *Riesz $s$-energy* was observed in [2]:

$$M^s(\omega_n^*; \mathbb{S}^1) = \frac{\mathcal{E}_s(\mathbb{S}^1; 2n)}{2n} - \frac{\mathcal{E}_s(\mathbb{S}^1; n)}{n},$$

where

$$\mathcal{E}_s(\mathbb{S}^1; n) := \inf_{\omega_n \in (\mathbb{S}^1)^n} \sum_{j=1}^{n} \sum_{\substack{k=1 \\ k\neq j}}^{n} \frac{1}{|z_j - z_k|^s}.$$

Thus, applying the asymptotic formulas for $\mathcal{E}_s(\mathbb{S}^1; n)$ given in [3], we obtain the dominant term of $M_n^s(\mathbb{S}^1)$ as $n \to \infty$:

$$M_n^s(\mathbb{S}^1) \sim \begin{cases} \dfrac{2\zeta(s)}{(2\pi)^s}\,(2^s-1)n^s\,, & s>1\,,\\[2ex] (1/\pi)\,n\log n\,, & s=1\,,\\[2ex] \dfrac{2^{-s}}{\sqrt{\pi}}\,\dfrac{\Gamma\big(\frac{1-s}{2}\big)}{\Gamma\big(1-\frac{s}{2}\big)}\,n\,, & s\in[0,1), \end{cases}$$

where $\zeta(s)$ denotes the classical Riemann zeta function and $a_n \sim b_n$ means that $\lim_{n\to\infty} a_n/b_n = 1$. These asymptotics, but for $M^s(\omega_n^*; \mathbb{S}^1)$, were stated in [2][1].

For $s$ an even integer, say $s = 2m$, the precise value of $M_n^{2m}(\mathbb{S}^1) = M^{2m}(\omega_n^*; \mathbb{S}^1)$ can be expressed in finite terms, as can be seen from formula (1.20) in [3].

**Corollary 3.** *We have*

$$M_n^{2m}(\mathbb{S}^1) = \frac{2}{(2\pi)^{2m}} \sum_{k=1}^{m} n^{2k}\zeta(2k)\alpha_{m-k}(2m)(2^{2k}-1), \quad m\in\mathbb{N},$$

*where $\alpha_j(s)$ is defined via the power series for* $\operatorname{sinc} z = (\sin \pi z)/(\pi z)$ :

$$(\operatorname{sinc} z)^{-s} = \sum_{j=0}^{\infty} \alpha_j(s) z^{2j}\,, \quad \alpha_0(s) = 1\,.$$

*In particular,*

$$M_n^2(\mathbb{S}^1) = \frac{2}{(2\pi)^2} n^2\zeta(2) = \frac{n^2}{4},$$

$$M_n^4(\mathbb{S}^1) = \frac{2}{(2\pi)^4}[n^2\zeta(2)\alpha_1(4)(2^2-1) + n^4\zeta(4)(2^4-1)] = \frac{n^2}{24} + \frac{n^4}{48},$$

$$\begin{aligned} M_n^6(\mathbb{S}^1) &= \frac{2}{(2\pi)^6}[n^2\zeta(2)\alpha_2(6)(2^2-1) + n^4\zeta(4)\alpha_1(6)(2^4-1) + n^6\zeta(6)(2^6-1)]\\ &= \frac{n^2}{120} + \frac{n^4}{192} + \frac{n^6}{480}, \end{aligned}$$

The case $s = 2$ of the above corollary was first proved in [1],[2] and the case $s = 4$ was first proved in [5]. We remark that an alternative formula for $\alpha_j(s)$ is

$$\alpha_j(s) = \frac{(-1)^j B_{2j}^{(s)}(s/2)}{(2j)!}(2\pi)^{2j}, \qquad j = 0, 1, 2, \dots\,,$$

[1] We remark that there is a factor of $2/(2\pi)^p$ missing in the asymptotics given in [2] for the case $p := s > 1$.

where $B_j^{(\alpha)}(x)$ denotes the generalized Bernoulli polynomial. Asymptotic formulas for $M_n^f(\mathbb{S}^1)$ for certain other functions $f$ can be obtained from the asymptotic formulas given in [4].

As other consequences of Theorem 1, we immediately deduce that equally spaced points are optimal for the following problems

$$\min_{\omega_n\in(\mathbb{S}^1)^n}\max_{z\in\mathbb{S}^1}\sum_{k=1}^{n}|z-z_k|^{\alpha}, \qquad (0<\alpha\le 1), \tag{7}$$

and

$$\max_{\omega_n\in(\mathbb{S}^1)^n}\min_{z\in\mathbb{S}^1}\sum_{k=1}^{n}\log\frac{1}{|z-z_k|}, \tag{8}$$

with the solution to (8) being well-known. Furthermore, various generalizations of the polarization problem for Riesz potentials for configurations on $\mathbb{S}^1$ are worthy of consideration, such as minimizing the potential on circles concentric with $\mathbb{S}^1$.

## 2. Proof of Theorem 1

For distinct points $z_1, z_2 \in \mathbb{S}^1$, we let $\widehat{z_1z_2}$ denote the closed subarc of $\mathbb{S}^1$ from $z_1$ to $z_2$ traversed in the counterclockwise direction. We further let $\gamma(\widehat{z_1z_2})$ denote the length of $\widehat{z_1z_2}$ (thus, $\gamma(\widehat{z_1z_2})$ equals either $d(z_1,z_2)$ or $2\pi - d(z_1,z_2)$). Observe that the points $z_1$ and $z_2$ partition $\mathbb{S}^1$ into two subarcs: $\widehat{z_1z_2}$ and $\widehat{z_2z_1}$. The following lemma (see proof of Lemma 1 in [2]) is a simple consequence of the convexity and monotonicity of the function $f$ and is used to show that any $n$-point configuration $\omega_n \subset \mathbb{S}^1$ such that $M^f(\omega_n) = M_n^f(\mathbb{S}^1)$ must have the property that any local minimum of $U^f(\omega_n;\cdot)$ is a global minimum of this function.

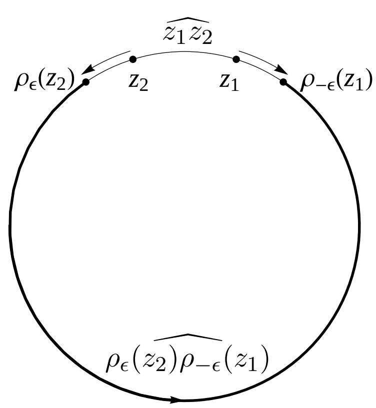


Figure 1. The points $z_1, z_2, \rho_{-\epsilon}(z_1), \rho_\epsilon(z_2)$ in Lemma 4. The potential increases at every point in the subarc $\rho_\epsilon(\widehat{z_2)\rho_{-\epsilon}}(z_1)$ when $(z_1,z_2)\to(\rho_{-\epsilon}(z_1),\rho_\epsilon(z_2))$; see (9).

For $\phi \in \mathbb{R}$ and $z \in \mathbb{S}^1$, we let $\rho_\phi(z) := e^{i\phi} z$ denote the counterclockwise rotation of $z$ by the angle $\phi$.

**Lemma 4** ([2])**.** *Let $z_1, z_2 \in \mathbb{S}^1$ and $0 < \epsilon < \gamma(\widehat{z_2 z_1})/2$. Then with $f$ as in Theorem 1,*

$$U^f((z_1, z_2); z) \le U^f((\rho_{-\epsilon}(z_1), \rho_\epsilon(z_2)); z), \tag{9}$$

*for $z$ in the subarc $\widehat{\rho_\epsilon(z_2)\rho_{-\epsilon}(z_1)}$, while the reverse inequality holds for $z$ in the subarc $\widehat{z_1 z_2}$. If $f$ is strictly convex on $(0, \pi]$, then these inequalities are strict. If $z_1 = z_2$, then we set $\widehat{z_1 z_2} = \{z_1\}$ and $\widehat{z_2 z_1} = \mathbb{S}^1$.*

We now assume that $\omega_n = (z_1, \dots, z_n)$ is ordered in a counterclockwise manner and also that the indexing is extended periodically so that $z_{k+n} = z_k$ for $k \in \mathbb{Z}$. For $1 \le k \le n$ and $\Delta \in \mathbb{R}$, we define $\tau_{k,\Delta} : (\mathbb{S}^1)^n \to (\mathbb{S}^1)^n$ by

$$\tau_{k,\Delta}(z_1, \dots, z_k, z_{k+1}, \dots, z_n) := (z_1, \dots, \rho_{-\Delta}(z_k), \rho_\Delta(z_{k+1}), \dots, z_n).$$

If $z_{k-1} \ne z_k$ and $z_{k+1} \ne z_{k+2}$, then $\tau_{k,\Delta}(\omega_n)$ retains the ordering of $\omega_n$ for $\Delta$ positive and sufficiently small. Given $\boldsymbol{\Delta} := (\Delta_1, \dots, \Delta_n)^T \in \mathbb{R}^n$, let $\tau_{\boldsymbol{\Delta}} := \tau_{n,\Delta_n} \circ \cdots \circ \tau_{2,\Delta_2} \circ \tau_{1,\Delta_1}$ and $\omega_n' := \tau_{\boldsymbol{\Delta}}(\omega_n)$. Letting $\alpha_k := \gamma(\widehat{z_k z_{k+1}})$ and $\alpha_k' := \gamma(\widehat{z_k' z_{k+1}'})$ for $k = 1, \dots, n$, we obtain the system of $n$ linear equations:

$$\alpha_k' = \alpha_k - \Delta_{k-1} + 2\Delta_k - \Delta_{k+1}, \qquad (1 \le k \le n), \tag{10}$$

which is satisfied as long as $\sum_{k=1}^n \alpha_k' = 2\pi$ or, equivalently, if $\omega_n'$ is ordered counterclockwise. Let

$$\operatorname{sep}(\omega_n) := \min_{1 \le \ell \le n} \alpha_\ell.$$

Then (10) holds if

$$\max_{1 \le k \le n} |\Delta_k| \le (1/4)\operatorname{sep}(\omega_n), \tag{11}$$

in which case, the configurations

$$\omega_{n,\boldsymbol{\Delta}}^{(\ell)} := \tau_{n,\Delta_\ell} \circ \cdots \circ \tau_{2,\Delta_2} \circ \tau_{1,\Delta_1}(\omega_n), \qquad (\ell = 1, \dots, n) \tag{12}$$

are all ordered counterclockwise. If the components of $\boldsymbol{\Delta}$ are nonnegative, then we may replace the '(1/4)' in (11) with '(1/2)'.

**Lemma 5.** *Suppose $\omega_n = (z_1, \dots, z_n)$ and $\omega_n' = (z_1', \dots, z_n')$ are $n$-point configurations on $\mathbb{S}^1$ ordered in a counterclockwise manner. Then there is a unique $\boldsymbol{\Delta}^* = (\Delta_1^*, \dots, \Delta_n^*) \in \mathbb{R}^n$ so that*

(a) $\Delta_k^* \ge 0$, $k = 1, \dots, n$,
(b) $\Delta_j^* = 0$ *for some* $j \in \{1, \dots, n\}$, *and*
(c) $\tau_{\boldsymbol{\Delta}^*}(\omega_n)$ *is a rotation of* $\omega_n'$.

*Proof.* The system (10) can be expressed in the form

$$A\boldsymbol{\Delta} = \boldsymbol{\beta}, \tag{13}$$

where

$$A := \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & -1 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & \cdots & -1 & 2 & -1 \\ -1 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}, \quad \boldsymbol{\Delta} := \begin{pmatrix} \Delta_1 \\ \Delta_2 \\ \vdots \\ \\ \Delta_n \end{pmatrix}, \text{ and } \boldsymbol{\beta} := \begin{pmatrix} \alpha_1' - \alpha_1 \\ \alpha_2' - \alpha_2 \\ \vdots \\ \\ \alpha_n' - \alpha_n \end{pmatrix}.$$

It is elementary to verify that $\ker A = (\text{range } A)^{\perp} = \text{span } (\mathbf{1})$, where $\mathbf{1} = (1, 1, \ldots, 1)^T$. Since $\boldsymbol{\beta}^T \mathbf{1} = \sum_{k=1}^{n} (\alpha_k' - \alpha_k) = 0$, the linear system (13) always has a solution $\boldsymbol{\Delta}$. Let $j \in \{1, \ldots, n\}$ satisfy $\Delta_j = \min_{1 \le k \le n} \Delta_k$. Then subtracting $\Delta_j \mathbf{1}$ from $\boldsymbol{\Delta}$, we obtain the desired $\boldsymbol{\Delta}^*$. Since $\ker A = \text{span } \mathbf{1}$, there is at most one solution of (13) satisfying properties (a) and (b), showing that $\boldsymbol{\Delta}^*$ is unique.

Part (c) holds as a direct result of the fact that both $\omega_n$ and $\omega_n'$ are ordered counterclockwise.

□

**Lemma 6.** *Let $\Omega_n = (z_1, \ldots, z_n)$ be a configuration of $n$ distinct points on $\mathbb{S}^1$ ordered counterclockwise, and with $f$ as in Theorem 1, suppose $\boldsymbol{\Delta} = (\Delta_1, \ldots, \Delta_n) \in \mathbb{R}^n$ is such that*

(a) $0 \le \Delta_k \le (1/2)\text{sep}(\Omega_n)$ *for $k = 1, \ldots, n$, and*
(b) *there is some $j \in \{1, \ldots, n\}$ for which $\Delta_j = 0$.*

*Let $\Omega_n' := \tau_{\boldsymbol{\Delta}}(\Omega_n) = (z_1', \ldots, z_n')$. Then $\widehat{z_j' z_{j+1}'} \subset \widehat{z_j z_{j+1}}$ and*

$$U^f(\Omega_n; z) \le U^f(\Omega_n'; z) \qquad (z \in \widehat{z_j' z_{j+1}'}). \tag{14}$$

*If $f$ is strictly convex on $(0, \pi]$ and $\Delta_k > 0$ for at least one $k$, then the inequality* (14) *is strict.*

We remark that $\Delta_k = 0$ for all $k = 1, \ldots, n$ is equivalent to saying that the points are equally spaced.

*Proof.* Recalling (12), it follows from condition (a) that $(z_1^{(\ell)}, \ldots, z_n^{(\ell)}) := \omega_{n,\boldsymbol{\Delta}}^{(\ell)}$ are counterclockwise ordered. Since $\Delta_j = 0$ and $\Delta_k \ge 0$ for $k = 1, \ldots, n$, the points $z_j^{(\ell)}$ and $z_{j+1}^{(\ell)}$ are moved at most once as $\ell$ varies from 1 to $n$ and move toward each other, while remaining in the complement of all other subarcs $\widehat{z_k^{(\ell)} z_{k+1}^{(\ell)}}$, i.e.,

$$\widehat{z_j' z_{j+1}'} = \widehat{z_j^{(n)} z_{j+1}^{(n)}} \subseteq \widehat{z_j^{(\ell)} z_{j+1}^{(\ell)}} \subseteq \widehat{z_{k+1}^{(\ell)} z_k^{(\ell)}},$$

for $k \in \{1, \ldots, n\} \setminus \{j\}$ and $\ell \in \{1, \ldots, n\}$. Lemma 4 implies that, for $\ell = 1, \ldots, n$, we have $U^f(\omega_n^{(\ell-1)}; z) \le U^f(\omega_n^{(\ell)}; z)$ for $z \in \widehat{z_j^{(\ell)} z_{j+1}^{(\ell)}}$ (where $\omega_n^{(0)} := \omega_n$) and the inequality is strict if $\Delta_\ell > 0$. Hence, (14) holds and the inequality is strict if $f$ is strictly convex and $\Delta_k > 0$ for some $k = 1, \ldots, n$. □

We now proceed with the proof of Theorem 1. Let $\omega_n = (z_1, \dots, z_n)$ be a non-equally spaced configuration of $n$ (not necessarily distinct) points on $\mathbb{S}^1$ ordered counterclockwise. By Lemma 5, there is some equally spaced configuration $\omega'_n$ (i.e., $\alpha'_k = 2\pi/n$ for $k = 1, \dots, n$) and some $\mathbf{\Delta^*} = (\Delta^*_1, \dots, \Delta^*_n)$ such that (a) $\omega'_n = \tau_{\mathbf{\Delta}^*}(\omega_n)$, (b) $\Delta^*_k \geq 0$ for $k = 1, \dots, n$, and (c) $\Delta^*_j = 0$ for some $j \in \{1, \dots, n\}$. Then (10) holds with $\alpha_k := \gamma(\widehat{z_k, z_{k+1}})$ and $\alpha'_k := 2\pi/n$. Since $\omega_n$ is not equally spaced, we have $\Delta^*_k > 0$ for at least one value of $k$.

For $0 \leq t \leq 1$, let $\omega^t_n := \tau_{(t\mathbf{\Delta}^*)}(\omega_n) = (z^t_1, \dots, z^t_n)$ and, for $k = 1, \dots, n$, let $\alpha^t_k := \gamma(\widehat{z^t_k z^t_{k+1}})$. Recalling (10), observe that

$$\begin{aligned}\alpha^t_k &= \alpha_k - t(\Delta_{k-1} + 2\Delta_k - \Delta_{k+1})\\ &= \alpha_k + t(2\pi/n - \alpha_k)\\ &= (1-t)\alpha_k + t(2\pi/n),\end{aligned}$$

for $0 \leq t \leq 1$ and $k = 1, \dots, n$, and so $\text{sep}(\omega^t_n) \geq t(2\pi/n)$. Now let $0 < t < s < \min(1, t(1 + \pi/(nD)))$, where $D := \max\{\Delta_k : 1 \leq k \leq n\}$. Then Lemma 6 (with $\Omega_n = \omega^t_n$, $\mathbf{\Delta} = (s-t)\mathbf{\Delta}^*$, and $\Omega'_n = \tau_{\mathbf{\Delta}}(\Omega_n) = \omega^s_n$) implies that $\widehat{z^s_j z^s_{j+1}} \subseteq \widehat{z^t_j z^t_{j+1}}$ and that

$$U^f(\omega^t_n; z) \leq U^f(\omega^s_n; z) \qquad (z \in \widehat{z^s_j z^s_{j+1}}), \tag{15}$$

where the inequality is sharp if $f$ is strictly convex.

Consider the function

$$h(t) := \min\,\{U^f(\omega^t_n; z) : z \in \widehat{z^t_j z^t_{j+1}}\}, \qquad (0 \leq t \leq 1).$$

Observe that

$$h(t) \leq \min\,\{U^f(\omega^t_n; z) : z \in \widehat{z^s_j z^s_{j+1}}\} \leq \min\{U^f(\omega^s_n; z) : z \in \widehat{z^s_j z^s_{j+1}}\} = h(s),$$

for $0 < t < s < \min(1, t(1 + \pi/(nD)))$. It is then easy to verify that $h$ is non-decreasing on $(0, 1)$. Since $\omega^t_n$ depends continuously on $t$, the function $h$ is continuous on $[0, 1]$ and thus $h$ is non-decreasing on $[0, 1]$.

We then obtain the desired inequality

$$M^f(\omega_n; \mathbb{S}^1) \leq h(0) \leq h(1) = M^f(\omega'_n; \mathbb{S}^1),$$

where the last equality is a consequence of the fact that $\omega'_n$ is an equally spaced configuration and so the minimum of $U^f(\omega'_n; z)$ over $\mathbb{S}^1$ is the same as the minimum over $\widehat{z'_j z'_{j+1}}$. If $f$ is strictly convex, then $h(0) < h(1)$ showing that any optimal $f$-polarization configuration must be equally spaced. This completes the proof of Theorem 1. □

**Acknowledgements:** We thank the referees for their helpful suggestions to improve the manuscript.

## References

[1] G. Ambrus, Analytic and Probabilistic Problems in Discrete Geometry, Ph.D. Thesis, University College London, 2009.

[2] G. Ambrus, K. Ball, and T. Erdélyi, Chebyshev constants for the unit circle, *Bull. London Math. Soc.* (to appear), arXiv:1006.5153

[3] J.S. Brauchart, D.P. Hardin, and E.B. Saff, The Riesz energy of the $N$th roots of unity: an asymptotic expansion for large $N$, *Bull. London Math. Soc.*, **41** (4), (2009), 621–633.

[4] J.S. Brauchart, D.P. Hardin and E.B. Saff, Discrete energy asymptotics on a Riemannian circle, *Uniform Distribution Theory*, **6** (2011), 77–108.

[5] T. Erdélyi and E.B. Saff, Polarization inequalities in higher dimensions, (submitted), J. Approx. Theory (to appear) arXiv:1206.4729v1

[6] N. Nikolov and R. Rafailov, On the sum of powered distances to certain sets of points on the circle, *Pacific J. Math.*, **253**(1), (2011), 157–168.

Center for Constructive Approximation, Department of Mathematics, Vanderbilt University, 1326 Stevenson Center, Nashville, TN, 37240, USA

*E-mail address*: doug.hardin@vanderbilt.edu

*E-mail address*: amos.p.kendall@vanderbilt.edu

*E-mail address*: edward.b.saff@vanderbilt.edu