ASYMPTOTICS FOR HESSENBERG MATRICES FOR THE BERGMAN SHIFT OPERATOR ON JORDAN REGIONS

EDWARD B. SAFF AND NIKOS STYLIANOPOULOS

Abstract. Let G be a bounded Jordan domain in the complex plane. The Bergman polynomials $\{p_n\}_{n=0}^{\infty}$ of G are the orthonormal polynomials with respect to the area measure over G. They are uniquely defined by the entries of an infinite upper Hessenberg matrix M. This matrix represents the Bergman shift operator of G. The main purpose of the paper is to describe and analyze a close relation between M and the Toeplitz matrix with symbol the normalized conformal map of the exterior of the unit circle onto the complement of $\overline{G}$. Our results are based on the strong asymptotics of p_n. As an application, we describe and analyze an algorithm for recovering the shape of G from its area moments.

1. Introduction

Let G be a bounded simply-connected domain in the complex plane $\mathbb{C}$, whose boundary $\Gamma := \partial G$ is a Jordan curve and let $\{p_n\}_{n=0}^{\infty}$ denote the sequence of Bergman polynomials of G. This is the unique sequence of polynomials

$$p_n(z) = \lambda_n z^n + \cdots, \quad \lambda_n > 0, \quad n = 0, 1, 2, \ldots, \tag{1.1}$$

that are orthonormal with respect to the inner product

$$\langle f, g\rangle := \int_G f(z)\overline{g(z)}dA(z),$$

where dA stands for the area measure. We denote by $L_a^2(G)$ the Hilbert space of all functions f analytic in G for which

$$\|f\|_{L^2(G)} := \langle f, f\rangle^{1/2} < \infty,$$

and recall (cf. [6]) that the polynomials $\{p_n\}_{n=0}^{\infty}$ form a complete orthonormal system for $L_a^2(G)$.

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Let $\Omega := \overline{\mathbb{C}} \setminus \overline{G}$ denote the complement of $\overline{G}$ in $\overline{\mathbb{C}}$ and let Φ denote the conformal map $\Omega \to \Delta := \{w : |w| > 1\}$, normalized so that near infinity

$$\Phi(z) = \gamma z + \gamma_0 + \frac{\gamma_1}{z} + \frac{\gamma_2}{z^2} + \cdots, \quad \gamma > 0. \tag{1.2}$$

Finally, let $\Psi := \Phi^{-1} : \Delta \to \Omega$ denote the inverse conformal map. Then,

$$\Psi(w) = bw + b_0 + \frac{b_1}{w} + \frac{b_2}{w^2} + \cdots, \quad |w| > 1, \tag{1.3}$$

with

$$b = 1/\gamma = \mathrm{cap}(\Gamma), \tag{1.4}$$

where $\mathrm{cap}(\Gamma)$ denotes the *(logarithmic) capacity* of Γ.

On $L_a^2(G)$ we consider the multiplication by z operator (also known as the *Bergman shift operator*) $\mathcal{M} : f \to zf$. Note that $\mathcal{M}$ defines a bounded, noncompact, linear operator on $L_a^2(G)$ and that

$$\sigma_{ess}(\mathcal{M}) = \Gamma; \tag{1.5}$$

see [1], where we use $\sigma_{ess}(L)$ to denote the *essential spectrum* of a bounded linear operator L; that is, the set of all $\lambda \in \mathbb{C}$ for which $L - \lambda I$ is not a Fredholm operator. For the operators we consider, the essential spectrum is the same as the continuous spectrum.

We also consider the matrix representation of $\mathcal{M}$ in terms of the orthonormal basis $\{p_n\}_{n=0}^{\infty}$. This induces the upper Hessenberg matrix

$$M = \begin{bmatrix} b_{00} & b_{01} & b_{02} & b_{03} & b_{04} & \cdots \\ b_{10} & b_{11} & b_{12} & b_{13} & b_{14} & \cdots \\ 0 & b_{21} & b_{22} & b_{23} & b_{24} & \cdots \\ 0 & 0 & b_{32} & b_{33} & b_{34} & \cdots \\ 0 & 0 & 0 & b_{43} & b_{44} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}, \tag{1.6}$$

where

$$b_{k,j} = \langle zp_j, p_k \rangle, \quad k \geq 0,\ j \geq 0. \tag{1.7}$$

Note that $b_{k,j} = 0$ for $k \geq j+2$ and that

$$zp_n(z) = \sum_{k=0}^{n+1} b_{k,n} p_k(z). \tag{1.8}$$

In particular,

$$b_{n+1,n} = \frac{\lambda_n}{\lambda_{n+1}} > 0, \quad n = 0, 1, \ldots. \tag{1.9}$$

It follows

$$b_{n+1,n} p_{n+1}(z) = zp_n(z) - \sum_{k=0}^{n} b_{k,n} p_k(z) \tag{1.10}$$

and, hence, the entries of M define uniquely the sequence of Bergman polynomials of G, in the sense that p_{n+1}, $n = 0, 1, \ldots$, can be computed recursively from (1.10).

It is shown in [11] and [9] (see also [12, Thm 2.4]) that except for some trivial cases, the matrix (1.6) is not banded; i.e., the p_n's do not satisfy a recurrence relation of bounded length. It is also well-known that the eigenvalues of the $n \times n$ principal submatrix of M coincide with the zeros of $p_n(z)$.

Our goal is to investigate the asymptotic behavior of the entries in the matrix M. In particular, we show that if the boundary of G is piecewise analytic without cusps, then all the diagonals (sub, super and main) have limits which are the coefficients of the Laurent expansion (1.3) of the inverse conformal map Ψ:

$$\lim_{n\to\infty} b_{n+1,n} = b \quad \text{and} \quad \lim_{n\to\infty} b_{n-k,n} = b_k, \quad k = 0, 1, \ldots. \tag{1.11}$$

A potential application of (1.11) is in the area of geometric tomography, where the following inverse problem arises: Given a finite number of complex moments

$$\mu_{kj} := \langle z^k, z^j \rangle = \int_G z^k \overline{z}^j \, dA(z), \quad k, j = 0, 1, \ldots, \tag{1.12}$$

how can one approximate the region G that generated these moments? Regarding existence and uniqueness, we note a result of Davis and Pollak [4] stating that the infinite matrix $[\mu_{m,k}]_{m,k=0}^{\infty}$ defines uniquely the curve Γ. By utilizing the given moments to compute Bergman polynomials, and thereby a principal submatrix of M, the subdiagonals of the submatrix will provide an approximation to the Laurent coefficients of the mapping of the unit circumference onto the boundary of G. We will discuss this procedure in Section 3.

We note that there is a one-to-one correspondence between the complex moments (1.12) and the real moments

$$\tau_{mn} := \int_G x^m y^n \, dxdy, \quad m, n = 0, 1, \ldots. \tag{1.13}$$

Namely,

$$\mu_{m,n} = \sum_{j=0}^{m} \sum_{k=0}^{n} i^{m-j} i^{n-k} \binom{m}{j} \binom{n}{k} \tau_{j+k, m+n-j-k}, \quad i := \sqrt{-1}, \tag{1.14}$$

or, in the inverse direction,

$$\tau_{m,n} = (-i)^n 2^{-m-n} \sum_{j=0}^{m} \sum_{k=0}^{n} \binom{m}{j} \binom{n}{k} \mu_{j+k, m+n-j-k}; \tag{1.15}$$

see [4]. Thus, the moments in (1.12) will uniquely determine the moments in (1.13) and vice-versa.

The Faber polynomials $\{F_n\}_{n=0}^{\infty}$ of G are defined as the polynomial part of the expansion of $\Phi^n(z)$, $n = 0, 1, \ldots$, near infinity, that is,

$$\Phi^n(z) = F_n(z) - E_n(z), \quad z \in \Omega, \tag{1.16}$$

where

$$F_n(z) = \gamma^n z^n + \cdots \quad \text{and} \quad E_n(z) = O\left(\frac{1}{z}\right), \quad z \to \infty. \tag{1.17}$$

The Faber polynomial of the 2nd kind, $G_n(z)$, is defined as the polynomial part of $\Phi^n(z)\Phi'(z)$, that is,

$$G_n(z) = \Phi^n(z)\Phi'(z) - H_n(z), \quad z \in \Omega, \tag{1.18}$$

where

$$G_n(z) = \gamma^{n+1} z^n + \cdots \quad \text{and} \quad H_n(z) = O\left(\frac{1}{z^2}\right), \quad z \to \infty. \tag{1.19}$$

It follows immediately from (1.16) and (1.18) that

$$G_n(z) = \frac{F'_{n+1}(z)}{n+1} \quad \text{and} \quad H_n(z) = \frac{E'_{n+1}(z)}{n+1}. \tag{1.20}$$

It is well-known that the Faber polynomials of the 2nd kind satisfy the following recurrence relation (see [5, p. 52]):

$$zG_n(z) = bG_{n+1}(z) + \sum_{j=0}^{n} b_j G_{n-j}(z), \quad G_0(z) \equiv b. \tag{1.21}$$

Consider now the Toeplitz (and upper Hessenberg) matrix T_Ψ defined by the continuous function $\Psi(w)$ on $\mathbb{T} := \{w : |w| = 1\}$, that is,

$$T_\Psi := \begin{bmatrix} b_0 & b_1 & b_2 & b_3 & b_4 & \cdots \\ b & b_0 & b_1 & b_2 & b_3 & \cdots \\ 0 & b & b_0 & b_1 & b_2 & \cdots \\ 0 & 0 & b & b_0 & b_1 & \cdots \\ 0 & 0 & 0 & b & b_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}. \tag{1.22}$$

It follows from (1.21) that the eigenvalues of the $n \times n$ principal submatrix of T_Ψ coincide with the zeros of $G_n(z)$; see also [15]. This is a relation similar to the one connecting the upper Hessenberg matrix M with the Bergman polynomials $\{p_n\}_{n=0}^\infty$.

In [13, §7.8] it is shown that if Γ is piecewise analytic without cusps, then

$$|b_n| \le c_1(\Gamma)\frac{1}{n^{1+\omega}}, \quad n \in \mathbb{N}, \tag{1.23}$$

where $\omega\pi$ $(0 < \omega < 2)$ is the smallest exterior angle of Γ. (Hereafter, we use $c_k(\Gamma)$, $k = 1, 2, \ldots$, to denote a non-negative constant that depends only on Γ.) Therefore, in this case, the symbol Ψ of the Toeplitz matrix T_Ψ belongs to the Wiener algebra, which leads to the conclusion that T_Ψ defines a bounded linear operator on the Hilbert space l^2 and that

$$\sigma_{ess}(T_\Psi) = \Gamma; \tag{1.24}$$

see e.g. [2, p. 1–10].

We end this section by noting a result, regarding a property of H_n, that we are going to use in Section 5. A proof can be found in [13, Lem. 2.1].

Lemma 1.1. *For any $n \in \mathbb{N}$, H_n is analytic and square integrable in Ω.*

2. Main results

In this section we state and discuss our main results. Their proofs are given in Section 5. Section 3 contains applications of our results to the recovery of planar regions.

From (1.5) and (1.24) it follows that

$$\sigma_{ess}(\mathcal{M}) = \sigma_{ess}(T_\Psi). \tag{2.1}$$

The next theorem shows that the connection between the matrices M and T_Ψ is much more substantial.

Theorem 2.1. *Assume that Γ is piecewise analytic without cusps. Then, it holds as $n \to \infty$,*

$$\sqrt{\frac{n+2}{n+1}}b_{n+1,n} = b + O\left(\frac{1}{n}\right), \tag{2.2}$$

and for $k \geq 0$,

$$\sqrt{\frac{n-k+1}{n+1}}b_{n-k,n} = b_k + O\left(\frac{1}{\sqrt{n}}\right), \tag{2.3}$$

where O depends on k but not on n. (See (5.17) for more precise estimates.)

Improvements in the order of convergence occur in cases when Γ is smooth. In order to state the corresponding results we need to introduce the smoothness class $C(q,\alpha)$ of Jordan curves. We say that Γ belongs to $C(q,\alpha)$, $q \in \mathbb{N}$, if Γ is defined by $z = g(s)$, where s denotes arclength, with $g^{(q)} \in \operatorname{Lip}\alpha$, for some $0 < \alpha < 1$. Then both Φ and $\Psi := \Phi^{-1}$ are q times continuously differentiable in $\overline{\Omega} \setminus \{\infty\}$ and $\overline{\Delta} \setminus \{\infty\}$ respectively, with $\Phi^{(q)}$ and $\Psi^{(q)}$ in $\operatorname{Lip}\alpha$: see, e.g., [14, p. 5].

Theorem 2.2. *Assume that $\Gamma \in C(p+1,\alpha)$, with $p+\alpha > 1/2$. Then, it holds as $n \to \infty$,*

$$\sqrt{\frac{n+2}{n+1}}b_{n+1,n} = b + O\left(\frac{1}{n^{2(p+\alpha)}}\right), \tag{2.4}$$

and for $k \geq 0$,

$$\sqrt{\frac{n-k+1}{n+1}}b_{n-k,n} = b_k + O\left(\frac{1}{n^{p+\alpha}}\right), \tag{2.5}$$

where O depends on k but not on n. (See (5.29) for more precise estimates.)

For the case of an analytic boundary Γ further improved asymptotic results can be obtained. To state these results we need to introduce some notation. For an analytic curve Γ the mapping Ψ can be analytically continued as a conformal map to the exterior of some disk $\{w : |w| < \varrho\}$, where

$0 < \varrho < 1$. We denote by L_σ the image of the circle $\{w : |w| = \sigma\}$ under the map Ψ. In other words,

$$L_\sigma := \{z \in \mathbb{C} : |\Phi(z)| = \sigma\}.$$

Theorem 2.3. * *Assume that the boundary Γ is analytic and let $\varrho < 1$ be the smallest index for which Φ is conformal in the exterior of L_ϱ. Then, it holds as $n \to \infty$,*

$$\sqrt{\frac{n+2}{n+1}}\, b_{n+1,n} = b + O(\varrho^{2n}), \tag{2.6}$$

and for $k \geq 0$,

$$\sqrt{\frac{n-k+1}{n+1}}\, b_{n-k,n} = b_k + O(\sqrt{n \log n}\varrho^n), \tag{2.7}$$

where O depends on k but not on n. (See (5.39) for more precise estimates.)

In the converse direction we have:

Theorem 2.4. *Assume that Γ is a Jordan curve without zero interior angles. If*

$$\limsup_{n\to\infty} \left| \sqrt{\frac{n+2}{n+1}} b_{n+1,n} - b \right|^{1/n} < 1, \tag{2.8}$$

then Γ is analytic.

The following example shows that the inverse statement does not make sense for the main diagonal of M.

Example 2.1. *Consider the case where the domain G has m-fold rotational symmetry about the origin, for some $m \geq 2$.*

This means that $\mathrm{e}^{i2\pi/m} z \in \Omega$, whenever $z \in \Omega$. Then, it is easy to see that

$$b_0 = 0 \quad \text{and} \quad b_{n,n} = 0, \quad n \geq m. \tag{2.9}$$

Indeed, by using symmetry arguments it follows

$$\Psi(\mathrm{e}^{i2\pi/m} w) = \mathrm{e}^{i2\pi/m}\Psi(w), \quad w \in \Omega, \tag{2.10}$$

and for $n = km + j$, with $j = 0, 1, \ldots, m-1$,

$$p_n(z) = z^j q_k(z^m), \quad \deg(q_k) = k. \tag{2.11}$$

The first relation in (2.9) follows at once from (2.10). For the second relation in (2.9), observe that (2.11) implies for $n \geq m$ that

$$p_n(z) = \lambda_n z^n + O(z^{n-m}),$$

which, in turn, yields $\langle z^{n+1}, p_n\rangle = 0$ and therefore $b_{n,n} = \langle z p_n, p_n\rangle = 0$.

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*This theorem, along with a sketch of its proof given in Section 5.3, was presented by the first author at the Joint Meeting of the AMS and MAA in Phoenix, January 2004.

3. A recovery algorithm

`Reconstruction Algorithm`

1. *Start with a finite set of complex moments* μ_{kj}, $k,j = 0,1,\ldots,n$*; see (1.12), or, equivalently from a finite set of real moments* τ_{kj}, $k,j = 0,1,\ldots,n$*; see (1.13).*
2. *Use the Arnoldi version of the Gram-Schmidt (GS) process, in the way indicated in* [13, §7.4], *to construct the Bergman polynomials* $\{p_k\}_{k=0}^n$ *from the moments* μ_{kj}, $k,j = 0,1,\ldots,n$*. This involves at the* k*-step the orthonormalization of the set* $\{p_0, p_1, \ldots, p_{k-1}, zp_{k-1}\}$*, rather than the set of monomials* $\{1, z, \ldots, z^{k-1}, z^k\}$*, as in the conventional GS. This process, in particular, yields the inner products*
$$b_{k,j} = \langle zp_j, p_k\rangle, \quad j = 0,1,\ldots,n, \;\; k = 0,\ldots,j+1.$$
3. *Choose a number* m*,* $1 < m < n$*, and set*
$$b^{(n)} := \sqrt{\frac{n+2}{n+1}}b_{n+1,n}, \quad b_k^{(n)} := \sqrt{\frac{n-k+1}{n+1}}b_{n-k,n}, \quad k = 0,1,\ldots,m. \tag{3.1}$$
(See Theorem 3.1 and Remark 3.2 below, for a suitable choice of m*.)*
4. *Form*
$$\Psi_m^{(n)}(w) := b^{(n)}w + b_0^{(n)} + \frac{b_1^{(n)}}{w} + \ldots + \frac{b_m^{(n)}}{w^m}. \tag{3.2}$$
5. *Approximate* Γ *by* $\Gamma_m^{(n)}$*, where*
$$\Gamma_m^{(n)} := \Psi_m^{(n)}(w), \quad w \in \mathbb{T}. \tag{3.3}$$

Remark 3.1. *We refer to* [13, §7.4] *for a discussion regarding the stability properties of the Arnoldi GS. In particular, we note that the Arnoldi GS does not suffer from the severe ill-conditioning associated with the conventional GS as reported, for instance, by theoretical and numerical evidence in* [10].

The following result justifies the use of the algorithm for analytic curves.

Theorem 3.1. *Assume that* Γ *is analytic, and let* $\varrho < 1$ *be the smallest index for which* Φ *is conformal in the exterior of* L_ϱ*. Set* $n = 2m$*. Then, for any* $|w| \geq 1$ *it holds that*
$$|\Psi(w) - \Psi_m^{(n)}(w)| \leq c_1(\Gamma)\sqrt{m\log m}\,\varrho^m + c_2(\Gamma)|w|\varrho^{4m}, \tag{3.4}$$
where the constants $c_1(\Gamma)$ *and* $c_2(\Gamma)$ *depend on* Γ *only.*

Remark 3.2. *Similar estimates, as in the above theorem, can be obtained for the case where* Γ *is piecewise analytic without cusps. However, these estimates are too pessimistic compared with actual numerical evidence; see Figure 2 below. We were only able to rigorously show that for an uniform error of order* $O(1/\sqrt{m})$ *we require the computation of the orthonormal polynomials up to degree* $m^{4+\omega}$*, where* $\omega\pi$ *is the smallest exterior angle of* Γ*.*

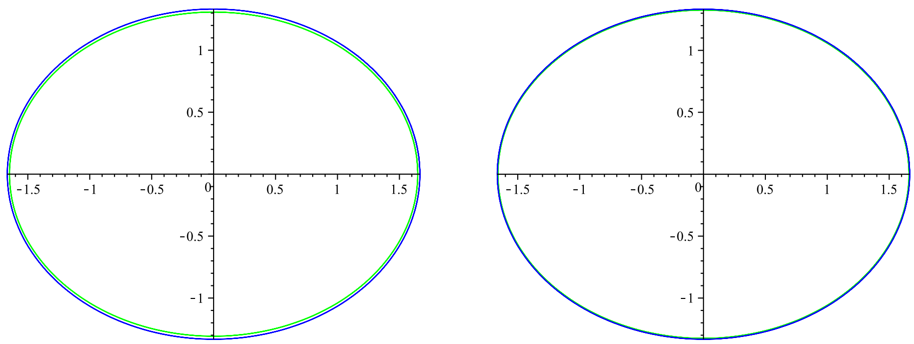

FIGURE 1. Recovery of an ellipse, with $n = 10$ (left) and $n = 20$ (right).

For applications to the 2D image reconstruction arising from tomographic data we refer to [8]. Here we highlight the performance of the reconstruction algorithm by applying it to the recovery of three curves, coming from different classes: an analytic curve, one curve with corners and one curve with cusps. For providing matter for comparison with the reconstruction algorithm of [13, §7.7] we have chosen to present results for exactly the same curves as in [13]. We note that the reconstruction algorithm of [13] is based on approximating first the exterior conformal mapping $w = \Phi(z)$ in terms of the ratio $p_{n+1}(z)/p_n(z)$, cf. the estimates (5.3)–(5.4) below, and then on inverting the so-formed Laurent series in order to compute an approximation of the inverse map $z = \Psi(w)$.

In each case we start by computing a finite set of complex moments (1.12) up to degree n, and then follow the steps 2–5 of the algorithm, taking $m = n/2$. In all three examples the complex moments are known explicitly. All computations were carried out on a desktop PC, using the computing environment MAPLE.

In Figures 1–3 we depict the computed approximation $\Gamma_m^{(n)}$ against the original curve Γ. The presented plots indicate that the above reconstruction algorithm constitutes a valid method for recovering a shape from its partial moments. Even in the cusped case, pictured in Figure 3, the fitting is remarkably close, despite the low degree of the moment matrix used.

In Figure 1 we illustrate the reconstruction of an ellipse, where, with the notation of Theorem 3.1, $\varrho = 1/3$.

In Figure 2 we reconstruct a square by using the complex moments up to the degree 16. We have chosen $n = 16$, so that the result can be compared with the recovery of a square, as shown on page 1067 of [7], obtained using the *Exponential Transform Algorithm*. This is another reconstruction algorithm based on moments.

In order to show that the proposed reconstruction algorithm works equally well for domains where the results of neither Theorem 3.1 nor that of Remark 3.2 apply, we use it for the recovery of the boundary of the 3-cusped

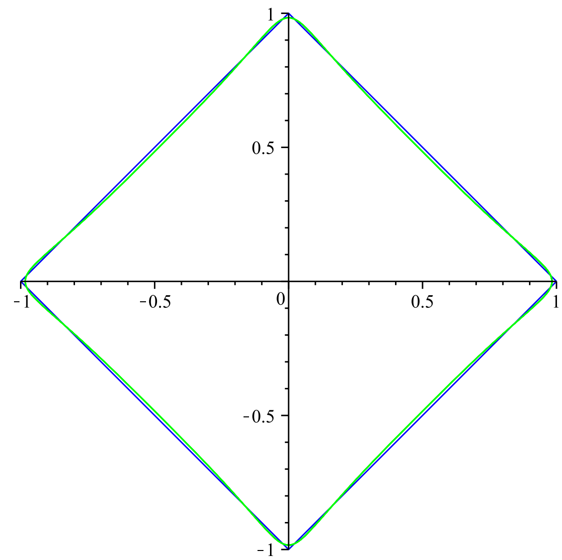

FIGURE 2. Recovery of a square, with $n = 16$.

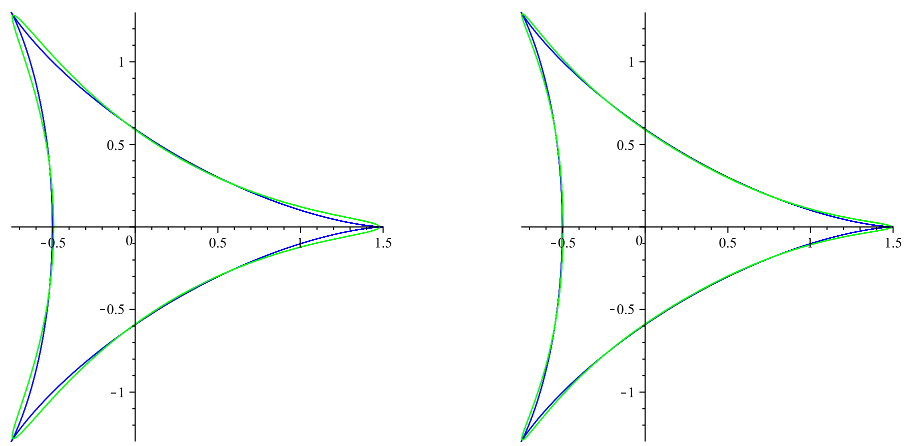

FIGURE 3. Recovery of a 3-cusped hypocycloid, with $n = 20$ (left) and $n = 30$ (right).

hypocycloid defined by

$$\Gamma := \{z = \Psi(w) = w + \frac{1}{2w^2}, \quad w \in \mathbb{T}\}.$$

The application of the algorithm with $n = 20$ and $n = 30$ is depicted in Figure 3.

Comparing the performance of the above algorithm with that of [13] for the cases of the ellipse and the hypocycloid, it appears that the latter algorithm performs slightly better. On the other hand, both algorithms perform better than the reconstruction algorithms of [7] for the case of the square. More definitive comparisons will require further experimentation and analysis of all three reconstruction algorithms.

4. Numerical Results

In this section we employ the first three steps of the reconstruction algorithm in order to present numerical results that illustrate the order of convergence in (2.2) and (2.3), that is the order in approximating b and b_2 by $b^{(n)}$ and $b_2^{(n)}$, respectively. We consider the case where Γ is the equilateral triangle Π_3 with vertices at 1, $e^{2i\pi/3}$, and $e^{4i\pi/3}$. Then, by using the Schwarz-Christoffel formula it is not difficult to see that the coefficients b_n of the associated conformal map (1.3) are given by $b_0 = 0$ and

$$b_n = \begin{cases} \operatorname{cap}(\Pi_3)(-1)^{m+1}\binom{2/3}{m}\frac{1}{n}, & \text{if } l = 1, \\ 0, & \text{if } l \neq 1, \end{cases} \tag{4.1}$$

for $n = 3m - l$, $m \in \mathbb{N}$ and $l \in \{0, 1, 2\}$, where $\binom{2/3}{m}$ denotes the binomial coefficient; see, e.g., [13, §7.8]. Furthermore, it follows by using the properties of hypergeometric functions that

$$b = \operatorname{cap}(\Pi_3) = \frac{3}{2}\frac{\Gamma(1/3)^3}{4\pi^2} = 0.730\,499\,243\,103\cdots, \tag{4.2}$$

where $\Gamma(x)$ denotes the Gamma function with argument x.

By using the rotational property of the equilateral triangle, as this is reflected in the relation (2.11), it is easy to see that

$$b_{n-k,n} = 0, \quad \text{if} \quad k \notin \{2, 5, 8, \ldots\}.$$

This is actually the reason why we consider the two approximations $b^{(n)}$ and $b_2^{(n)}$. Accordingly, we let $t^{(n)}$ and $t_2^{(n)}$ denote the two errors

$$t^{(n)} := b - b^{(n)} \quad \text{and} \quad t_2^{(n)} := b_2 - b_2^{(n)}. \tag{4.3}$$

Then, from Theorem 2.1 we have that

$$|t^{(n)}| \leq c(\Gamma)\frac{1}{n} \quad \text{and} \quad |t_2^{(n)}| \leq c(\Gamma)\frac{1}{\sqrt{n}}, \quad n \in \mathbb{N}. \tag{4.4}$$

In Tables 4.1 and 4.2 we report the computed values of $b^{(n)}$, $t^{(n)}$ and $b_2^{(n)}$, $t_2^{(n)}$, with n varying from 100 to 200. We also report the values of the parameter s, which is designed to test the two hypotheses

$$|t^{(n)}| \approx 1/n^s \quad \text{and} \quad |t_2^{(n)}| \approx 1/n^s.$$

This was done by estimating s by means of the two formulae

$$s_n := \log\left(\frac{|t^{(n)}|}{|t^{(n+10)}|}\right) / \log\left(\frac{n+10}{10}\right) \quad \text{and} \quad s_n := \log\left(\frac{|t_2^{(n)}|}{|t_2^{(n+10)}|}\right) / \log\left(\frac{n+10}{10}\right).$$

In view of Remark 3.1, regarding the stability properties of the Arnoldi GS process, we expect all the figures quoted in the tables to be correct.

n	$b^{(n)}$	$t^{(n)}$	s
100	0.730 487 539	1.17e-05	1.9627
110	0.730 489 536	9.70e-06	1.9659
120	0.730 491 062	8.18e-06	1.9685
130	0.730 492 255	6.98e-06	1.9708
140	0.730 493 204	6.03e-06	1.9728
150	0.730 493 973	5.26e-06	1.9745
160	0.730 494 603	4.63e-06	1.9761
170	0.730 495 127	4.11e-06	1.9774
180	0.730 495 567	3.67e-06	1.9786
190	0.730 495 940	3.30e-06	1.9799
200	0.730 496 259	2.98e-06	—

TABLE 4.1. Equilateral triangle: Errors and rates in approximating $b = 0.730\,499\,243\,103\cdots$ by $b^{(n)}$.

n	$b_2^{(n)}$	$t_2^{(n)}$	s
100	0.243 555 903	-5.61e-05	1.9873
110	0.243 546 213	-4.64e-05	1.9886
120	0.243 538 830	-3.90e-05	1.9897
130	0.243 533 076	-3.33e-05	1.9907
140	0.243 528 504	-2.87e-05	1.9914
150	0.243 524 812	-2.50e-05	1.9921
160	0.243 521 788	-2.20e-05	1.9926
170	0.243 519 280	-1.95e-05	1.9931
180	0.243 517 177	-1.74e-05	1.9936
190	0.243 515 396	-1.56e-05	1.9939
200	0.243 513 875	-1.41e-05	—

TABLE 4.2. Equilateral triangle: Errors and rates in approximating $b_2 := 0.243\,499\,747\,701\cdots$ by $b_2^{(n)}$.

It is interesting to note the following regarding the presented results:

- The values of $b^{(n)}$ decay monotonically to b.
- The values of $b_2^{(n)}$ increase monotonically b_2.
- The values of the parameter s indicate clearly that

  $$|t^{(n)}| \approx 1/n^2 \quad \text{and} \quad |t_2^{(n)}| \approx 1/n^2,$$

  This suggests that the two estimates

  $$|t^{(n)}| \leq c(\Gamma)\frac{1}{n} \quad \text{and} \quad |t_2^{(n)}| \leq c(\Gamma)\frac{1}{\sqrt{n}}, \quad n \in \mathbb{N},$$

predicted by Theorem 2.1 are pessimistic.

5. Proofs

5.1. **Proof of Theorem 2.1.** The derivation in the case where Γ is piecewise analytic without cusps is based on results from [12] and [13]. In particular, we utilize the following fact (see [12, Thm 1.1]):

$$\frac{n+1}{\pi}\frac{\gamma^{2(n+1)}}{\lambda_n^2}=1-\alpha_n, \tag{5.1}$$

where

$$0\le\alpha_n\le c_1(\Gamma)\,\frac{1}{n}. \tag{5.2}$$

We also note the following estimate [12, Thm 1.2]:

$$p_n(z)=\sqrt{\frac{n+1}{\pi}}\,\Phi^n(z)\Phi'(z)\,\{1+A_n(z)\},\quad z\in\Omega, \tag{5.3}$$

where

$$|A_n(z)|\le\frac{c_2(\Gamma)}{\text{dist}(z,\Gamma)\,|\Phi'(z)|}\,\frac{1}{\sqrt{n}}+c_3(\Gamma)\,\frac{1}{n}. \tag{5.4}$$

Recall that we use $c_k(\Gamma)$, $k=1,2,\ldots$, to denote a non-negative constant that depends on Γ only.

The result (2.2) follows immediately from (1.9) and (5.1)–(5.2). For the general case (2.3), our proof relies on the use of the auxiliary polynomial

$$q_{n-1}(z):=G_n(z)-\frac{\gamma^{n+1}}{\lambda_n}p_n(z),\quad n\in\mathbb{N}. \tag{5.5}$$

This is a polynomial of degree at most $n-1$, but it can be identically zero, as the special case when G is a disk shows.

Next we fix $k=0,1,2,\ldots$. Then from (1.21), in conjunction with (5.5) and the orthogonality of p_n, we deduce for any $n=k,k+1,k+2,\ldots$, that

$$\begin{aligned}\frac{\gamma^{n+1}}{\lambda_n}b_{n-k,n}&=\langle z\frac{\gamma^{n+1}}{\lambda_n}p_n,p_{n-k}\rangle=\langle zG_n-zq_{n-1},p_{n-k}\rangle\\&=\langle zG_n,p_{n-k}\rangle-\langle zq_{n-1},p_{n-k}\rangle\\&=b\langle G_{n+1},p_{n-k}\rangle+\sum_{j=0}^{k}b_j\langle G_{n-j},p_{n-k}\rangle-\langle zq_{n-1},p_{n-k}\rangle.\end{aligned} \tag{5.6}$$

Thus, it remains to estimate the two different types of inner products appearing in (5.6), namely $\langle p_l,G_m\rangle$ and $\langle zq_m,p_l\rangle$. This is the objective of the following two lemmas.

Lemma 5.1. *Assume that Γ is piecewise analytic without cusps. Then, for $l=0,1,2,\ldots$, it holds that*

$$\langle p_l,G_m\rangle=\begin{cases}\gamma^{m+1}/\lambda_m, & m=l,\\ \xi_m, & m=l+1,l+2,\ldots,\end{cases} \tag{5.7}$$

where

$$|\xi_m| \le c_1(\Gamma)\frac{1}{m}. \tag{5.8}$$

Proof. For the special case where $m = l$ the result is a trivial consequence of the orthonormality property of the polynomial p_m and the fact that G_m is a polynomial of exact degree m with leading coefficient γ^{m+1}. That is,

$$\begin{aligned}\langle p_m, G_m\rangle &= \langle p_m, \gamma^{m+1}z^m + \cdots\rangle = \langle p_m, \gamma^{m+1}z^m\rangle \\ &= \gamma^{m+1}\langle p_m, \frac{1}{\lambda_m}p_m\rangle = \frac{\gamma^{m+1}}{\lambda_m}. \end{aligned} \tag{5.9}$$

Assume now that $m \in \{l+1, l+2, \dots\}$. Then, an application of Green's formula, the splitting (1.18) and the residue theorem give:

$$\begin{aligned}\langle p_l, G_m\rangle &= \int_G p_l(z)\overline{G_m(z)}dA(z) = \int_G p_l(z)\overline{\frac{F'_{m+1}(z)}{m+1}}dA(z) \\ &= \frac{\pi}{m+1}\left\{\frac{1}{2\pi i}\int_\Gamma p_l(z)\overline{F_{m+1}(z)}dz\right\} \\ &= \frac{\pi}{m+1}\left\{\frac{1}{2\pi i}\int_\Gamma p_l(z)\overline{\Phi^{m+1}(z)}dz + \frac{1}{2\pi i}\int_\Gamma p_l(z)\overline{E_{m+1}(z)}dz\right\} \\ &= \frac{\pi}{m+1}\left\{\frac{1}{2\pi i}\int_\Gamma \frac{p_l(z)}{\Phi^{m+1}(z)}dz + \frac{1}{2\pi i}\int_\Gamma p_l(z)\overline{E_{m+1}(z)}dz\right\} \\ &= \frac{1}{2(m+1)i}\int_\Gamma p_l(z)\overline{E_{m+1}(z)}dz. \end{aligned} \tag{5.10}$$

To conclude the proof we use the estimate given in [13, Lem. 2.5], to obtain

$$\left|\frac{1}{2i}\int_\Gamma p_l(z)\overline{E_{m+1}(z)}dz\right| \le c_2(\Gamma)\|p_l\|_{L^2(G)}\left[\int_\Omega |E'_{m+1}(z)|^2 dA(z)\right]^{1/2}, \tag{5.11}$$

where we made use of the fact that $E'_{m+1} \in L^2(\Omega)$ (see Lemma 1.1) and that a piecewise analytic without cusps Jordan curve is quasiconformal and rectifiable.

Therefore, from (5.10), the second relation in (1.20) and (5.11), we have

$$|\langle p_l, G_m\rangle| \le c_3(\Gamma)\left[\int_\Omega |H_m(z)|^2 dA(z)\right]^{1/2}, \tag{5.12}$$

and the required result follows, because the last integral is $O(1/m^2)$; see [13, Thm 2.4]. ∎

Lemma 5.2. *Assume that Γ is piecewise analytic without cusps. Then, for every $m \in \mathbb{N}$ and $l = 0, 1, 2, \dots$, it holds that*

$$|\langle zq_m, p_l\rangle| \le c_1(\Gamma)\frac{1}{m}. \tag{5.13}$$

Proof. The result is a simple consequence of Corollary 2.1 in [13] which states

$$\|q_m\|_{L^2(G)} \le c_2(\Gamma)\frac{1}{m},$$

and the Cauchy-Schwarz inequality:

$$|\langle zq_m, p_l\rangle| \le \|zq_m\|_{L^2(G)}\|p_l\|_{L^2(G)} \le \max\{|z| : z\in\Gamma\}\,\|q_m\|_{L^2(G)}.$$

□

Returning to the proof of Theorem 2.1, we apply the results of the two previous lemmas to (5.6) and use (1.23) to obtain:

$$\begin{aligned}\frac{\gamma^{n+1}}{\lambda_n}b_{n-k,n} &= b\langle G_{n+1},p_{n-k}\rangle + \sum_{j=0}^{k-1} b_j\langle G_{n-j},p_{n-k}\rangle + b_k\frac{\gamma^{n-k+1}}{\lambda_{n-k}} - \langle zq_{n-1},p_{n-k}\rangle \\ &= O\left(\frac{1}{n}\right) + \sum_{j=1}^{k-1} O\left(\frac{1}{(n-j)\,j^{1+\omega}}\right) + b_k\frac{\gamma^{n-k+1}}{\lambda_{n-k}}, \qquad (5.14)\end{aligned}$$

where $0<\omega<2$, and O does not depend on n or k. Furthermore, from (5.1)–(5.2) we have:

$$\frac{\gamma^{n+1}}{\lambda_n} = \sqrt{\frac{\pi}{n+1}}\left[1+O\left(\frac{1}{n}\right)\right] \qquad (5.15)$$

and

$$\frac{\lambda_{n-k}}{\gamma^{n-k+1}} = \sqrt{\frac{n-k+1}{\pi}}\left[1+O\left(\frac{1}{n-k+1}\right)\right]. \qquad (5.16)$$

Thus, by multiplying both sides of (5.14) by $\lambda_{n-k}/\gamma^{n-k+1}$ we get

$$\frac{\lambda_{n-k}}{\gamma^{n-k+1}}\frac{\gamma^{n+1}}{\lambda_n}b_{n-k,n} = b_k + \frac{\lambda_{n-k}}{\gamma^{n-k+1}}\left[O\left(\frac{1}{n}\right) + \sum_{j=1}^{k-1} O\left(\frac{1}{(n-j)\,j^{1+\omega}}\right)\right],$$

which, in view of the estimates (5.15)–(5.16), yields for $n\ge k\ge 0$, $n\ge 1$, that

$$\begin{aligned}\sqrt{\frac{n-k+1}{n+1}}b_{n-k,n} &= b_k\left[1+O\left(\frac{1}{n-k+1}\right)\right] \\ &\quad + O(\sqrt{n-k+1})\left[O\left(\frac{1}{n}\right) + \sum_{j=1}^{k-1} O\left(\frac{1}{(n-j)\,j^{1+\omega}}\right)\right], \qquad (5.17)\end{aligned}$$

where an empty sum equals zero. This leads, for fixed k and $n\to\infty$, to the required estimate (2.3), where now O depends on k but not on n. □

5.2. **Proof of Theorem 2.2.** If $\Gamma \in C(p+\alpha)$, with $p+\alpha > 1/2$, then the following asymptotic formulas hold as $n \to \infty$, see [14, p. 19–20]:

$$\sqrt{\frac{n+1}{\pi}}\frac{\gamma^{n+1}}{\lambda_n} = 1 + O\left(\frac{1}{n^{2(p+\alpha)}}\right) \tag{5.18}$$

and

$$p_n(z) = \sqrt{\frac{n+1}{\pi}}\,\Phi^n(z)\Phi'(z)\left\{1 + O\left(\frac{\log n}{n^{p+\alpha}}\right)\right\}, \quad z \in \overline{\Omega}. \tag{5.19}$$

The proof of the theorem goes along similar lines as the proof of Theorem 2.1 given above. More precisely, for deriving the result for $b_{n+1,n}$ we use the estimate (5.18) in the place of (5.1)–(5.2).

For the general case $k = 0, 1, \ldots$, we need estimates for the inner products $\langle p_l, G_m\rangle$ and $\langle zq_m, p_l\rangle$. This is done in the following two lemmas, which play the role of Lemma 5.1 and Lemma 5.2 in the proof of Theorem 2.1.

Lemma 5.3. *Assume that $\Gamma \in C(p+1,\alpha)$, with $p+\alpha > 1/2$, then for $l = 0, 1, 2, \ldots$, it holds that*

$$\langle p_l, G_m\rangle = \begin{cases} \gamma^{m+1}/\lambda_m, & m = l, \\ \xi_m, & m = l+1, l+2, \ldots, \end{cases} \tag{5.20}$$

where

$$|\xi_m| \le c_1(\Gamma)\frac{1}{m^{p+\alpha+1/2}}. \tag{5.21}$$

Proof. The result for $m = l$ is established in Lemma 5.1. Hence, we only consider the case $m = l+1, l+2, \ldots$.

The following estimate has been obtained by Suetin for $\Gamma \in C(p+\alpha)$; see [14, Lem. 1.5]:

$$\left|\frac{1}{2\pi i}\int_\Gamma H_m(z)\overline{E_{m+1}(z)}dz\right| \le c_2(\Gamma)\frac{1}{m^{2(p+\alpha)}}. \tag{5.22}$$

By using Green's formula in the unbounded domain Ω, together with (1.20), it is readily seen that

$$\frac{1}{2\pi i}\int_\Gamma H_m(z)\overline{E_{m+1}(z)}dz = -\frac{m+1}{\pi}\int_\Omega |H_m(z)|^2 dA(z). \tag{5.23}$$

Hence, from (5.22),

$$\int_\Omega |H_m(z)|^2 dA(z) \le c_3(\Gamma)\frac{1}{m^{2(p+\alpha)+1}}, \tag{5.24}$$

and the result (5.21) follows from the estimate (5.12), which is applicable in this case because any smooth Jordan curve is also quasiconformal and rectifiable. $\square$

Lemma 5.4. *Assume that $\Gamma \in C(p+1,\alpha)$. Then for every $m \in \mathbb{N}$ and $l = 0, 1, 2, \ldots$, it holds that*

$$|\langle zq_m, p_l\rangle| \le c_1(\Gamma)\frac{1}{m^{p+\alpha+1/2}}. \tag{5.25}$$

Proof. As in the proof of Lemma 5.2 we have

$$|\langle zq_m, p_l\rangle| \leq \max\{|z| : z\in\Gamma\}\, \|q_m\|_{L^2(G)}.$$

The result of the lemma then follows from (5.24) and the estimate

$$\|q_m\|_{L^2(G)} \leq c_2(\Gamma)\left[\int_\Omega |H_m(z)|^2 dA(z)\right]^{1/2},$$

established in [13, Thm 2.1] for domains bounded by a quasiconformal and rectifiable boundary. □

In order to conclude the proof of the theorem, we need an estimate for the decay of the coefficients b_n, when the boundary Γ belongs to the class $C(p+1,\alpha)$, with $p+\alpha>1/2$. This is done in [13, Cor. 1.1], where it is shown that

$$|b_n| \leq c_3(\Gamma)\frac{1}{n^{p+\alpha+1/2}}, \quad n\in\mathbb{N}. \tag{5.26}$$

Therefore, by using the results for $\langle p_l, G_m\rangle$ and $\langle zq_m, p_l\rangle$, obtained in the previous two lemmas, together with (5.18) and (5.6), we see that

$$\frac{\gamma^{n+1}}{\lambda_n}b_{n-k,n} = O\left(\frac{1}{n^{p+\alpha+1/2}}\right) + \sum_{j=1}^{k-1} O\left(\frac{1}{(j(n-j))^{p+\alpha+1/2}}\right) + b_k\frac{\gamma^{n-k+1}}{\lambda_{n-k}},$$

where O does not depend on n or k. Furthermore, from (5.18) we get

$$\frac{\gamma^{n+1}}{\lambda_n} = \sqrt{\frac{\pi}{n+1}}\left[1+O\left(\frac{1}{n^{2(p+\alpha)}}\right)\right] \tag{5.27}$$

and

$$\frac{\lambda_{n-k}}{\gamma^{n-k+1}} = \sqrt{\frac{n-k+1}{\pi}}\left[1+O\left(\frac{1}{(n-k+1)^{2(p+\alpha)}}\right)\right]. \tag{5.28}$$

The above yield, for $n\geq k\geq 0$, $n\geq 1$, that

$$\begin{aligned}\sqrt{\frac{n-k+1}{n+1}}b_{n-k,n} = b_k&\left[1+O\left(\frac{1}{(n-k+1)^{2(p+\alpha)}}\right)\right] + O(\sqrt{n-k+1})\\ &\times\left[O\left(\frac{1}{n^{p+\alpha+1/2}}\right) + \sum_{j=1}^{k-1} O\left(\frac{1}{(j(n-j))^{p+\alpha+1/2}}\right)\right],\end{aligned} \tag{5.29}$$

where a empty sum equals zero. This leads, for fixed k and $n\to\infty$, to the required estimate (2.5), where now O depends on k but not on n. □

5.3. **Proof of Theorem 2.3.** Assume that $\Gamma := \partial G$ is an analytic Jordan curve. Then the conformal map Φ has an analytic and univalent continuation across Γ in G. Let $\varrho<1$ be defined by

$$\varrho := \inf\{r :\ \Phi \text{ is analytic and univalent in } \operatorname{ext}(L_\varrho)\setminus\infty\}.$$

Then the following asymptotic formulas of Carleman [3] hold as $n \to \infty$:

$$\sqrt{\frac{n+1}{\pi}}\frac{\gamma^{n+1}}{\lambda_n} = 1 + O(\varrho^{2n}) \tag{5.30}$$

and

$$p_n(z) = \sqrt{\frac{n+1}{\pi}}\,\Phi^n(z)\Phi'(z)\left\{1 + O(\sqrt{n}\varrho^n)\right\}, \quad z \in \overline{\Omega}, \tag{5.31}$$

see [6, p. 12]. In particular,

$$p_n(z) = \frac{\lambda_n}{\gamma^{n+1}}\,\Phi^n(z)\Phi'(z)\left\{1 + \omega_n(z)\right\}, \tag{5.32}$$

where

$$\omega_n(z) = \sum_{\nu=1}^{n} \nu A_\nu w^{\nu-1-n} - \sum_{\nu=1}^{\infty} \nu a_\nu w^{-\nu-1-n}, \quad w = \Phi(z), \tag{5.33}$$

with

$$\sum_{\nu=1}^{n} \nu|A_\nu|^2 + \sum_{\nu=1}^{\infty} \nu|a_\nu|^2\varrho^{-2\nu} \leq \frac{\varrho^{2n+2}}{(n+1)(1-\varrho^{2n})}, \quad n \in \mathbb{N}; \tag{5.34}$$

see [6, p. 15].

We fix two different points z and z_0 on Γ and define

$$Q_{j+1}(z) := \int_{z_0}^{z} p_j(\zeta)d\zeta + \frac{\lambda_j}{(j+1)\gamma^{j+1}}\Phi^{j+1}(z_0).$$

Then, by using integration by parts and the change of variable $w = \Phi(\zeta)$, we have from (5.32) and (5.33) that, for any $j \in \mathbb{N}$,

$$\begin{aligned} Q_{j+1}(z) &= \frac{\lambda_j}{(j+1)\gamma^{j+1}}\Phi^{j+1}(z) + \frac{\lambda_j}{\gamma^{j+1}}\int_{z_0}^{z} \Phi^j(\zeta)\Phi'(\zeta)\omega_j(\zeta)d\zeta \\ &= \frac{\lambda_j}{(j+1)\gamma^{j+1}}\Phi^{j+1}(z) + \frac{\lambda_j}{\gamma^{j+1}}\int_{w_0}^{\Phi(z)} w^j\omega_j(\Psi(w))dw \\ &= \frac{\lambda_j}{(j+1)\gamma^{j+1}}\Phi^{j+1}(z) + \frac{\lambda_j}{\gamma^{j+1}}\left[\sum_{\nu=1}^{j} A_\nu w^\nu + \sum_{\nu=1}^{\infty} a_\nu w^{-\nu}\right]_{w_0}^{\Phi(z)}, \end{aligned} \tag{5.35}$$

where $w_0 = \Phi(z_0)$. We claim that for $|w| = 1$ there holds

$$\left|\sum_{\nu=1}^{j} A_\nu w^\nu + \sum_{\nu=1}^{\infty} a_\nu w^{-\nu}\right| = O\left(\sqrt{\frac{\log(j+1)}{j+1}}\varrho^j\right). \tag{5.36}$$

Indeed,

$$\left|\sum_{\nu=1}^{j} A_\nu w^\nu + \sum_{\nu=1}^{\infty} a_\nu w^{-\nu}\right| \leq \sum_{\nu=1}^{j} |A_\nu| + \sum_{\nu=1}^{\infty} |a_\nu|$$

$$\leq \sqrt{\sum_{\nu=1}^{j} \nu |A_\nu|^2} \sqrt{\sum_{\nu=1}^{j} \frac{1}{\nu}} + \sqrt{\sum_{\nu=1}^{\infty} \nu |a_\nu|^2 \varrho^{-2\nu}} \sqrt{\sum_{\nu=1}^{\infty} \frac{\varrho^{2\nu}}{\nu}}$$

$$\leq c_1(\Gamma)\sqrt{\log(j+1)} \sqrt{\sum_{\nu=1}^{j} \nu |A_\nu|^2} + c_2(\Gamma) \sqrt{\sum_{\nu=1}^{\infty} \nu |a_\nu|^2 \varrho^{-2\nu}}$$

$$\leq c_3(\Gamma) \sqrt{\frac{\log(j+1)}{j+1}} \varrho^j, \tag{5.37}$$

by (5.34), which establishes the claim.

Hence, using the estimate (5.30) we get

$$Q_{j+1}(z) = \frac{\Phi^{j+1}(z)}{\sqrt{\pi(j+1)}} \left\{ 1 + O\left(\sqrt{(j+1)\log(j+1)}\right) \varrho^j \right\}, \quad z \in \Gamma. \tag{5.38}$$

Next, by Green's formula we have for fixed $k = 0, 1, \ldots$ and $n \geq k+1$:

$$\begin{aligned}
2\pi i \sqrt{\frac{n-k+1}{n+1}} b_{n-k,n} &= 2\pi i \sqrt{\frac{n-k+1}{n+1}} \langle z p_n, p_{n-k} \rangle \\
&= \frac{2\pi i}{2i} \sqrt{\frac{n-k+1}{n+1}} \int_\Gamma z p_n(z) \overline{Q_{n-k+1}(z)} dz \\
&= \int_\Gamma z \Phi^n(z) \Phi'(z) \overline{\Phi^{n-k+1}(z)} dz + h_n \\
&= \int_\Gamma \frac{\Phi^n(z)}{\Phi^{n-k+1}(z)} \Phi'(z) z dz + h_n \\
&= \int_{|w|=1} \frac{w^n}{w^{n-k+1}} \Psi(w) dw + h_n \\
&= 2\pi i b_k + h_n,
\end{aligned} \tag{5.39}$$

where

$$h_n = O(\sqrt{n})\varrho^n + O\left(\sqrt{(n-k+1)\log(n-k+1)}\right) \varrho^{n-k} \{1 + O(\sqrt{n})\varrho^n\}. \tag{5.40}$$

Thus, for $k \geq 0$ fixed and $\varrho < 1$,

$$\sqrt{\frac{n-k+1}{n+1}} b_{n-k,n} = b_k + O(\sqrt{n \log n}\, \varrho^n), \quad \text{as } n \to \infty. \tag{5.41}$$

It remains to prove (2.6). This follows at once from the strong asymptotics for the leading coefficient (5.30) and the relation (1.9). □

5.4. **Proof of Theorem 2.4.** We first note that our assumption (2.8), combined with (1.9), implies that

$$\limsup_{n\to\infty} \left| \sqrt{\frac{n+2}{n+1}} \frac{\lambda_n}{\lambda_{n+1}} - b \right|^{1/n} < 1. \tag{5.42}$$

Now set

$$\xi_n := \sqrt{\frac{n+2}{n+1}}\frac{\lambda_n}{\lambda_{n+1}}\frac{1}{b} - 1,$$

so that

$$\limsup_{n\to\infty} |\xi_n|^{1/n} < 1. \tag{5.43}$$

At the other hand, we have from (5.1) and (1.4) that

$$(1+\xi_n)^2 = \frac{1-\alpha_{n+1}}{1-\alpha_n}.$$

Hence,

$$\xi_n = \frac{\alpha_n - \alpha_{n+1}}{(1-\alpha_n)(2+\xi_n)},$$

and by using the fact that $\xi_n \to 0$, as $n \to \infty$ together with $0 \leq \alpha_n < 1$ and $\alpha_n \to 0$, as $n \to \infty$, we obtain the double inequality

$$c_1|\alpha_n - \alpha_{n+1}| \leq |\xi_n| \leq c_2|\alpha_n - \alpha_{n+1}|, \tag{5.44}$$

for some positive constants c_1 and c_2.

Now, by expanding α_n in the telescoping series

$$\alpha_n = (\alpha_n - \alpha_{n+1}) + (\alpha_{n+1} - \alpha_{n+2}) + \cdots,$$

we conclude, in view of (5.43)–(5.44), that

$$\limsup_{n\to\infty} \alpha_n^{1/n} < 1, \tag{5.45}$$

and this, in view of Theorem 1.3 in [13] leads to

$$\limsup_{n\to\infty} |b_n|^{1/n} < 1.$$

The last inequality implies that the conformal map $\Psi(w)$ has an analytic continuation across $\mathbb{T}$ into $\mathbb{D}$ (see (1.3)) and thus Γ is the analytic image of $\mathbb{T}$. Therefore, around any $w_0 \in \mathbb{T}$, the map Ψ can be represented by a Taylor series expansion of the form

$$\Psi(w) = \Psi(w_0) + a_1(w-w_0) + a_2(w-w_0)^2 + a_3(z-z_0)^3 \cdots.$$

If we had $\Psi'(w_0) = 0$, then

$$\Psi(w) = \Psi(w_0) + a_2(w-w_0)^2 + \cdots,$$

with $a_2 \neq 0$, because Ψ is univalent in Δ. These show that w_0 would be mapped by Ψ onto an exterior pointing cusp on Γ. Since, by assumption, this cannot happen, we see that $\Psi'(w) \neq 0$, $w \in \mathbb{T}$, which yields the required property that Γ is an analytic Jordan curve. $\square$

5.5. **Proof of Theorem 3.1.** Recall that $n := 2m$. On $|w| = R$, where $\varrho < 1 \leq R < \infty$, we have from (1.3) and (3.2)

$$|\Psi(w) - \Psi_m^{(n)}(w)| \leq |b^{(n)} - b|R + \sum_{k=0}^{m} \frac{|b_k^{(n)} - b_k|}{R^k} + \sum_{k=m+1}^{\infty} \frac{|b_k|}{R^k}.$$

Therefore, by using the result of Theorem 2.3 (see also (5.39)) and the estimate

$$|b_k| \leq c_1(\Gamma)\frac{\varrho^k}{\sqrt{k}}, \quad k \in \mathbb{N};$$

see [13, Cor. 1.1] we get

$$|\Psi(w) - \Psi_m^{(n)}(w)| \leq c_2(\Gamma)\varrho^{4m}R + c_3(\Gamma)\sqrt{m\log m}\varrho^m + c_4(\Gamma)\left(\frac{\varrho}{R}\right)^m, \quad (5.46)$$

which yields the desired estimate. □

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Center for Constructive Approximation, Department of Mathematics, Vanderbilt University, 1326 Stevenson Center, 37240 Nashville, USA
E-mail address: edward.b.saff@Vanderbilt.Edu
URL: http://www.math.vanderbilt.edu/~esaff/

Department of Mathematics and Statistics, University of Cyprus, P.O. Box 20537, 1678 Nicosia, Cyprus
E-mail address: nikos@ucy.ac.cy
URL: http://ucy.ac.cy/~nikos