REMARKS ON RELATIVE ASYMPTOTICS FOR GENERAL ORTHOGONAL POLYNOMIALS

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Abstract. Using a nonlinear integral characterization of orthogonal polynomials in the complex plane, we provide a simple method for deducing a weak form of relative asymptotics exterior to the convex hull of the common support of the generating measures. The simplicity of the approach makes it a natural precursor for the presentation of Szegő theory.

Dedicated to Guillermo López Lagomasino on the occasion of his 60th birthday

1. Introduction

Let μ denote a finite positive Borel measure with compact support $S_\mu := \mathrm{supp}(\mu)$ in the complex plane $\mathbb{C}$ and consider the associated inner product and norm

$$< f, g >_\mu := \int_{S_\mu} f(t)\overline{g(t)}\mathrm{d}\mu(t), \quad \|f\|_\mu := \sqrt{< f, f >_\mu}.$$

If S_μ contains at least $N+1$ points, we denote by

$$P_n(z) = P_n(z;\mu) = \kappa_n z^n + \cdots, \kappa_n > 0, \quad n = 0, 1, \ldots, N, \tag{1.1}$$

the unique sequence of polynomials of respective degrees n with positive leading coefficients that are orthonormal with respect to $\mathrm{d}\mu$; that is, $< P_m, P_n >_\mu = \delta_{m,n}$.

We remark that when $S_\mu \subset \mathbb{R}$, the P_n's satisfy a three-term recurrence relation, which is a useful tool in establishing asymptotic properties (as $n \to \infty$) of these polynomials when S_μ has infinite cardinality. Furthermore, if S_μ is a subset of the unit circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1|\}$, then the P_n's satisfy a two-term recurrence relation (cf. [6]) involving the inverse polynomials $P_n^*(z) := z^n\overline{P_n(1/\overline{z})}$. However, finite-term recurrences of fixed length do not in general hold for the P_n's when S_μ is not a subset of a circle or straight line (cf. [4], [3] for the case when μ is area measure over a bounded Jordan region). The goal of the present note is to show how a simple nonlinear characterization of orthogonal polynomials can be used to establish relative asymptotics (more precisely, *comparison estimates*) for general sequences of orthonormal polynomials. For comparison estimates for orthogonal polynomials on an interval, see, for example, results of J. Korous described in [6], Section 7.1.

2. A Nonlinear Characterization

We begin with the simple observation that for any polynomial Q of degree at most n, and any point $z_0 \in \mathbb{C}$, we have

$$\int_{S_\mu} \overline{P_n(t)} \left[\frac{Q(zt/z_0) - Q(z)}{t - z_0} \right] \mathrm{d}\mu(t) = 0, \tag{2.1}$$

since the expression in brackets is a polynomial in t of degree at most $n-1$. Consequently, if $z_0 \notin S_\mu$ and $\deg(Q) \le n$,

$$Q(z) \int_{S_\mu} \frac{\overline{P_n(t)}}{t - z_0} \mathrm{d}\mu(t) = \int_{S_\mu} \frac{\overline{P_n(t)} Q(zt/z_0)}{t - z_0} \mathrm{d}\mu(t). \tag{2.2}$$

Introducing notation for the Cauchy kernel,

$$k(t, z) := \frac{1}{t - z},$$

we deduce for $z_0 = z$ that

$$Q(z) < k(\cdot, z), P_n >_\mu = < Qk(\cdot, z), P_n >_\mu, \quad z \notin S_\mu, \quad \deg(Q) \le n, \tag{2.3}$$

which, for $Q = P_n$, yields

$$P_n(z) < k(\cdot, z), P_n >_\mu = < k(\cdot, z), |P_n|^2 >_\mu, \quad z \notin S_\mu. \tag{2.4}$$

Combining (2.3) and (2.4), we further obtain (for Q not identically zero)

$$\frac{P_n(z)}{Q(z)} < Qk(\cdot, z), P_n >_\mu = < k(\cdot, z), |P_n|^2 >_\mu, \quad z \notin S_\mu, \quad \deg(Q) \le n. \tag{2.5}$$

We remark that the restriction "$z \notin S_\mu$" in formulas (2.3)-(2.5) is imposed only to ensure that the integrals in these formulas exist; hence for certain measures μ such as area measure this restriction can be removed, i.e. these formulas hold for all $z \in \mathbb{C}$. We now show that the identity (2.4) characterizes the orthonormal polynomial P_n.

Proposition 2.1. *Let $q_n(z) = \tau_n z^n + \cdots, \tau_n > 0$, be a polynomial of degree n normalized so that $\|q_n\|_\mu = 1$. If, for $|z|$ large,*

$$q_n(z) < k(\cdot, z), q_n >_\mu = < k(\cdot, z), |q_n|^2 >_\mu, \tag{2.6}$$

then $q_n = P_n(\cdot; \mu)$.

Proof. Expanding $k(t, z) = 1/(t - z)$ in powers of $1/z$ we deduce from (2.6) that

$$-\sum_{k=0}^{\infty} \frac{1}{z^{k+1}} < \cdot^k, q_n >_\mu = \frac{1}{q_n(z)} < k(\cdot, z), |q_n|^2 >_\mu = \mathcal{O}(\frac{1}{z^{n+1}}) \quad \text{as} \quad z \to \infty.$$

Consequently, $< \cdot^k, q_n >_\mu = 0$ for $k = 0, \ldots, n-1$, which, taking into account the imposed normalizations, establishes the claim of the proposition. □

We note that Bender and Ben-Naim [1] consider similar nonlinear characterizations of $P_n(z; \mu)$; however, they do not mention the more general identities (2.2) and (2.3).

Using (2.4) we provide a short proof of the well-known result of Fejér (see, e.g. [2]) concerning the zeros of $P_n(z; \mu)$.

Proposition 2.2. *$P_n(z;\mu)$ has no zeros that lie outside the convex hull $\mathrm{Co}(S_\mu)$ of S_μ.*

Proof. Suppose $P_n(z_0) = 0$ for some $z_0 \notin \mathrm{Co}(S_\mu)$. Without loss of generality we assume that $z_0 \in \mathbb{R}_+$ and that the imaginary axis separates S_μ from z_0. Then, from (2.4), we have

$$< k(\cdot, z_0), |P_n|^2 >_\mu = 0. \tag{2.7}$$

But Re $k(t, z_0) < 0$ for all $t \in S_\mu$, and so $< k(\cdot, z_0), |P_n|^2 >_\mu$ has negative real part, which contradicts (2.7). □

In view of (2.4), the same proof gives

Corollary 2.3. *For $P_n = P_n(\cdot;\mu)$ we have*

$$\chi_n(z) := < k(\cdot, z), P_n >_\mu \neq 0, \quad z \notin \mathrm{Co}(S_\mu). \tag{2.8}$$

Remark 2.4. If $z \in \mathrm{Co}(S_\mu)$, then (2.8) need no longer be true. Indeed, if we take $\mathrm{d}\mu(t) = (2\pi)^{-1}\mathrm{d}\theta$ on the unit circle $\mathbb{T}$, then $P_n(z) = z^n$, and

$$\chi_n(z) = \int_{\mathbb{T}} \frac{\overline{P_n(t)}}{t - z}\mathrm{d}\mu(t) = \frac{1}{2\pi}\int_{\mathbb{T}} \frac{\bar{t}^n}{t - z}|\mathrm{d}t| = \frac{1}{2\pi}\int_{\mathbb{T}} \frac{|\mathrm{d}t|}{t^n(t - z)} = 0, \quad |z| < 1,$$

for each $n \geq 0$.

3. Applications to Relative Asymptotics

Hereafter we assume that the supports of the measures of orthogonality contain infinitely many points. In this section we show how formula (2.5) can be utilized to obtain results on the comparative growth of polynomials that are orthonormal with respect to varying weights. We begin with

Proposition 3.1. *Let $q_n(z) = \lambda_{n,n}z^n + \cdots, \lambda_{n,n} > 0$, be orthonormal with respect to the finite positive measure ν_n and $p_n(z) = \tau_{n,n}z^n + \cdots, \tau_{n,n} > 0$, be orthonormal with respect to $\rho_n(z)d\nu_n(z)$, where the supports S_{ν_n} are all contained in a compact set $K \subset \mathbb{C}$. If there exist positive constants m_1, m_2 such that $m_1 \leq \rho_n(z) \leq m_2$ for all $z \in S_{\nu_n}, n = 0, 1, \ldots,$ then for any closed set $E \subset \overline{\mathbb{C}} \setminus \mathrm{Co}(K)$ there exist positive constants c_1, c_2 such that*

$$c_1 \leq |p_n(z)/q_n(z)| \leq c_2, \quad z \in E, \quad n = 0, 1, \ldots. \tag{3.1}$$

Proof. First we observe that the assumption on the ρ_n's implies that

$$\sqrt{m_1} \leq \frac{\lambda_{n,n}}{\tau_{n,n}} \leq \sqrt{m_2}, \quad n = 0, 1, \ldots; \tag{3.2}$$

indeed, $\lambda_{n,n}/\tau_{n,n} = < q_n, p_n >_{\rho_n\nu_n}$, from which it follows by the Cauchy-Schwarz inequality that

$$\lambda_{n,n}/\tau_{n,n} \leq \|q_n\|_{\rho_n\nu_n}\|p_n\|_{\rho_n\nu_n} \leq \sqrt{m_2}\|q_n\|_{\nu_n} = \sqrt{m_2}.$$

Similarly, from the fact that $\tau_{n,n}/\lambda_{n,n} = < p_n, q_n >_{\nu_n}$ we deduce the lower bound in (3.2).

From (2.5) with $P_n = p_n$, $Q = q_n$ and $\mu = \rho_n\nu_n$ we obtain that, for $z \notin K$,

$$\frac{p_n(z)}{q_n(z)} < q_n k(\cdot, z), p_n >_{\rho_n\nu_n} = < k(\cdot, z), |p_n|^2 >_{\rho_n\nu_n}. \tag{3.3}$$

Select a point z_0 in K and define the functions

$$f_n(z) := < (z_0 - z) q_n k(\cdot, z), p_n >_{\rho_n \nu_n}, \quad g_n(z) := < (z_0 - z) k(\cdot, z), |p_n|^2 >_{\rho_n \nu_n} .$$

Observe that each f_n and each g_n is analytic in $\overline{\mathbb{C}} \setminus K$, even at infinity. Moreover, for each n, we have $g_n(\infty) = 1$ and, from (3.3), it follows that $f_n(\infty) = \lambda_{n,n}/\tau_{n,n}$.

It is evident from the definition of the g_n's that these functions are uniformly bounded on any closed set $F \subset \overline{\mathbb{C}} \setminus K$ by the constant $M_F := \max\{|(z_0 - z)/(t - z)| : z \in F, t \in K\}$ and thus they form a normal family of analytic functions in $\overline{\mathbb{C}} \setminus K$. The same is true for the f_n's since, by the Cauchy-Schwarz inequality we have, for $z \in F$, the following estimate:

$$|f_n(z)| \leq \|(z_0 - z) q_n k(\cdot, z)\|_{\rho_n \nu_n} \leq M_F \|q_n\|_{\rho_n \nu_n} \leq M_F \sqrt{m_2}.$$

Finally, we observe from Proposition 2.2, Corollary 2.3 and (3.3) that the f_n's and g_n's are zero-free in the domain $\overline{\mathbb{C}} \setminus \mathrm{Co}(K)$. Hence, by Hurwitz's theorem, any limit function of these normal families is either identically zero or never zero in $\overline{\mathbb{C}} \setminus \mathrm{Co}(K)$. But, in view of (3.2) and of the previously noted values of $f_n(\infty)$ and $g_n(\infty)$, the former possibility cannot occur. Consequently, from (3.3), we deduce that the ratios $p_n(z)/q_n(z)$ form a normal family of zero-free analytic functions in $\overline{\mathbb{C}} \setminus \mathrm{Co}(K)$ and that every limit function of this family is zero-free in this domain, from which the conclusion (3.1) follows. □

3.0.1. ***Measures supported on the unit circle and on the disk.*** In this subsection we consider polynomials that are orthonormal with respect to varying measures that are either supported on the unit circle $\mathbb{T}$ or are absolutely continuous with respect to area measure over the open unit disk $\mathbb{D}$.

Suppose that for $n = 0, 1, \ldots,$ the polynomial $p_n(z) = \tau_{n,n} z^n + \cdots$ is orthonormal with respect to the measure ν_n supported on the unit circle. Identity (2.5) with $Q(z) = z^n$ and $P_n = p_n$ becomes (2.5)

$$\frac{p_n(z)}{z^n} < \cdot^n k(\cdot, z), p_n >_{\nu_n} = < k(\cdot, z), |p_n|^2 >_{\nu_n} . \tag{3.4}$$

Setting

$$\widetilde{f}_n(z) := < -z \cdot^n k(\cdot, z), p_n >_{\nu_n}, \quad \widetilde{g}_n(z) := < -z k(\cdot, z), |p_n|^2 >_{\nu_n}, \tag{3.5}$$

we observe that $\widetilde{f}_n(z)$ can be written in terms of the inverse polynomial p_n^* as

$$\widetilde{f}_n(z) = < -z k(\cdot, z), \overline{p_n^*} >_{\nu_n} .$$

Using the fact that $|p_n(t)| = |p_n^*(t)|$ for $t \in S_{\nu_n}$ we deduce that the $\widetilde{f}_n$'s form a normal family of nonzero analytic functions in $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ *provided* the total masses $\|\nu_n\|$ are uniformly bounded. Moreover, from (3.4), we see that $\widetilde{f}_n(\infty) = 1/\tau_{n,n}$, which will be uniformly bounded below by a positive constant provided the $\tau_{n,n}$'s are bounded above. Clearly the $\widetilde{g}_n$'s form a normal family in $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$, and so by arguing as above we obtain the following result for polynomials orthogonal on the unit circle with varying weights.

Proposition 3.2. *If $p_n(z) = \tau_{n,n} z^n + \cdots, \tau_{n,n} > 0$, is orthonormal with respect to the measure ν_n with $S_{\nu_n} \subset \mathbb{T}$ and the sequences $\{\|\nu_n\|\}$ and $\{\tau_{n,n}\}$ are bounded* *above, then for each closed set $E \subset \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ there exist positive constants c_1, c_2 such that*

$$c_1 \leq \left| \frac{p_n(z)}{z^n} \right| \leq c_2, \quad z \in E, \quad n = 0, 1, \ldots. \tag{3.6}$$

In particular, (3.6) *holds if $d\nu_n(z) = \rho_n(z)|dz|$ on $\mathbb{T}$ and there exist positive constants m_1, m_2 such that $\int_{\mathbb{T}} \rho_n(z)|dz| \leq m_1$ and $\rho_n(z) \geq m_2$ for all $z \in \mathbb{T}$, $n = 0, 1, \ldots$.*

Remark: For the case when $\nu_n = \nu$ is independent of n, the condition that the coefficients $\tau_{n,n} = \tau_n$ are bounded is equivalent to ν being in the Szegő class, for which it is well-known (cf.[6], Theorem 12.1.1) that the sequence $\{p_n(z)/z^n\}$ converges to the Szegő function in $E \subset \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$. The above proposition can therefore be regarded as a weak form of that famous result.

Next we consider measures that are absolutely continuous with respect to area measure dA over the unit disk $\mathbb{D}$. We begin with the observation that for any radially symmetric density function $w(z) = w(|z|)$ on $\mathbb{D}$, the polynomials $\alpha_n(w) z^n$ are orthonormal with respect to $w(z)\mathrm{d}A$, where

$$\alpha_n(w) = [2\pi \int_0^1 r^{2n+1} w(r) \mathrm{d}r]^{-1/2}.$$

For example, if $w = w_\epsilon$ is the characteristic function of the annulus $\{z : 1 - \epsilon \leq |z| \leq 1\}$, then

$$\alpha_n(w_\epsilon) = \sqrt{\frac{n+1}{\pi}} [1 - (1-\epsilon)^{2n+2}]^{-1/2}. \tag{3.7}$$

Proposition 3.3. *Let $Q_n(z) = \kappa_{n,n} z^n + \cdots$, $\kappa_{n,n} > 0$, be orthonormal with respect to the measure $d\mu_n(z) = \rho_n(z) dA(z)$ on $\mathbb{D}$. If there exist positive constants M_1, M_2 such that*

$$\kappa_{n,n} \leq M_1 \sqrt{n} \quad \text{and} \quad \rho_n(z) \leq M_2, \ \ z \in \mathbb{D}, \tag{3.8}$$

then, for any closed set $E \subset \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$, there exist positive constants c_1, c_2 such that

$$c_1 \leq \left| \frac{Q_n(z)}{\sqrt{n} z^n} \right| \leq c_2, \quad z \in E, \quad n = 1, 2, \ldots. \tag{3.9}$$

In particular, (3.9) *holds if there exist positive constants m_1, M_2 and ϵ such that $\rho_n(z) \leq M_2, z \in \mathbb{D}$, and $\rho_n(z) \geq m_1$ for $1 - \epsilon \leq |z| \leq 1, n = 1, 2, \ldots$.*

Proof. Again from (2.5) we have

$$\frac{Q_n(z)}{\sqrt{n} z^n} < -z\sqrt{n} \cdot^n k(\cdot, z), Q_n >_{\mu_n} = < -zk(\cdot, z), |Q_n|^2 >_{\mu_n}, \tag{3.10}$$

and so (3.9) will follow if we show that

$$\widehat{f}_n(z) := < -z\sqrt{n} \cdot^n k(\cdot, z), Q_n >_{\mu_n}$$

is uniformly bounded on closed subsets of $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ and $\widehat{f}_n(\infty) = \sqrt{n}/\kappa_{n,n}$ is uniformly bounded below by a positive constant. The latter condition is immediate from (3.8);

the former property follows from the fact that on any closed subset E of $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$,

$$|\widehat{f}_n(z)|^2 \leq \| - z\sqrt{n} \cdot^n k(\cdot, z)\|^2_{\mu_n} \leq K_1 n \int_{\mathbb{D}} |t|^{2n} \rho_n(t) \mathrm{d}A(t) \leq K_2 n \int_0^1 r^{2n+1} \mathrm{d}r \leq K_3,$$

where the K_i's are constants.

Regarding the last statement of the proposition, it suffices to note that

$$\frac{\kappa_{n,n}}{\alpha_n(w_\epsilon)} =< Q_n, \alpha_n(w_\epsilon) \cdot^n >_{w_\epsilon dA} \leq \|Q_n\|_{w_\epsilon dA} \leq \frac{1}{\sqrt{m_1}},$$

which, in view of (3.7), implies the first condition in (3.8). □

3.0.2. ***Ratio asymptotics***. A normal families argument can also be used to obtain a weak form of ratio asymptotics for the polynomials $P_n(\cdot;\mu)$ by selecting $z_0 \in S_\mu$ and applying (2.5) with $Q(z) = (z - z_0)P_{n-1}(z)$.

Proposition 3.4. *Let $P_n(z) = \kappa_n z^n + \cdots, \kappa_n > 0$, be orthonormal with respect to $d\mu$ and let $z_0 \in S_\mu$. If the ratios $\{\kappa_n/\kappa_{n-1}\}$ are bounded from above for some subsequence of integers $\mathcal{N}$, then for any closed set $E \subset \overline{\mathbb{C}} \setminus \mathrm{Co}(S_\mu)$ there exist positive constants c_1, c_2 such that*

$$c_1 \leq \frac{\kappa_{n-1}}{\kappa_n} \left| \frac{P_n(z)}{(z - z_0)P_{n-1}(z)} \right| \leq c_2, \quad z \in E, \, n \in \mathcal{N}. \tag{3.11}$$

If, in addition, $S_\mu \subset \overline{\mathbb{D}}$, then for any closed set $E \subset \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ there exist positive constants c_3, c_4 such that

$$c_3 \leq \left| \frac{P_n(z)}{zP_{n-1}(z)} \right| \leq c_4, \quad z \in E, \, n \in \mathcal{N}. \tag{3.12}$$

We remark that for $S_\mu \subset \overline{\mathbb{D}}$, the characterization

$$\frac{1}{\kappa_n} = \min_{p = z^n + \cdots} \|p\|_\mu,$$

can be used to show that $\kappa_n/\kappa_{n-1} \geq 1$, so that (3.12) follows from (3.11).

We conclude with the following generalized form of a conjecture of B. Simon concerning possible extensions of Rakhmanov's theorem (cf. [5]).

Conjecture: If $\mathrm{d}\mu(z) = \rho(z)\mathrm{d}A$, $z \in \mathbb{D}$, where $\rho(z) > 0$ a.e. in an annulus $A_\epsilon := \{z : 1 - \epsilon \leq |z| \leq 1\}$, then

$$\frac{P_n(z)}{zP_{n-1}(z)} \to 1, \quad z \in \overline{\mathbb{C}} \setminus \overline{\mathbb{D}};$$

in particular, $\kappa_n/\kappa_{n-1} \to 1$ as $n \to \infty$.

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