

A FASCINATING POLYNOMIAL SEQUENCE ARISING FROM AN ELECTROSTATICS PROBLEM ON THE SPHERE

J. S. BRAUCHART^{1,*}, P. D. DRAGNEV^{2,†}, E. B. SAFF^{3,‡,§} and
C. E. VAN DE WOESTIJNE^{4,¶}

¹School of Mathematics and Statistics, University of New South Wales, Sydney,
NSW, 2052, Australia
e-mail: j.brauchart@unsw.edu.au

²Department of Mathematical Sciences, Indiana-Purdue University, Fort Wayne, IN 46805, USA
e-mail: dragnevp@ipfw.edu

³Center for Constructive Approximation, Department of Mathematics, Vanderbilt University,
Nashville, TN 37240, USA
e-mail: Edward.B.Saff@Vanderbilt.Edu

⁴Lehrstuhl für Mathematik und Statistik, Montanuniversität Leoben, Franz-Josef-Straße 18,
8700 Leoben, Austria
e-mail: c.vandewoestijne@unileoben.ac.at

(Received July 12, 2011; revised September 21, 2011; accepted October 11, 2011)

Abstract. A positive unit point charge approaching from infinity a perfectly spherical isolated conductor carrying a total charge of +1 will eventually cause a negatively charged spherical cap to appear. The determination of the smallest distance $\rho(d)$ (d is the dimension of the unit sphere) from the point charge to the sphere where still all of the sphere is positively charged is known as Gonchar's problem. Using classical potential theory for the harmonic case, we show that $1 + \rho(d)$ is equal to the largest positive zero of a certain sequence of monic polynomials of degree $2d - 1$ with integer coefficients which we call Gonchar polynomials. Rather surprisingly, $\rho(2)$ is the Golden ratio and $\rho(4)$ the lesser known Plastic number. But Gonchar polynomials have other interesting properties. We discuss their factorizations, investigate their zeros and present some challenging conjectures.

* The research of this author was supported, in part, by the Austrian Science Foundation (FWF) under grant S9603-N13. The author is recipient of an APART-fellowship of the Austrian Academy of Sciences at Vanderbilt (2009) and at UNSW (2010).

† The research of this author was supported, in part, by a Research-in-aid Grant from ORES, IPFW.

‡ The research of this author was supported, in part, by U.S. National Science Foundation Grant DMS-0808093.

§ Corresponding author.

¶ The research of this author was supported by the Austrian Science Foundation FWF under grants S9606 and S9611.

Key words and phrases: Coulomb potential, electrostatics problem, Golden ratio, Gonchar problem, Gonchar polynomial, Plastic number, signed equilibrium, sphere.

2010 Mathematics Subject Classification: primary 30C10, secondary 31B10.

1. Introduction

Let \mathbb{S}^d denote the unit sphere in the Euclidean space \mathbb{R}^{d+1} . Suppose that it is insulated and has a total positive charge of $+1$. In the absence of an external field the charge will distribute uniformly with respect to the normalized surface area measure (unit Lebesgue measure) σ_d . Now we introduce a positive unit point charge exterior to the sphere that repels the charge on the sphere in accordance with the Newtonian potential $1/r^{d-1}$, where r represents the distance between point charges. If this point charge is very close to the sphere, then one would expect it to cause a negatively charged spherical cap to appear, while if the point charge is very far from the sphere its influence is negligible and the charge on the sphere will be everywhere positive and nearly uniformly distributed over the entire sphere. We consider the following question: what is the smallest distance from the unit point charge to \mathbb{S}^d such that the distribution of the positive charge on the sphere covers all of the sphere? We will denote this critical distance by $\rho(d)$. As we shall show, $1 + \rho(d)$ equals the largest positive root of the following polynomial equation of degree $2d - 1$:

$$(1) \quad G(d; z) := [(z - 1)^d - z - 1]z^{d-1} + (z - 1)^d = 0, \quad d = 1, 2, 3, \dots$$

As the question above was communicated to the authors by A. A. Gonchar, we shall refer to $G(d; z)$ as *Gonchar polynomials*.

Rather surprisingly, $\rho(2)$ turns out to be the *Golden ratio*, which is the limit of the ratio of successive terms in the *Fibonacci sequence* $F_n = F_{n-1} + F_{n-2}$, $F_0 = F_1 = 1$ and $\rho(4)$ is the so-called *Plastic number* [7], which is the limit of ratios of successive terms for the less known *Padovan sequence* $P_n = P_{n-2} + P_{n-3}$, $P_0 = P_1 = P_2 = 1$ (sequence A000931 in Sloane's OEIS [6]). That is,

$$\rho(2) = \lim_{n \rightarrow \infty} \frac{F_{n+1}}{F_n} = \frac{1 + \sqrt{5}}{2} = 1.618033988 \dots,$$

$$\rho(4) = \lim_{n \rightarrow \infty} \frac{P_{n+1}}{P_n} = \frac{(9 - \sqrt{69})^{1/3} + (9 + \sqrt{69})^{1/3}}{2^{1/3}3^{2/3}} = 1.3247179572 \dots$$

In addition to this curious coincidence, the polynomials $G(d; z)$ exhibit rather fascinating properties with regard to their irreducibility over the ring of polynomials with integer coefficients, as well as the asymptotic behavior of their zeros. Our goal in the next section is to show how the Gonchar polynomials are derived and then, in Section 3, to explore some of their properties and draw the reader's attention to some related conjectures.

2. Signed equilibrium

A general charge distribution on $\mathbb{S}^d = \{ \mathbf{x} \in \mathbb{R}^{d+1} : |\mathbf{x}| = 1 \}$ will be modelled by a signed measure η supported on \mathbb{S}^d with $\eta(\mathbb{S}^d) = 1$. The corresponding *Newtonian potential* and *energy* are given by

$$V^\eta(\mathbf{x}) = \int_{\mathbb{S}^d} \frac{1}{|\mathbf{x} - \mathbf{y}|^{d-1}} d\eta(\mathbf{y}), \quad \mathcal{I}(\eta) = \int_{\mathbb{S}^d} V^\eta(\mathbf{x}) d\eta(\mathbf{x}).$$

For example, $V^\eta(\mathbf{x})$ is the familiar Coulomb potential when $d = 2$. For $\eta = \sigma_d$ we expect from the rotational invariance that the potential V^{σ_d} is constant on \mathbb{S}^d and on concentric spheres. This is a well-known fact in potential theory, but for completeness we provide a proof below.

LEMMA 1. *The potential $V^{\sigma_d}(\mathbf{x})$ satisfies*

$$(2) \quad V^{\sigma_d}(\mathbf{x}) = \begin{cases} 1 & \text{for } |\mathbf{x}| \leq 1, \\ 1/|\mathbf{x}|^{d-1} & \text{for } |\mathbf{x}| > 1. \end{cases}$$

PROOF. Indeed, for any fixed \mathbf{x} the Newtonian kernel $f(\mathbf{y}) = |\mathbf{x} - \mathbf{y}|^{1-d}$ is a harmonic function in $\mathbb{R}^{d+1} \setminus \{\mathbf{x}\}$; that is, $\Delta f(\mathbf{y}) = \sum_{i=1}^{d+1} f_{y_i y_i}(\mathbf{y}) \equiv 0$. Therefore, by the mean value property for harmonic functions, we have

$$(3) \quad V^{\sigma_d}(\mathbf{x}) = \int_{\mathbb{S}^d} f(\mathbf{y}) d\sigma_d(\mathbf{y}) = f(\mathbf{0}) = \frac{1}{|\mathbf{x}|^{d-1}} \quad \text{for } |\mathbf{x}| > 1.$$

Considering the sequence of functions $f_n(\mathbf{y}) = |(n + 1)\mathbf{x}/n - \mathbf{y}|^{1-d}$ and applying the monotone convergence theorem allows us to extend this to the unit sphere giving $V^{\sigma_d}(\mathbf{x}) \equiv 1$ for $|\mathbf{x}| = 1$. Since the potential $V^{\sigma_d}(\mathbf{x})$ is a harmonic function in $\mathbb{C} \setminus \mathbb{S}^d$, and because it is constant on the boundary \mathbb{S}^d , the maximum principle for harmonic functions implies that it is constant everywhere in the closed unit ball (“Faraday cage effect”). (One can also verify this directly by applying the identity $|\mathbf{x} - \mathbf{y}|^2 = |\mathbf{x}|^2 \left| \frac{\mathbf{x}}{|\mathbf{x}|^2} - \mathbf{y} \right|^2$ and (3).) \square

In the classical Coulomb case ($d = 2$), a standard electrostatics problem (see [3, Ch. 2]) is to find the charge distribution on a charged, insulated, conducting sphere in the presence of an external field (such as generated by a positive point charge q off the sphere). This motivates the following definition.

DEFINITION 2. Given a continuous function $Q(\mathbf{x})$ on \mathbb{S}^d , we call a signed measure η_Q supported on \mathbb{S}^d and of total charge $\eta_Q(\mathbb{S}^d) = 1$ a *signed equi-*

librium associated with Q if its weighted Newtonian potential is constant on \mathbb{S}^d ; that is, for some constant F_Q ,

$$V^{\eta_Q}(\mathbf{x}) + Q(\mathbf{x}) = F_Q \quad \text{everywhere on } \mathbb{S}^d.$$

In the above definition, the function Q is referred to as an *external field*. Thus Lemma 1 establishes that σ_d is the (signed) equilibrium measure on \mathbb{S}^d in the absence of an external field. The uniqueness of the signed equilibrium is settled by the following known proposition [1, Lemma 23], whose proof is presented in the Appendix.

PROPOSITION 3. *If a signed equilibrium η_Q associated with an external field Q on \mathbb{S}^d exists, then it is unique.*

Here, we are concerned with the external field generated by a positive point charge of amount q located at $\mathbf{a} = R\mathbf{p}$, $R > 1$, where \mathbf{p} is the North Pole of \mathbb{S}^d ; that is, $\mathbf{p} = (0, 0, \dots, 0, 1)$. Such a field is given by

$$(4) \quad Q_{\mathbf{a},q}(\mathbf{x}) := \frac{q}{|\mathbf{x} - \mathbf{a}|^{d-1}}, \quad \mathbf{x} \in \mathbb{R}^{d+1},$$

and the associated signed equilibrium is described in the next result. (For the general result, see [1].)

LEMMA 4. *The signed equilibrium η_Q associated with the external field $Q = Q_{\mathbf{a},q}$ of (4) is absolutely continuous with respect to the (normalized) surface area measure on \mathbb{S}^d ; that is, $d\eta_Q(\mathbf{x}) = \eta'_{R,q}(\mathbf{x})d\sigma_d(\mathbf{x})$, and its density is given by*

$$(5) \quad \eta'_{R,q}(\mathbf{x}) = 1 + \frac{q}{R^{d-1}} - \frac{q(R^2 - 1)}{|\mathbf{x} - \mathbf{a}|^{d+1}}, \quad \mathbf{x} \in \mathbb{S}^d.$$

For the classical Coulomb case this relation is well-known from elementary physics (cf. [3, p. 61]).

PROOF. We note first that if $|\mathbf{x}| > 1$, then the function $u(\mathbf{z}) := |\mathbf{z} - \mathbf{x}|^{1-d}$ is harmonic in $|\mathbf{z}| \leq 1$, and since the Poisson integral formula [2] preserves harmonic functions, we have

$$(6) \quad \frac{1}{|\mathbf{z} - \mathbf{x}|^{d-1}} = \int_{\mathbb{S}^d} \frac{1}{|\mathbf{y} - \mathbf{x}|^{d-1}} \frac{1 - |\mathbf{z}|^2}{|\mathbf{z} - \mathbf{y}|^{d+1}} d\sigma_d(\mathbf{y}), \quad |\mathbf{z}| < 1.$$

Using a monotone convergence theorem argument as in Lemma 1 we extend (6) to $|\mathbf{x}| = 1$. We shall make use of the identity $|R\mathbf{y} - \mathbf{x}| = |\mathbf{y} - R\mathbf{x}|$ for any $|\mathbf{x}| = |\mathbf{y}| = 1$. Then from (6) we obtain with $\mathbf{z} = \mathbf{p}/R$ and $|\mathbf{x}| = 1$,

$$(7) \quad Q(\mathbf{x}) = \frac{q}{|R\mathbf{p} - \mathbf{x}|^{d-1}} = \frac{q}{|\mathbf{p} - R\mathbf{x}|^{d-1}} = \frac{q}{R^{d-1}|\mathbf{z} - \mathbf{x}|^{d-1}}$$

$$\begin{aligned}
 &= \int_{\mathbb{S}^d} \frac{q}{R^{d-1}|\mathbf{y} - \mathbf{x}|^{d-1}} \frac{1 - |\mathbf{z}|^2}{|\mathbf{z} - \mathbf{y}|^{d+1}} d\sigma_d(\mathbf{y}) \\
 &= \int_{\mathbb{S}^d} \frac{q}{|\mathbf{y} - \mathbf{x}|^{d-1}} \frac{R^2 - 1}{|\mathbf{p} - R\mathbf{y}|^{d+1}} d\sigma_d(\mathbf{y}) = \int_{\mathbb{S}^d} \frac{1}{|\mathbf{y} - \mathbf{x}|^{d-1}} \frac{q(R^2 - 1)}{|\mathbf{a} - \mathbf{y}|^{d+1}} d\sigma_d(\mathbf{y}).
 \end{aligned}$$

The Poisson integral formula applied to the constant function $u(\mathbf{z}) \equiv q/R^{d-1}$ and $\mathbf{z} = \mathbf{p}/R$ yields

$$\frac{q}{R^{d-1}} = \int_{\mathbb{S}^d} \frac{q(R^2 - 1)}{|\mathbf{a} - \mathbf{y}|^{d+1}} d\sigma_d(\mathbf{y}).$$

This together with (7) implies that for $d\nu := \eta'_{R,q} d\sigma_d$ we have

$$\int_{\mathbb{S}^d} d\nu = 1 \quad \text{and} \quad V^\nu(\mathbf{x}) + Q(\mathbf{x}) = 1 + \frac{q}{R^{d-1}}, \quad \mathbf{x} \in \mathbb{S}^d,$$

which proves the lemma. \square

REMARK. A surprising aspect of the equilibrium support, which varies with R , is illustrated in Fig. 1 for \mathbb{S}^2 and $q = 1$. Writing $\eta_{Q_{a,1}} = \eta_Q = \eta_Q^+ - \eta_Q^-$ in terms of its Jordan decomposition, it is clear on geometrical and physical grounds that the support of η_Q^+ is a spherical cap centered at the South Pole, while its complement (a spherical cap centered at the North Pole) is the support of η_Q^- , see Fig. 1(a). Although somewhat counter-intuitive, we see from Fig. 1(b) that, as the unit charge approaches the sphere ($R \rightarrow 1$), from a certain distance on, the support of η_Q^+ occupies increasingly more of the sphere. This phenomenon can be explained by the geometry of the sphere, where an increasingly ‘needle-like’ negative sink forming at the North Pole is balanced by an increasingly uniform positive part so that the total charge is always one.

From formula (5) we observe that the minimum value of the density $\eta'_{R,q}(\mathbf{x})$ is attained at the North Pole \mathbf{p} , and its value there is

$$(8) \quad \eta'_{R,q}(\mathbf{p}) = 1 + \frac{q}{R^{d-1}} - \frac{q(R^2 - 1)}{|\mathbf{a} - \mathbf{p}|^{d+1}} = 1 + \frac{q}{R^{d-1}} - \frac{q(R^2 - 1)}{(R - 1)^{d+1}},$$

which we write in the equivalent form

$$\eta'_{R,q}(\mathbf{p}) = \frac{q}{R^{d-1}(R - 1)^d} G(d, q; R),$$

where

$$(9) \quad G(d, q; z) := [(z - 1)^d/q - z - 1] z^{d-1} + (z - 1)^d.$$

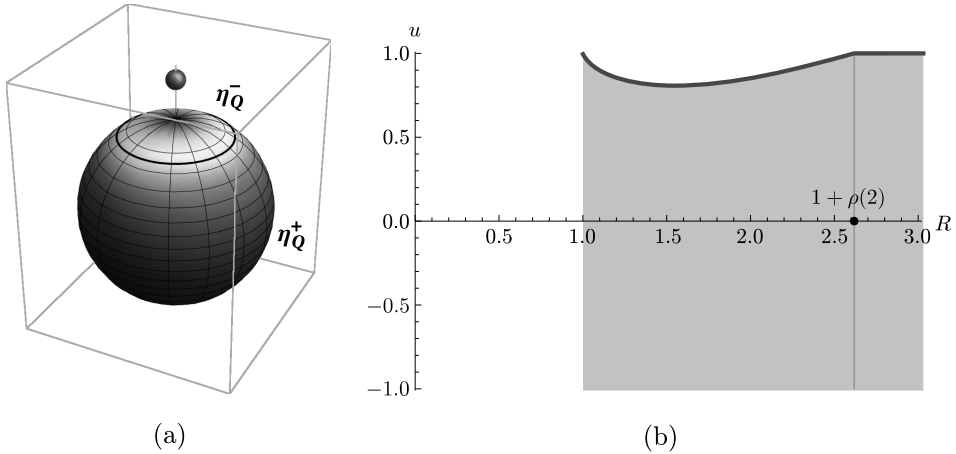


Fig. 1: Signed equilibrium η_Q on \mathbb{S}^2 (left) and projection onto the polar axis of the positive part η_Q^+

This leads to the following theorem.

THEOREM 5. *For the external field $Q_{\mathbf{a},q}(\mathbf{x})$ of (4) with $\mathbf{a} = R\mathbf{p}$, the signed equilibrium is a positive measure on all of \mathbb{S}^d if and only if $R \geq R_q$, where R_q is the unique (real) zero in $(1, +\infty)$ of the polynomial $G(d, q; z)$.*

In particular, the solution to Gonchar’s problem is given by $\rho(d) = R_1 - 1$.

PROOF. We use the fact that the support of $\eta_{Q_{\mathbf{a},q}}$ is all of \mathbb{S}^d if and only if $\eta'_{R,q}(\mathbf{p}) \geq 0$ in (8) or equivalently that $G(d, q; R) \geq 0$. Hence we seek the number $R > 1$ such that $G(d, q; R) = 0$. Observing that $G(d, q; 1) < 0$ and $G(d, q; x) > 0$ for $x > 1$ sufficiently large, there exists at least one value R_q such that $G(d, q; R_q) = 0$. Moreover, this root in $(1, \infty)$ is unique as can be seen by applying Descartes’ Rule of Signs to

$$G(d, q; 1 + w) = \frac{1}{q} \sum_{m=0}^{d-1} \binom{d-1}{m} w^{m+d} - \sum_{m=0}^{d-1} \left[\binom{d}{m} + \binom{d-1}{m} \right] w^m. \quad \square$$

Curiously, for $d = 2$ and $d = 4$ we obtain the Golden ratio and the Plastic number as answers to Gonchar’s problem as mentioned in the Introduction. Furthermore, for large values of d the asymptotic analysis (provided in the Appendix) shows that

$$(10) \quad R_q = 2 + [\log(3q)]/d + \mathcal{O}(1/d^2) \quad \text{as } d \rightarrow \infty.$$

The appearance of 2 as the limit as $d \rightarrow \infty$ of the critical distances R_q ($q > 0$ fixed) can be explained by studying the asymptotic behavior of $G(d, q; R)$ as

$d \rightarrow \infty$. Indeed, for $R \geq 2 + \varepsilon$ ($\varepsilon > 0$ fixed) it goes to $+\infty$ as $d \rightarrow \infty$, while for $1 < R \leq 2 - \varepsilon$ ($\varepsilon > 0$ fixed) it goes to $-\infty$, leaving the number 2 as the only candidate.

3. The polynomials $G(d; z)$

In the following we investigate the family of Gonchar polynomials $G(d, z) = G(d, 1; z)$ given in (1). Aside from the solution to Gonchar’s problem, these polynomials are interesting in themselves and their distinctive properties merit further study.

A polynomial P with real coefficients is called (*self-*)*reciprocal* if its *reciprocal polynomial* $P^*(z) := z^{\deg P} P(1/z)$ coincides with $P(z)$. In other words, the coefficients of z^k and of $z^{\deg P - k}$ in $P(z)$ are the same. Notice that $G(d; z)$ is *self-reciprocal* for even d since

$$z^{2d-1}G(d; 1/z) = [(1 - z)^d - z - 1] z^{d-1} + (1 - z)^d.$$

Consequently, if ζ is a zero of $G(d; z)$, then so is $1/\zeta$ for even d . For odd d we infer from

$$\begin{aligned} G(d; z) + G^*(d; z) &= [1 + (-1)^d] (z - 1)^d (z^{d-1} + 1) - 2(z^d + z^{d-1}) \\ &= -2(z^d + z^{d-1}) \end{aligned}$$

that the coefficients of z^k and z^{2d-1-k} in $G(d; z)$ sum to zero except for the ‘innermost’ pair.

3.1. Factorizations and irreducibility. With the aid of symbolic computation programs one can find factorizations and check irreducibility of explicitly given polynomials. For $d = 1, 2, \dots, 7$, we thereby obtain the following factorizations over the integers of $G(d; z)$.

$$\begin{aligned} G(1; z) &= z - 3, \\ G(2; z) &= (z + 1)(z^2 - 3z + 1), \\ G(3; z) &= z^5 - 3z^4 + 3z^3 - 5z^2 + 3z - 1, \\ G(4; z) &= (z + 1)(z^3 - 3z^2 + 2z - 1)(z^3 - 2z^2 + 3z - 1), \\ G(5; z) &= z^9 - 5z^8 + 10z^7 - 10z^6 + 5z^5 - 7z^4 + 10z^3 - 10z^2 + 5z - 1, \\ G(6; z) &= (z + 1)(z^2 - z + 1)(z^8 - 6z^7 + 15z^6 - 21z^5 + 21z^4 - 21z^3 \\ &\quad + 15z^2 - 6z + 1), \end{aligned}$$

$$G(7; z) = z^{13} - 7z^{12} + 21z^{11} - 35z^{10} + 35z^9 - 21z^8 + 7z^7 - 9z^6 + 21z^5 - 35z^4 + 35z^3 - 21z^2 + 7z - 1.$$

One can easily verify that $(z + 1)$ divides $G(d; z)$ if and only if d is even. Furthermore, the factor $z^2 - z + 1$ arises in the following cases.

PROPOSITION 6. *The cyclotomic polynomial $z^2 - z + 1$ divides $G(d; z)$ if and only if 6 divides d .*

PROOF. Note that if ζ is a zero of $z^2 - z + 1$, then $\zeta^2 = \zeta - 1$ and $\zeta^3 = -1$. Using formula (1), it readily follows that $G(d; \zeta) = 0$ whenever $d \equiv 0 \pmod{6}$ and $G(d; \zeta) \neq 0$ otherwise. \square

A general irreducibility result has so far eluded the authors. The Eisenstein criterion and, in general, reduction to finite fields, seem not to be effective tools for studying the polynomials $G(d; z)$. However, using Mathematica, we verified the following conjecture for d up to 500.

CONJECTURE 1. *Set $\ell(d; z) \equiv 1$ for d odd, $\ell(d; z) := z + 1$ for d even but not divisible by 6, and $\ell(d; z) := (z + 1)(z^2 - z + 1)$ if 6 divides d . Then $G(d; z)/\ell(d; z)$ is irreducible over the rationals except for $d = 4, 8$ and 12.*

Regarding the exceptional cases we record that, in addition to $G(4; z)$ as given above,

$$G(8; z) = (z + 1)(z^4 - 3z^3 + 3z^2 - 3z + 1)(z^{10} - 6z^9 + 16z^8 - 24z^7 + 24z^6 - 21z^5 + 24z^4 - 24z^3 + 16z^2 - 6z + 1),$$

$$G(12; z) = (z + 1)(z^2 - z + 1)(z^6 - 4z^5 + 5z^4 - 3z^3 + 5z^2 - 4z + 1) \times (z^{14} - 8z^{13} + 29z^{12} - 62z^{11} + 85z^{10} - 77z^9 + 48z^8 - 33z^7 + 48z^6 - 77z^5 + 85z^4 - 62z^3 + 29z^2 - 8z + 1).$$

3.2. Zeros of Gonchar polynomials. Fig. 2 illustrates some features of $G(d; z)$ on the real line and suggests some of its general properties. Note the qualitatively different behavior for even and odd d .

Depending on the parity of d , the polynomial equation $G(d; z) = 0$ has either one (d odd) or three (d even) real simple roots. More precisely, the following holds.

PROPOSITION 7. *If d is odd, then $G(d; z)$ has precisely one real zero, which is simple and lies in the interval $(2, 3]$. If d is even, then $G(d; z)$ has exactly three real zeros: one at -1 , one in the interval $(1/3, 1/2)$ and one in the interval $(2, 3)$; all these zeros are simple.*

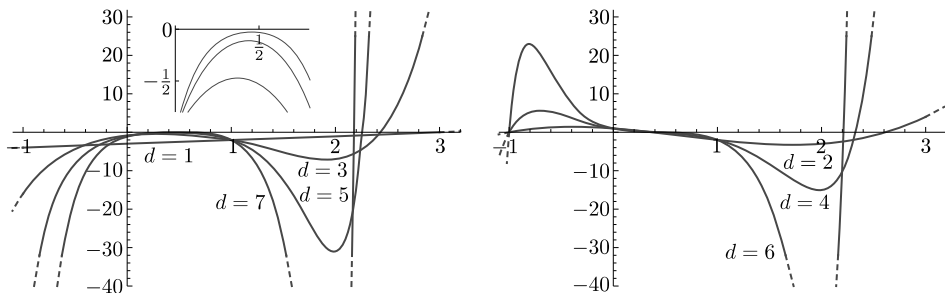


Fig. 2: The polynomial $G(d; z)$ on the real line for $d = 1, 2, \dots, 7$

PROOF. Note that for $d \geq 2$ there holds:

$$G(d; -1) = (-2)^d [1 - (-1)^d], \quad G(d; 0) = (-1)^d,$$

$$G(d; 1) = -2, \quad G(d; 2) = 1 - 2^d.$$

The assertion of the proposition is trivial for $G(1; z) = z - 3$. So let $d \geq 2$. Since $G(d; 2) < 0$ and $G(d; 3) > 0$, the polynomial $G(d; z)$ has at least one real zero in the interval $(2, 3)$ and, as we observed in the proof of Theorem 5, this is its only zero on $[1, \infty)$ and must be simple. For odd d and $x < 1$ we have $[(x - 1)^d - x - 1] x^{d-1} \leq 0$ and $(x - 1)^d < 0$, so $G(d; x) < 0$; thus $G(d; x)$ has no real zeros outside $(2, 3)$. For even d , the self-reciprocity of $G(d; x)$ implies that to each zero $\xi(d)$ of $G(d; x)$ in $[1, \infty)$ there is a zero $\xi^*(d) = 1/\xi(d)$ in $(0, 1]$. Consequently, by the first part of the proof, $G(d; x)$ has one and only one zero in $(0, 1)$ and this zero is simple and lies in the interval $(1/3, 1/2)$. It remains only to consider the interval $(-\infty, 0]$. Clearly $G(d; -1) = 0$, and by computing $G'(d; x)$ and analyzing its sign (in particular, $G'(d; -1) > 0$), one can show that $G(d; x)$ is strictly increasing on $(-\infty, -1)$. Thus $G(d; x) < 0$ on $(-\infty, -1)$ and, by self-reciprocity, $G(d; x) > 0$ on $(-1, 0)$. \square

Regarding the behavior as d increases of the zeros in $(2, 3)$, M. Lamprecht [4] proved the following.

PROPOSITION 8. *The zeros $\xi(d)$ of $G(d; z)$ in the interval $(2, 3]$ form a strictly monotonically decreasing sequence with limit point 2 (compare with (10)).*

We now turn to the study of the complex zeros of $G(d; z)$. Observe that by Proposition 7, $G(d; z)$ has either $d - 1$ (if d is odd) or $d - 2$ (if d is even) pairs of complex conjugated zeros (counting multiplicity). In Fig. 3 we have plotted the zeros of $G(d; z)$ for $d = 9, 10, 11$, and 12 along with the two circles $\mathcal{C}_0, \mathcal{C}_1$ each of radius 1 centered, respectively, at 0 and 1. Notice that

these circles intersect at the points $(1 \pm i\sqrt{3})/2$ and, by Proposition 6, these points are zeros of $G(d; z)$ if and only if d is a multiple of 6. The zeros of $G(d; z)$ seem to occur roughly into three categories: zeros close to C_0 (indicated by \times), zeros close to C_1 (indicated by $+$) and zeros close to the vertical line $x = 1/2$ (indicated by \blacklozenge). The numbers N_1, N_2, N_3 of zeros in each of the categories are listed in Table 1 for $d = 1, \dots, 12, 42$, from which it appears that these numbers are nearly the same. We will discuss this further in Section 3.3 (see Conjecture 2).

Fig. 3 suggests that for even d the polynomial $G(d; z)$ may have zeros lying precisely on C_0 . This, in fact, is the case as we now prove.

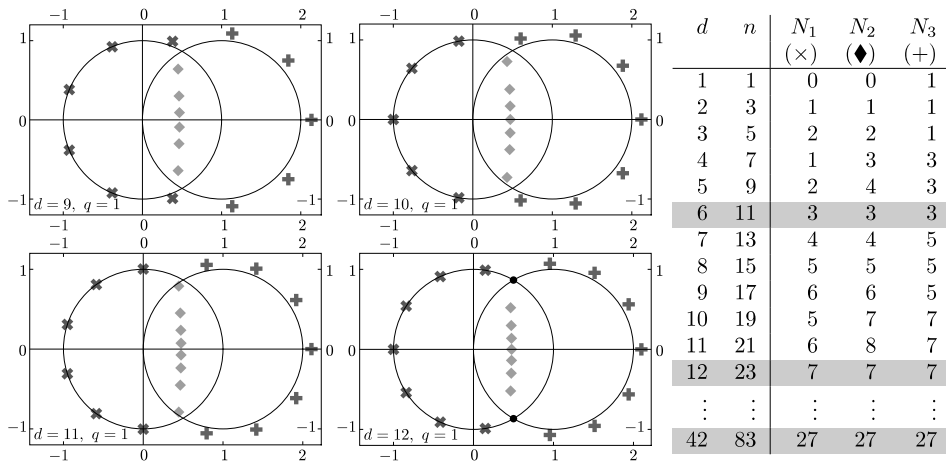


Fig. 3: Counting the zeros of $G(d; z)$, $n = 2d - 1$. Two additional zeros appear when 6 divides d , see shaded entries in table

PROPOSITION 9. If d is even, $G(d; z)$ has exactly $4\lfloor d/6 \rfloor + 1$ zeros on the unit circle C_0 ; all are simple and satisfy $\text{Re } z \leq 1/2$. Their positions are determined by the solutions of the equation $g(\theta) = f(\theta)$ given in (11) below.

If d is odd, there are no zeros of $G(d; z)$ on the unit circle C_0 .

PROOF. Since $z = 1$ is not a zero of $G(d; z)$, the equation $G(d; z) = 0$ is equivalent to

$$z^{d-1} + 1 = \frac{z + 1}{z - 1} \frac{z^{d-1}}{(z - 1)^{d-1}}.$$

Substituting $z = e^{i\theta}$ and changing to trigonometric functions we arrive at

$$(11) \quad g(\theta) := (-1)^{d/2} \cos\left(\frac{d-1}{2}\theta\right) = (-1)^{d/2} \frac{\cos(\theta/2)}{[2i \sin(\theta/2)]^d} =: f(\theta).$$

Suppose d is even. Then $f(\theta) = \cos(\theta/2)/[2 \sin(\theta/2)]^d$ is real-valued. Since the complex zeros of $G(d; z)$ occur in complex conjugate pairs, we may assume $0 \leq \theta \leq \pi$. Using elementary calculus one shows that the function $f(\theta)$ is monotone decreasing and convex on $(0, \pi)$. Hence,

$$f(\theta) \geq f(\pi/3) + f'(\pi/3)(\theta - \pi/3) = \frac{\sqrt{3}}{2} + \frac{3d+1}{4} \left(\frac{\pi}{3} - \theta\right), \quad \theta \in (0, \pi/3).$$

This implies that $f(\theta) > 1$ for $\theta \in (0, \pi/3 - \alpha)$, where $\alpha = (4 - 2\sqrt{3})/(3d + 1)$. On the other hand, on $[\pi/3 - \alpha, \pi/3)$ one can show that $g(\theta) < \sqrt{3}/2 < f(\theta)$. This follows easily from the fact that α is smaller than a sixth of the period of $g(\theta)$ and an inspection of the three canonical cases in Fig. 4. Hence, in (11) we have only to consider the range $[\pi/3, \pi]$ whereupon $f(\theta) \leq f(\pi/3) = \sqrt{3}/2 < 1$.

The function g is a cosine function with period $4\pi/(d - 1)$, where the sign factor ensures that g has a positive derivative at $\theta = \pi$ (which is also an intersection point of f and g). The other intersection points of f and g occur in the half-periods where $g \geq 0$. On such a half-period I the convex function f and the concave function g (when restricted to I) can intersect in at most two points counting multiplicity, as can be seen by applying Rolle’s theorem to the strictly convex function $f - g$. That there are at least two such points on I can be seen from the intermediate value theorem applied to the same function ($f - g > 0$ at the endpoints of I and $f - g < 0$ at the midpoint of I) as illustrated in Fig. 4.

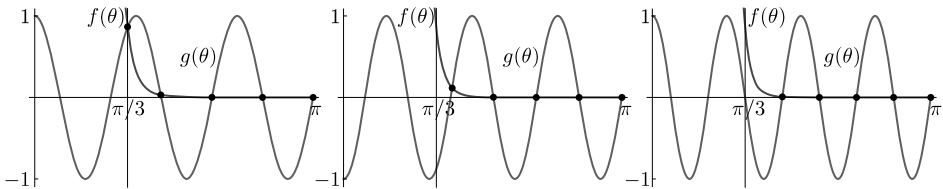


Fig. 4: Graphs of $f(\theta)$ and $g(\theta)$; typical cases when solving equation (11)

Since there are $\lfloor (d - 1)/6 \rfloor$ full periods of g in $[\pi/3, \pi]$ plus one more period (partially contained in $[\pi/3, \pi]$) whenever 6 divides d , we have $2\lfloor d/6 \rfloor$ zeros in the upper half-plane and that many conjugate (nonreal) zeros of $G(d; z)$ in the lower half-plane.

For odd d , equation (11) has no real solution in $(0, \pi]$. Since 1 is not a zero of $G(d; z)$, there are no zeros of $G(d; z)$ on \mathcal{C}_0 . \square

THEOREM 10. *All zeros of $G(d; z)$ are simple for each $d \geq 1$.*

PROOF. The cases $d = 1$ and $d = 2$ are obvious, so assume $d \geq 3$. Let ζ be a zero of $G(d; z)$. Then ζ is simple if $G'(d; \zeta) \neq 0$. By means of some helpful substitutions for the expressions in braces¹, we find that

$$\begin{aligned} \zeta(\zeta - 1)G'(d; \zeta) &= d\zeta\{(\zeta - 1)^d(\zeta^{d-1} + 1)\} - \zeta^d(\zeta - 1) + (d - 1) \\ &\quad \times \{[(\zeta - 1)^d - \zeta - 1]\zeta^{d-1}\}(\zeta - 1) \\ &= (d - 1)[\zeta^{d+1} - (\zeta - 1)^{d+1}] + (d + 1)\zeta^d \\ &= (d - 1)\zeta\{\zeta^d - (\zeta - 1)^d\} + (d - 1)(\zeta - 1)^d + (d + 1)\zeta^d \\ &= (d - 1)(\zeta - 1)^d(\zeta^d + 1) + 2\zeta^d =: P(d; \zeta). \end{aligned}$$

Suppose to the contrary that $P(d; \zeta) = 0$. Then on replacing $(\zeta - 1)^d$ by $-2\zeta^d/[(d - 1)(\zeta^d + 1)]$ in the formula (1) for $G(d; \zeta)$ we get

$$(12) \quad -(\zeta^d + 1)G(d; \zeta) = \zeta^{d-1} \left(\zeta^{d+1} + \frac{d+1}{d-1}\zeta^d + \frac{d+1}{d-1}\zeta + 1 \right).$$

The polynomial $Q(d; z)$ of degree $d + 1$ obtained by replacing ζ by z in the second parenthetical expression in (12) has three real zeros (at -1) and $d - 2$ complex zeros for even d and two negative zeros ('near' -1) and $d - 1$ complex zeros for odd d .² This can be seen from the facts that

$$Q(d; -1) = -\frac{2[1 - (-1)^d]}{d - 1}, \quad Q'(d; -1) = [1 - (-1)^d] \frac{d + 1}{d - 1},$$

$$Q''(d; -1) = 0$$

and Descartes' Rule of Signs. Substituting $w = e^{i\phi}$ and using trigonometric functions, we arrive at

$$Q(d; e^{i\phi}) = \frac{4e^{i[(d+1)/2]\phi}}{d - 1} \cos \frac{\phi}{2} \cos \frac{d\phi}{2} \left(d + \tan \frac{\phi}{2} \tan \frac{d\phi}{2} \right).$$

¹ The replacements are $(\zeta - 1)^d(\zeta^{d-1} + 1) \mapsto \zeta^d + \zeta^{d-1}$, $[(\zeta - 1)^d - \zeta - 1]\zeta^{d-1} \mapsto -(\zeta - 1)^d$ and $\zeta^d - (\zeta - 1)^d \mapsto \zeta^{d-1}[(\zeta - 1)^d - 1]$.

² Interestingly, if $\frac{d+1}{d-1}$ is changed to $\frac{d-3}{d-1}$ in $Q(d; z)$, then the new polynomial has all its zeros on the unit circle.

By symmetry, the number of solutions in the open set $(0, 2\pi) \setminus \{\pi\}$ of the equation

$$(13) \quad \tan \frac{\phi}{2} \tan \frac{d\phi}{2} = -d$$

equals the number of zeros of $\tan(d\phi/2)$ in $(0, 2\pi)$ (cf. Fig. 5), which is $d - 2$ for even d and $d - 1$ for odd d . Thus, all zeros of $Q(d; z)$ are accounted for.

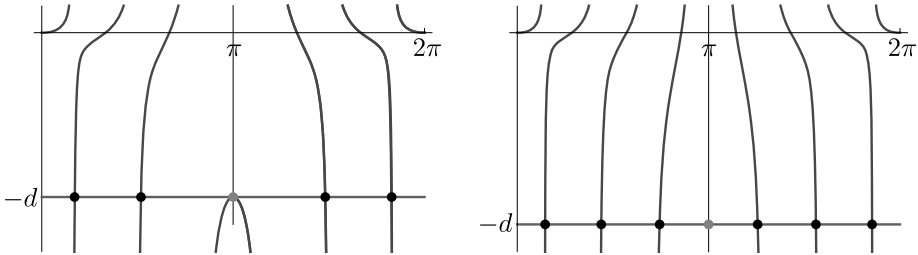


Fig. 5: Typical cases when solving equation (13)

They are either negative or complex conjugate pairs of zeros located on the unit circle.

For odd d , no zero ζ of $G(d; z)$ is on the unit circle (Proposition 9) or negative (Proposition 7), so $Q(d; \zeta) \neq 0$, which contradicts (12). Suppose d is even. If ζ is on the unit circle \mathcal{C}_0 , then it is simple (Proposition 9). If ζ is not on \mathcal{C}_0 , then $Q(d; \zeta) \neq 0$, which again contradicts (12). \square

3.3. Asymptotics of Gonchar polynomials. Numerically computing the zeros for $G(d; z)$ for small values of d , we observe (cf. Figs. 3 and 6 and Table 1) that they essentially form three groups separated by the sets

$$(14a) \quad A_1 := \{z \in \mathbb{C} : \operatorname{Re} z < 1/2, |z - 1| > 1\},$$

$$(14b) \quad A_2 := \{z \in \mathbb{C} : |z| < 1, |z - 1| < 1\},$$

$$(14c) \quad A_3 := \{z \in \mathbb{C} : \operatorname{Re} z > 1/2, |z| > 1\}.$$

For the purpose of asymptotic analysis (large d) we rewrite the equation $G(d; z) = 0$ in three different ways to emphasize an exponentially decaying right-hand side when considering zeros of $G(d; z)$ from the indicated part of the complex plane: for $\operatorname{Re} z < 1/2$

$$(15a) \quad z^{d-1} + 1 = \frac{z + 1}{z - 1} \left(\frac{z}{z - 1} \right)^{d-1},$$

for $|z - 1| |z| < 1$

$$(15b) \quad (z - 1)^d - (z + 1)z^{d-1} = -(z - 1)[(z - 1)z]^{d-1},$$

and for $\text{Re } z > 1/2$

$$(15c) \quad (z - 1)^d - z - 1 = -(z - 1) \left(\frac{z - 1}{z} \right)^{d-1}.$$

The following theorem concerning the limit behavior of the zeros of $G(d; z)$ as $d \rightarrow \infty$ is illustrated in Fig. 6.

THEOREM 11. *Let Γ be the set consisting of the boundary of the union of the two unit disks centered at 0 and 1 and the line-segment connecting the intersection points as indicated in Fig. 6. Then, as $d \rightarrow \infty$, all the zeros of $G(d; z)$ tend to Γ , and every point on Γ attracts zeros of these polynomials.*

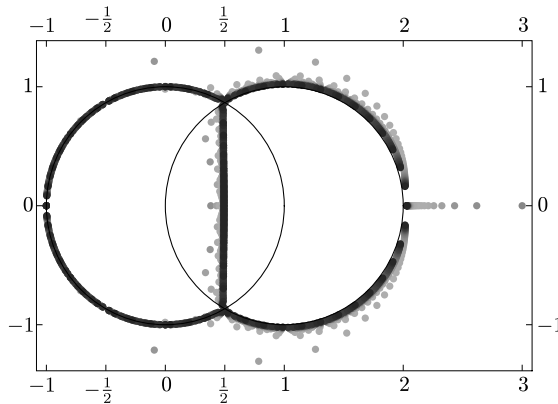


Fig. 6: Zeros of $G(d; z)$ for $d = 1, 2, \dots, 40$

PROOF. First, we observe that a closed set K in $\mathbb{C} \setminus \Gamma$ is free of zeros of $G(d; z)$ for sufficiently large d as can be seen from the following relations obtained from (15a):

$$\begin{aligned} & \lim_{d \rightarrow \infty} \left| 1 + \frac{1}{z^{d-1}} - \frac{z + 1}{z - 1} \left(\frac{1}{z - 1} \right)^{d-1} \right|^{1/d} \\ &= \begin{cases} 1 & \text{if } |z| > 1, |z - 1| > 1, \\ 1/|z - 1| & \text{if } |z| > 1, |z - 1| < 1, \end{cases} \end{aligned}$$

$$\begin{aligned}
& \lim_{d \rightarrow \infty} \left| \frac{z+1}{z-1} - \left(\frac{z-1}{z} \right)^{d-1} (z^{d-1} + 1) \right|^{1/d} \\
&= \begin{cases} 1 & \text{if } |z| < 1, \left| \frac{z-1}{z} \right| < 1 \text{ (Re } z > 1/2), \\ \left| \frac{z-1}{z} \right| & \text{if } |z| < 1, \left| \frac{z-1}{z} \right| > 1 \text{ (Re } z < 1/2), \end{cases} \\
& \lim_{d \rightarrow \infty} \left| z^{d-1} + 1 - \frac{z+1}{z-1} \left(\frac{z}{z-1} \right)^{d-1} \right|^{1/d} \\
&= \begin{cases} |z| & \text{if } |z| > 1, \left| \frac{z}{z-1} \right| < 1 \text{ (Re } z < 1/2), \\ 1 & \text{if } |z| < 1, \left| \frac{z}{z-1} \right| < 1 \text{ (Re } z < 1/2). \end{cases}
\end{aligned}$$

The second part of the assertion, that every point of Γ attracts zeros, is proved by contradiction. Given a supposedly non-attracting point w on Γ , there is a sufficiently small³ open disk D_w centered at w containing no zeros of $G(d; z)$ for all sufficiently large d . It is possible to then define a single-valued analytic branch of the d -th root of any of the rational functions whose moduli appear after the lim signs above. Thereby, we obtain sequences of functions which are analytic and uniformly bounded in D_w . Such sequences form normal families in D_w . According to the right-hand sides above, at least one limit function of these families (which is necessarily analytic in D_w) will have the property that its modulus is 1 in one part and is non-constant in the other part of D_w which is separated by Γ . This gives the desired contradiction, since an analytic function in a domain that has constant modulus on a subdomain must be constant throughout the whole domain. Consequently, each point of Γ attracts zeros of $G(d; z)$ as $d \rightarrow \infty$. \square

It is inviting to compare the zeros of $G(d; z)$ with the ones of the polynomials given at the left-hand sides of (15a), (15b) and (15c). Such comparisons will likely lead to a finer analysis of the properties of the zeros of $G(d; z)$.

We conclude this note with some challenging conjectures.

³The disk is small enough that its intersection with Γ is contained either in C_0 , C_1 or the open line-segment connecting the intersection points of C_0 and C_1 .

CONJECTURE 2. *For every positive integer d , the zeros of $G(d; z)$ form three groups separated by the sets A_1 , A_2 and A_3 except when 6 divides d in which case one also has the zeros $(1 \pm i\sqrt{3})/2$.*

When counting the zeros in the sets A_1 , A_2 and A_3 a very regular pattern emerges, which can be seen from Table 1. In fact, inspection of this table shows that the values of column N_1 (increased by 2 when 6 divides d) agree with the number of zeros on \mathcal{C}_0 obtained in Proposition 9. Assuming that this is true for all $d \geq 1$, by self-reciprocity of $G(d; z)$, it would follow that $N_1 = N_2 = N_3 = 4k - 1$ if $d = 6k$. *We expect that the zero counting scheme indicated in the Table 1 generalizes to all $d \geq 1$.*

Numerically, the zeros in A_1 and A_3 can be found near the respective unit circle \mathcal{C}_0 and \mathcal{C}_1 .

CONJECTURE 3. *If d is even, all the zeros of $G(d; z)$ in A_1 are the zeros on the unit circle \mathcal{C}_0 given in Proposition 9. If d is odd, the zeros of $G(d; z)$ in A_1 alternately lie inside and outside \mathcal{C}_0 . Furthermore, the zeros of $G(d; z)$ in A_3 are always outside of \mathcal{C}_1 .*

CONJECTURE 4. *The zeros of $G(d; z)$ in A_2 are located on a curve which is convex from the left.*

Acknowledgement. The first author is grateful to Don Zagier and Wadim Zudilin for inspiring this work when attending the workshop “Geometry and Arithmetic around Hypergeometric Functions” from September 28th–October 4th, 2008, which was made possible by the Mathematisches Forschungsinstitut Oberwolfach (MFO) and the Oberwolfach–Leibniz–Fellow Programme (OWFL).

Appendix A

PROOF OF PROPOSITION 3. Suppose η_1 and η_2 are two signed equilibria on \mathbb{S}^d associated with the same external field Q . Then

$$V^{\eta_1}(\mathbf{x}) + Q(\mathbf{x}) = F_1, \quad V^{\eta_2}(\mathbf{x}) + Q(\mathbf{x}) = F_2 \quad \text{for all } \mathbf{x} \in \mathbb{S}^d.$$

Subtracting the two equations and integrating with respect to $\eta := \eta_1 - \eta_2$ we obtain

$$\mathcal{I}(\eta) = \int V^\eta(\mathbf{x}) d\eta(\mathbf{x}) = \int \int \frac{1}{|\mathbf{x} - \mathbf{y}|^{d-1}} d\eta(\mathbf{x}) d\eta(\mathbf{y}) = 0,$$

which, from the strict positivity of the Newtonian kernel (cf. [5, Theorem 1.15]), implies $\eta = 0$. \square

PROOF OF ASYMPTOTIC (10). Let ζ_d denote the largest real zero of $G(d, q; z)$ (that is $R_q = \zeta_d$). We may assume (see discussion regarding appearance of the number 2) that

$$\zeta_d = 2 + f_d \quad \text{with} \quad f_d = f(\zeta_d) \rightarrow 0 \quad \text{as} \quad d \rightarrow \infty.$$

Rewriting the equation $G(d, q; \zeta_d) = 0$ in the following way using the above relation and exploiting the exponential decay of the right-hand side yields (as $d \rightarrow \infty$)

$$(1 + f_d)^d - q(3 + f_d) = -q(1 + f_d) \left(\frac{1 + f_d}{2 + f_d} \right)^{d-1} = q \mathcal{O}((2/3)^d),$$

or equivalently,

$$\log(1 + f_d) = \frac{1}{d} \log(3q) + \frac{1}{d} \log \left(1 + \frac{f_d}{3} + \mathcal{O}((2/3)^d) \right) \quad \text{as} \quad d \rightarrow \infty.$$

Now we can use the series expansion of the logarithm function to get a relation for f_d :

$$f_d + \mathcal{O}(f_d^2) = \frac{1}{d} \log(3q) + \frac{1}{d} \mathcal{O}(f_d) \quad \text{as} \quad d \rightarrow \infty.$$

The asymptotic form (10) follows. \square

References

- [1] J. S. Brauchart, P. D. Dragnev and E. B. Saff, Riesz extremal measures on the sphere for axis-supported external fields, *J. Math. Anal. Appl.*, **356** (2009), 769–792.
- [2] L. L. Helms, *Potential Theory*, Universitext, Springer-Verlag London Ltd. (London, 2009).
- [3] J. D. Jackson, *Classical Electrodynamics*, 3rd ed., John Wiley & Sons Inc. (New York, 1998).
- [4] M. Lamprecht, personal communication.
- [5] N. S. Landkof, *Foundations of Modern Potential Theory*, Die Grundlehren der mathematischen Wissenschaften **180**, Springer-Verlag (New York, 1972).
- [6] N. J. A. Sloane, The on-line encyclopedia of integer sequences, *Notices Amer. Math. Soc.*, **50** (2003), 912–915, <http://www.research.att.com/~njas/sequences/>.
- [7] I. Stewart, Tales of a neglected number, *Sci. Amer.*, **274** (1996), 102–103.