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Riesz polarization inequalities in higher dimensions

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Abstract

We derive bounds and asymptotics for the maximum Riesz polarization quantity

$$M_n^p(A) := \max_{\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n\in A} \min_{\mathbf{x}\in A} \sum_{j=1}^{n} \frac{1}{|\mathbf{x}-\mathbf{x}_j|^p}$$

(which is n times the Chebyshev constant) for quite general sets $A \subset \mathbb{R}^m$ with special focus on the unit sphere and unit ball. We combine elementary averaging arguments with potential theoretic tools to formulate and prove our results. We also give a discrete version of the recent result of Hardin, Kendall, and Saff which solves the Riesz polarization problem for the case when A is the unit circle and $p > 0$, as well as provide an independent proof of their result for $p = 4$ that exploits classical polynomial inequalities and yields new estimates. Furthermore, we raise some challenging conjectures.

Keywords: Chebyshev constants; Polarization inequalities; Riesz energy; Potentials

1. Introduction

For $n \in \mathbb{N}$, let $\omega_n = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ denote n (not necessarily distinct) points in m-dimensional Euclidean space $\mathbb{R}^m$. We define for $p > 0$ and a compact set $A \subset \mathbb{R}^m$, the *Riesz*

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polarization quantities

$$M^p(\omega_n, A) := \min_{\mathbf{x}\in A} \sum_{j=1}^{n} \frac{1}{|\mathbf{x}-\mathbf{x}_j|^p}, \qquad M_n^p(A) := \max_{\omega_n\subset A} M^p(\omega_n, A). \tag{1.1}$$

Such max–min quantities for potentials were first introduced by M. Ohtsuka who explored (for very general kernels) their relationship to various definitions of capacity that arise in electrostatics (see [17]). In particular, he showed that for any compact set $A \subset \mathbb{R}^m$ the following limit, called the *Chebyshev constant* of A, exists as an extended real number:

$$\mathcal{M}^p(A) := \lim_{n\to\infty} \frac{M_n^p(A)}{n}. \tag{1.2}$$

Moreover, he showed that $\mathcal{M}^p(A)$ is not smaller than the Wiener constant $W_p(A)$ for A (see Section 2). In this paper we primarily focus on results when the set A is the unit sphere or the unit ball and consider both the cases when the limit (1.2) is finite and when it is infinite.

In his Ph.D. dissertation [1], G. Ambrus proved the following basic result for the case when $A \subset \mathbb{R}^2$ is the unit circle $\mathbb{S}^1$ and $p = 2$.

Theorem 1.1. *We have*

$$M_n^2(\mathbb{S}^1) = \frac{n^2}{4}, \quad n \ge 1, \tag{1.3}$$

and $M^2(\omega_n, \mathbb{S}^1) = n^2/4, \omega_n \subset \mathbb{S}^1$, *if and only if the* n *points of* ω_n *are distinct and equally spaced on* $\mathbb{S}^1$.

In [2], Ambrus's rather technical proof along with a simpler proof based on Bernstein's inequality [3] for entire functions are presented. Bernstein's inequality was also used in [2] to provide an equally simple proof of the following estimates for the unit circle.

Theorem 1.2. *For* $n \ge 2$ *we have*

$$M_n^p(\mathbb{S}^1) \le \begin{cases} c_p n^p, & p > 1, \\ c_1 n \log n, & p = 1, \\ \dfrac{c_0 n}{1-p}, & p \in [0,1), \end{cases}$$

for some constants $c_p > 0$ *depending only on* $p \ge 1$ *and an absolute constant* $c_0 > 0$.

In Section 2 we use minimum energy methods and potential theory to obtain estimates for $M_n^p(A)$ for a large class of sets $A \subset \mathbb{R}^m$. In Section 3 we apply the results of Section 2 to obtain higher dimensional analogs of Theorem 1.2 for the unit sphere as well as for the unit ball.

In Section 4 we return to the case of the unit circle of the complex plane. For all $p > 0$, it is conjectured in [2] that the maximum polarization on $\mathbb{S}^1$ occurs for the n-th roots of unity $\omega_n^* := \{e^{i2\pi k/n} : k = 1, 2, \dots, n\}$; that is,

$$M_n^p(\mathbb{S}^1) = M^p(\omega_n^*, \mathbb{S}^1). \tag{1.4}$$

This conjecture was recently proved by Hardin, Kendall, and Saff in [9]. Here, we provide some additional consequences of their argument. Furthermore, by exploring connections to classical polynomial inequalities, we provide an independent proof of the conjecture for $p = 4$, namely that

$$M_n^4(\mathbb{S}^1) = \frac{n^4}{48} + \frac{n^2}{24}, \tag{1.5}$$

where the maximum is attained for n distinct equally spaced points on the unit circle. Although our argument (obtained prior to the general result in [9]) is not brief, it does yield additional inequalities for the discrete Riesz potential in this special case.

In Section 5, we provide the proofs of results stated in Sections 2 and 3.

We call the reader's attention to two recent articles [15,16] that contain somewhat related results for the extrema of sums of certain powered distances to finite point sets.

2. Polarization inequalities via energy methods

For a set $\omega_n = \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$ of $n(\geq 2)$ distinct points in $\mathbb{R}^m$, we define the *Riesz p-energy* of ω_n by

$$E_p(\omega_n) := \sum_{j \neq k} \frac{1}{|\mathbf{x}_j - \mathbf{x}_k|^p} = 2 \sum_{1 \leq j < k \leq n} \frac{1}{|\mathbf{x}_j - \mathbf{x}_k|^p},$$

and we consider the *minimum n-point Riesz p-energy* of an infinite compact set $A \subset \mathbb{R}^m$ defined by

$$\mathcal{E}_p(A; n) := \min\{E_p(\omega_n) : \omega_n \subset A, |\omega_n| = n\}. \tag{2.1}$$

We denote by $\omega_{n,p}^* = \{\mathbf{x}_1^*, \mathbf{x}_2^*, \dots, \mathbf{x}_n^*\}$ an n-point p-energy minimizing configuration on A; i.e., $E_p(\omega_{n,p}^*) = \mathcal{E}_p(A; n)$. Further we denote by $U_{n,p}^*(\mathbf{x})$ the potential function associated with $\omega_{n,p}^*$; i.e.,

$$U_{n,p}^*(\mathbf{x}) := \sum_{j=1}^{n} |\mathbf{x} - \mathbf{x}_j^*|^{-p}.$$

It is well-known (and easy to show) that

$$(n-1)\mathcal{E}_p(A; n+1) \geq (n+1)\mathcal{E}_p(A; n), \tag{2.2}$$

from which it follows that

$$C^*(A, n, p) := \min\{U_{n,p}^*(\mathbf{x}) : \mathbf{x} \in A\} \geq \frac{1}{n-1} \mathcal{E}_p(A; n); \tag{2.3}$$

indeed, we have

$$2C^*(A, n, p) + \mathcal{E}_p(A; n) \geq \mathcal{E}_p(A; n+1),$$

and after multiplying this inequality by $n - 1$ and applying (2.2), we get (2.3). Thus lower estimates for $\mathcal{E}_p(A; n)$ yield lower estimates for $M_n^p(A)$.

We next mention some known asymptotic results for $\mathcal{E}_p(A; n)$ as $n \to \infty$. The following theorem appearing in [10,4] has been referred to as the *Poppy-seed Bagel Theorem* because of its interpretation for distributing points on a torus.

Theorem 2.1. *Let $d \in \mathbb{N}$ and $A \subset \mathbb{R}^m$ be an infinite compact d-rectifiable set. Then for $p > d$ we have*

$$\lim_{n\to\infty} \frac{\mathcal{E}_p(A;n)}{n^{1+p/d}} = \frac{C_{p,d}}{\mathcal{H}_d(A)^{p/d}}, \tag{2.4}$$

where $C_{p,d}$ is a finite positive constant (independent of A and m) and $\mathcal{H}_d(\cdot)$ denotes the d-dimensional Hausdorff measure in $\mathbb{R}^m$ normalized so that an embedded d-dimensional unit cube has measure 1.

By a *d-rectifiable set* we mean the Lipschitz image of a bounded set in $\mathbb{R}^d$.

In [13, Theorem 3.1] it is shown that $C_{p,1}$ can be expressed in terms of the classical Riemann zeta function; namely $C_{p,1} = 2\zeta(p)$. For $d \geq 2$ the precise value of $C_{p,d}$ is not known. The significance (and difficulty) of determining $C_{p,d}$ is deeply rooted in its connection to densest sphere packings in $\mathbb{R}^d$. For $d = 2$ it is conjectured in [11] that $C_{p,2} = (\sqrt{3}/2)^{p/2}\zeta_L(p)$, where L denotes the planar hexagonal lattice of points $m(1,0) + n(1/2, \sqrt{3}/2)$, $m, n \in \mathbb{Z}$, and ζ_L is the Epstein zeta function $\zeta_L(p) := \sum_{X\in L, X\neq 0} |X|^{-p}$.

Concerning lower estimates for $C_{p,d}$, it follows from [6, Proposition 4] that, for $p > d \geq 2$ and $\frac{1}{2}(p-d)$ not an integer,

$$C_{p,d} \geq \frac{d\pi^{p/2}}{p-d} \left(\frac{\Gamma\left(1 + \frac{p-d}{2}\right)}{\Gamma\left(1+\frac{p}{2}\right)} \right)^{p/d}. \tag{2.5}$$

For the case $p = d$, the minimum p-energy grows like $n^2 \log n$. The following result is given in [10].

Theorem 2.2. *Let $d \in \mathbb{N}$ and A be an infinite compact subset of a d-dimensional C^1-manifold embedded in $\mathbb{R}^m$. Then*

$$\lim_{n\to\infty} \frac{\mathcal{E}_d(A;n)}{n^2 \log n} = \frac{\beta_d}{\mathcal{H}_d(A)},$$

where β_d is the volume of the d-dimensional unit ball.

For the case when $0 < p < d := \dim(A)$, the Hausdorff dimension of A, a theorem from classical potential theory (cf., e.g. [12]) asserts that

$$\lim_{n\to\infty} \frac{\mathcal{E}_p(A;n)}{n^2} = W_p(A), \tag{2.6}$$

where $W_p(A)$ is the so-called *Wiener constant* defined by

$$W_p(A) := \inf \iint \frac{1}{|\mathbf{x}-\mathbf{y}|^p}\, d\mu(\mathbf{x})\, d\mu(\mathbf{y}),$$

the infimum being taken over all Borel probability measures μ supported on A.

From the above results and observations we immediately obtain the following.

Theorem 2.3. *If $A \subset \mathbb{R}^m$ is an infinite compact set, then*

$$M_n^p(A) \geq \frac{1}{n-1}\, \mathcal{E}_p(A;n), \quad n \geq 2. \tag{2.7}$$

Let $d \in \mathbb{N}$. If A is d-rectifiable, then

$$\liminf_{n\to\infty} \frac{M_n^p(A)}{n^{p/d}} \geq \frac{C_{p,d}}{\mathcal{H}_d(A)^{p/d}}, \quad p > d, \tag{2.8}$$

where the constant $C_{p,d}$ is given in Theorem 2.1.

If A is any infinite compact subset of a d-dimensional C^1-manifold, then

$$\liminf_{n\to\infty} \frac{M_n^d(A)}{n \log n} \geq \frac{\beta_d}{\mathcal{H}_d(A)}, \quad p = d. \tag{2.9}$$

If A is any infinite compact subset of $\mathbb{R}^m$, then

$$\mathcal{M}^p(A) = \lim_{n\to\infty} \frac{M_n^p(A)}{n} \geq W_p(A), \quad 0 < p < d = \dim(A). \tag{2.10}$$

We remark that inequality (2.7) appears in [7,8]. Also, as previously mentioned, the inequality (2.10) is proved in [17]. Moreover, it follows from [7, Theorem 11] that equality holds in (2.10) whenever the maximum principle is satisfied on A for Riesz potentials having kernel $K(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}|^{-p}$.

Regarding upper bounds for $M_n^p(A)$, standard arguments (see Section 5) yield the following.

Theorem 2.4. *Let $A \subset \mathbb{R}^m$ be an infinite compact set. If $\mathcal{H}_d(A) > 0$, then there exists a constant $c_p > 0$ depending only on p such that*

$$M_n^p(A) \leq \frac{c_p}{p-d} n^{p/d}, \quad p > d, \; n \geq 1, \tag{2.11}$$

and there exists an absolute constant $c_1 > 0$ such that

$$M_n^d(A) \leq c_1 n \log n, \quad p = d, \; n \geq 2. \tag{2.12}$$

If there exists a probability measure μ_A supported on A whose p-potential is bounded on A, say

$$\int \frac{1}{|\mathbf{x} - \mathbf{y}|^p} \, d\mu_A(\mathbf{y}) \leq w_p, \quad \mathbf{x} \in A,$$

then

$$M_n^p(A) \leq n w_p, \quad p > 0, \; n \geq 1. \tag{2.13}$$

The essential property used in the proof of Theorem 2.4 given in Section 5 is that A is upper d-regular with respect to a Borel probability measure μ supported on A; that is, there exists a positive constant C_0 such that for any open ball $B^m(\mathbf{x}, r) \subset \mathbb{R}^m$ with center $\mathbf{x} \in A$ and radius $r > 0$ there holds

$$\mu(B^m(\mathbf{x}, r) \cap A) \leq C_0 r^d. \tag{2.14}$$

This property is a consequence of Frostman's Lemma (see [14, Chapter 8]).

3. Polarization inequalities for the unit sphere and unit ball

Let

$$\mathbb{S}^d := \{\mathbf{x} \in \mathbb{R}^{d+1} : |\mathbf{x}| = 1\} \quad \text{and} \quad \mathbb{B}^d := \{\mathbf{x} \in \mathbb{R}^d : |\mathbf{x}| \le 1\}. \tag{3.1}$$

Utilizing the results of Section 2 together with the known facts (cf. [12]) that

$$\begin{aligned} W_p(\mathbb{S}^d) &= \iint \frac{1}{|\mathbf{x}-\mathbf{y}|^p}\, d\sigma_d(\mathbf{x})\, d\sigma_d(\mathbf{y}) \\ &= 2^{d-p-1} \frac{\Gamma\left(\frac{d+1}{2}\right)\Gamma\left(\frac{d-p}{2}\right)}{\sqrt{\pi}\,\Gamma\left(d-\frac{p}{2}\right)}, \quad 0 < p < d, \end{aligned} \tag{3.2}$$

where σ_d denotes the normalized surface area on $\mathbb{S}^d$, and

$$W_p(\mathbb{B}^d) = \frac{\Gamma\left(\frac{d-p}{2}\right)\Gamma\left(\frac{p}{2}+1\right)}{\Gamma\left(\frac{d}{2}\right)}, \quad d-2 \le p < d,\ p > 0, \tag{3.3}$$

we shall prove the following two theorems.

Theorem 3.1. *For the sphere $\mathbb{S}^d$, $d \ge 2$, we have*

$$\liminf_{n\to\infty} \frac{M_n^p(\mathbb{S}^d)}{n^{p/d}} \ge C_{p,d} \left(\frac{\Gamma\left(\frac{d+1}{2}\right)}{2\pi^{(d+1)/2}}\right)^{p/d}, \quad p > d; \tag{3.4}$$

$$\lim_{n\to\infty} \frac{M_n^p(\mathbb{S}^d)}{n\log n} = \frac{1}{d}\frac{\Gamma\left(\frac{d+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{d}{2}\right)} =: \tau_d, \quad p = d; \tag{3.5}$$

$$\lim_{n\to\infty} \frac{M_n^p(\mathbb{S}^d)}{n} = 2^{d-p-1} \frac{\Gamma\left(\frac{d+1}{2}\right)\Gamma\left(\frac{d-p}{2}\right)}{\sqrt{\pi}\,\Gamma\left(d-\frac{p}{2}\right)}, \quad 0 < p < d. \tag{3.6}$$

Furthermore, the following upper estimates hold for all $n \ge 3$.

$$M_n^p(\mathbb{S}^d) \le \begin{cases} \left(\dfrac{np\tau_d}{p-d}\right)^{p/d}, & p > d, \\ \tau_d \dfrac{n[\log n + \log(\log n) + \log(2^d\tau_d)]}{1-(\log n)^{-1}}, & p = d, \\ n2^{d-p-1}\dfrac{\Gamma\left(\frac{d+1}{2}\right)\Gamma\left(\frac{d-p}{2}\right)}{\sqrt{\pi}\,\Gamma\left(d-\frac{p}{2}\right)}, & 0 < p < d. \end{cases} \tag{3.7}$$

Theorem 3.2. *For the unit ball $\mathbb{B}^d$, we have*

$$\liminf_{n\to\infty} \frac{M_n^p(\mathbb{B}^d)}{n^{p/d}} \ge C_{p,d} \left(\frac{\Gamma\left(\frac{d}{2}+1\right)}{\pi^{d/2}}\right)^{p/d}, \quad p > d; \tag{3.8}$$

$$\lim_{n\to\infty} \frac{M_n^p(\mathbb{B}^d)}{n\log n} = 1, \quad p = d; \tag{3.9}$$

$$\lim_{n\to\infty} \frac{M_n^p(\mathbb{B}^d)}{n} = \frac{\Gamma\left(\frac{d-p}{2}\right)\Gamma\left(\frac{p}{2}+1\right)}{\Gamma\left(\frac{d}{2}\right)}, \quad 0 \le d-2 < p < d, \ p > 0; \tag{3.10}$$

$$\frac{M_n^p(\mathbb{B}^d)}{n} = 1, \quad 0 < p \le d-2, \ n = 1, 2, \ldots. \tag{3.11}$$

Furthermore, the following upper estimates hold for all $n \ge 3$:

$$M_n^p(\mathbb{B}^d) \le \begin{cases} \left(\dfrac{pn}{p-d}\right)^{p/d}, & p > d, \\ \dfrac{n[\log n + \log(\log n) + d\log 2]}{1 - (\log n)^{-1}}, & p = d, \\ \dfrac{n\Gamma\left(\frac{d-p}{2}\right)\Gamma\left(\frac{p}{2}+1\right)}{\Gamma\left(\frac{d}{2}\right)} & d-2 < p < d, \ p > 0. \end{cases} \tag{3.12}$$

Remark 1. It is easily seen that for $p > d$ and $n \ge 2^d$, we have $M_n^p(\mathbb{B}^d) \ge 4^{-p} n^{p/d}$. Indeed, let $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\}$ be a maximal δ-net in $\mathbb{B}^d$ with $\delta := 4n^{-1/d}$. Then

$$m\beta_d(\delta/2)^d \le \beta_d(1+\delta/2)^d,$$

so

$$m \le \left(\frac{1+\delta/2}{\delta/2}\right)^d \le \left(\frac{4}{\delta}\right)^d \le n.$$

Also, for every $\mathbf{x} \in \mathbb{B}^d$, there is an $\mathbf{x}_k \in \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\}$ such that $|\mathbf{x} - \mathbf{x}_k| \le \delta$. Therefore,

$$\sum_{j=1}^m |\mathbf{x} - \mathbf{x}_j|^{-p} \ge |\mathbf{x} - \mathbf{x}_k|^{-p} \ge \delta^{-p} = 4^{-p} n^{p/d}.$$

Observe further that for the case $0 < p < d$, we have $M_n^p(\mathbb{B}^d) \ge n$ since we can take all the points $\mathbf{x}_j$ equal to $\mathbf{0}$, the center of the unit ball $\mathbb{B}^d$, and, moreover, such points are optimal in the case when $0 < p \le d-2$ (see the proof of (3.11) in Section 5).

Remark 2. For the case $p > d$ the above theorems establish the asymptotically sharp order (namely $n^{p/d}$) but not the sharp coefficient for the unit sphere and unit ball. Note, however, from the lower estimates in (2.5), (3.4) and (3.8) that, for $A = \mathbb{B}^d$ or $A = \mathbb{S}^d$, we have

$$\lim_{p\to d^+} \left(\liminf_{n\to\infty} \frac{M_n^p(A)}{n^{p/d}}\right) = \infty.$$

This is clearly consistent with the upper bounds provided in Theorems 3.1 and 3.2 for the case $p > d$.

We conclude this section with the following conjectures, which would be analogs of Theorems 2.1 and 2.2.

Conjecture 1. *Let $p > d$ and $m \geq d$, where p and m are integers. We conjecture that for every infinite compact d-rectifiable set A in $\mathbb{R}^m$, there should hold*

$$\lim_{n\to\infty} \frac{M_n^p(A)}{n^{p/d}} = \frac{\sigma_{p,d}}{\mathcal{H}_d(A)^{p/d}}, \tag{3.13}$$

where $\sigma_{p,d}$ is a positive and finite constant independent of A and m.

We further conjecture that if A is d-rectifiable with $\mathcal{H}_d(A) > 0$, then any sequence $\{\omega_n^*\}_{n=2}^{\infty}$ of p-polarization maximizing configurations on A is asymptotically uniformly distributed on A with respect to $\mathcal{H}_d$.

In particular, (1.4) implies that the constant $\sigma_{p,1}$ appearing in this conjecture would have to equal $2(2^p - 1)\zeta(p)$.

Conjecture 2. *Let $d \in \mathbb{N}$ and A be an infinite compact subset of a d-dimensional C^1-manifold embedded in $\mathbb{R}^m$. Then we conjecture that*

$$\lim_{n\to\infty} \frac{M_n^p(A)}{n \log n} = \frac{\beta_d}{\mathcal{H}_d(A)}, \tag{3.14}$$

where β_d is the volume of the d-dimensional unit ball.

The results of this section assert that (3.14) holds for spheres and balls.

4. Polarization on the unit circle

In this section we explore some connections between polynomial inequalities and the polarization inequality recently proved in [9]. Let $g$be a positive-valued even function defined on $\mathbb{R} \setminus (2\pi\mathbb{Z})$ that is periodic with period 2π. We denote by Ω_n the collection of all sets

$$\omega_n := \{t_1 < t_2 < \cdots < t_n\} \subset [0, 2\pi)$$

and put

$$\widetilde{\omega}_n := \{\widetilde{t}_1 < \widetilde{t}_2 < \cdots < \widetilde{t}_n\} \subset [0, 2\pi)$$

with

$$\widetilde{t}_j := 2(j-1)\pi/n, \quad j = 1, 2, \ldots, n.$$

We introduce the notation

$$P_{\omega_n(t)} := \sum_{j=1}^{n} g(t - t_j), \qquad P_{\widetilde{\omega}_n}(t) := \sum_{j=1}^{n} g(t - \widetilde{t}_j).$$

In [9] the following theorem is proved.

Theorem 4.1. *Let g be a positive-valued even function defined on $\mathbb{R} \setminus (2\pi\mathbb{Z})$ that is periodic with period 2π. If g is non-increasing and strictly convex on $(0, \pi]$, then*

$$\max_{\omega_n\in\Omega_n} \left\{ \min_{t\in[-\pi,\pi)} P_{\omega_n}(t) \right\} = P_{\widetilde{\omega}_n}(\pi/n).$$

In fact, a closer look at the proof of the main result in [9] shows that the following Riesz lemma type improvement also holds.

Theorem 4.2. *Let g be a positive-valued even function defined on $\mathbb{R} \setminus (2\pi\mathbb{Z})$ that is periodic with period 2π. If g is non-increasing and strictly convex on $(0, \pi]$, then there is a number $\gamma \in [0, 2\pi)$ (depending on ω_n) such that*

$$P_{\omega_n}(t) \le P_{\widetilde{\omega}_n}(t - \gamma), \quad t \in (\gamma, \gamma + 2\pi/n),$$

for every $\omega_n \in \Omega_n$.

A consequence of Theorem 4.2 is the following discrete version of Theorem 4.1.

Theorem 4.3. *Let g be a positive-valued even function defined on $\mathbb{R} \setminus (2\pi\mathbb{Z})$ that is periodic with period 2π. If g is non-increasing and strictly convex on $(0, \pi]$, then*

$$\max_{\omega_n \in \Omega_n} \left\{ \min_{t \in \widetilde{\omega}_{2n}} P_{\omega_n}(t) \right\} = P_{\widetilde{\omega}_n}(\pi/(2n)),$$

and equality holds when $\omega_n = \omega_n^* = \{t_1^* < t_2^* < \cdots < t_n^*\}$ with

$$t_j^* = \frac{\pi}{2n} + \frac{2(j-1)\pi}{n}, \quad j = 1, 2, \ldots, n.$$

Proof. Let γ be the number guaranteed by Theorem 4.2. Observe that $\widetilde{\omega}_{2n}$ has exactly two points in the interval $(\gamma, \gamma + 2\pi/n)$ (mod 2π). Denote these points by α and $\beta = \alpha + \pi/n$. Due to the fact that $P_{\widetilde{\omega}_n}$ is non-increasing on $(0, \pi/n)$ and

$$P_{\widetilde{\omega}_n}(t) = P_{\widetilde{\omega}_n}(2\pi/n - t), \quad t \in (0, 2\pi/n),$$

we have

$$\min\{P_{\widetilde{\omega}_n}(\alpha - \gamma), P_{\widetilde{\omega}_n}(\beta - \gamma)\} \le P_{\widetilde{\omega}_n}(\pi/(2n)),$$

which finishes the proof of the inequality of the theorem. The fact that equality holds in the case described in the theorem is obvious. □

Associated with $\omega_n := \{t_1 < t_2 < \cdots < t_n\} \subset [0, 2\pi)$ let

$$Q_{\omega_n}(t) := \prod_{j=1}^{n} \sin\left(\frac{t - t_j}{2}\right).$$

Let

$$T_n(t) := Q_{\widetilde{\omega}_n}(t) = 2^{1-n} \sin\left(\frac{nt}{2}\right).$$

Our next three theorems are consequences of Theorems 4.2 and 4.3.

Theorem 4.4. *There is a number $\gamma \in [0, 2\pi)$ (depending on ω_n) such that*

$$-(\log|Q_{\omega_n}|)^{(m)}(t) \le -(\log|T_n|)^{(m)}(t), \quad t \in (\gamma, \gamma + 2\pi/n),$$

for every $\omega_n \in \Omega_n$ and for every even integer $m > 0$.

Theorem 4.5. *Let*

$$E(\omega_n) := [0, 2\pi) \setminus \bigcup_{j=1}^{n} \left(t_j - \pi/n, t_j + \pi/n\right) \pmod{2\pi}.$$

We have

$$\max_{\omega_n \in \Omega_n} \left\{ \min_{t \in E(\omega_n)} -(\log |Q_{\omega_n}|)^{(m)}(t) \right\} = -(\log |T_n|)^{(m)}(\pi/n)$$

for every even integer $m > 0$.

Theorem 4.6. *We have*

$$\max_{\omega_n \in \Omega_n} \left\{ \min_{t \in \widetilde{\omega}_{2n}} -(\log |Q_{\omega_n}|)^{(m)}(t) \right\} = -(\log |T_n|)^{(m)}(\pi/(2n)),$$

for every even integer $m > 0$, and equality holds when $\omega_n = \omega_n^* = \{t_1^* < t_2^* < \cdots < t_n^*\}$ with

$$t_j^* = \frac{\pi}{2n} + \frac{2(j-1)\pi}{n}, \quad j = 1, 2, \ldots, n.$$

Proof of Theorem 4.4. For the sake of brevity let $Q := Q_{\omega_n}(t)$. Let $t \notin \omega_n \pmod{2\pi}$. We have

$$(\log |Q|)''(t) = \left(\frac{Q'}{Q}\right)'(t) = \frac{d}{dt}\left(\frac{1}{2}\sum_{j=1}^{n} \cot\left(\frac{t - t_j}{2}\right)\right) = -\frac{1}{4}\sum_{j=1}^{n} \csc^2\left(\frac{t - t_j}{2}\right),$$

and hence

$$-(\log |Q|)^{(m)}(t) = \frac{1}{4}\sum_{j=1}^{n} f^{(m-2)}(t - t_j) = \sum_{j=1}^{n} g_m(t - t_j),$$

where $f(t) := \csc^2(t/2)$ and $g_m(t) := \frac{1}{4} f^{(m-2)}(t)$. It is well known and elementary to check that

$$\tan t = \sum_{j=1}^{\infty} a_j t^j, \quad t \in (-\pi/2, \pi/2),$$

with each $a_j \geq 0$, $j = 1, 2, \ldots$. Hence, if $h(t) = \tan(t/2)$, then

$$h^{(k)}(t) > 0, \quad t \in (0, \pi),\ k = 0, 1, \ldots.$$

Now observe that

$$f(t) = \csc^2\left(\frac{t}{2}\right) = \sec^2\left(\frac{\pi - t}{2}\right) = 2h'(\pi - t),$$

and hence,

$$(-1)^k f^{(k)}(t) = 2h^{(k+1)}(\pi - t) > 0, \quad t \in (0, \pi).$$

This implies that if $m > 0$ is an even integer, $g_m(t) = \frac{1}{4} f^{(m-2)}(t)$ is a positive, decreasing, strictly convex function on $(0, \pi)$. It is also clear that if m is even, then g_m is even since f is even. Now we can apply Theorem 4.2 to deduce that there is a number $\gamma \in [0, 2\pi)$ (depending on ω_n) such that

$$-(\log |Q_{\omega_n}|)^{(m)}(t) = \sum_{j=1}^{n} g_m(t - t_j) \le -(\log |T_n|)^{(m)}(t), \quad t \in [\gamma, \gamma + 2\pi/n),$$

and the proof is finished. □

Proof of Theorem 4.5. The theorem follows from Theorem 4.4 immediately. □

Proof of Theorem 4.6. We use the notation and the observations in the proof of Theorem 4.4. However, at the end of the proof we use Theorem 4.3 to deduce that

$$\min_{t \in \widetilde{\omega}_{2n}} Q_{\omega_n}(t) \le T_n(\pi/(2n)),$$

and equality holds when $Q_{\omega_n} = T_n$. □

We conclude this section by giving an independent proof of the unit circle polarization conjecture in [2] for the case $p = 4$, where we show that, for $z_1, z_2, \ldots, z_n \in \mathbb{S}^1$, a "good polarization point" $z_0 \in \mathbb{S}^1$ can be chosen so that

$$\prod_{j=1}^{n} |z_0 - z_j| = \max_{z \in \mathbb{S}^1} \prod_{j=1}^{n} |z - z_j|. \tag{4.1}$$

Theorem 4.7. *If $z_1, z_2, \ldots, z_n \in \mathbb{S}^1$, then*

$$\min_{z \in \mathbb{S}^1} \sum_{j=1}^{n} \frac{1}{|z - z_j|^4} \le \frac{n^4}{48} + \frac{n^2}{24}, \quad n \ge 1,$$

and equality holds when the points z_j are distinct and equally spaced on $\mathbb{S}^1$; that is, (1.5) holds. Moreover, if $z_1, z_2, \ldots, z_n \in \mathbb{S}^1$, and $z_0 \in \mathbb{S}^1$ is chosen so that (4.1) holds, then

$$\sum_{j=1}^{n} \frac{1}{|z_0 - z_j|^4} \le \frac{n^4}{48} + \frac{n^2}{24}, \quad n \ge 1.$$

This result naturally suggests the following open question.

Problem. For what values of $p \in (0, \infty)$ is it true that

$$\sum_{j=1}^{n} \frac{1}{|z_0 - z_j|^p} \le M_n^p(\mathbb{S}_1)$$

whenever $z_1, z_2, \ldots, z_n \in \mathbb{S}_1$ and $z_0 \in \mathbb{S}_1$ satisfies (4.1)?

In addition to the value $p = 4$, a closer look at the main result in [2] shows that $p = 2$ is also such a value.

Proof of Theorem 4.7. Write $z_j = e^{it_j}$, $t_j \in [0, 2\pi)$, $j = 1, 2, \ldots, n$, and set

$$Q_n(t) := \prod_{j=1}^{n} \sin\left(\frac{t - t_j}{2}\right).$$

Then H_n defined by $H_n(t) := Q_n(2t)$ is a real trigonometric polynomial of degree n. We have the following identities:

$$\frac{Q_n'(t)}{Q_n(t)} = \frac{1}{2}\sum_{j=1}^{n} \cot\left(\frac{t-t_j}{2}\right),$$

$$\left(\frac{Q_n'}{Q_n}\right)'(t) = -\frac{1}{4}\sum_{j=1}^{n} \csc^2\left(\frac{t-t_j}{2}\right) = -\frac{1}{4}\sum_{j=1}^{n} \sin^{-2}\left(\frac{t-t_j}{2}\right),$$

$$\left(\frac{Q_n'}{Q_n}\right)''(t) = \frac{1}{4}\sum_{j=1}^{n} \cos\left(\frac{t-t_j}{2}\right)\sin^{-3}\left(\frac{t-t_j}{2}\right),$$

$$\left(\frac{Q_n'}{Q_n}\right)'''(t) = \frac{1}{4}\sum_{j=1}^{n} \left(\sin^{-2}\left(\frac{t-t_j}{2}\right) - \frac{3}{2}\sin^{-4}\left(\frac{t-t_j}{2}\right)\right),$$

so

$$\frac{3}{8}\sum_{j=1}^{n} \sin^{-4}\left(\frac{t-t_j}{2}\right) = -\left(\frac{Q_n'}{Q_n}\right)'''(t) - \left(\frac{Q_n'}{Q_n}\right)'(t).$$

On the other hand,

$$\left(\frac{Q_n'}{Q_n}\right)''' = \frac{Q_n^{(4)}}{Q_n} - 3Q_n'''\frac{Q_n'}{Q_n^2} - 3Q_n''\left(\frac{Q_n''Q_n^2 - 2Q_nQ_n'Q_n'}{Q_n^4}\right) + Q_n'\left(\frac{1}{Q_n}\right)'''$$

and

$$\left(\frac{Q_n'}{Q_n}\right)' = \frac{Q_n''}{Q_n} - \left(\frac{Q_n'}{Q_n}\right)^2.$$

Hence

$$\left(\frac{Q_n'}{Q_n}\right)'''(t_0) = \frac{Q_n^{(4)}}{Q_n}(t_0) - 3\left(\frac{Q_n''}{Q_n}\right)^2(t_0) \quad \text{and} \quad \left(\frac{Q_n'}{Q_n}\right)'(t_0) = \frac{Q_n''}{Q_n}(t_0)$$

at every point t_0 such that $Q_n'(t_0) = 0$. So if $z_0 = e^{it_0} \in \mathbb{S}^1$ is chosen so that

$$|Q_n(t_0)| = \max_{t\in[-\pi,\pi]} |Q_n(t)|,$$

then

$$6\sum_{j=1}^{n} \frac{1}{|z_0 - z_j|^4} = \left(3\left(\frac{Q_n''}{Q_n}\right)^2 - \frac{Q_n^{(4)}}{Q_n} - \frac{Q_n''}{Q_n}\right)(t_0)$$
$$= \left(\frac{3}{16}\left(\frac{H_n''}{H_n}\right)^2 - \frac{1}{16}\frac{H_n^{(4)}}{H_n} - \frac{1}{4}\frac{H_n''}{H_n}\right)\left(\frac{t_0}{2}\right).$$

Without loss of generality we may assume that $t_0 = 0$ and $z_0 = 1$.

Set

$$F(H_n) := \left(\frac{3}{16}(H_n'')^2 - \frac{1}{16}H_n^{(4)} - \frac{1}{4}H_n''\right)(0)$$

and let $\mathcal{A}_n$ be the set of all real trigonometric polynomials H_n of degree at most n such that

$$H_n(0) = \max_{t\in[-\pi,\pi]} |H_n(t)| = 1.$$

A simple compactness argument shows that there is a $\widetilde{H}_n \in \mathcal{A}_n$ such that

$$F(\widetilde{H}_n) = \sup_{H_n\in\mathcal{A}_n} F(H_n).$$

Let

$$\widetilde{U}_n(t) := \frac{1}{2}(\widetilde{H}_n(t) + \widetilde{H}_n(-t)).$$

Then $\widetilde{U}_n \in \mathcal{A}_n$ is even and $F(\widetilde{U}_n) = F(\widetilde{H}_n)$. Since $\widetilde{U}_n \in \mathcal{A}_n$ is even, it is of the form

$$\widetilde{U}_n(t) =: \widetilde{P}_n(\cos t)$$

for a $\widetilde{P}_n \in \mathcal{P}_n$ satisfying

$$\widetilde{P}_n(1) = \max_{x\in[-1,1]} |\widetilde{P}_n(x)| = 1,$$

where $\mathcal{P}_n$ denotes the set of all real algebraic polynomials of degree at most n.

Observe that $U_n \in \mathcal{A}_n$ is even if and only if it is of the form

$$U_n(t) =: P_n(\cos t)$$

for a $P_n \in \mathcal{P}_n$ satisfying

$$P_n(1) = \max_{x\in[-1,1]} |P_n(x)| = 1.$$

A simple calculation shows that

$$U_n(0) = P_n(1), \qquad U_n''(0) = -P_n'(1), \qquad U_n^{(4)}(0) = 3P_n''(1) + P_n'(1).$$

Let

$$G(P_n) := F(U_n) = \left(\frac{3}{16}(U_n'')^2 - \frac{1}{16}U_n^{(4)} - \frac{1}{4}U_n''\right)(0)$$
$$= \frac{3}{16}((P_n')^2 - P_n'' + P_n')(1).$$

We have

$$G(P_n) = F(U_n) \le F(\widetilde{H}_n) = F(\widetilde{U}_n) = G(\widetilde{P}_n)$$

for every $P_n \in \mathcal{P}_n$ such that

$$P_n(1) = \max_{x\in[-1,1]} |P_n(x)| = 1.$$

Next we show by a simple variational method that $\widetilde{P}_n \in \mathcal{P}_n$ equioscillates between -1 and 1 at least n times on $[-1, 1]$. That is, there are

$$-1 \le y_n < y_{n-1} < \cdots < y_1 = 1$$

such that

$$\widetilde{P}_n(y_j) = (-1)^{j-1}, \quad j = 1, 2, \ldots, n.$$

To show this, first we observe that $\widetilde{P}_n'(1) > 0$ since $\widetilde{P}_n'(1) \geq 0$, and Markov's inequality for the second derivative (see p. 249 of [5]) together with $\widetilde{P}_n'(1) = 0$ would imply that

$$\begin{aligned} G(\widetilde{P}_n) &= \frac{3}{16}((\widetilde{P}_n')^2 - \widetilde{P}_n'' + \widetilde{P}_n')(1) = \frac{-3}{16}\widetilde{P}_n''(1) \\ &\leq \frac{3}{16}T_n''(1) = \frac{3}{16}\,\frac{n^2(n^2-1)}{3} < \frac{1}{16}(2n^4 + 4n^2) = G(T_n), \end{aligned}$$

where T_n is the Chebyshev polynomial of degree n defined by $T_n(\cos t) = \cos(nt)$, and this contradicts the extremal property of $\widetilde{P}_n$. Now let

$$E := \{y \in [-1,1] : |\widetilde{P}(y)| = 1\}.$$

We list the elements of E as

$$E = \{1 = y_1 > y_2 > \cdots > y_\mu\},$$

where

$$\widetilde{P}_n(y_{k_j}) = \widetilde{P}_n(y_{k_j+1}) = \cdots = \widetilde{P}_n(y_{k_{j+1}-1}), \qquad j = 0, 1, \ldots, m-1,$$

and

$$\widetilde{P}_n(y_{k_j}) = -\widetilde{P}_n(y_{k_j-1}) = (-1)^j, \qquad j = 1, 2, \ldots, m-1,$$

for some

$$1 = k_0 < k_1 < \cdots < k_m = \mu + 1.$$

Now we pick

$$\alpha_j \in (y_{k_j}, y_{k_j-1}), \qquad j = 1, 2, \ldots, m-1.$$

Assume that $m \leq n-1$. For the polynomial $R_n \in \mathcal{P}_n$ defined by

$$R_n(x) := (x-1)^2 \prod_{j=1}^{m-1}(x - \alpha_j)$$

we have

$$\begin{aligned} &R_n(y)\widetilde{P}_n(y) > 0, \qquad y \in E \setminus \{1\}, \\ &R_n(1) = R_n'(1) = 0 \quad \text{and} \quad R_n''(1) > 0. \end{aligned}$$

These properties together with $\widetilde{P}_n'(1) > 0$ imply that for a sufficiently small value of $\varepsilon > 0$ the polynomial

$$S_n = \widetilde{P}_n - \varepsilon R_n \in \mathcal{P}_n$$

satisfies

$$S_n(1) = \max_{x \in [-1,1]} |S_n(x)| = 1$$

and $G(S_n) > G(\widetilde{P}_n)$, so $S_n \in \mathcal{P}_n$ contradicts the extremal property of $\widetilde{P}_n$. This finishes the proof of the fact that $\widetilde{P}_n \in \mathcal{P}_n$ equioscillates between -1 and 1 at least n times on $[-1,1]$, as we claimed.

As a consequence, the Intermediate Value Theorem implies that $\widetilde{P}_n$ has at least $n-1$ zeros in $(-1,1)$, say

$$(-1<)x_{n-1} < x_{n-2} < \cdots < x_1(<1).$$

Observe that the polynomial $\widetilde{P}_n \in \mathcal{P}_n$ has an odd number of zeros (by counting multiplicities) in each of the intervals (y_{j+1}, y_j) for $j = 1, 2, \dots, n-1$; hence x_j is the only (simple) zero of $\widetilde{P}_n$ in (y_{j+1}, y_j) for each $j = 1, 2, \dots, n-1$. Therefore $\widetilde{P}_n$ has only real zeros and it is of the form

$$\widetilde{P}_n(x) = c\prod_{j=1}^{\mu}(x - x_j)$$

with either $\mu = n-1$ or $\mu = n$, and in the case $\mu = n$ we have $x_n \in \mathbb{R} \setminus [y_n, 1]$.

Note that

$$\frac{\widetilde{P}_n'(x)}{\widetilde{P}_n(x)} = \sum_{j=1}^{\mu}\frac{1}{x-x_j}, \qquad \left(\frac{\widetilde{P}_n'(x)}{\widetilde{P}_n(x)}\right)' = -\sum_{j=1}^{\mu}\frac{1}{(x-x_j)^2},$$

and

$$G(\widetilde{P}_n) = \frac{3}{16}\left(\frac{(\widetilde{P}_n')^2 - \widetilde{P}_n''\widetilde{P}_n}{(\widetilde{P}_n)^2} + \frac{\widetilde{P}_n'}{\widetilde{P}_n}\right)(1) = \frac{3}{16}\left(\sum_{j=1}^{\mu}\frac{1}{(1-x_j)^2} + \sum_{j=1}^{\mu}\frac{1}{(1-x_j)}\right).$$

If $\mu = n-1$, then $\widetilde{P}_n$ equioscillates between -1 and 1 on $[-1,1]$ the maximum number of times, so $\widetilde{P}_n \equiv T_{n-1}$, where T_{n-1} is the Chebyshev polynomial of degree $n-1$ defined by $T_{n-1}(\cos t) = \cos((n-1)t)$. Hence

$$\begin{aligned} G(\widetilde{P}_n) &= \frac{3}{16}\left(\sum_{j=1}^{\mu}\frac{1}{(1-x_j)^2} + \sum_{j=1}^{\mu}\frac{1}{1-x_j}\right) \\ &= \frac{3}{16}\left(\frac{(T_{n-1}')^2 - T_{n-1}''T_{n-1}}{T_{n-1}^2} + \frac{T_{n-1}'}{T_{n-1}}\right)(1) \\ &= \frac{3}{16}\left((n-1)^4 - \frac{(n-1)^2((n-1)^2-1)}{3} + (n-1)^2\right) \\ &= \frac{1}{8}(n-1)^4 + \frac{1}{4}(n-1)^2. \end{aligned}$$

If $\mu = n$ we must have $x_n \in (-\infty, y_n) \cup (1, \infty)$. However, $1 < x_n$ would imply that

$$Y_n(x) := -c(x - (2 - x_n))\prod_{j=1}^{n-1}(x - x_j)$$

satisfies

$$Y_n(1) = \max_{x\in[-1,1]}|Y_n(x)| = 1 \quad \text{and} \quad G(Y_n) = G(\widetilde{P}_n),$$

and hence $Y_n \in \mathcal{P}_n$ also shares the extremal property of $\widetilde{P}_n$ while it has all its zeros in $(-\infty, 1)$. Hence $x_n < y_n < x_{n-1}$. But then $\widetilde{P}_n$ is just the Chebyshev polynomial T_n transformed linearly from the interval $[-1, 1]$ to $[\eta, 1]$ for some $\eta \le -1$. This implies that

$$\begin{aligned} G(\widetilde{P}_n) &= \frac{3}{16}\left(\sum_{j=1}^{\mu}\frac{1}{(1-x_j)^2} + \sum_{j=1}^{\mu}\frac{1}{1-x_j}\right) \\ &= \frac{3}{16}\left(\left(\frac{2}{1-\eta}\right)^2 \frac{(T_n')^2 - T_n''T_n}{T_n^2} + \frac{2}{1-\eta}\frac{T_n'}{T_n}\right)(1) \\ &\le \frac{3}{16}\left(n^4 - \frac{n^2(n^2-1)}{3} + n^2\right) = \frac{1}{8}n^4 + \frac{1}{4}n^2. \end{aligned}$$

Now we conclude that

$$G(\widetilde{P}_n) \le G(T_n) = \frac{1}{8}n^4 + \frac{1}{4}n^2,$$

and hence

$$F(\widetilde{H}_n) = G(\widetilde{P}_n) \le G(T_n) = \frac{1}{8}n^4 + \frac{1}{4}n^2.$$

Therefore

$$6\sum_{j=1}^{n}\frac{1}{|z_0 - z_j|^4} = F(H_n) \le F(\widetilde{H}_n) \le G(T_n) = \frac{1}{8}n^4 + \frac{1}{4}n^2,$$

and this completes the proof. □

We conclude this section by mentioning two formulas that may be useful for future investigation of the polarization problem for the unit circle. Let

$$A_p(t) := \sum_{j=1}^{n}\frac{1}{|e^{it} - z_j|^p}, \qquad p > 0,$$

where $z_j = e^{it_j} \in \mathbb{S}^1,\ j = 1, 2, \ldots, n$. Then a straightforward calculation yields the following:

$$A_2(t) = \frac{(Q_n'(t))^2 - Q_n''(t)Q_n(t)}{(Q_n(t))^2} \quad \text{with } Q_n(t) := \prod_{j=1}^{n}\sin\left(\frac{t-t_j}{2}\right),$$

and

$$A_{p+2}(t) = \frac{1}{p^2+p}\left(A_p''(t) + \frac{p^2}{4}A_p(t)\right), \qquad p > 0.$$

5. Proofs of Theorems 2.4, 3.1 and 3.2

Proof of Theorem 2.4. We proceed with an argument similar to that in [11]. Let $\omega_n = \{\mathbf{x}_j\}_{j=1}^n \subset A$. Setting

$$r_n := (2nC_0)^{-1/d}, \qquad D_j := A \setminus B^m(\mathbf{x}_j, r_n), \quad \mathcal{D} := \cap_{j=1}^n D_j,$$

we have from (2.14) that

$$\mu(\mathcal{D}) \ge 1 - \sum_{j=1}^{n}\mu(B^m(\mathbf{x}_j, r_n) \cap A) \ge 1 - nC_0 r_n^d = \frac{1}{2}.$$

Thus, for

$$f_n(\mathbf{x}) := \sum_{j=1}^{n} |\mathbf{x} - \mathbf{x}_j|^{-p},$$

we obtain

$$M^p(\omega_n, A) \le \frac{1}{\mu(\mathcal{D})} \int_{\mathcal{D}} f_n(\mathbf{x})\, d\mu(\mathbf{x}) \le 2 \sum_{j=1}^{n} \int_{D_j} |\mathbf{x} - \mathbf{x}_j|^{-p}\, d\mu(\mathbf{x}). \tag{5.1}$$

Next, we bound the integrals over D_j utilizing (2.14):

$$\begin{aligned}\int_{D_j} |\mathbf{x} - \mathbf{x}_j|^{-p}\, d\mu(\mathbf{x}) &= \int_0^\infty \mu\{\mathbf{x} \in D_j : |\mathbf{x} - \mathbf{x}_j|^{-p} > t\}\, dt \\ &\le 1 + \int_1^{r_n^{-p}} \mu(B^m(\mathbf{x}_j, t^{-1/p}) \cap A)\, dt \\ &\le 1 + C_0 \int_1^{r_n^{-p}} \frac{1}{t^{d/p}}\, dt,\end{aligned}$$

where we assume that n is sufficiently large so that $r_n^{-p} > 1$. Thus from (5.1) it follows that

$$M^p(\omega_n, A) \le 2n \left(1 + C_0 \int_1^{r_n^{-p}} \frac{1}{t^{d/p}}\, dt\right). \tag{5.2}$$

Consequently, for $p > d$ we get

$$M^p(\omega_n, A) \le 2n \left(1 + C_0 \frac{p}{p-d} [r_n^{d-p} - 1]\right) \le \frac{c_p}{p-d} n^{p/d} \tag{5.3}$$

and for $p = d$ we obtain

$$M^d(\omega_n, A) \le 2n[1 + C_0 \log(r_n^{-d})] = 2n[1 + C_0 \log(2nC_0)] \le c_1 n \log n. \tag{5.4}$$

This completes the proof of parts (2.11) and (2.12) of Theorem 2.4, while (2.13) follows immediately upon integration of $f_n(\mathbf{x})$ with respect to $d\mu_A$. □

Proof of Theorem 3.1. Inequality (3.4) is an immediate consequence of (2.8), while Eq. (3.6) follows from (3.2), (2.10), and the last assertion in Theorem 2.4, since

$$\int |\mathbf{x} - \mathbf{y}|^{-p}\, d\sigma_d(y) = W_p(\mathbb{S}^d), \quad \mathbf{x} \in \mathbb{S}^d,\ p < d. \tag{5.5}$$

To prove Eq. (3.5), we first note that from (2.9) we have

$$\liminf_{n\to\infty} \frac{M_n^p(\mathbb{S}^d)}{n \log n} \ge \frac{\beta_d}{\mathcal{H}_d(\mathbb{S}^d)} = \frac{1}{d} \frac{\Gamma\left(\frac{d+1}{2}\right)}{\sqrt{\pi}\, \Gamma\left(\frac{d}{2}\right)} = \tau_d.$$

Hence, if we establish the upper estimate in (3.7) for $p = d$, then (3.5) will follow. For this purpose, we refine the argument used in the proof of Theorem 2.4. With $\mu = \sigma_d$, the following estimates are known for $\mathbf{x} \in \mathbb{S}^d$ (cf. [11]):

$$\sigma_d(B^{d+1}(\mathbf{x}, r) \cap \mathbb{S}^d) \le \tau_d r^d, \tag{5.6}$$

and

$$\int_{\mathbb{S}^d\setminus B^{d+1}(\mathbf{x},r)} |\mathbf{x}-\mathbf{y}|^{-d}\,d\sigma_d(\mathbf{y}) = d\tau_d 2^{-d/2}\int_{-1}^{1-r^2/2}(1-t)^{-1}(1+t)^{\frac{d}{2}-1}\,dt \le d\tau_d\log(2/r),$$

for $0<r<2$. Utilizing these estimates and using (5.1) with $r_n=(\tau_d n\log n)^{-1/d}$, $D_j=\mathbb{S}^d\setminus B^{d+1}(\mathbf{x}_j,r_n)$, and $n\ge 3$, we obtain

$$\begin{aligned} M^d(\omega_n,A) &\le \frac{1}{1-n\tau_d r_n^d}\sum_{j=1}^{n}\int_{D_j}|\mathbf{x}-\mathbf{x}_j|^{-d}\,d\sigma_d(\mathbf{x}) \le \frac{nd}{1-n\tau_d r_n^d}\tau_d\log(2/r_n)\\ &= \frac{nd}{1-(\log n)^{-1}}\tau_d\left(\log 2+\frac{1}{d}\log(\tau_d n\log n)\right)\\ &= \tau_d\frac{n[\log n+\log(\log n)+\log(2^d\tau_d)]}{1-(\log n)^{-1}}. \end{aligned}$$

This completes the proof of (3.5) as well as the upper bound in (3.7) for the case $p=d$.

It remains to establish (3.7) for the cases $p<d$ and $p>d$. But, as observed above, the former is a consequence of (2.13) and (5.5). So hereafter we assume $p>d$. From the estimate

$$\begin{aligned} \int_{\mathbb{S}^d\setminus B^{d+1}(\mathbf{x},r)} |\mathbf{x}-\mathbf{y}|^{-p}\,d\sigma_d(\mathbf{y}) &= d\tau_d 2^{-p/2}\int_{-1}^{1-r^2/2}(1-t)^{-\frac{p}{2}+\frac{d}{2}-1}(1+t)^{\frac{d}{2}-1}\,dt\\ &\le d\tau_d 2^{-\frac{p}{2}+\frac{d}{2}-1}\int_{-1}^{1-r^2/2}(1-t)^{-\frac{p}{2}+\frac{d}{2}-1}\,dt\\ &= \frac{d\tau_d}{p-d}[r^{-p+d}-2^{-p+d}],\quad r<2, \end{aligned}$$

and inequality (5.5), we deduce (as above) that

$$M^p(\omega_n,A)\le\frac{n}{1-n\tau_d r^d}\left(\frac{d\tau_d}{p-d}\right)r^{-p+d}. \tag{5.7}$$

In this case, an optimal choice for r is

$$r=r_n=\left(\frac{p-d}{np\tau_d}\right)^{1/d},$$

which when substituted in (5.7) yields the estimate stated in inequality (3.7) for the case $p>d$. □

Proof of Theorem 3.2. Assertion (3.8) is immediate from (2.8). Also the upper bounds in (3.12) for the cases $p>d$ and $p=d$, can be established in the same way as in the proof of Theorem 3.1, with the measure σ_d replaced by normalized d-dimensional Lebesgue measure (volume measure). We leave the details for the reader. Furthermore, (3.9) follows from (3.12) together with Theorem 2.2.

For the case $d-2<p<d$, $p>0$, the upper estimate in (3.12) follows from (3.3), (2.13), and the fact that

$$\int\frac{1}{|\mathbf{x}-\mathbf{y}|^p}\,d\mu_p(\mathbf{y})\le W_p(\mathbb{B}^d),\quad \mathbf{x}\in\mathbb{B}^d,$$

where μ_p is the p-equilibrium probability measure on $\mathbb{B}^d$ (cf. [12]). Together with (2.10), we also deduce (3.10). (Alternatively, one can apply the result of [7, Theorem 11] mentioned in Section 2 to deduce (3.10).)

It remains to establish (3.11). For this purpose observe that for the range $0 < p < d-2$, the kernel $K(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}|^{-p}$ is superharmonic, so that the minimum principle applies. Let $\omega_n = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ be a list of n points (not necessarily distinct) in $\mathbb{B}^d$ and set

$$U(\mathbf{x}) := \sum_{k=1}^{n} \frac{1}{|\mathbf{x} - \mathbf{x}_k|^p}.$$

We claim that

$$M^p(\omega_n, \mathbb{B}^d) = \min\{U(\mathbf{x}) : \mathbf{x} \in \mathbb{B}^d\} \le n, \tag{5.8}$$

from which (3.11) will follow, since on taking all points $\mathbf{x}_k$ to be at zero, we get that $M_n^p(\mathbb{B}^d) \ge n$. To establish (5.8), let σ_{d-1} denote normalized surface area measure on the boundary $\mathbb{S}^{d-1}$ of $\mathbb{B}^d$. By the minimum principle we have

$$M^p(\omega_n, \mathbb{B}^d) = \min\{U(\mathbf{x}) : \mathbf{x} \in \mathbb{S}^{d-1}\} \le \int_{\mathbb{S}^{d-1}} U(\mathbf{x})\, d\sigma_{d-1}(\mathbf{x}). \tag{5.9}$$

Again applying the minimum principle, it follows that

$$V(\mathbf{y}) := \int_{\mathbb{S}^{d-1}} \frac{1}{|\mathbf{x} - \mathbf{y}|^p}\, d\sigma_{d-1}(\mathbf{x})$$

satisfies $1 = V(\mathbf{0}) \ge \min\{V(\mathbf{y}) : |\mathbf{y}| = r\}$ for each $0 \le r \le 1$. But as is easily seen, V is constant on each sphere $|\mathbf{y}| = r$, from which we deduce that $1 \ge V(\mathbf{y})$ for all $\mathbf{y} \in \mathbb{B}^d$. Therefore, from (5.9) we obtain

$$M^p(\omega_n, \mathbb{B}^d) \le \sum_{k=1}^{n} V(\mathbf{x}_k) \le n,$$

which establishes the claim and completes the proof. □

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