Abstract

We derive bounds and asymptotics for the maximum Riesz polarization quantity

$$M_P^n(A) := \max_{x_1, x_2, \ldots, x_n \in A} \min_{x \in A} \sum_{j=1}^{n} \frac{1}{|x - x_j|^p}$$

(which is $n$ times the Chebyshev constant) for quite general sets $A \subset \mathbb{R}^m$ with special focus on the unit sphere and unit ball. We combine elementary averaging arguments with potential theoretic tools to formulate and prove our results. We also give a discrete version of the recent result of Hardin, Kendall, and Saff which solves the Riesz polarization problem for the case when $A$ is the unit circle and $p > 0$, as well as provide an independent proof of their result for $p = 4$ that exploits classical polynomial inequalities and yields new estimates. Furthermore, we raise some challenging conjectures.

Keywords: Chebyshev constants; Polarization inequalities; Riesz energy; Potentials

1. Introduction

For $n \in \mathbb{N}$, let $\omega_n = \{x_1, x_2, \ldots, x_n\}$ denote $n$ (not necessarily distinct) points in $m$-dimensional Euclidean space $\mathbb{R}^m$. We define for $p > 0$ and a compact set $A \subset \mathbb{R}^m$, the Riesz
polarization quantities

\[ M^p(\omega_n, A) := \min_{x \in A} \sum_{j=1}^{n} \frac{1}{|x - x_j|^p}, \quad M^p_n(A) := \max_{\omega_n \subset A} M^p(\omega_n, A). \] (1.1)

Such max–min quantities for potentials were first introduced by M. Ohtsuka who explored (for very general kernels) their relationship to various definitions of capacity that arise in electrostatics (see [17]). In particular, he showed that for any compact set \( A \subset \mathbb{R}^m \) the following limit, called the Chebyshev constant of \( A \), exists as an extended real number:

\[ M^p(A) := \lim_{n \to \infty} \frac{M^p_n(A)}{n}. \] (1.2)

Moreover, he showed that \( M^p(A) \) is not smaller than the Wiener constant \( W^p(A) \) for \( A \) (see Section 2). In this paper we primarily focus on results when the set \( A \) is the unit sphere or the unit ball and consider both the cases when the limit (1.2) is finite and when it is infinite.

In his Ph.D. dissertation [1], G. Ambrus proved the following basic result for the case when \( A \subset \mathbb{R}^2 \) is the unit circle \( S^1 \) and \( p = 2 \).

**Theorem 1.1.** We have

\[ M_n^2(S^1) = \frac{n^2}{4}, \quad n \geq 1, \] (1.3)

and \( M^2(\omega_n, S^1) = n^2/4, \omega_n \subset S^1, \) if and only if the \( n \) points of \( \omega_n \) are distinct and equally spaced on \( S^1 \).

In [2], Ambrus’s rather technical proof along with a simpler proof based on Bernstein’s inequality [3] for entire functions are presented. Bernstein’s inequality was also used in [2] to provide an equally simple proof of the following estimates for the unit circle.

**Theorem 1.2.** For \( n \geq 2 \) we have

\[ M_n^p(S^1) \leq \begin{cases} c_p n^p, & p > 1, \\ c_1 n \log n, & p = 1, \\ c_0 n \frac{1 - p}{1 - p}, & p \in [0, 1), \end{cases} \]

for some constants \( c_p > 0 \) depending only on \( p \geq 1 \) and an absolute constant \( c_0 > 0 \).

In Section 2 we use minimum energy methods and potential theory to obtain estimates for \( M_n^p(A) \) for a large class of sets \( A \subset \mathbb{R}^m \). In Section 3 we apply the results of Section 2 to obtain higher dimensional analogs of Theorem 1.2 for the unit sphere as well as for the unit ball.

In Section 4 we return to the case of the unit circle of the complex plane. For all \( p > 0 \), it is conjectured in [2] that the maximum polarization on \( S^1 \) occurs for the \( n \)-th roots of unity \( \omega_n^* := \{e^{2\pi k/n} : k = 1, 2, \ldots, n\} \); that is,

\[ M_n^p(S^1) = M^p(\omega_n^*, S^1). \] (1.4)

This conjecture was recently proved by Hardin, Kendall, and Saff in [9]. Here, we provide some additional consequences of their argument. Furthermore, by exploring connections to classical
polynomial inequalities, we provide an independent proof of the conjecture for \( p = 4 \), namely that
\[
M^4_\alpha(S^1) = \frac{n^4}{48} + \frac{n^2}{24},
\]
where the maximum is attained for \( n \) distinct equally spaced points on the unit circle. Although our argument (obtained prior to the general result in [9]) is not brief, it does yield additional inequalities for the discrete Riesz potential in this special case.

In Section 5, we provide the proofs of results stated in Sections 2 and 3.

We call the reader’s attention to two recent articles [15,16] that contain somewhat related results for the extrema of sums of certain powered distances to finite point sets.

2. Polarization inequalities via energy methods

For a set \( \omega_n = \{x_1, x_2, \ldots, x_n\} \) of \( n \geq 2 \) distinct points in \( \mathbb{R}^m \), we define the \( \text{Riesz p-energy} \) of \( \omega_n \) by
\[
E_p(\omega_n) := \sum_{j \neq k} \frac{1}{|x_j - x_k|^p} = 2 \sum_{1 \leq j < k \leq n} \frac{1}{|x_j - x_k|^p},
\]
and we consider the \textit{minimum n-point Riesz p-energy} of an infinite compact set \( A \subset \mathbb{R}^m \) defined by
\[
\mathcal{E}_p(A; n) := \min \{ E_p(\omega_n) : \omega_n \subset A, |\omega_n| = n \}. \tag{2.1}
\]
We denote by \( \omega_{n,p}^* = \{x_1^*, x_2^*, \ldots, x_n^*\} \) an \( n \)-point \( p \)-energy minimizing configuration on \( A \); i.e., \( E_p(\omega_{n,p}^*) = \mathcal{E}_p(A; n) \). Further we denote by \( U_{n,p}^*(x) \) the potential function associated with \( \omega_{n,p}^* \); i.e.,
\[
U_{n,p}^*(x) := \sum_{j=1}^{n} |x - x_j^*|^{-p}.
\]
It is well-known (and easy to show) that
\[
(n - 1)\mathcal{E}_p(A; n + 1) \geq (n + 1)\mathcal{E}_p(A; n), \tag{2.2}
\]
from which it follows that
\[
C^*(A, n, p) := \min \{ U_{n,p}^*(x) : x \in A \} \geq \frac{1}{n - 1} \mathcal{E}_p(A; n); \tag{2.3}
\]
indeed, we have
\[
2C^*(A, n, p) + \mathcal{E}_p(A; n) \geq \mathcal{E}_p(A; n + 1),
\]
and after multiplying this inequality by \( n - 1 \) and applying (2.2), we get (2.3). Thus lower estimates for \( \mathcal{E}_p(A; n) \) yield lower estimates for \( M^p_\alpha(A) \).

We next mention some known asymptotic results for \( \mathcal{E}_p(A; n) \) as \( n \to \infty \). The following theorem appearing in [10,4] has been referred to as the \textit{Poppy-seed Bagel Theorem} because of its interpretation for distributing points on a torus.
Theorem 2.1. Let $d \in \mathbb{N}$ and $A \subset \mathbb{R}^m$ be an infinite compact $d$-rectifiable set. Then for $p > d$ we have
\[
\lim_{n \to \infty} \frac{\mathcal{E}_p(A; n)}{n^{1 + p/d}} = \frac{C_{p,d}}{\mathcal{H}_d(A)^{p/d}},
\] (2.4)
where $C_{p,d}$ is a finite positive constant (independent of $A$ and $m$) and $\mathcal{H}_d(\cdot)$ denotes the $d$-dimensional Hausdorff measure in $\mathbb{R}^m$ normalized so that an embedded $d$-dimensional unit cube has measure 1.

By a $d$-rectifiable set we mean the Lipschitz image of a bounded set in $\mathbb{R}^d$.

In [13, Theorem 3.1] it is shown that $C_{p,1}$ can be expressed in terms of the classical Riemann zeta function; namely $C_{p,1} = 2\zeta(p)$. For $d \geq 2$ the precise value of $C_{p,d}$ is not known. The significance (and difficulty) of determining $C_{p,d}$ is deeply rooted in its connection to densest sphere packings in $\mathbb{R}^d$. For $d = 2$ it is conjectured in [11] that $C_{p,2} = (\sqrt{3}/2)^{p/2}\zeta_L(p)$, where $L$ denotes the planar hexagonal lattice of points $m(1,0) + n(1/2, \sqrt{3}/2)$, $m, n \in \mathbb{Z}$, and $\zeta_L$ is the Epstein zeta function $\zeta_L(p) := \sum_{X \in L, X \neq 0} |X|^{-p}$.

Concerning lower estimates for $C_{p,d}$, it follows from [6, Proposition 4] that, for $p > d \geq 2$ and $\frac{1}{2}(p - d)$ not an integer,
\[
C_{p,d} \geq \frac{d\pi^{p/2}}{p - d} \left( \frac{\Gamma\left(1 + \frac{p-d}{2}\right)}{\Gamma\left(1 + \frac{p}{2}\right)} \right)^{p/d}.
\] (2.5)

For the case $p = d$, the minimum $p$-energy grows like $n^2 \log n$. The following result is given in [10].

Theorem 2.2. Let $d \in \mathbb{N}$ and $A$ be an infinite compact subset of a $d$-dimensional $C^1$-manifold embedded in $\mathbb{R}^m$. Then
\[
\lim_{n \to \infty} \frac{\mathcal{E}_d(A; n)}{n^2 \log n} = \frac{\beta_d}{\mathcal{H}_d(A)},
\]
where $\beta_d$ is the volume of the $d$-dimensional unit ball.

For the case when $0 < p < d := \dim(A)$, the Hausdorff dimension of $A$, a theorem from classical potential theory (cf., e.g. [12]) asserts that
\[
\lim_{n \to \infty} \frac{\mathcal{E}_p(A; n)}{n^2} = W_p(A),
\] (2.6)
where $W_p(A)$ is the so-called Wiener constant defined by
\[
W_p(A) := \inf \iint \frac{1}{|x - y|^p} \, d\mu(x) \, d\mu(y),
\]
the infimum being taken over all Borel probability measures $\mu$ supported on $A$.

From the above results and observations we immediately obtain the following.

Theorem 2.3. If $A \subset \mathbb{R}^m$ is an infinite compact set, then
\[
M_n^p(A) \geq \frac{1}{n-1} \mathcal{E}_p(A; n), \quad n \geq 2.
\] (2.7)
Let $d \in \mathbb{N}$. If $A$ is $d$-rectifiable, then

$$\liminf_{n \to \infty} \frac{M^p_n(A)}{n^{p/d}} \geq \frac{C_{p,d}}{\mathcal{H}_d(A)^{p/d}}, \quad p > d,$$

where the constant $C_{p,d}$ is given in Theorem 2.1.

If $A$ is any infinite compact subset of a $d$-dimensional $C^1$-manifold, then

$$\liminf_{n \to \infty} \frac{M^d_n(A)}{n \log n} \geq \frac{\beta_d}{\mathcal{H}_d(A)}, \quad p = d. \tag{2.9}$$

If $A$ is any infinite compact subset of $\mathbb{R}^m$, then

$$\mathcal{M}^p(A) = \lim_{n \to \infty} \frac{M^p_n(A)}{n} \geq W_p(A), \quad 0 < p < d = \dim(A). \tag{2.10}$$

We remark that inequality (2.7) appears in [7,8]. Also, as previously mentioned, the inequality (2.10) is proved in [17]. Moreover, it follows from [7, Theorem 11] that equality holds in (2.10) whenever the maximum principle is satisfied on $A$ for Riesz potentials having kernel $K(x, y) = |x - y|^{-p}$.

Regarding upper bounds for $M^p_n(A)$, standard arguments (see Section 5) yield the following.

**Theorem 2.4.** Let $A \subset \mathbb{R}^m$ be an infinite compact set. If $\mathcal{H}_d(A) > 0$, then there exists a constant $c_p > 0$ depending only on $p$ such that

$$M^p_n(A) \leq \frac{c_p}{p - d} n^{p/d}, \quad p > d, \quad n \geq 1, \tag{2.11}$$

and there exists an absolute constant $c_1 > 0$ such that

$$M^d_n(A) \leq c_1 n \log n, \quad p = d, \quad n \geq 2. \tag{2.12}$$

If there exists a probability measure $\mu_A$ supported on $A$ whose $p$-potential is bounded on $A$, say

$$\int \frac{1}{|x - y|^p} \, d\mu_A(y) \leq w_p, \quad x \in A,$$

then

$$M^p_n(A) \leq nw_p, \quad p > 0, \quad n \geq 1. \tag{2.13}$$

The essential property used in the proof of Theorem 2.4 given in Section 5 is that $A$ is upper $d$-regular with respect to a Borel probability measure $\mu$ supported on $A$; that is, there exists a positive constant $C_0$ such that for any open ball $B^m(x, r) \subset \mathbb{R}^m$ with center $x \in A$ and radius $r > 0$ there holds

$$\mu(B^m(x, r) \cap A) \leq C_0 r^d. \tag{2.14}$$

This property is a consequence of Frostman’s Lemma (see [14, Chapter 8]).
3. Polarization inequalities for the unit sphere and unit ball

Let
\[ S^d := \{ x \in \mathbb{R}^{d+1} : |x| = 1 \} \quad \text{and} \quad B^d := \{ x \in \mathbb{R}^d : |x| \leq 1 \}. \quad (3.1) \]

Utilizing the results of Section 2 together with the known facts (cf. [12]) that
\[
W_p(S^d) = \int \int \frac{1}{|x-y|^p} \, d\sigma_d(x) \, d\sigma_d(y) = 2^{d-p-1} \frac{\Gamma \left( \frac{d+1}{2} \right) \Gamma \left( \frac{d-p}{2} \right)}{\sqrt{\pi} \Gamma \left( d - \frac{p}{2} \right)}, \quad 0 < p < d, \quad (3.2)
\]
where \( \sigma_d \) denotes the normalized surface area on \( S^d \), and
\[
W_p(B^d) = \frac{\Gamma \left( \frac{d-p}{2} \right) \Gamma \left( \frac{p}{2} + 1 \right)}{\Gamma \left( \frac{d}{2} \right)}, \quad d - 2 \leq p < d, \quad p > 0, \quad (3.3)
\]
we shall prove the following two theorems.

**Theorem 3.1.** For the sphere \( S^d, \ d \geq 2 \), we have
\[
\liminf_{n \to \infty} \frac{M_n^p(S^d)}{n^{p/d}} \geq C_{p,d} \left( \frac{\Gamma \left( \frac{d+1}{2} \right)}{2\pi^{(d+1)/2}} \right)^{p/d}, \quad p > d; \quad (3.4)
\]
\[
\lim_{n \to \infty} \frac{M_n^p(S^d)}{n} = \frac{1}{d} \frac{\Gamma \left( \frac{d+1}{2} \right)}{\sqrt{\pi} \Gamma \left( \frac{d}{2} \right)} =: \tau_d, \quad p = d; \quad (3.5)
\]
\[
\lim_{n \to \infty} \frac{M_n^p(S^d)}{n^2} = 2^{d-p-1} \frac{\Gamma \left( \frac{d+1}{2} \right) \Gamma \left( \frac{d-p}{2} \right)}{\sqrt{\pi} \Gamma \left( d - \frac{p}{2} \right)}, \quad 0 < p < d. \quad (3.6)
\]
Furthermore, the following upper estimates hold for all \( n \geq 3 \).
\[
M_n^p(S^d) \leq \begin{cases} 
\left( \frac{np \tau_d}{p-d} \right)^{p/d}, & p > d, \\
n[\log n + \log(\log n) + \log(2^d \tau_d)] \frac{1}{1 - (\log n)^{-1}}, & p = d, \\
n2^{d-p-1} \frac{\Gamma \left( \frac{d+1}{2} \right) \Gamma \left( \frac{d-p}{2} \right)}{\sqrt{\pi} \Gamma \left( d - \frac{p}{2} \right)}, & 0 < p < d. 
\end{cases} \quad (3.7)
\]

**Theorem 3.2.** For the unit ball \( B^d \), we have
\[
\liminf_{n \to \infty} \frac{M_n^p(B^d)}{n^{p/d}} \geq C_{p,d} \left( \frac{\Gamma \left( \frac{d}{2} + 1 \right)}{\pi^{d/2}} \right)^{p/d}, \quad p > d; \quad (3.8)
\]
\[
\lim_{n \to \infty} \frac{M_p^n(B^d)}{n^\frac{1}{2} \log n} = 1, \quad p = d; \quad (3.9)
\]
\[
\lim_{n \to \infty} \frac{M_p^n(B^d)}{n} = \frac{\Gamma\left(\frac{d-p}{2}\right) \Gamma\left(\frac{p}{2} + 1\right)}{\Gamma\left(\frac{d}{2}\right)}, \quad 0 \leq d - p < d, \quad p > 0; \quad (3.10)
\]
\[
\frac{M_p^n(B^d)}{n} = 1, \quad 0 < p \leq d - 2, \quad n = 1, 2, \ldots. \quad (3.11)
\]

Furthermore, the following upper estimates hold for all \( n \geq 3 \):
\[
M_p^n(B^d) \leq \begin{cases} 
\left(\frac{pn}{p - d}\right)^{p/d}, & p > d, \\
\frac{n[\log n + \log(\log n) + d \log 2]}{1 - (\log n)^{-1}}, & p = d, \\
\frac{n \Gamma\left(\frac{d-p}{2}\right) \Gamma\left(\frac{p}{2} + 1\right)}{\Gamma\left(\frac{d}{2}\right)}, & d - 2 < p < d, \quad p > 0.
\end{cases} \quad (3.12)
\]

**Remark 1.** It is easily seen that for \( p > d \) and \( n \geq 2^d \), we have \( M_p^n(B^d) \geq 4^{-p} n^{p/d} \). Indeed, let \( \{x_1, x_2, \ldots, x_m\} \) be a maximal \( \delta \)-net in \( B^d \) with \( \delta := 4n^{-1/d} \). Then
\[
m \beta_d(\delta/2)^d \leq \beta_d(1 + \delta/2)^d.
\]
so
\[
m \leq \left(\frac{1 + \delta/2}{\delta/2}\right)^d \leq \left(\frac{4}{\delta}\right)^d \leq n.
\]

Also, for every \( x \in B^d \), there is an \( x_k \in \{x_1, x_2, \ldots, x_m\} \) such that \( |x - x_k| \leq \delta \). Therefore,
\[
\sum_{j=1}^{m} |x - x_j|^{-p} \geq |x - x_k|^{-p} \geq \delta^{-p} = 4^{-p} n^{p/d}.
\]

Observe further that for the case \( 0 < p < d \), we have \( M_p^n(B^d) \geq n \) since we can take all the points \( x_j \) equal to \( 0 \), the center of the unit ball \( B^d \), and, moreover, such points are optimal in the case when \( 0 < p \leq d - 2 \) (see the proof of (3.11) in Section 5).

**Remark 2.** For the case \( p > d \) the above theorems establish the asymptotically sharp order (namely \( n^{p/d} \)) but not the sharp coefficient for the unit sphere and unit ball. Note, however, from the lower estimates in (2.5), (3.4) and (3.8) that, for \( A = B^d \) or \( A = \mathbb{S}^d \), we have
\[
\lim_{p \to d^+} \left(\liminf_{n \to \infty} \frac{M_p^n(A)}{n^{p/d}}\right) = \infty.
\]
This is clearly consistent with the upper bounds provided in Theorems 3.1 and 3.2 for the case \( p > d \).

We conclude this section with the following conjectures, which would be analogs of Theorems 2.1 and 2.2.
**Conjecture 1.** Let $p > d$ and $m \geq d$, where $p$ and $m$ are integers. We conjecture that for every infinite compact $d$-rectifiable set $A$ in $\mathbb{R}^m$, there should hold
\[
\lim_{n \to \infty} \frac{M_p^n(A)}{n^{p/d}} = \frac{\sigma_{p,d}}{\mathcal{H}_d(A)^{p/d}},
\]
where $\sigma_{p,d}$ is a positive and finite constant independent of $A$ and $m$.

We further conjecture that if $A$ is $d$-rectifiable with $\mathcal{H}_d(A) > 0$, then any sequence $\{\omega_n^*\}_{n=2}^\infty$ of $p$-polarization maximizing configurations on $A$ is asymptotically uniformly distributed on $A$ with respect to $\mathcal{H}_d$.

In particular, (1.4) implies that the constant $\sigma_{p,1}$ appearing in this conjecture would have to equal $2(2^p - 1)\zeta(p)$.

**Conjecture 2.** Let $d \in \mathbb{N}$ and $A$ be an infinite compact subset of a $d$-dimensional $C^1$-manifold embedded in $\mathbb{R}^m$. Then we conjecture that
\[
\lim_{n \to \infty} \frac{M_p^n(A)}{n \log n} = \frac{\beta_d}{\mathcal{H}_d(A)},
\]
where $\beta_d$ is the volume of the $d$-dimensional unit ball.

The results of this section assert that (3.14) holds for spheres and balls.

### 4. Polarization on the unit circle

In this section we explore some connections between polynomial inequalities and the polarization inequality recently proved in [9]. Let $g$ be a positive-valued even function defined on $\mathbb{R} \setminus (2\pi \mathbb{Z})$ that is periodic with period $2\pi$. We denote by $\Omega_n$ the collection of all sets
\[
\omega_n := \{t_1 < t_2 < \cdots < t_n\} \subset [0, 2\pi)
\]
and put
\[
\tilde{\omega}_n := \{\tilde{t}_1 < \tilde{t}_2 < \cdots < \tilde{t}_n\} \subset [0, 2\pi)
\]
with
\[
\tilde{t}_j := 2(j - 1)\pi/n, \quad j = 1, 2, \ldots, n.
\]

We introduce the notation
\[
P_{\omega_n}(t) := \sum_{j=1}^n g(t - t_j), \quad P_{\tilde{\omega}_n}(t) := \sum_{j=1}^n g(t - \tilde{t}_j).
\]

In [9] the following theorem is proved.

**Theorem 4.1.** Let $g$ be a positive-valued even function defined on $\mathbb{R} \setminus (2\pi \mathbb{Z})$ that is periodic with period $2\pi$. If $g$ is non-increasing and strictly convex on $(0, \pi]$, then
\[
\max_{\omega_n \in \Omega_n} \left\{ \min_{t \in [-\pi, \pi]} P_{\omega_n}(t) \right\} = P_{\tilde{\omega}_n}(\pi/n).
\]
In fact, a closer look at the proof of the main result in [9] shows that the following Riesz lemma type improvement also holds.

**Theorem 4.2.** Let $g$ be a positive-valued even function defined on $\mathbb{R} \setminus (2\pi \mathbb{Z})$ that is periodic with period $2\pi$. If $g$ is non-increasing and strictly convex on $(0, \pi]$, then there is a number $\gamma \in [0, 2\pi)$ (depending on $\omega_n$) such that

$$P_{\omega_n}(t) \leq P_{\omega_n}(t - \gamma), \quad t \in (\gamma, \gamma + 2\pi/n),$$

for every $\omega_n \in \Omega_n$.

A consequence of Theorem 4.2 is the following discrete version of Theorem 4.1.

**Theorem 4.3.** Let $g$ be a positive-valued even function defined on $\mathbb{R} \setminus (2\pi \mathbb{Z})$ that is periodic with period $2\pi$. If $g$ is non-increasing and strictly convex on $(0, \pi]$, then

$$\max_{\omega_n \in \Omega_n} \left\{ \min_{t \in \omega_{2n}} P_{\omega_n}(t) \right\} = P_{\omega_n}(\pi/(2n)),$$

and equality holds when $\omega_n = \omega^*_n = \{t_1^* < t_2^* < \cdots < t_n^*\}$ with

$$t_j^* = \frac{\pi}{2n} + \frac{2(j - 1)\pi}{n}, \quad j = 1, 2, \ldots, n.$$

**Proof.** Let $\gamma$ be the number guaranteed by Theorem 4.2. Observe that $\omega_{2n}$ has exactly two points in the interval $(\gamma, \gamma + 2\pi/n)$ (mod $2\pi$). Denote these points by $\alpha$ and $\beta = \alpha + \pi/n$. Due to the fact that $P_{\omega_n}$ is non-increasing on $(0, \pi/n)$ and

$$P_{\omega_n}(t) = P_{\omega_n}(2\pi/n - t), \quad t \in (0, 2\pi/n),$$

we have

$$\min\{P_{\omega_n}(\alpha - \gamma), P_{\omega_n}(\beta - \gamma)\} \leq P_{\omega_n}(\pi/(2n)),$$

which finishes the proof of the inequality of the theorem. The fact that equality holds in the case described in the theorem is obvious. □

Associated with $\omega_n := \{t_1 < t_2 < \cdots < t_n\} \subset [0, 2\pi)$ let

$$Q_{\omega_n}(t) := \prod_{j=1}^{n} \sin \left( \frac{t - t_j}{2} \right).$$

Let

$$T_n(t) := Q_{\omega_n}(t) = 2^{1-n} \sin \left( \frac{nt}{2} \right).$$

Our next three theorems are consequences of Theorems 4.2 and 4.3.

**Theorem 4.4.** There is a number $\gamma \in [0, 2\pi)$ (depending on $\omega_n$) such that

$$-(\log |Q_{\omega_n}|^{(m)}(t)) \leq -(\log |T_n|^{(m)}(t)), \quad t \in (\gamma, \gamma + 2\pi/n),$$

for every $\omega_n \in \Omega_n$ and for every even integer $m > 0$. 
Theorem 4.5. Let

\[ E(\omega_n) := [0, 2\pi) \setminus \bigcup_{j=1}^{n} (t_j - \pi/n, t_j + \pi/n) \mod 2\pi. \]

We have

\[
\max_{\omega_n \in \mathcal{L}_n} \left\{ \min_{t \in E(\omega_n)} - (\log |Q_{\omega_n}|)^{(m)}(t) \right\} = - (\log |T_n|)^{(m)}(\pi/n)
\]

for every even integer \( m > 0 \).

Theorem 4.6. We have

\[
\max_{\omega_n \in \mathcal{L}_n} \left\{ \min_{t \in \omega_n} - (\log |Q_{\omega_n}|)^{(m)}(t) \right\} = - (\log |T_n|)^{(m)}(\pi/(2n)),
\]

for every even integer \( m > 0 \), and equality holds when \( \omega_n = \omega_n^* = \{ t_1^*, t_2^* < \cdots < t_n^* \} \) with

\[ t_j^* = \frac{\pi}{2n} + \frac{2(j - 1)\pi}{n}, \quad j = 1, 2, \ldots, n. \]

Proof of Theorem 4.4. For the sake of brevity let \( Q := Q_{\omega_n}(t) \). Let \( t \not\equiv \omega_n \mod 2\pi \). We have

\[
(\log |Q|)''(t) = \left( \frac{Q'}{Q} \right)'(t) = \frac{d}{dt} \left( \frac{1}{2} \sum_{j=1}^{n} \cot \left( \frac{t - t_j}{2} \right) \right) = -\frac{1}{4} \sum_{j=1}^{n} \csc^2 \left( \frac{t - t_j}{2} \right),
\]

and hence

\[-(\log |Q|)^{(m)}(t) = \frac{1}{4} \sum_{j=1}^{n} f^{(m-2)}(t - t_j) = \sum_{j=1}^{n} g_m(t - t_j), \]

where \( f(t) := \csc^2(t/2) \) and \( g_m(t) := \frac{1}{4} f^{(m-2)}(t) \). It is well known and elementary to check that

\[ \tan t = \sum_{j=1}^{\infty} a_j t^j, \quad t \in (-\pi/2, \pi/2), \]

with each \( a_j \geq 0, \quad j = 1, 2, \ldots \). Hence, if \( h(t) = \tan(t/2) \), then

\[ h^{(k)}(t) > 0, \quad t \in (0, \pi), \quad k = 0, 1, \ldots. \]

Now observe that

\[ f(t) = \csc^2 \left( \frac{t}{2} \right) = \sec^2 \left( \frac{\pi/2 - t}{2} \right) = 2h'(\pi - t), \]

and hence,

\[ (-1)^k f^{(k)}(t) = 2h^{(k+1)}(\pi - t) > 0, \quad t \in (0, \pi). \]

This implies that if \( m > 0 \) is an even integer, \( g_m(t) = \frac{1}{4} f^{(m-2)}(t) \) is a positive, decreasing, strictly convex function on \((0, \pi)\). It is also clear that if \( m \) is even, then \( g_m \) is even since \( f \) is
even. Now we can apply Theorem 4.2 to deduce that there is a number \( \gamma \in [0, 2\pi) \) (depending on \( \omega_n \)) such that

\[-(\log |Q_{\omega_n}|)^{(m)}(t) = \sum_{j=1}^{n} g_{m}(t - t_j) \leq -(\log |T_n|)^{(m)}(t), \quad t \in [\gamma, \gamma + 2\pi/n),\]

and the proof is finished. \( \square \)

**Proof of Theorem 4.5.** The theorem follows from Theorem 4.4 immediately. \( \square \)

**Proof of Theorem 4.6.** We use the notation and the observations in the proof of Theorem 4.4. However, at the end of the proof we use Theorem 4.3 to deduce that

\[\min_{t \in \omega_n} Q_{\omega_n}(t) \leq T_n(\pi/(2n)),\]

and equality holds when \( Q_{\omega_n} = T_n \). \( \square \)

We conclude this section by giving an independent proof of the unit circle polarization conjecture in [2] for the case \( p = 4 \), where we show that, for \( z_1, z_2, \ldots, z_n \in S^1 \), a “good polarization point” \( z_0 \in S^1 \) can be chosen so that

\[\prod_{j=1}^{n} |z_0 - z_j| = \max_{z \in S^1} \prod_{j=1}^{n} |z - z_j|. \quad (4.1)\]

**Theorem 4.7.** If \( z_1, z_2, \ldots, z_n \in S^1 \), then

\[\min_{z \in S^1} \sum_{j=1}^{n} \frac{1}{|z - z_j|^4} \leq \frac{n^4}{48} + \frac{n^2}{24}, \quad n \geq 1,\]

and equality holds when the points \( z_j \) are distinct and equally spaced on \( S^1 \); that is, (1.5) holds. Moreover, if \( z_1, z_2, \ldots, z_n \in S^1 \), and \( z_0 \in S^1 \) is chosen so that (4.1) holds, then

\[\sum_{j=1}^{n} \frac{1}{|z_0 - z_j|^4} \leq \frac{n^4}{48} + \frac{n^2}{24}, \quad n \geq 1.\]

This result naturally suggests the following open question.

**Problem.** For what values of \( p \in (0, \infty) \) is it true that

\[\sum_{j=1}^{n} \frac{1}{|z_0 - z_j|^p} \leq M_n^p(S_1)\]

whenever \( z_1, z_2, \ldots, z_n \in S_1 \) and \( z_0 \in S_1 \) satisfies (4.1)?

In addition to the value \( p = 4 \), a closer look at the main result in [2] shows that \( p = 2 \) is also such a value.

**Proof of Theorem 4.7.** Write \( z_j = e^{it_j}, t_j \in [0, 2\pi), j = 1, 2, \ldots, n, \) and set

\[Q_n(t) := \prod_{j=1}^{n} \sin \left( \frac{t - t_j}{2} \right).\]
Then $H_n$ defined by $H_n(t) := Q_n(2t)$ is a real trigonometric polynomial of degree $n$. We have the following identities:

$$
\frac{Q_n'(t)}{Q_n(t)} = \frac{1}{2} \sum_{j=1}^{n} \cot \left( \frac{t - t_j}{2} \right),
$$

$$
\left( \frac{Q_n'}{Q_n} \right)'(t) = -\frac{1}{4} \sum_{j=1}^{n} \csc^2 \left( \frac{t - t_j}{2} \right) = -\frac{1}{4} \sum_{j=1}^{n} \sin^{-2} \left( \frac{t - t_j}{2} \right),
$$

$$
\left( \frac{Q_n'}{Q_n} \right)''(t) = \frac{1}{4} \sum_{j=1}^{n} \cos \left( \frac{t - t_j}{2} \right) \sin^{-3} \left( \frac{t - t_j}{2} \right),
$$

$$
\left( \frac{Q_n'}{Q_n} \right)'''(t) = \frac{1}{4} \sum_{j=1}^{n} \left( \sin^{-2} \left( \frac{t - t_j}{2} \right) - \frac{3}{2} \sin^{-4} \left( \frac{t - t_j}{2} \right) \right),
$$

so

$$
\frac{3}{8} \sum_{j=1}^{n} \sin^{-4} \left( \frac{t - t_j}{2} \right) = -\left( \frac{Q_n'}{Q_n} \right)'''(t) - \left( \frac{Q_n'}{Q_n} \right)'(t).
$$

On the other hand,

$$
\left( \frac{Q_n'}{Q_n} \right)''' = \frac{Q_n^{(4)}}{Q_n} - 3 Q_n'' \left( \frac{Q_n'}{Q_n} \right) - 3 Q_n'' \left( \frac{Q_n Q_n' - 2 Q_n Q_n' Q_n'}{Q_n^4} \right) + Q_n' \left( \frac{1}{Q_n} \right)''
$$

and

$$
\left( \frac{Q_n'}{Q_n} \right)' = \frac{Q_n''}{Q_n} - \left( \frac{Q_n'}{Q_n} \right)^2.
$$

Hence

$$
\left( \frac{Q_n'}{Q_n} \right)'''(t_0) = \frac{Q_n^{(4)}}{Q_n}(t_0) - 3 \left( \frac{Q_n''}{Q_n} \right)^2(t_0) \quad \text{and} \quad \left( \frac{Q_n'}{Q_n} \right)'(t_0) = \frac{Q_n''}{Q_n}(t_0)
$$

at every point $t_0$ such that $Q_n'(t_0) = 0$. So if $z_0 = e^{it_0} \in S^1$ is chosen so that

$$
|Q_n(t)| = \max_{t \in [-\pi, \pi]} |Q_n(t)|,
$$

then

$$
6 \sum_{j=1}^{n} \frac{1}{|z_0 - z_j|^4} = \left( 3 \left( \frac{Q_n''}{Q_n} \right)^2 - \frac{Q_n^{(4)}}{Q_n} - \frac{Q_n''}{Q_n} \right)(t_0)
$$

$$
= \left( \frac{3}{16} \left( \frac{H_n''}{H_n} \right)^2 - \frac{1}{16} H_n^{(4)} - \frac{1}{4} H_n'' \right) \left( \frac{t_0}{2} \right).
$$

Without loss of generality we may assume that $t_0 = 0$ and $z_0 = 1$.

Set

$$
F(H_n) := \left( \frac{3}{16} \left( H_n'' \right)^2 - \frac{1}{16} H_n^{(4)} - \frac{1}{4} H_n'' \right)(0)
$$
and let $\mathcal{A}_n$ be the set of all real trigonometric polynomials $H_n$ of degree at most $n$ such that

$$H_n(0) = \max_{t \in [-\pi, \pi]} |H_n(t)| = 1.$$ 

A simple compactness argument shows that there is a $\tilde{H}_n \in \mathcal{A}_n$ such that

$$F(\tilde{H}_n) = \sup_{H_n \in \mathcal{A}_n} F(H_n).$$

Let

$$\tilde{U}_n(t) := \frac{1}{2}(\tilde{H}_n(t) + \tilde{H}_n(-t)).$$

Then $\tilde{U}_n \in \mathcal{A}_n$ is even and $F(\tilde{U}_n) = F(\tilde{H}_n)$. Since $\tilde{U}_n \in \mathcal{A}_n$ is even, it is of the form

$$\tilde{U}_n(t) = \tilde{P}_n(\cos t)$$

for a $\tilde{P}_n \in \mathcal{P}_n$ satisfying

$$\tilde{P}_n(1) = \max_{x \in [-1, 1]} |\tilde{P}_n(x)| = 1,$$

where $\mathcal{P}_n$ denotes the set of all real algebraic polynomials of degree at most $n$.

Observe that $U_n \in \mathcal{A}_n$ is even if and only if it is of the form

$$U_n(t) = P_n(\cos t)$$

for a $P_n \in \mathcal{P}_n$ satisfying

$$P_n(1) = \max_{x \in [-1, 1]} |P_n(x)| = 1.$$

A simple calculation shows that

$$U_n(0) = P_n(1), \quad U_n''(0) = -P_n'(1), \quad U_n^{(4)}(0) = 3P_n''(1) + P_n'(1).$$

Let

$$G(P_n) := F(U_n) = \left(\frac{3}{16}(U_n'')^2 - \frac{1}{16}U_n^{(4)} - \frac{1}{4}U_n''\right)(0)$$

$$= \frac{3}{16}((P_n')^2 - P_n'' + P_n')(1).$$

We have

$$G(P_n) = F(U_n) \leq F(\tilde{H}_n) = F(\tilde{U}_n) = G(\tilde{P}_n)$$

for every $P_n \in \mathcal{P}_n$ such that

$$P_n(1) = \max_{x \in [-1, 1]} |P_n(x)| = 1.$$

Next we show by a simple variational method that $\tilde{P}_n \in \mathcal{P}_n$ equioscillates between $-1$ and $1$ at least $n$ times on $[-1, 1]$. That is, there are

$$-1 \leq y_n < y_{n-1} < \cdots < y_1 = 1$$

such that

$$\tilde{P}_n(y_j) = (-1)^{j-1}, \quad j = 1, 2, \ldots, n.$$
To show this, first we observe that \( \tilde{P}_n'(1) > 0 \) since \( \tilde{P}_n'(1) \geq 0 \), and Markov’s inequality for the second derivative (see p. 249 of [5]) together with \( \tilde{P}_n'(1) = 0 \) would imply that

\[
G(\tilde{P}_n) = \frac{3}{16}((\tilde{P}_n')^2 - \tilde{P}_n'' + \tilde{P}_n')(1) = \frac{-3}{16} \tilde{P}_n''(1)
\]

\[
\leq \frac{3}{16} T_n''(1) = \frac{3}{16} \frac{n^2(n^2 - 1)}{3} < \frac{1}{16}(2n^4 + 4n^2) = G(T_n),
\]

where \( T_n \) is the Chebyshev polynomial of degree \( n \) defined by \( T_n(\cos t) = \cos(nt) \), and this contradicts the extremal property of \( \tilde{P}_n \). Now let

\[
E := \{ y \in [-1, 1] : |\tilde{P}(y)| = 1 \}.
\]

We list the elements of \( E \) as

\[
E = \{ 1 = y_1 > y_2 > \cdots > y_m \},
\]

where

\[
\tilde{P}_n(y_{k_j}) = \tilde{P}_n(y_{k_j+1}) = \cdots = \tilde{P}_n(y_{k_j+1-1}), \quad j = 0, 1, \ldots, m - 1,
\]

and

\[
\tilde{P}_n(y_{k_j}) = -\tilde{P}_n(y_{k_j-1}) = (-1)^j, \quad j = 1, 2, \ldots, m - 1,
\]

for some

\[
1 = k_0 < k_1 < \cdots < k_m = \mu + 1.
\]

Now we pick

\[
\alpha_j \in (y_{k_j}, y_{k_j-1}), \quad j = 1, 2, \ldots, m - 1.
\]

Assume that \( m \leq n - 1 \). For the polynomial \( R_n \in \mathcal{P}_n \) defined by

\[
R_n(x) := (x - 1)^2 \prod_{j=1}^{m-1} (x - \alpha_j)
\]

we have

\[
R_n(y) \tilde{P}_n(y) > 0, \quad y \in E \setminus \{1\},
\]

\[
R_n(1) = R_n'(1) = 0 \quad \text{and} \quad R_n''(1) > 0.
\]

These properties together with \( \tilde{P}_n'(1) > 0 \) imply that for a sufficiently small value of \( \varepsilon > 0 \) the polynomial

\[
S_n = \tilde{P}_n - \varepsilon R_n \in \mathcal{P}_n
\]

satisfies

\[
S_n(1) = \max_{x \in [-1, 1]} |S_n(x)| = 1
\]

and \( G(S_n) > G(\tilde{P}_n) \), so \( S_n \in \mathcal{P}_n \) contradicts the extremal property of \( \tilde{P}_n \). This finishes the proof of the fact that \( \tilde{P}_n \in \mathcal{P}_n \) equioscillates between \(-1\) and \(1\) at least \( n \) times on \([-1, 1]\), as we claimed.
As a consequence, the Intermediate Value Theorem implies that \( \widetilde{P}_n \) has at least \( n - 1 \) zeros in \((-1, 1)\), say

\[-1 < x_{n-1} < x_{n-2} < \cdots < x_1 < 1.
\]

Observe that the polynomial \( \widetilde{P}_n \in P_n \) has an odd number of zeros (by counting multiplicities) in each of the intervals \((y_{j+1}, y_j)\) for \( j = 1, 2, \ldots, n - 1 \); hence \( x_j \) is the only (simple) zero of \( \widetilde{P}_n \) in \((y_{j+1}, y_j)\) for each \( j = 1, 2, \ldots, n - 1 \). Therefore \( \widetilde{P}_n \) has only real zeros and it is of the form

\[ \widetilde{P}_n(x) = c \prod_{j=1}^{\mu} (x - x_j) \]

with either \( \mu = n - 1 \) or \( \mu = n \), and in the case \( \mu = n \) we have \( x_n \in \mathbb{R} \setminus [y_n, 1] \).

Note that

\[ \frac{\widetilde{P}_n'}{\widetilde{P}_n}(x) = \sum_{j=1}^{\mu} \frac{1}{x - x_j}, \quad \left( \frac{\widetilde{P}_n'}{\widetilde{P}_n}(x) \right)' = -\sum_{j=1}^{\mu} \frac{1}{(x - x_j)^2}, \]

and

\[ G(\widetilde{P}_n) = \frac{3}{16} \left( \frac{(\widetilde{P}_n')^2 - \widetilde{P}_n'' \widetilde{P}_n}{(\widetilde{P}_n)^2} + \frac{\widetilde{P}_n'}{\widetilde{P}_n} \right)(1) = \frac{3}{16} \left( \sum_{j=1}^{\mu} \frac{1}{(1 - x_j)^2} + \sum_{j=1}^{\mu} \frac{1}{1 - x_j} \right). \]

If \( \mu = n - 1 \), then \( \widetilde{P}_n \) equioscillates between \(-1\) and \(1\) on \([-1, 1]\) the maximum number of times, so \( \widetilde{P}_n \equiv T_{n-1} \), where \( T_{n-1} \) is the Chebyshev polynomial of degree \( n - 1 \) defined by \( T_{n-1}(\cos t) = \cos((n - 1)t) \). Hence

\[ G(\widetilde{P}_n) = \frac{3}{16} \left( \sum_{j=1}^{\mu} \frac{1}{(1 - x_j)^2} + \sum_{j=1}^{\mu} \frac{1}{1 - x_j} \right) = \frac{3}{16} \left( \frac{(T_{n-1}')^2 - T_{n-1}'' T_{n-1}}{T_{n-1}^2} + \frac{T_{n-1}'}{T_{n-1}} \right)(1) = \frac{3}{16} \left( (n - 1)^4 - \frac{(n - 1)^2((n - 1)^2 - 1)}{3} + (n - 1)^2 \right) = \frac{1}{8}(n - 1)^4 + \frac{1}{4}(n - 1)^2. \]

If \( \mu = n \) we must have \( x_n \in (\infty, y_n) \cup (1, \infty) \). However, \( 1 < x_n \) would imply that

\[ Y_n(x) := -c(x - (2 - x_n)) \prod_{j=1}^{n-1} (x - x_j) \]

satisfies

\[ Y_n(1) = \max_{x \in [-1, 1]} |Y_n(x)| = 1 \quad \text{and} \quad G(Y_n) = G(\widetilde{P}_n), \]

and hence \( Y_n \in P_n \) also shares the extremal property of \( \widetilde{P}_n \) while it has all its zeros in \((\infty, 1)\). Hence \( x_n < y_n < x_{n-1} \). But then \( \widetilde{P}_n \) is just the Chebyshev polynomial \( T_n \) transformed linearly.
from the interval \([-1, 1]\) to \([\eta, 1]\) for some \(\eta \leq -1\). This implies that
\[
G(\tilde{P}_n) = \frac{3}{16} \left( \sum_{j=1}^{\mu} \frac{1}{(1-x_j)^2} + \sum_{j=1}^{\mu} \frac{1}{1-x_j} \right)
= \frac{3}{16} \left( \left( \frac{2}{1-\eta} \right)^2 \frac{(T'_n)^2}{T_n^2} - \frac{2}{1-\eta} \frac{T'_n}{T_n} \right)
\leq \frac{3}{16} \left( n^4 - \frac{n^2(n^2-1)}{3} + n^2 \right) = \frac{1}{8} n^4 + \frac{1}{4} n^2.
\]

Now we conclude that
\[
G(\tilde{P}_n) \leq G(T_n) = \frac{1}{8} n^4 + \frac{1}{4} n^2,
\]
and hence
\[
F(\tilde{H}_n) = G(\tilde{P}_n) \leq G(T_n) = \frac{1}{8} n^4 + \frac{1}{4} n^2.
\]
Therefore
\[
6 \sum_{j=1}^{n} \frac{1}{|z_0 - z_j|^4} = F(H_n) \leq F(\tilde{H}_n) \leq G(T_n) = \frac{1}{8} n^4 + \frac{1}{4} n^2,
\]
and this completes the proof.  

We conclude this section by mentioning two formulas that may be useful for future investigation of the polarization problem for the unit circle. Let
\[
A_p(t) := \sum_{j=1}^{n} \frac{1}{|e^{it} - z_j|^p}, \quad p > 0,
\]
where \(z_j = e^{it_j} \in S^1, \ j = 1, 2, \ldots, n\). Then a straightforward calculation yields the following:
\[
A_2(t) = \frac{(Q'_n(t))^2 - Q''_n(t)Q_n(t)}{(Q_n(t))^2} \quad \text{with} \quad Q_n(t) := \prod_{j=1}^{n} \sin \left( \frac{t - t_j}{2} \right),
\]
and
\[
A_{p+2}(t) = \frac{1}{p^2 + p} \left( A''_p(t) + \frac{p^2}{4} A_p(t) \right), \quad p > 0.
\]

5. Proofs of Theorems 2.4, 3.1 and 3.2

Proof of Theorem 2.4. We proceed with an argument similar to that in [11]. Let \(\omega_n = \{x_j\}_{j=1}^{n} \subset A\). Setting
\[
r_n := (2nC_0)^{-1/d}, \quad D_j := A \setminus B^m(x_j, r_n), \quad D := \bigcap_{j=1}^{n} D_j,
\]
we have from (2.14) that
\[
\mu(D) \geq 1 - \sum_{j=1}^{n} \mu(B^m(x_j, r_n) \cap A) \geq 1 - nC_0 r_n^d = \frac{1}{2}.
\]
Thus, for
\[ f_n(x) := \sum_{j=1}^{n} |x - x_j|^{-p}, \]
we obtain
\begin{equation}
M^p(\omega_n, A) \leq \frac{1}{\mu(D)} \int_D f_n(x) \, d\mu(x) \leq 2 \sum_{j=1}^{n} \int_{D_j} |x - x_j|^{-p} \, d\mu(x). \tag{5.1}
\end{equation}

Next, we bound the integrals over \( D_j \) utilizing (2.14):
\begin{align*}
\int_{D_j} |x - x_j|^{-p} \, d\mu(x) &= \int_{0}^{\infty} \mu\{x \in D_j : |x - x_j|^{-p} > t\} \, dt \\
&\leq 1 + \int_1^{r_n^{-p}} \mu(B^m(x_j, t^{-1/p}) \cap A) \, dt \\
&\leq 1 + C_0 \int_1^{r_n^{-p}} \frac{1}{t^{d/p}} \, dt,
\end{align*}
where we assume that \( n \) is sufficiently large so that \( r_n^{-p} > 1 \). Thus from (5.1) it follows that
\begin{equation}
M^p(\omega_n, A) \leq 2n \left( 1 + C_0 \int_1^{r_n^{-p}} \frac{1}{t^{d/p}} \, dt \right). \tag{5.2}
\end{equation}
Consequently, for \( p > d \) we get
\begin{equation}
M^p(\omega_n, A) \leq 2n \left( 1 + C_0 \frac{p}{p - d} [r_n^{d-p} - 1] \right) \leq \frac{c_p}{p - d} n^{p/d} \tag{5.3}
\end{equation}
and for \( p = d \) we obtain
\begin{equation}
M^d(\omega_n, A) \leq 2n[1 + C_0 \log(r_n^{-d})] = 2n[1 + C_0 \log(2nC_0)] \leq c_1 n \log n. \tag{5.4}
\end{equation}
This completes the proof of parts (2.11) and (2.12) of Theorem 2.4, while (2.13) follows immediately upon integration of \( f_n(x) \) with respect to \( d\mu_A \). \( \square \)

**Proof of Theorem 3.1.** Inequality (3.4) is an immediate consequence of (2.8), while Eq. (3.6) follows from (3.2), (2.10), and the last assertion in Theorem 2.4, since
\begin{equation}
\int |x - y|^{-p} \, d\sigma_d(y) = W_p(\mathbb{S}^d), \quad x \in \mathbb{S}^d, \quad p < d. \tag{5.5}
\end{equation}
To prove Eq. (3.5), we first note that from (2.9) we have
\begin{equation}
\liminf_{n \to \infty} \frac{M_n^p(\mathbb{S}^d)}{n \log n} \geq \frac{\beta_d}{\mathcal{H}_d(\mathbb{S}^d)} = \frac{1}{d} \frac{\Gamma\left(d + \frac{1}{2}\right)}{\sqrt{\pi} \Gamma\left(\frac{d}{2}\right)} = \tau_d.
\end{equation}
Hence, if we establish the upper estimate in (3.7) for \( p = d \), then (3.5) will follow. For this purpose, we refine the argument used in the proof of Theorem 2.4. With \( \mu = \sigma_d \), the following estimates are known for \( x \in \mathbb{S}^d \) (cf. [11]):
\begin{equation}
\sigma_d(B^{d+1}(x, r) \cap \mathbb{S}^d) \leq \tau_d r^d, \tag{5.6}
\end{equation}
and

\[
\int_{B^{d+1}(x, r)} |x - y|^{-d} \, d\sigma_d(y) = d \tau_d 2^{-d/2} \int_{-1}^{1} (1 - t)^{-1/2} (1 + t)^{d/2 - 1} \, dt \\
\leq d \tau_d \log(2/r),
\]

for 0 < r < 2. Utilizing these estimates and using (5.1) with \(r_n = (\tau_d n \log n)^{-1/d}, D_j = S^d \setminus B^{d+1}(x_j, r_n), \) and \(n \geq 3,\) we obtain

\[
M^d(\omega_n, A) \leq \frac{1}{1 - n \tau_d r_n^d} \sum_{j=1}^{n} \int_{D_j} |x - x_j|^{-d} \, d\sigma_d(x) \leq \frac{nd}{1 - n \tau_d r_n^d} \tau_d \log(2/r_n)
\]

\[
= \frac{nd}{1 - (\log n)^{-1}} \tau_d \left( \log 2 + \frac{1}{d} \log(\tau_d n \log n) \right)
\]

\[
= \frac{\tau_d}{1 - (\log n)^{-1}} n[\log n + \log(\log n) + \log(2 \tau_d)].
\]

This completes the proof of (3.5) as well as the upper bound in (3.7) for the case \(p = d.\)

It remains to establish (3.7) for the cases \(p < d\) and \(p > d.\) But, as observed above, the former is a consequence of (2.13) and (5.5). So hereafter we assume \(p > d.\) From the estimate

\[
\int_{B^{d+1}(x, r)} |x - y|^{-p} \, d\sigma_d(y) = d \tau_d 2^{-p/2} \int_{-1}^{1} (1 - t)^{-\frac{d}{2} + \frac{q}{2} - 1} (1 + t)^{\frac{d}{2} - 1} \, dt \\
\leq d \tau_d 2^{-\frac{d}{2} + \frac{q}{2} - 1} \int_{-1}^{1} (1 - t)^{-\frac{p}{2} + \frac{q}{2} - 1} \, dt
\]

\[
= \frac{d \tau_d}{p - d} [r^{-p+d} - 2^{-p+d}], \quad r < 2,
\]

and inequality (5.5), we deduce (as above) that

\[
M^p(\omega_n, A) \leq \frac{n}{1 - n \tau_d r^d} \left( \frac{d \tau_d}{p - d} \right) r^{-p+d}.
\]  

(5.7)

In this case, an optimal choice for \(r\) is

\[
r = r_n = \left( \frac{p - d}{np \tau_d} \right)^{1/d},
\]

which when substituted in (5.7) yields the estimate stated in inequality (3.7) for the case \(p > d.\)

**Proof of Theorem 3.2.** Assertion (3.8) is immediate from (2.8). Also the upper bounds in (3.12) for the cases \(p > d\) and \(p = d,\) can be established in the same way as in the proof of Theorem 3.1, with the measure \(\sigma_d\) replaced by normalized \(d\)-dimensional Lebesgue measure (volume measure). We leave the details for the reader. Furthermore, (3.9) follows from (3.12) together with Theorem 2.2.

For the case \(d - 2 < p < d, \) \(p > 0,\) the upper estimate in (3.12) follows from (3.3), (2.13), and the fact that

\[
\int \frac{1}{|x - y|^p} \, d\mu_p(y) \leq W_p(B^d), \quad x \in B^d,
\]
where $\mu_p$ is the $p$-equilibrium probability measure on $\mathbb{B}^d$ (cf. [12]). Together with (2.10), we also deduce (3.10). (Alternatively, one can apply the result of [7, Theorem 11] mentioned in Section 2 to deduce (3.10).)

It remains to establish (3.11). For this purpose observe that for the range $0 < p < d - 2$, the kernel $K(x, y) = |x - y|^{-p}$ is superharmonic, so that the minimum principle applies. Let $\omega_n = \{x_1, x_2, \ldots, x_n\}$ be a list of $n$ points (not necessarily distinct) in $\mathbb{B}^d$ and set

$$U(x) := \sum_{k=1}^{n} \frac{1}{|x - x_k|^p}.$$  

We claim that

$$M^p(\omega_n, \mathbb{B}^d) = \min\{U(x) : x \in \mathbb{B}^d\} \leq n, \tag{5.8}$$

from which (3.11) will follow, since on taking all points $x_k$ to be at zero, we get that $M^p_n(\mathbb{B}^d) \geq n$.

To establish (5.8), let $\sigma_{d-1}$ denote normalized surface area measure on the boundary $\mathbb{S}^d$ of $\mathbb{B}^d$. By the minimum principle we have

$$M^p(\omega_n, \mathbb{B}^d) = \min\{U(x) : x \in \mathbb{S}^{d-1}\} \leq \int_{\mathbb{S}^{d-1}} U(x) d\sigma_{d-1}(x). \tag{5.9}$$

Again applying the minimum principle, it follows that

$$V(y) := \int_{\mathbb{S}^{d-1}} \frac{1}{|x - y|^p} d\sigma_{d-1}(x)$$

satisfies $1 = V(0) \geq \min\{V(y) : |y| = r\}$ for each $0 \leq r \leq 1$. But as is easily seen, $V$ is constant on each sphere $|y| = r$, from which we deduce that $1 \geq V(y)$ for all $y \in \mathbb{B}^d$. Therefore, from (5.9) we obtain

$$M^p(\omega_n, \mathbb{B}^d) \leq \sum_{k=1}^{n} V(x_k) \leq n,$$

which establishes the claim and completes the proof. \qed

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