Asymptotics of Greedy Energy Points

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Abstract: For a symmetric kernel $k : X \times X \to \mathbb{R} \cup \{+\infty\}$ on a locally compact metric space X, we investigate the asymptotic behavior of greedy k-energy points $\{a_i\}_1^\infty$ for a compact subset $A \subset X$ that are defined inductively by selecting $a_1 \in A$ arbitrarily and a_{n+1} so that $\sum_{i=1}^n k(a_{n+1}, a_i) = \inf_{x \in A} \sum_{i=1}^n k(x, a_i)$. We give sufficient conditions under which these points (also known as Leja points) are asymptotically energy minimizing (i.e. have energy $\sum_{i \neq j}^N k(a_i, a_j)$ as $N \to \infty$ that is asymptotically the same as $\mathcal{E}(A, N) := \min\{\sum_{i \neq j} k(x_i, x_j) : x_1, \ldots, x_N \in A\}$), and have asymptotic distribution equal to the equilibrium measure for A. For the case of Riesz kernels $k_s(x, y) := |x - y|^{-s}$, $s > 0$, we show that if A is a rectifiable Jordan arc or closed curve in $\mathbb{R}^p$ and $s > 1$, then greedy k_s-energy points are *not* asymptotically energy minimizing, in contrast to the case $s < 1$. (In fact we show that *no* sequence of points can be asymptotically energy minimizing for $s > 1$.) Additional results are obtained for greedy k_s-energy points on a sphere, for greedy best-packing points (the case $s = \infty$), and for weighted Riesz kernels. For greedy best-packing points we provide a simple counterexample to a conjecture attributed to L. Bos.

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1. Introduction, background results and notation

The aim of this paper is to study asymptotic properties of special types of extremal point configurations which we shall call *greedy energy points*. As the name suggests, these configurations are generated by a greedy algorithm which is, in fact, an energy minimizing construction. The notion of energy that we refer to will be specified shortly. We focus on two aspects: the asymptotic behavior of their energy and their limiting distributions, as their cardinality approaches infinity. In many aspects they are similar to minimal (non-greedy) energy configurations, which are those with smallest possible energy. But we will also show that in some situations the behavior of greedy points differs significantly from that of minimal energy points.

Part of the results in this paper are presented in the abstract setting of locally compact metric spaces. Potential theory on more general spaces (such as locally compact Hausdorff spaces) was developed by Choquet [9],

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[10], Fuglede [17] and Ohtsuka [27]. Recently Zorii [31], [32] has studied properties of potentials with external fields in this context.

We also investigate greedy configurations in $\mathbb{R}^p$, interacting through the so-called Riesz potential $V = 1/r^s$, where $s > 0$ and r denotes Euclidean distance, as well as greedy 'best-packing' points that are chosen to maximize the minimum distance to previously selected points.

We next introduce the basic notions necessary to describe our results. We will also present in this section some background material.

Let X denote a locally compact metric space containing infinitely many points. A *kernel* in X is, by definition, a lower semicontinuous function (l.s.c.) $k : X \times X \to \mathbb{R} \cup \{+\infty\}$. It is called *positive* if $k(x, y) \geq 0$ for all $x, y \in X$.

For a set $\omega_N = \{x_1, \ldots, x_N\}$ of N ($N \geq 2$) points in X, not necessarily distinct, we write $\mathrm{card}(\omega_N) = N$ and define the *discrete energy* of ω_N by

$$E(\omega_N) := \sum_{1 \leq i \neq j \leq N} k(x_i, x_j) = \sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} k(x_i, x_j) \,.$$

If the kernel is *symmetric*, i.e., $k(x, y) = k(y, x)$ for all $x, y \in X$, we may also write

$$E(\omega_N) = 2 \sum_{1 \leq i < j \leq N} k(x_i, x_j) \,.$$

For a set $A \subset X$, the *N-point energy* of A is given by

(1) $$\mathcal{E}(A, N) := \inf\{E(\omega_N) : \omega_N \subset A, \ \mathrm{card}(\omega_N) = N\} \,.$$

We say that $\omega_N^* \subset A$ is an *optimal N-point configuration* on A if

$$E(\omega_N^*) = \mathcal{E}(A, N) \,.$$

When A is compact, such a configuration always exists by the lower semicontinuity of k. In order to study the asymptotic behavior of the sequence $E(\omega_N^*)$ we need to introduce the continuous counterparts of the above notions.

Let $\mathcal{M}(A)$ denote the linear space of all real-valued Radon measures that are compactly supported on A, and let $\mathcal{M}^+(A) := \{\mu \in \mathcal{M}(A) : \mu \geq 0\}$. We also introduce the class $\mathcal{M}_1(A) := \{\mu \in \mathcal{M}^+(A) : \mu(X) = 1\}$. Given a measure $\mu \in \mathcal{M}(A)$, the *continuous energy* of μ is the double integral

(2) $$W(\mu) := \int \int k(x, y) \, d\mu(x) \, d\mu(y) \,.$$

The function

(3) $$U^\mu(x) := \int k(x, y) \, d\mu(y)$$

is called the *potential* of μ. Since any l.s.c. function is bounded below on compact sets, the above integrals are well-defined, although they may attain the value $+\infty$.

We say that k satisfies the *maximum principle* if for every measure $\mu \in \mathcal{M}_1(A)$,

$$(4) \qquad \sup_{x \in \text{supp}(\mu)} U^\mu(x) = \sup_{x \in X} U^\mu(x) .$$

The quantity $w(A) := \inf\{W(\mu) : \mu \in \mathcal{M}_1(A)\}$ plays an important role in potential theory and is called the *Wiener energy* of A. The *capacity* of A is defined as $\text{cap}_k(A) := w(A)^{-1}$ if k is positive, and otherwise, it is defined as $\text{cap}_k(A) := \exp(-w(A))$. A property is said to hold *quasi-everywhere* (q.e.), if the exceptional set has Wiener energy $+\infty$.

Given a net $\{\mu_\alpha\} \subset \mathcal{M}(A)$, we say that $\{\mu_\alpha\}$ converges in the *weak-star topology* to a measure $\mu \in \mathcal{M}(A)$ when

$$\lim_\alpha \int f \, d\mu_\alpha = \int f \, d\mu , \qquad \text{for all} \quad f \in C_c(A) ,$$

where $C_c(A)$ denotes the space of compactly supported continuous functions on A. We will use the notation

$$\mu_\alpha \xrightarrow{\ *\ } \mu$$

to denote the weak-star convergence of measures. If A is compact, we know by the Banach-Alaoglu theorem that $\mathcal{M}_1(A)$ equipped with the weak-star topology is compact.

If A is compact and $w(A) < \infty$, a measure $\mu \in \mathcal{M}_1(A)$ satisfying the property $W(\mu) = w(A)$ is called an *equilibrium measure*. The existence of such a measure is guaranteed by the lower semicontinuity of k and the compactness of $\mathcal{M}_1(A)$ (see Theorem 2.3 in [17]). However, uniqueness does not always hold.

The following result is due to G. Choquet [10], and it is central in this theory.

Theorem 1.1. *Let k be an arbitrary kernel and $A \subset X$ be a compact set. If $\{\omega_N^*\}$ is a sequence of optimal N-point configurations on A, then*

$$(5) \qquad \lim_{N \to \infty} \frac{E(\omega_N^*)}{N^2} = w(A) .$$

The following variation of Theorem 1.1 was obtained by Farkas and Nagy [16].

Theorem 1.2. *Assume that the kernel k is positive and is finite on the diagonal, i.e., $k(x, x) < +\infty$ for all $x \in X$. Then for arbitrary sets $A \subset X$,*

$$\lim_{N \to \infty} \frac{\mathcal{E}(A, N)}{N^2} = w(A) ,$$

where $\mathcal{E}(A, N)$ is the N-point energy of A (see (1)).

In this paper we study an alternative construction of points obtained by means of a "greedy" algorithm.

Definition 1.3. Let $k : X \times X \to \mathbb{R} \cup \{+\infty\}$ be a symmetric kernel on a locally compact metric space X, and let $A \subset X$ be a compact set. A sequence $(a_n)_{n=1}^\infty \subset A$ is called a *greedy k-energy sequence* on A if it is generated in the following way:

- a_1 is selected arbitrarily on A.
- Assuming that $a_1, \ldots, a_n$ have been selected, a_{n+1} is chosen to satisfy

$$(6) \qquad \sum_{i=1}^{n} k(a_{n+1}, a_i) = \inf_{x \in A} \sum_{i=1}^{n} k(x, a_i),$$

for every $n \geq 1$.

We remark that the choice of a_{n+1} is not unique in general. We will use the notation

$$\alpha_{N,k} := \{a_1, \ldots, a_N\}$$

to denote the set of the first N points of this sequence. It is significantly easier to obtain numerically these configurations rather than optimal N-point configurations, since in order to obtain the former we have to minimize a functional of one variable instead of N variables.

It was shown by Fuglede (see Theorem 2.4 in [17]) that if k is symmetric and $A \subset X$ is compact, every $\mu \in \mathcal{M}_1(A)$ that has minimal energy satisfies the inequality $U^\mu(x) \leq w(A)$ for all $x \in \mathrm{supp}(\mu)$. The *essential support* of μ is the set

$$(7) \qquad S_\mu^* := \{x \in A : U^\mu(x) \leq w(A)\}.$$

Hence $\mathrm{supp}(\mu) \subset S_\mu^*$.

The following is a restricted version of Definition 1.3.

Definition 1.4. Under the same assumptions as in Definition 1.3, assume that $w(A) < \infty$, and let $\mu \in \mathcal{M}_1(A)$ be an equilibrium measure. A sequence $(a_n = a_{n,k,\mu})_{n=1}^\infty \subset A$ is called a *greedy (k, μ)-energy sequence* on A if it is generated in the following way:

- a_1 is selected arbitrarily on S_μ^*.
- Assuming that $a_1, \ldots, a_n$ have been selected, a_{n+1} is chosen to satisfy $a_{n+1} \in S_\mu^*$ and

$$\sum_{i=1}^{n} k(a_{n+1}, a_i) = \inf_{x \in S_\mu^*} \sum_{i=1}^{n} k(x, a_i)$$

for every $n \geq 1$.

The set of the first N points of this sequence is denoted by $\alpha_{N,k,\mu}$.

Albert Edrei [14] was probably the first person who studied the point configurations $\alpha_{N,k}$ in the particular case $X = \mathbb{C}$ and $k(x, y) = -\log(|x-y|)$. However, in the literature these configurations are often called *Leja points*, in recognition of Leja's article [23]. When the kernel employed is the Green function or the Newtonian kernel $k(x, y) = 1/|x - y|$ in the unit sphere

S^2, the configurations $\alpha_{N,k}$ are also referred to as *Leja-Górski points* (see [19] and references therein). In [1], certain configurations known as *fast Leja points* are introduced, and an algorithm is presented to compute them. These configurations are defined over discretizations of planar sets and the kernel employed is the logarithmic kernel. In [11] a constrained energy problem for this kernel is considered and associated *constrained Leja points* are introduced. We remark that Leja points are important in interpolation theory because they provide a Newton-type interpolation point scheme on the real line or complex plane.

A very relevant class of kernels is the so-called *M. Riesz kernels* in $X = \mathbb{R}^p$, which depend on a parameter s in $[0, +\infty)$. It is defined as follows:

$$k_s(x, y) := K(|x - y|; s), \qquad x, y \in \mathbb{R}^p,$$

where $|\cdot|$ denotes the Euclidean norm and

$$(8) \qquad K(t; s) := \begin{cases} t^{-s}, & \text{if } s > 0, \\ -\log(t), & \text{if } s = 0. \end{cases}$$

We shall use the notations $I_s(\mu)$ and U_s^μ to denote the energy (2) and potential (3) of a measure $\mu \in \mathcal{M}(A)$ with respect to the Riesz s-kernel, and $w_s(A)$ to denote the Wiener energy of a set A in this new setting. We will also use $E_s(\omega_N)$ to represent the discrete energy of an N-point configuration $\omega_N \subset \mathbb{R}^p$, and

$$(9) \qquad \mathcal{E}_s(A, N) := \inf\{E_s(\omega_N) : \omega_N \subset A, \ \mathrm{card}(\omega_N) = N\}$$

to denote the N-point Riesz s-energy of a compact set $A \subset \mathbb{R}^p$. Additionally, greedy k_s-energy configurations will be denoted by $\alpha_{N,s}$.

A few words about Riesz s-kernels are needed at this point. Let $A \subset \mathbb{R}^p$ be compact, and $0 \leq s < \dim_\mathcal{H}(A)$, where $\dim_\mathcal{H}(A)$ denotes the Hausdorff dimension of A (which will be denoted by d throughout the rest of this section). Then there is a unique equilibrium measure $\lambda_{A,s} \in \mathcal{M}_1(A)$ with finite energy, i.e., $I_s(\lambda_{A,s}) = w_s(A) < +\infty$. On the other hand, if $s \geq d$, then $I_s(\mu) = +\infty$ for all $\mu \in \mathcal{M}_1(A)$. We refer the reader to Theorems 8.5 and 8.9 in [26] for justifications of these facts.

For $s < d$, Theorem 1.1 asserts that

$$(10) \qquad \lim_{N \to \infty} \frac{E_s(\omega_{N,s}^*)}{N^2} = I_s(\lambda_{A,s}),$$

where $\{\omega_{N,s}^*\}$ denotes any sequence of optimal N-point configurations on A with respect to the Riesz s-kernel. In addition (see [22]),

$$\frac{1}{N} \sum_{x \in \omega_{N,s}^*} \delta_x \xrightarrow{*} \lambda_{A,s}, \qquad N \to \infty,$$

where δ_x is the Dirac unit measure concentrated at x. If $s \geq d$, then Theorem 1.1 tells us that

$$\lim_{N \to \infty} \frac{E_s(\omega_{N,s}^*)}{N^2} = +\infty,$$

so the order of growth of $E_s(\omega^*_{N,s})$ is greater than N^2.

Throughout the rest of the paper we denote by $\mathrm{Vol}(B^d)$ the volume of the unit ball B^d in $\mathbb{R}^d$, and $\mathcal{H}_d$ represents d-dimensional Hausdorff measure in $\mathbb{R}^p$ (normalized by the condition $\mathcal{H}_d([0,1]^d) = 1$, where $[0,1]^d$ denotes here the embedding of the d-dimensional unit cube in $\mathbb{R}^p$). Regarding the case $s \geq d$, in [20] and [4] geometric measure theoretic tools were employed to obtain the following result.

Theorem 1.5. *Let A be a compact subset of a d-dimensional C^1-manifold in $\mathbb{R}^p$. If $\{\omega^*_{N,d}\}$ is any sequence of optimal N-point configurations on A for $s = d$, then*

$$(11) \qquad \lim_{N \to \infty} \frac{E_d(\omega^*_{N,d})}{N^2 \log N} = \frac{\mathrm{Vol}(B^d)}{\mathcal{H}_d(A)} \, .$$

Furthermore, if $\mathcal{H}_d(A) > 0$, any sequence $\{\widetilde{\omega}_N\}$ of configurations on A whose energies satisfy (11) is uniformly distributed with respect to $\mathcal{H}_d$ in the sense that

$$(12) \qquad \frac{1}{N} \sum_{x \in \widetilde{\omega}_N} \delta_x \xrightarrow{\;*\;} \frac{\mathcal{H}_d|_A}{\mathcal{H}_d(A)} \, , \qquad N \to \infty \, .$$

Assume now that $A \subset \mathbb{R}^p$ is a d-rectifiable compact set, i.e., A is the image of a bounded set in $\mathbb{R}^d$ under a Lipschitz mapping. If $\{\omega^*_{N,s}\}$ is any sequence of optimal N-point configurations on A for $s > d$, there holds

$$(13) \qquad \lim_{N \to \infty} \frac{E_s(\omega^*_{N,s})}{N^{1+s/d}} = \frac{C_{s,d}}{\mathcal{H}_d(A)^{s/d}} \, ,$$

where $C_{s,d} > 0$ is a constant independent of A and p. In addition, if $\mathcal{H}_d(A) > 0$, any sequence of configurations on A whose energies satisfy (13) is uniformly distributed with respect to $\mathcal{H}_d$.

We remark that the constant $C_{s,d}$ equals $2\zeta(s)$ when $d = 1$, where $\zeta(s)$ is the classical Riemann zeta function (cf. [25]).

Definition 1.6. Let A be a compact set of Hausdorff dimension d. A sequence of point sets $\omega_N \subset A$, is said to be *asymptotically s-energy minimizing* on A, and we shall write $\{\omega_N\}_N \in \mathrm{AEM}(A; s)$, if it satisfies, with $\omega^*_{N,s}$ replaced by ω_N, the limit relation (10), (11) or (13), according to whether $s < d$, $s = d$, or $s > d$.

We conclude this section by illustrating in Figures 1-4 the first 200 points of four approximate greedy k_s-energy sequences on the unit square $A = [0,1]^2$ for four different values of s (for better visualization we have deleted the coordinate axes). The initial point is always selected to be the origin. The points in Figures 2-4 were obtained by minimizing over a discretization of $[0,1]^2$ formed by the set

$$\{(i/100, j/100) : 0 \leq i, j \leq 100\} \, ,$$

whereas, in the case of Figure 1, the points were obtained using a discretization of the boundary of $[0,1]^2$ consisting of 4000 equally spaced points. We remark that if $s = 0$, it follows from the maximum modulus principle that all greedy energy points will lie on the boundary of the square and thus only the boundary was discretized in this case.
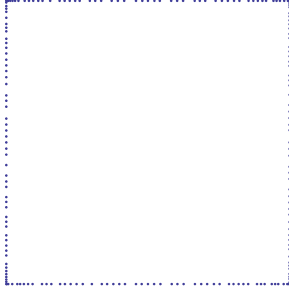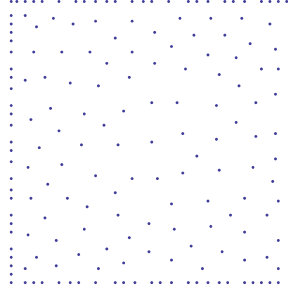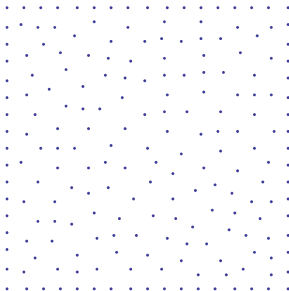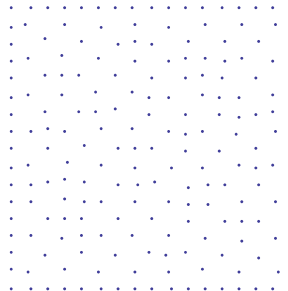

FIGURE 1. $s = 0$.

FIGURE 2. $s = 1$.

FIGURE 3. $s = 2$.

FIGURE 4. $s = 4$.

In Section 2 we state and discuss our main results. Their proofs are given in subsequent sections.

2. MAIN RESULTS

2.1. The Potential theoretic case: Sets of positive capacity. Let

$$U_n(x) := \sum_{j=1}^{n-1} k(x, a_j), \qquad n \geq 2.$$

Our first result on the asymptotic behavior of greedy sequences is the following.

Theorem 2.1. *Let $k : X \times X \to \mathbb{R} \cup \{+\infty\}$ be a symmetric kernel on a locally compact metric space X that satisfies the maximum principle. Assume $A \subset X$ is a compact set and $\{\alpha_{N,k}\}$ is a greedy k-energy sequence on A. Then*

(i) *the following limit holds:*

$$(14) \qquad \lim_{N\to\infty} \frac{E(\alpha_{N,k})}{N^2} = w(A);$$

(ii) *if $w(A) < \infty$ and the equilibrium measure $\mu \in \mathcal{M}_1(A)$ is unique, it follows that*

$$(15) \qquad \frac{1}{N} \sum_{a\in\alpha_{N,k}} \delta_a \xrightarrow{\ *\ } \mu, \qquad N \to \infty;$$

(iii) *if $w(A) < \infty$, there holds*

$$(16) \qquad \lim_{n\to\infty} \frac{U_n(a_n)}{n} = w(A),$$

where a_n is the n-th element of the greedy k-energy sequence.

Furthermore, if $w(A) < \infty$, the analogues of assertions $(i), (ii),$ and (iii) hold for any greedy (k,μ)-energy sequence on A without assuming the maximum principle.

Theorem 2.1 generalizes a result due to Siciak [30] (see Lemma 3.1) stated for Riesz potentials. For sets of positive capacity, his result asserts that if $A \subset \mathbb{R}^p$ is a compact set, $p - 2 \le s < p$, $p \ge 2$, and $\{\alpha_{N,s}\}$ is a greedy k_s-energy sequence on A, then (16) holds for $k = k_s$.

As a consequence of Theorem 2.1, we deduce the following corollaries for Riesz kernels. Throughout this paper we denote the d-dimensional unit sphere in $\mathbb{R}^{d+1}$ by S^d.

Corollary 2.2. *Let d be a positive integer and $s \in [0,d)$. Then any greedy k_s-energy sequence $\alpha_{N,s} \subset S^d$ is $\mathrm{AEM}(S^d; s)$ and the asymptotic formula*[3]

$$(17) \quad \lim_{N\to\infty} \frac{E_s(\alpha_{N,s})}{N^2} = \begin{cases} \frac{\Gamma((d+1)/2)\Gamma(d-s)}{\Gamma((d-s+1)/2)\Gamma(d-s/2)}, & \text{if} \quad 0 < s < d, \\[2mm] -\log(2) + \frac{1}{2}(\psi(d) - \psi(d/2)), & \text{if} \quad s = 0, \end{cases}$$

holds, where $\psi(x) := \Gamma'(x)/\Gamma(x)$ denotes the digamma function. In addition,

$$(18) \qquad \frac{1}{N} \sum_{a\in\alpha_{N,s}} \delta_a \xrightarrow{\ *\ } \sigma_d, \qquad N \to \infty,$$

where σ_d is the normalized Lebesgue measure on S^d.

Corollary 2.3. *Let $\alpha_{N,s}$ be any greedy k_s-energy sequence on $[-1,1]$. For $s \in [0,1)$, this sequence is $\mathrm{AEM}([-1,1]; s)$, which means that*

$$(19) \qquad \lim_{N\to\infty} \frac{E_s(\alpha_{N,s})}{N^2} = \begin{cases} \frac{\sqrt{\pi}\,\Gamma(1+s/2)}{\cos(\pi s/2)\Gamma((1+s)/2)}, & \text{if} \quad 0 < s < 1, \\[2mm] \log(2), & \text{if} \quad s = 0. \end{cases}$$

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[3]We remark that for $d = 1$ and $s = 0$ we have $\mathcal{E}_0(S^1, N) = -N\log(N)$, $N \ge 2$, (cf. [6]).

Furthermore,

$$\frac{1}{N} \sum_{a \in \alpha_{N,s}} \delta_a \xrightarrow{\quad * \quad} \frac{c_s}{(1 - x^2)^{(1-s)/2}} \, dx \,, \qquad x \in [-1, 1], \quad N \to \infty \,,$$

where c_s is a normalizing constant.

Our next result concerns second-order asymptotics for Riesz energy on the unit circle. It is known that if $s \in (0, 1)$, then the following limit holds (see [6]).

$$(20) \qquad \lim_{N \to \infty} \frac{\mathcal{E}_s(S^1, N) - I_s(\sigma)N^2}{N^{1+s}} = \frac{2\zeta(s)}{(2\pi)^s} \,,$$

where $\mathcal{E}_s(S^1, N)$ denotes (see (9)) the N-point minimal Riesz s-energy of S^1, and $\zeta(s)$ is the analytic extension of the classical Riemann zeta function. We know by Corollary 2.2 that all greedy k_s-energy sequences are $\mathrm{AEM}(S^1; s)$ when $s \in (0, 1)$. Nevertheless, the expression (21) below shows that in terms of second-order asymptotics greedy k_s-energy sequences and optimal N-point configurations for $s \in (0, 1)$ behave differently.

Proposition 2.4. *Let $s \in (0, 1)$ and consider an arbitrary greedy k_s-energy sequence $\{\alpha_{N,s}\}_N$ on S^1. Then the following next order asymptotics holds:*

$$(21) \qquad \lim_{n \to \infty} \frac{E_s(\alpha_{3 \cdot 2^n, s}) - I_s(\sigma)(3 \cdot 2^n)^2}{(3 \cdot 2^n)^{1+s}} = f(s)\frac{2\zeta(s)}{(2\pi)^s} \,,$$

where $f(s) = \frac{1}{2}(\frac{4}{3})^{1+s} + (\frac{1}{3})^{1+s} < 1$ for $s \in (0, 1)$, $\zeta(s)$ is the analytic extension of the classical Riemann zeta function, and σ is the normalized arclength measure on S^1.

If $s \in (0, 1)$, then $\zeta(s) < 0$, and therefore $f(s)\frac{2\zeta(s)}{(2\pi)^s} > \frac{2\zeta(s)}{(2\pi)^s}$. Hence we obtain the following

Corollary 2.5. *For all $s \in (0, 1)$ and for any greedy k_s-energy sequence $\{\alpha_{N,s}\}_N$ on S^1, the sequence*

$$\frac{E_s(\alpha_{N,s}) - I_s(\sigma)N^2}{N^{1+s}}$$

is not convergent.

Remark: It is well-known that on S^1 the minimal N-point Riesz s-energy $\mathcal{E}_s(S^1, N)$ is attained only by configurations consisting of N equally spaced points, and this property holds for every $s \geq 0$. We will show (see Lemma 4.2) that for such s greedy configurations $\alpha_{2^n, s}$ on S^1 are formed by 2^n equally spaced points.

2.2. **Sets of Capacity Zero.**

2.2.1. *Greedy k_s-energy sequences on S^1.* In this subsection we present some results about the asymptotic behavior of $E_s(\alpha_{N,s})$ for greedy k_s-energy sequences on S^1 when $s \geq 1$. As we shall see in Theorem 2.6, greedy k_s-energy sequences on S^1 are not AEM$(S^1; s)$ for $s > 1$, which is perhaps a surprising result. We conclude that the behavior of $E_s(\alpha_{N,s})$ exhibits a transition at $s = 1$, the Hausdorff dimension of S^1, since as we saw in the previous section greedy k_s-energy sequences are AEM$(S^1; s)$ for $s < 1$.

Remark: It follows from the geometric lemmas proved in Section 4 that greedy k_s-energy sequences $\alpha_{N,s}$ on S^1 are independent of s, i.e., once the points $a_1, \ldots, a_n$ have been selected, the choice of a_{n+1} is independent of the value of s and depends only on the position of the first n points of the sequence. As a consequence we will denote greedy k_s-energy sequences on S^1 by α_N instead of $\alpha_{N,s}$.

In [25] (see Theorem 3.1) it was proved that if Γ is a rectifiable Jordan arc, then for $s > 1$,

$$(22) \qquad \lim_{N\to\infty} \frac{E_s(\omega^*_{N,s})}{N^{1+s}} = \frac{2\zeta(s)}{\mathcal{H}_1(\Gamma)^s},$$

and if $s = 1$,

$$(23) \qquad \lim_{N\to\infty} \frac{E_1(\omega^*_{N,1})}{N^2 \log N} = \frac{2}{\mathcal{H}_1(\Gamma)},$$

where $\{\omega^*_{N,s}\}_N$ is any sequence of optimal N-point configurations with respect to the Riesz s-kernel.

We remind the reader that by $\mathcal{E}_s(S^1, N)$ we denote the N-point Riesz s-energy of S^1 (see (9)). As it was observed previously, optimal N-point configurations on S^1 consist precisely of N equally spaced points, and this property holds for all values of $s \in [0, \infty)$. From (22) we have

$$(24) \qquad \lim_{N\to\infty} \frac{\mathcal{E}_s(S^1, N)}{N^{1+s}} = \frac{2\zeta(s)}{(2\pi)^s}.$$

By Corollary 2.2 and Theorem 2.18 (see Subsection 2.2.3) we know that if $s \in [0, d]$, then any greedy k_s-energy sequence $\{\alpha_{N,s}\}$ on S^d is AEM$(S^d; s)$. However the situation changes when $s > 1$ on S^1.

Proposition 2.6. *For $s > 1$, any greedy k_s-energy sequence $\{\alpha_{N,s}\}_N$ on S^1 is not asymptotically s-energy minimizing. In fact, the subsequence $\alpha_{3\cdot 2^n, s}$ satisfies*

$$\lim_{n\to\infty} \frac{E_s(\alpha_{3\cdot 2^n, s})}{(3 \cdot 2^n)^{1+s}} = f(s) \frac{2\zeta(s)}{(2\pi)^s},$$

where $f(s) = \frac{1}{2}(\frac{4}{3})^{1+s} + (\frac{1}{3})^{1+s} > 1$ for all $s > 1$.

As in the previous section, we want to describe the difference in terms of second-order asymptotics between greedy k_s-energy sequences and optimal

N-point configurations when $s = 1$. The following formula holds (see [6]):

$$(25) \qquad \lim_{N \to \infty} \frac{\mathcal{E}_1(S^1, N) - \frac{1}{\pi} N^2 \log N}{N^2} = \frac{1}{\pi} (\gamma - \log(\pi/2)),$$

where $\gamma = \lim_{M \to \infty} (1 + \frac{1}{2} + \cdots + \frac{1}{M} - \log M)$ denotes the Euler-Mascheroni constant.

Proposition 2.7. *For any greedy k_1-energy sequence $\{\alpha_N\}_N$ on S^1 we have*

$$(26) \quad \lim_{n \to \infty} \frac{E_1(\alpha_{3 \cdot 2^n}) - \frac{1}{\pi} (3 \cdot 2^n)^2 \log(3 \cdot 2^n)}{(3 \cdot 2^n)^2} = \frac{1}{\pi} (\gamma - \log(\pi/2) + \log(2^{\frac{16}{9}}/3)).$$

Corollary 2.8. *For any greedy k_1-energy sequence $\{\alpha_N\}_N$ on S^1, the sequence*

$$\frac{E_1(\alpha_N) - \frac{1}{\pi} N^2 \log N}{N^2}$$

is not convergent.

2.2.2. k_s-*Energy of sequences on Jordan arcs or curves in $\mathbb{R}^p$ for $s \geq 1$ and best-packing.* Throughout this subsection, by a Jordan arc in $\mathbb{R}^p$ we understand a set homeomorphic to a closed segment. A closed Jordan curve refers to a set homeomorphic to a circle.

Our main result states that for $s > 1$ it is *not* possible to find *any* sequence of points on a Jordan arc or curve that is asymptotically s-energy minimizing.

Theorem 2.9. *Let $\{x_k\}_{k=0}^{\infty} \subset \Gamma$ be an arbitrary sequence of distinct points, where Γ is a rectifiable Jordan arc or closed Jordan curve in $\mathbb{R}^p$. Set $\mathcal{X}_n := \{x_k\}_{k=0}^{n}$. Then $\{\mathcal{X}_n\}_n \notin \mathrm{AEM}(\Gamma; s)$ for all $s > 1$. In particular, $\{\alpha_{N,s}\} \notin \mathrm{AEM}(\Gamma; s)$ for any greedy k_s-energy sequence on Γ when $s > 1$.*

The next result shows that, in contrast to the case $s > 1$, for $s = 1$ greedy k_1-energy sequences on S^1 are $\mathrm{AEM}(S^1; 1)$. More generally, we shall prove this fact for *smooth* Jordan arcs or curves Γ by which we mean that the natural parametrization $\Phi : [0, L] \longrightarrow \Gamma$, where $L = \mathcal{H}_1(\Gamma)$, is of class C^1 and $\Phi'(t) \neq \mathbf{0}$ for all $t \in [0, L]$.

Theorem 2.10. *Let $\Gamma \subset \mathbb{R}^p$ be a smooth Jordan arc or closed curve, and let $s = d = 1$. Then any greedy k_1-energy sequence $\{\alpha_{N,1}\}$ on Γ is $\mathrm{AEM}(\Gamma; 1)$, i.e.*

$$(27) \qquad \lim_{N \to \infty} \frac{E_1(\alpha_{N,1})}{N^2 \log N} = \frac{2}{\mathcal{H}_1(\Gamma)}.$$

Furthermore,

$$(28) \qquad \frac{1}{N} \sum_{a \in \alpha_{N,1}} \delta_a \xrightarrow{*} \frac{\mathcal{H}_1|_{\Gamma}}{\mathcal{H}_1(\Gamma)}, \qquad N \to \infty.$$

For the analogous result for greedy k_d-energy on the unit sphere $S^d \subset \mathbb{R}^{d+1}$, see Theorem 2.18.

We next consider best-packing configurations. For a collection of N distinct points $\omega_N = \{x_1, \ldots, x_N\} \subset \mathbb{R}^p$ we set

$$\delta(\omega_N) := \min_{1 \leq i \neq j \leq N} |x_i - x_j|,$$

and for an infinite set $A \subset \mathbb{R}^p$, we let

$$\delta_N(A) := \sup\{\delta(\omega_N) : \omega_N \subset A, \, \text{card}(\omega_N) = N\}$$

be the *best-packing distance* of N-point configurations on A. In [5] it is shown (see Theorem 2.2) that if $A = \Gamma$ is a rectifiable Jordan curve or arc in $\mathbb{R}^p$,

$$\lim_{N \to \infty} N\delta_N(\Gamma) = \mathcal{H}_1(\Gamma).$$

This fact leads us to the following.

Definition 2.11. Let $\Gamma \subset \mathbb{R}^p$ be a Jordan arc or curve, and let $\omega_N \subset \Gamma$ be a sequence of N-point configurations. We say that $\{\omega_N\} \in \text{AEM}(\Gamma, \infty)$ if

$$\lim_{N \to \infty} N\delta(\omega_N) = \mathcal{H}_1(\Gamma).$$

Theorem 2.12. *Let $\Gamma \subset \mathbb{R}^p$ be a rectifiable Jordan arc or curve with length $L = \mathcal{H}_1(\Gamma)$, and let $\{x_k\}_{k=0}^\infty \subset \Gamma$ be an arbitrary infinite sequence such that $x_i \neq x_j$ if $i \neq j$. Set $\mathcal{X}_n := \{x_0, \ldots, x_n\}$. Then $\{\mathcal{X}_n\} \notin \text{AEM}(\Gamma, \infty)$. In fact,*

$$(29) \qquad \liminf_{n \to \infty} n\,\delta(\mathcal{X}_n) \leq \frac{4 + 3\sqrt{2}}{4 + 4\sqrt{2}} L < L.$$

Moreover, if $c := \limsup_{n \to \infty} n\,\delta(\mathcal{X}_n) > \frac{2 + \sqrt{2}}{4} L$, then

$$(30) \qquad \liminf_{n \to \infty} n\,\delta(\mathcal{X}_n) \leq \frac{L}{2} + \sqrt{c\,(L - c)} < c.$$

In particular, if $\limsup_{n \to \infty} n\,\delta(\mathcal{X}_n) = L$, then $\liminf_{n \to \infty} n\,\delta(\mathcal{X}_n) \leq L/2$.

In analogy with finite s, we define *greedy best-packing configurations* on a compact set $A \subset \mathbb{R}^p$ by selecting $a_0 \in A$ and choosing $a_n \in A$ so that

$$\min_{0 \leq i \leq n-1} |a_n - a_i| = \max_{x \in A} \min_{0 \leq i \leq n-1} |x - a_i|.$$

Such points are referred to in [12] as *Leja-Bos* points. Theorem 2.12 shows that such points are not asymptotically optimal on rectifiable Jordan arcs or curves.

In [12] there appears a conjecture attributed to L. Bos stating that if A is a compact domain of $\mathbb{C}$, every Leja-Bos sequence $\{a_n\}_{n=0}^\infty$ on A with $|a_0| = \max\{|x| : x \in A\}$ is asymptotically uniformly distributed. We wish to point out that this conjecture is false as the following result asserts (see also Figure 1 in Section 5).

Proposition 2.13. *There exist greedy best-packing sequences on $[0, 1]$ and $[0, 1]^2$ that are not asymptotically uniformly distributed.*

It is obvious, however, that greedy best-packing sequences are *dense* in A.

2.2.3. *Weighted Riesz potentials.* In this subsection we will consider the notion of weighted discrete Riesz energy introduced in [4]. We reproduce here the main definitions.

Definition 2.14. Let $A \subset \mathbb{R}^p$ be an infinite compact set whose d-dimensional Hausdorff measure $\mathcal{H}_d(A)$ is finite. A symmetric function $w : A \times A \longrightarrow [0, \infty]$ is called a CPD-*weight function* on $A \times A$ if

- w is continuous (as a function on $A \times A$) at $\mathcal{H}_d$-almost every point of the diagonal $D(A) := \{(x, x) : x \in A\}$,
- there is some neighborhood G of $D(A)$ (relative to $A \times A$) such that $\inf_G w > 0$, and
- w is bounded on any closed subset $B \subset A \times A$ such that $B \cap D(A) = \emptyset$.

The term CPD stands for (almost) continuous and positive on the diagonal.

Definition 2.15. Let $s > 0$. Given a collection of N ($N \geq 2$) points $\omega_N := \{x_1, \ldots, x_N\} \subset A$, the *weighted Riesz s-energy of ω_N* is defined by

$$E_s^w(\omega_N) := \sum_{1 \leq i \neq j \leq N} \frac{w(x_i, x_j)}{|x_i - x_j|^s},$$

while the *N-point weighted Riesz s-energy of A* is given by

$$\mathcal{E}_s^w(A, N) := \inf\{E_s^w(\omega_N) : \omega_N \subset A, \ \mathrm{card}(\omega_N) = N\}.$$

The *weighted Hausdorff measure $\mathcal{H}_d^{s,w}$* on Borel sets $B \subset A$ is defined by

$$\mathcal{H}_d^{s,w}(B) := \int_B (w(x, x))^{-d/s} d\mathcal{H}_d(x).$$

The following result about the asymptotic behavior of $\{\mathcal{E}_s^w(A, N)\}_N$ was obtained in [4].

Theorem 2.16. *Let A be a compact subset of a d-dimensional C^1-manifold in $\mathbb{R}^p$ and assume that $w : A \times A \to [0, \infty]$ is a CPD-weight function on $A \times A$. Then*

$$(31) \qquad \lim_{N \to \infty} \frac{\mathcal{E}_d^w(A, N)}{N^2 \log N} = \frac{\mathrm{Vol}(B^d)}{\mathcal{H}_d^{d,w}(A)},$$

Furthermore, if $\mathcal{H}_d(A) > 0$ and $\{\widetilde{\omega}_N\}$ is a sequence of configurations on A satisfying (31), with $\mathcal{E}_d^w(A, N)$ replaced by $E_d^w(\widetilde{\omega}_N)$, then

$$(32) \qquad \frac{1}{N} \sum_{x \in \widetilde{\omega}_N} \delta_x \xrightarrow{\ *\ } \frac{\mathcal{H}_d^{d,w}|_A}{\mathcal{H}_d^{d,w}(A)}, \qquad N \to \infty.$$

Assume now that $A \subset \mathbb{R}^p$ is a d-rectifiable set. Then for $s > d$,

$$(33) \qquad \lim_{N \to \infty} \frac{\mathcal{E}_s^w(A, N)}{N^{1+s/d}} = \frac{C_{s,d}}{[\mathcal{H}_d^{s,w}(A)]^{s/d}},$$

where $C_{s,d}$ is the same positive constant that appears in Theorem 1.5. In addition, if $\mathcal{H}_d(A) > 0$, any sequence $\{\widetilde{\omega}_N\}$ of configurations on A satisfying (33) with $\mathcal{E}_s^w(A, N)$ replaced by $E_s^w(\widetilde{\omega}_N)$ also satisfies (32).

Definition 2.17. Let w be a lower semicontinuous CPD-weight function on $A \times A$. A sequence $(a_n)_{n=1}^\infty \subset A$ is called a *greedy (w, s)-energy sequence* on A if it is generated in the same way as generated in Definition 1.3, with $k(x, y) := w(x, y)/|x - y|^s$.

The next result concerns greedy (w, d)-energy points on the unit sphere $S^d \subset \mathbb{R}^{d+1}$.

Theorem 2.18. *Assume that $w : S^d \times S^d \to [0, \infty)$ is a continuous function such that $w(x, x) > 0$ for all $x \in S^d$. Let $\{\alpha_{N,d}^w\}_N$ be an arbitrary greedy (w, d)-energy sequence on S^d, $d \geq 1$. Then*

$$(34) \qquad \lim_{N \to \infty} \frac{E_d^w(\alpha_{N,d}^w)}{N^2 \log N} = \frac{\mathrm{Vol}(B^d)}{\mathcal{H}_d^{d,w}(S^d)},$$

and therefore

$$\frac{1}{N} \sum_{a \in \alpha_{N,d}^w} \delta_a \xrightarrow{\ *\ } \frac{\mathcal{H}_d^{d,w}|_{S^d}}{\mathcal{H}_d^{d,w}(S^d)}, \qquad N \to \infty.$$

In particular, any greedy k_d-energy sequence $\{\alpha_{N,d}\}_N$ on S^d is $\mathrm{AEM}(S^d, d)$ and satisfies (18) for $s = d$.

In the following result we consider greedy (w, p)-energy sequences on sets in $\mathbb{R}^p$ with positive Lebesgue measure.

Theorem 2.19. *Let $A \subset \mathbb{R}^p$ be a compact set such that $\mathcal{H}_p(A) > 0$, and let $\{\alpha_{N,p}^w\}_N$ be an arbitrary greedy (w, p)-energy sequence on A. Assume that $w : A \times A \to [0, \infty)$ is a continuous function such that $w(x, x) > 0$ for all $x \in A$. Then*

$$(35) \qquad \lim_{N \to \infty} \frac{E_p^w(\alpha_{N,p}^w)}{N^2 \log N} = \frac{\mathrm{Vol}(B^p)}{\mathcal{H}_p^{p,w}(A)},$$

and therefore

$$(36) \qquad \frac{1}{N} \sum_{a \in \alpha_{N,p}^w} \delta_a \xrightarrow{\ *\ } \frac{\mathcal{H}_p^{p,w}|_A}{\mathcal{H}_p^{p,w}(A)}, \qquad N \to \infty.$$

In particular, any greedy k_p-energy sequence $\{\alpha_{N,p}\}_N$ on A is $\mathrm{AEM}(A; p)$ and is asymptotically uniformly distributed with respect to $\mathcal{H}_p$.

In view of Proposition 2.6, it is not in general possible to extend Theorem 2.18 to $s > d$. However, for any compact set $A \subset \mathbb{R}^p$ with $\mathcal{H}_\delta(A) > 0$ (where $\delta > 0$ is arbitrary, not necessarily an integer), we can show that the order of growth of $E_s^w(\alpha_{N,s}^w)$ when $s > \delta$ ($s = \delta$) is at most $N^{1+s/\delta}$ ($N^2 \log N$). Let

$$\mathcal{H}_\delta^\infty(A) := \inf\{\sum_i (\operatorname{diam} G_i)^\delta : A \subset \bigcup_i G_i\}, \qquad \delta > 0.$$

Theorem 2.20. *Let $0 < \delta \le p$. Assume that $A \subset \mathbb{R}^p$ is a compact set such that $\mathcal{H}_\delta(A) > 0$. Let w be a bounded lower semicontinuous CPD-weight function on $A \times A$. Consider an arbitrary greedy (w,s)-energy sequence $\{\alpha_{N,s}^w\}_N \subset A$, for $s \ge \delta$. Then, for $N \ge 2$*

$$E_s^w(\alpha_{N,s}^w) \le \begin{cases} M_{s,\delta,A} \, \|w\| \, \mathcal{H}_\delta^\infty(A)^{-s/\delta} N^{1+s/\delta}, & \text{if } s > \delta, \\[2mm] M_{\delta,A} \, \|w\| \, \mathcal{H}_\delta^\infty(A)^{-1} N^2 \log N, & \text{if } s = \delta, \end{cases}$$

where the constants $M_{s,\delta,A} > 0$ and $M_{\delta,A} > 0$ are independent of w and N, and $\|w\| := \sup\{w(x,y) : x,y \in A\}$.

Corollary 2.21. *Let $A \subset \mathbb{R}^p$ be a d-rectifiable set. Suppose $s > d$ and w is a bounded lower semicontinuous CPD-weight function on $A \times A$. Consider an arbitrary greedy (w,s)-energy sequence $\{\alpha_{N,s}^w\}_N \subset A$. Then there are constants $C_1, C_2 > 0$ such that*

$$(37) \qquad C_1 \, N^{1+s/d} \le E_s^w(\alpha_{N,s}^w) \le C_2 \, N^{1+s/d}.$$

If $s = d$ and A is assumed to be a compact subset of a d-dimensional C^1-manifold, then there are constants $C_3, C_4 > 0$ such that

$$(38) \qquad C_3 \, N^2 \log N \le E_d^w(\alpha_{N,d}^w) \le C_4 \, N^2 \log N,$$

for any greedy (w,d)-energy sequence $\{\alpha_{N,d}^w\}_N \subset A$.

Corollary 2.22. *Let $A \subset \mathbb{R}^p$ be a d-rectifiable set. Suppose $s > d$ and w is a bounded lower semicontinuous CPD-weight function on $A \times A$. Consider an arbitrary greedy (w,s)-energy sequence $\{a_n\}_{n=1}^\infty \subset A$. Then $\{a_n\}_{n=1}^\infty$ is dense in A. If $s = d$ and A is assumed to be a compact subset of a d-dimensional C^1-manifold, the same conclusion holds for any greedy (w,d)-energy sequence. Taking $w \equiv 1$ the result is applicable to greedy k_s-energy sequences.*

We can slightly improve the density result in certain cases like a real interval.

Proposition 2.23. *Let $[a,b] \subset \mathbb{R}$ and $s > 1$. Assume that w is a bounded lower semicontinuous CPD-weight function on $[a,b] \times [a,b]$, and $(a_n)_{n=1}^\infty$ is a greedy (w,s)-energy sequence on $[a,b]$. If I is any closed subinterval of $[a,b]$, then*

$$(39) \qquad \liminf_{N \to \infty} \frac{(\operatorname{card}\{1 \le n \le N : a_n \in I\})^{1+\frac{1}{s}}}{N} > 0.$$

We conclude this section by remarking that some results about greedy sequences in the context of external fields have been obtained by the first author and will appear in a separate work.

3. Proofs of results from Section 2.1

Proof of Theorem 2.1. Assume first that $w(A) < \infty$, $\mu \in \mathcal{M}_1(A)$ is an equilibrium measure, and $\{\alpha_{N,k,\mu}\}$ is an arbitrary greedy (k,μ)-energy sequence on A. If a_n is the n-th element of this sequence, it follows by definition that

$$U_n(a_n) \leq U_n(x), \qquad \text{for all} \quad x \in S_\mu^*, \quad n \geq 2.$$

Hence, for any $x \in S_\mu^*$,

$$E(\alpha_{N,k,\mu}) = \sum_{1 \leq i \neq j \leq N} k(a_i, a_j) = 2 \sum_{j=2}^{N} \sum_{i=1}^{j-1} k(a_i, a_j)$$

$$= 2 \sum_{j=2}^{N} U_j(a_j) \leq 2 \sum_{j=2}^{N} U_j(x) = 2 \sum_{j=2}^{N} \sum_{i=1}^{j-1} k(x, a_i).$$

We now integrate the above inequality with respect to μ to obtain

$$E(\alpha_{N,k,\mu}) \leq 2 \sum_{j=2}^{N} \sum_{i=1}^{j-1} U^\mu(a_i).$$

Taking into account that $U^\mu(a_i) \leq w(A)$ for all i $(a_i \in S_\mu^*)$ it follows that

$$(40) \qquad E(\alpha_{N,k,\mu}) \leq N(N-1)\, w(A).$$

Now, if $\{\omega_N^*\}$ is a sequence of optimal N-point configurations on A, then $E(\omega_N^*) \leq E(\alpha_{N,k,\mu})$ for all N. Therefore (14) for $\alpha_{N,k,\mu}$ is a consequence of (40) and (5).

Consider the sequence of normalized counting measures

$$\nu_N := \frac{1}{N} \sum_{a \in \alpha_{N,k,\mu}} \delta_a,$$

and assume that the equilibrium measure μ is unique. Let $g_n : A \times A \to \mathbb{R}$ be a non-decreasing sequence of continuous functions that converges pointwise to k on $A \times A$ (recall that k is lower semicontinuous and hence lower bounded on $A \times A$). We have

$$\int \int g_n(x,y)\, d\nu_N(x) d\nu_N(y) = \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} g_n(a_i, a_j)$$

$$= \frac{1}{N^2} \Big(\sum_{i=1}^{N} g_n(a_i, a_i) + \sum_{1 \leq i \neq j \leq N} g_n(a_i, a_j) \Big)$$

$$\leq \frac{1}{N^2} \left(\sum_{i=1}^{N} g_n(a_i, a_i) + \sum_{1 \leq i \neq j \leq N} k(a_i, a_j) \right);$$

hence

$$(41) \qquad \int \int g_n(x, y) d\nu_N(x) d\nu_N(y) \leq \frac{\sum_{i=1}^{N} g_n(a_i, a_i) + E(\alpha_{N,k,\mu})}{N^2}.$$

By the compactness of A and the continuity of g_n, there exists a constant $M_n > 0$ such that

$$\sum_{i=1}^{N} |g_n(a_i, a_i)| \leq N M_n.$$

Therefore, for each fixed n we have

$$(42) \qquad \lim_{N \to \infty} \frac{\sum_{i=1}^{N} g_n(a_i, a_i) + E(\alpha_{N,k,\mu})}{N^2} = w(A).$$

Now let $\{\nu_N\}_{N \in \mathcal{N}}$ be a subsequence that converges in the weak-star topology to a measure $\lambda \in \mathcal{M}_1(A)$. Since $\nu_N \times \nu_N$ converges weak-star to $\lambda \times \lambda$, we have

$$\lim_{N \to \infty} \int \int g_n(x, y) d\nu_N(x) d\nu_N(y) = \int \int g_n(x, y) d\lambda(x) d\lambda(y).$$

Thus from (41) and (42) we conclude that

$$\int \int g_n(x, y) d\lambda(x) d\lambda(y) \leq w(A),$$

and letting $n \to \infty$ we obtain

$$W(\lambda) = \int \int k(x, y) d\lambda(x) d\lambda(y) \leq w(A).$$

It follows that λ is an equilibrium measure. By hypothesis there is only one equilibrium measure. Thus $\lambda = \mu$ and (15) is proved for $\alpha_{N,k,\mu}$.

We next show (16) for $\alpha_{N,k,\mu}$. It is not assumed now that the equilibrium measure is unique, and $\alpha_{N,k,\mu}$ denotes a greedy (k, μ)-energy sequence associated with a certain equilibrium measure μ. We know from the first part of the proof that

$$(43) \qquad \lim_{N \to \infty} \frac{E(\alpha_{N,k,\mu})}{N^2} = \lim_{N \to \infty} \frac{2 \sum_{i=2}^{N} U_i(a_i)}{N^2} = w(A).$$

For every $n \geq 1$,

$$\frac{U_{n+1}(a_{n+1})}{n} = \inf_{x \in S_\mu^*} \frac{1}{n} \sum_{i=1}^{n} k(x, a_i).$$

Integrating this equality with respect to μ and recalling that $a_i \in S_\mu^*$, we get

$$(44) \qquad \frac{U_{n+1}(a_{n+1})}{n} \leq \frac{1}{n} \sum_{i=1}^{n} \int k(x, a_i) \, d\mu(x) = \frac{1}{n} \sum_{i=1}^{n} U^\mu(a_i) \leq w(A).$$

On the other hand, for every $n \geq 2$,

$$(45) \qquad U_{n+1}(a_{n+1}) \geq U_n(a_n) + L \,,$$

where $L := \inf\{k(x, y) : x, y \in S_\mu^*\}$. By rescaling k we may assume that $L \leq -1$.

Let $\epsilon \in (0, 1)$. Assume that m is an integer such that

$$(46) \qquad \frac{U_{m+1}(a_{m+1})}{m} < w(A) - \epsilon \,.$$

We shall show that this inequality can hold only for finitely many integers m. Applying (45) repeatedly we obtain for $(1 + \epsilon/(3L)) \, m \leq i \leq m$,

$$\frac{U_{i+1}(a_{i+1})}{m} \leq w(A) - \epsilon - \frac{(m-i)L}{m} \leq w(A) - \epsilon + \frac{\epsilon/3}{1 + \epsilon/(3L)} \leq w(A) - \frac{\epsilon}{2} \,,$$

and so

$$\frac{U_{i+1}(a_{i+1})}{i} \leq \frac{m}{i}(w(A) - \epsilon/2) \leq \frac{m}{i} \, w(A) - \frac{\epsilon}{2} \,.$$

Taking into account (44) and the last inequality,

$$(47) \qquad \frac{2}{(m+1)\,m} \sum_{i=1}^{m} U_{i+1}(a_{i+1}) \leq \frac{2}{(m+1)\,m} \sum_{1 \leq i < (1+\epsilon/(3L))m} i \, w(A)$$

$$+ \frac{2}{(m+1)\,m} \sum_{(1+\epsilon/(3L))m \leq i \leq m} m \, w(A) - \frac{\epsilon}{2} \frac{2}{(m+1)\,m} \sum_{(1+\epsilon/(3L))m \leq i \leq m} i \,.$$

Furthermore, it is easy to see that

$$(48) \qquad -\frac{\epsilon}{2} \frac{2}{(m+1)\,m} \sum_{(1+\epsilon/(3L))m \leq i \leq m} i \leq \frac{\epsilon^2}{6L(m+1)}\left(1 + 2m + \frac{m\epsilon}{3L}\right)$$

$$\leq \frac{\epsilon^2(1 + \epsilon/(3L))}{6L} \,.$$

If $w(A) \leq 0$, then

$$\frac{2}{(m+1)\,m}\left\{\sum_{1 \leq i < (1+\epsilon/(3L))m} i \, w(A) + \sum_{(1+\epsilon/(3L))m \leq i \leq m} m \, w(A)\right\} \leq w(A)$$

and hence it follows from (47) and (48) that

$$(49) \qquad \frac{2}{(m+1)\,m} \sum_{i=1}^{m} U_{i+1}(a_{i+1}) \leq w(A) + \frac{\epsilon^2(1 + \epsilon/(3L))}{6L} \,.$$

Since the right-hand side of (49) is a constant strictly less than $w(A)$, by (43) it follows that there are only finitely many integers m satisfying (46). This implies with (44) that (16) holds.

Now assume that $w(A) > 0$. It is easy to verify that

$$\frac{2}{(m+1)\,m}\left\{\sum_{1 \leq i < (1+\epsilon/(3L))m} i \, w(A) + \sum_{(1+\epsilon/(3L))m \leq i \leq m} m \, w(A)\right\}$$

$$\leq \left(1 + \frac{2}{m+1} + \frac{\epsilon}{3L(m+1)} + \frac{\epsilon^2 m}{9(m+1)L^2}\right) w(A),$$

and so, from (47) and (48), we deduce that

$$\frac{2}{(m+1)\,m} \sum_{i=1}^{m} U_{i+1}(a_{i+1}) \leq \left(1 + \frac{2}{m+1} + \frac{\epsilon}{3L(m+1)} + \frac{\epsilon^2 m}{9(m+1)L^2}\right) w(A)$$

$$+ \frac{\epsilon^2(1 + \epsilon/(3L))}{6L}\,.$$

If we assume that there are infinitely many integers m satisfying (46), then applying the last inequality we obtain

$$(50) \qquad \limsup_{N \to \infty} \frac{2\sum_{i=2}^{N} U_i(a_i)}{N^2} \leq w(A) + \frac{\epsilon^2 w(A)}{9\,L^2} + \frac{\epsilon^2(1 + \epsilon/(3L))}{6L}\,.$$

We may assume without loss of generality that $L \leq -1$ also satisfies $L < -(1 + 2w(A))/3$. Then the right-hand side of (50) is a constant strictly less than $w(A)$, which contradicts (43). This concludes the proof of (16) for $\alpha_{N,k,\mu}$.

If k satisfies the maximum principle, we know by Fuglede's result (see paragraph after Definition 1.3) and (4) that $U^\mu(x) \leq w(A)$ for all $x \in A$. Therefore the assertions (14)-(16) follow (replacing S_μ^* by A) for any greedy k-energy sequence $\{\alpha_{N,k}\}$ on A by using the argument presented above. □

Proof of Corollary 2.2. It is well-known (see for example [22]) that for any $s < d$ the equilibrium measure associated with the Riesz kernel k_s is unique and coincides with σ_d. Since $\operatorname{supp}(\sigma_d) = S^d$, any greedy k_s-energy sequence $\{\alpha_{N,s}\}_N \subset S^d$ is a greedy (k_s, σ_d)-energy sequence. Therefore by (14) we obtain that $\{\alpha_{N,s}\}_N \in \operatorname{AEM}(S^d; s)$. The values on the right-hand side of (17) are the values of $I_s(\sigma_d)$. The case $s > 0$ follows from formula (1.2) of [21] and the case $s = 0$ from formula (2.26) of [8]. Finally (18) follows from (15). □

Proof of Corollary 2.3. It is shown in [22] that for $s < 1$ the equilibrium measure associated with the Riesz kernel k_s is

$$\frac{c_s}{(1 - x^2)^{(1-s)/2}}\,dx\,, \qquad x \in (-1, 1)\,,$$

and its energy is given by the value on the right-hand side of (19). □

Proof of Proposition 2.4. We have
(51)
$$\frac{E_s(\alpha_{3\cdot 2^n, s}) - I_s(\sigma)(3 \cdot 2^n)^2}{(3 \cdot 2^n)^{1+s}} = \frac{1}{3^{1+s}} \frac{E_s(\alpha_{3\cdot 2^n, s}) - I_s(\sigma)(2^n)^2 - I_s(\sigma)2^{2n+3}}{(2^n)^{1+s}}\,.$$

As will be justified in Section 4 (see Lemma 4.3), the relation

$$E_s(\alpha_{3\cdot 2^n, s}) = \frac{1}{2} \mathcal{E}_s(S^1, 2^{n+2}) + \mathcal{E}_s(S^1, 2^n)$$

holds. Therefore, from (51), it follows that

$$\frac{E_s(\alpha_{3\cdot 2^n,s}) - I_s(\sigma)(3\cdot 2^n)^2}{(3\cdot 2^n)^{1+s}}$$

$$= \frac{1}{3^{1+s}}\left(\frac{\mathcal{E}_s(S^1, 2^n) - I_s(\sigma)(2^n)^2}{(2^n)^{1+s}} + \frac{4^{1+s}}{2}\frac{\mathcal{E}_s(S^1, 2^{n+2}) - I_s(\sigma)(2^{n+2})^2}{(2^{n+2})^{1+s}}\right).$$

Applying now (20) we get

$$\lim_{n\to\infty}\frac{E_s(\alpha_{3\cdot 2^n,s}) - I_s(\sigma)(3\cdot 2^n)^2}{(3\cdot 2^n)^{1+s}} = \left(\frac{1}{2}\left(\frac{4}{3}\right)^{1+s} + \left(\frac{1}{3}\right)^{1+s}\right)\frac{2\zeta(s)}{(2\pi)^s}.$$

Finally, it is easy to check that $f(s) = \frac{1}{2}(\frac{4}{3})^{1+s} + (\frac{1}{3})^{1+s} < 1$ for all $s \in (0,1)$.
□

Proof of Corollary 2.5. Since $\alpha_{2^n,s}$ consists of 2^n equally spaced points (see Lemma 4.2 below), $E_s(\alpha_{2^n,s}) = \mathcal{E}_s(S^1, 2^n)$, and therefore

$$\lim_{n\to\infty}\frac{E_s(\alpha_{2^n,s}) - I_s(\sigma)2^{2n}}{2^{n(1+s)}} = \frac{2\zeta(s)}{(2\pi)^s},$$

but the subsequence $\{\alpha_{3\cdot 2^n,s}\}_n$ provides a different limit value, given by (21).
□

4. Proofs of results from Subsection 2.2.1

In order to prove Proposition 2.6 we need some auxiliary lemmas that give a geometric description of greedy k_s-energy sequences on S^1. The proofs of Lemmas 4.1 and 4.2 below are straightforward; cf. [24] for the details.

Lemma 4.1. *Let $s \geq 0$ and consider two points $x_1, x_2 \in S^1$. Set*

$$f(x) := K(|x - x_1|; s) + K(|x - x_2|; s), \qquad x \in S^1,$$

where K is defined in (8). Then on each arc determined by x_1 and x_2 the function f has only one minimum and it is attained at the midpoint of the arc.

Lemma 4.2. *Let $s \geq 0$ and assume that $(a_n)_{n=1}^{\infty}$ is an arbitrary greedy k_s-energy sequence on S^1. Then*

(i) *for every positive integer m, the set $\alpha_{2^m,s}$ consists of 2^m equally spaced points, that is,*

$$\alpha_{2^m,s} = \{a_1 e^{i\frac{2\pi n}{2^m}}\}_{n=1}^{2^m};$$

(ii) *for every positive integer m, the set $\alpha_{3\cdot 2^m}$ can be written as*

(52) $$\alpha_{3\cdot 2^m} = S_{2^{m+2}} \setminus S_{2^m},$$

where $S_{2^{m+2}}$ and S_{2^m} are formed, respectively, by 2^{m+2} and 2^m equally spaced points, and $S_{2^m} \subset S_{2^{m+2}}$;

(iii) *the choice of any point a_n is independent of s.*

Since greedy k_s-energy sequences $\{\alpha_{N,s}\}$ on the unit circle S^1 are independent of s, we will denote them simply by α_N.

Lemma 4.3. *Let $s \geq 0$. Then given any greedy k_s-energy sequence $\{\alpha_N\}_N$ on S^1 the following relation holds for every $n \geq 1$:*

$$(53) \qquad E_s(\alpha_{3\cdot 2^n}) = \frac{1}{2} \mathcal{E}_s(S^1, 2^{n+2}) + \mathcal{E}_s(S^1, 2^n).$$

Proof. If $\{x_k\}_{k=1}^N \subset S^1$ is an arbitrary collection of N equally spaced points, then using the simple equality $|e^{i\xi} - e^{i\theta}| = 2|\sin(\frac{\xi-\theta}{2})|$, we conclude that for $s > 0$,

$$(54) \qquad \mathcal{E}_s(S^1, N) = E_s(\{x_k\}_{k=1}^N) = 2^{-s} N \sum_{n=1}^{N-1} \sin\left(\frac{\pi n}{N}\right)^{-s}.$$

Consider any greedy k_s-energy sequence $(\alpha_N)_{N=1}^\infty$ on S^1. We claim that

$$E_s(\alpha_{3\cdot 2^n}) = E_s(S_{2^{n+2}}) - 2^{n+1} \cdot 2^{-s} \sum_{k=1}^{2^{n+2}-1} \sin\left(\frac{\pi k}{2^{n+2}}\right)^{-s} + E_s(S_{2^n}),$$

where $\alpha_{3\cdot 2^n} = S_{2^{n+2}} \setminus S_{2^n}$ is as in (52). To see this, notice that $E_s(\alpha_{3\cdot 2^n})$ is obtained by removing twice from $E_s(S_{2^{n+2}})$ all terms $|e^{i\xi} - e^{i\theta}|^{-s}$ where either $e^{i\xi} \in S_{2^n}$ or $e^{i\theta} \in S_{2^n}$.

Since

$$E_s(S_{2^{n+2}}) = \mathcal{E}_s(S^1, 2^{n+2}), \qquad E_s(S_{2^n}) = \mathcal{E}_s(S^1, 2^n),$$

(53) follows by applying (54). The case $s = 0$ is proved similarly. $\qquad\square$

Proof of Proposition 2.6. Using (53) we obtain

$$\frac{E_s(\alpha_{3\cdot 2^n})}{3^{1+s} 2^{n(1+s)}} = \frac{1}{3^{1+s}} \frac{1}{2} \frac{2^{(n+2)(1+s)}}{2^{n(1+s)}} \frac{\mathcal{E}_s(S^1, 2^{n+2})}{2^{(n+2)(1+s)}} + \frac{1}{3^{1+s}} \frac{\mathcal{E}_s(S^1, 2^n)}{2^{n(1+s)}}.$$

Simplifying the above expression and applying (24) we conclude that

$$\lim_{n\to\infty} \frac{E_s(\alpha_{3\cdot 2^n})}{(3\cdot 2^n)^{1+s}} = \left(\frac{1}{2}\left(\frac{4}{3}\right)^{1+s} + \left(\frac{1}{3}\right)^{1+s}\right) \frac{2\zeta(s)}{(2\pi)^s}.$$

It is straightforward to check that $f(s) = \frac{1}{2}\left(\frac{4}{3}\right)^{1+s} + \left(\frac{1}{3}\right)^{1+s} > 1$ for all $s > 1$.
$\square$

Proof of Proposition 2.7. First observe that

$$\frac{E_1(\alpha_{3\cdot 2^n}) - \frac{1}{\pi}(3\cdot 2^n)^2 \log(3\cdot 2^n)}{(3\cdot 2^n)^2}$$

$$= \frac{1}{9}\left(\frac{(1/2)\,\mathcal{E}_1(S^1, 2^{n+2}) + \mathcal{E}_1(S^1, 2^n) - \frac{1}{\pi}(3\cdot 2^n)^2 \log(3\cdot 2^n)}{2^{2n}}\right).$$

We add and subtract $(1/\pi)2^{2n} \log(2^n)$ to obtain

$$(55) \qquad \frac{E_1(\alpha_{3\cdot 2^n}) - \frac{1}{\pi}(3\cdot 2^n)^2 \log(3\cdot 2^n)}{(3\cdot 2^n)^2}$$

$$= \frac{1}{9}\left(\frac{\mathcal{E}_1(S^1, 2^n) - \frac{1}{\pi}2^{2n} \log(2^n)}{2^{2n}} + 16\frac{(1/2)\,\mathcal{E}_1(S^1, 2^{n+2}) - \frac{1}{\pi}\Lambda_n}{2^{2(n+2)}}\right)$$

where $\Lambda_n = (3 \cdot 2^n)^2 \log(3 \cdot 2^n) - 2^{2n} \log(2^n)$. Taking into account that

$$\Lambda_n = \frac{2^{2(n+2)}}{2} \log(2^{n+2}) + \log(3)(3 \cdot 2^n)^2 - 8 \log(4) 2^{2n}$$

it follows that

$$(56) \qquad 16 \frac{(1/2)\,\mathcal{E}_1(S^1, 2^{n+2}) - \frac{1}{\pi}\Lambda_n}{2^{2(n+2)}}$$

$$= 8 \frac{\mathcal{E}_1(S^1, 2^{n+2}) - \frac{1}{\pi} 2^{2(n+2)} \log(2^{n+2})}{2^{2(n+2)}} + \frac{1}{\pi}(8 \log(4) - 9 \log(3))\,.$$

Applying (25), (55) and (56) we conclude that

$$\lim_{n \to \infty} \frac{E_1(\alpha_{3 \cdot 2^n}) - \frac{1}{\pi}(3 \cdot 2^n)^2 \log(3 \cdot 2^n)}{(3 \cdot 2^n)^2}$$

$$= \frac{1}{\pi}(\gamma - \log(\pi/2)) + \frac{1}{\pi}\Big(\frac{8}{9} \log(4) - \log(3)\Big) = \frac{1}{\pi}\Big(\gamma - \log(\pi/2) + \log(2^{\frac{16}{9}}/3)\Big)\,.$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Proof of Corollary 2.8. Since $E_1(\alpha_{2^n}) = \mathcal{E}_1(S^1, 2^n)$ for all n, the result follows from (25) and (26). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

5. Proofs of results from subsection 2.2.2

Proof of Theorem 2.9. Assume first that Γ is a Jordan arc. If $x_1, x_2 \in \Gamma$, we denote by (x_1, x_2) the subarc joining x_1 and x_2, and by $l(x_1, x_2)$ its length.

Let $\mathcal{X}_n := \{x_{k,n}\}_{k=0}^n$ be a sequence of configurations on Γ, where we assume that the points $x_{k,n}$ are located in successive order. Set

$$(57) \qquad\qquad d_{k,n} := l(x_{k-1,n}, x_{k,n})\,, \qquad k = 1, \ldots, n\,.$$

In [25] the following result was proved:

Theorem 5.1. *Let Γ be a rectifiable Jordan arc in $\mathbb{R}^p$. If $s > 1$ and $\{\mathcal{X}_n\}_n \in \mathrm{AEM}(\Gamma; s)$, then*

$$(58) \qquad\qquad \lim_{n \to \infty} \sum_{k=1}^n \Big| d_{k,n} - \frac{L}{n} \Big| = 0\,, \qquad L := \mathcal{H}_1(\Gamma)\,.$$

We prove Theorem 2.9 by contradiction. Let $\{x_k\}_{k=0}^\infty \subset \Gamma$ be an arbitrary sequence of distinct points and set $\mathcal{X}_n := \{x_k\}_{k=0}^n$. We will use the notation $\mathcal{X}_n = \{x_{0,n}, \ldots, x_{n,n}\}$. Assume that $\{\mathcal{X}_n\}_n \in \mathrm{AEM}(\Gamma; s)$. Let $\delta > 0$ and consider the sets

$$A_n^\delta := \{k :\in \{1, \ldots, n\} : \frac{L-\delta}{n} < d_{k,n} < \frac{L+\delta}{n}\}\,, \qquad B_n^\delta := \{1, \ldots, n\} \backslash A_n^\delta\,.$$

Let $\epsilon > 0$ be a fixed number. Then from (58) there exists $N = N(\epsilon) \in \mathbb{N}$ such that, if $n \geq N$,

$$(59) \qquad\qquad \sum_{k=1}^n \Big| d_{k,n} - \frac{L}{n} \Big| \leq \epsilon\,.$$

If $k \in B_n^\delta$, then $|d_{k,n} - L/n| \geq \delta/n$, and from (59) it follows that

$$\text{card}(B_n^\delta)\frac{\delta}{n} \leq \epsilon, \qquad n \geq N.$$

Therefore,

$$\text{card}(A_n^\delta) = n - \text{card}(B_n^\delta) \geq n\left(1 - \frac{\epsilon}{\delta}\right), \qquad n \geq N.$$

There are exactly n subarcs $(x_{k-1,n}, x_{k,n})$, and when we add the next $n/2$ points (we may assume that n is even) to the configuration $\mathcal{X}_n$, obviously at most $n/2$ of these new points will lie in the subarcs $(x_{k-1,n}, x_{k,n})$ where $k \in A_n^\delta$. Setting

$$C_n^\delta := \{k \in A_n^\delta : (x_{k-1,n}, x_{k,n}) \text{ does not contain a new point}\},$$

we have

$$\text{card}(C_n^\delta) \geq n\left(1 - \frac{\epsilon}{\delta}\right) - \frac{n}{2} = n\left(\frac{1}{2} - \frac{\epsilon}{\delta}\right).$$

Now since the intervals $(x_{k-1,n}, x_{k,n})$ with $k \in C_n^\delta$ do not contain a new point, there are at least $\text{card}(C_n^\delta)$ values of k' in $\{1, \ldots, 3n/2\}$ such that $d_{k',3n/2} = d_{k,n}$ for some $k \in C_n^\delta$. For these values of k' and the corresponding values of k, we have

$$\left|d_{k',3n/2} - \frac{L}{3n/2}\right| = \left|d_{k,n} - \frac{L}{n} + \frac{L}{3n}\right|.$$

Now we choose δ to be any fixed value less than $L/3$, say $\delta := L/6$. Then for $k \in C_n^\delta$,

$$\left|d_{k,n} - \frac{L}{n} + \frac{L}{3n}\right| \geq \left|\frac{L}{3n} - \left|\frac{L}{n} - d_{k,n}\right|\right| = \frac{L}{3n} - \left|\frac{L}{n} - d_{k,n}\right| > \frac{L}{3n} - \frac{L}{6n} = \frac{L}{6n}.$$

Finally,

$$\sum_{k'=1}^{3n/2}\left|d_{k',3n/2} - \frac{L}{3n/2}\right| \geq n\left(\frac{1}{2} - \frac{\epsilon}{\delta}\right)\frac{L}{6n} = \left(\frac{1}{2} - \frac{6\,\epsilon}{L}\right)\frac{L}{6}.$$

But the above estimate contradicts (59) since we can select ϵ sufficiently small so that

$$\left(\frac{1}{2} - \frac{6\,\epsilon}{L}\right)\frac{L}{6} > \epsilon.$$

If Γ is a closed Jordan curve, we select an orientation for it. Then the above reasoning used to prove the result in the case of Jordan arcs is also applicable. We only have to define $(x_{k-1,n}, x_{k,n})$ as the subarc joining $x_{k-1,n}$ and $x_{k,n}$ on which a particle moves from $x_{k-1,n}$ to $x_{k,n}$ following the orientation prescribed. The details of the argument are left to the reader. □

Proof of Theorem 2.10. We first assume that Γ is a smooth Jordan arc of length L. We will reduce the problem of asymptotics of $\alpha_{N,1}$ on Γ to a weighted problem on $[0, L]$ and then apply Theorem 2.19. Let $\Phi : [0, L] \longrightarrow$

Γ be the natural parametrization of Γ and define $w : [0, L] \times [0, L] \longrightarrow [0, \infty)$ by

$$(60) \qquad w(x, y) := \frac{|x - y|}{|\Phi(x) - \Phi(y)|} \, .$$

Let $\Psi = \Phi^{-1}$ be the inverse function of Φ. If a_n is the n-th element of the greedy k_1-energy sequence on Γ, let $b_n := \Psi(a_n) \in [0, L]$ and $\beta_N := \{b_1, \ldots, b_N\}$. Since for $t = \Phi(x), x \in [0, L]$,

$$\inf_{t \in \Gamma} \sum_{i=1}^{n-1} \frac{1}{|t - a_i|} = \inf_{x \in [0,L]} \sum_{i=1}^{n-1} \frac{1}{|\Phi(x) - \Phi(b_i)|} = \inf_{x \in [0,L]} \sum_{i=1}^{n-1} \frac{w(x, b_i)}{|x - b_i|} \, ,$$

it follows that $\{\beta_N\}$ is a greedy $(w, 1)$-energy sequence on $[0, L]$ (see Definition 2.17) associated with the weight function (60). Notice that

$$\mathcal{H}_1^{1,w}([0, L]) = \int_0^L w(x, x)^{-1} \, dx = \int_0^L |\Phi'(x)| \, dx = L \, .$$

Applying Theorem 2.19 we obtain that

$$\lim_{N \to \infty} \frac{E_1(\alpha_{N,1})}{N^2 \log N} = \lim_{N \to \infty} \frac{E_1^w(\beta_N)}{N^2 \log N} = \frac{2}{\mathcal{H}_1^{1,w}([0, L])} = \frac{2}{L} \, .$$

If Γ is a smooth Jordan closed curve and $\Phi : [0, L] \longrightarrow \Gamma$ is the natural parametrization of Γ ($\Phi(0) = \Phi(L), \Phi'(0) = \Phi'(L)$), we set

$$\overline{w}(z, \xi) := \frac{|z - \xi|}{|\Phi(x) - \Phi(y)|} \, , \qquad z = e^{2\pi i x/L}, \xi = e^{2\pi i y/L}; \quad x, y \in [0, L] \, ,$$

and apply (with the aid of Theorem 2.18) a similar argument as above on the unit circle S^1.

In both cases, (28) is a consequence of (27) and Theorem 1.5. $\qquad \square$

Proof of Theorem 2.12. Let $p > 1$ be a rational number and let $n \in \mathbb{Z}_+$ be such that n/p is an integer. We denote the first $n+1$ points of the sequence $\{x_k\}_{k=0}^\infty$ by $\mathcal{X}_n = \{x_{0,n}, \ldots, x_{n,n}\}$, where as in the proof of Theorem 2.9 the points $x_{k,n}$ are located on Γ in successive order. There are exactly n subarcs $(x_{i,n}, x_{i+1,n})$. We add to $\mathcal{X}_n$ the next n/p points of the sequence $\{x_k\}$. Then there are at least $(p-1)\, n/p$ subarcs $(x_{i,n}, x_{i+1,n})$ not containing a new point. These subarcs have length at least $\delta(\mathcal{X}_n)$. We select $(p - 1)\, n/p$ of those.

On the other hand, there are $2\, n/p$ subarcs $(x_{i,(p+1)n/p}, x_{i+1,(p+1)n/p})$ remaining with length at least $\delta(\mathcal{X}_{(p+1)n/p})$. Consequently,

$$(61) \qquad \frac{(p - 1)\, n}{p} \delta(\mathcal{X}_n) + \frac{2\, n}{p} \delta(\mathcal{X}_{(p+1)n/p}) \le L \, .$$

Thus

$$(62) \qquad \liminf_{n \to \infty} n\, \delta(\mathcal{X}_n) \le \frac{p^2 + p}{p^2 + 2p - 1} L \, .$$

Letting $f(p)$ denote the right-hand side of (62), we see that for $p > 1$ the function f attains its minimum when $p = 1 + \sqrt{2}$, and $f(1 + \sqrt{2}) = \frac{4 + 3\sqrt{2}}{4 + 4\sqrt{2}} L$, which establishes (29).

Let $\mathcal{X}_{n_k}$ be a subsequence of configurations such that $\lim_{k \to \infty} n_k \, \delta(\mathcal{X}_{n_k}) = c$. Notice that we cannot apply (61) directly because we cannot assume that n_k/p is an integer. Let $\lfloor x \rfloor$ denote the integral part of x and let $\{x\} := x - \lfloor x \rfloor$. Then we get

(63) $$\left(n_k - \left\lfloor \frac{n_k}{p} \right\rfloor \right) \delta(\mathcal{X}_{n_k}) + 2 \left\lfloor \frac{n_k}{p} \right\rfloor \delta(\mathcal{X}_{n_k + \lfloor n_k/p \rfloor}) \leq L.$$

Since

$$\left| \left(n_k - \left\lfloor \frac{n_k}{p} \right\rfloor \right) \delta(\mathcal{X}_{n_k}) - \frac{(p-1)}{p} n_k \, \delta(\mathcal{X}_{n_k}) \right| = \left\{ \frac{n_k}{p} \right\} \delta(\mathcal{X}_{n_k}) \leq \delta(\mathcal{X}_{n_k}),$$

it follows that

(64) $$\lim_{k \to \infty} \left(n_k - \left\lfloor \frac{n_k}{p} \right\rfloor \right) \delta(\mathcal{X}_{n_k}) = \frac{(p-1)}{p} c.$$

Similarly,

$$\left| (p+1) \left\lfloor \frac{n_k}{p} \right\rfloor \delta(\mathcal{X}_{n_k + \lfloor n_k/p \rfloor}) - \left(n_k + \left\lfloor \frac{n_k}{p} \right\rfloor \right) \delta(\mathcal{X}_{n_k + \lfloor n_k/p \rfloor}) \right| \leq p \, \delta(\mathcal{X}_{n_k + \lfloor n_k/p \rfloor})$$

and thus

(65) $$\liminf_{k \to \infty} \left(n_k + \left\lfloor \frac{n_k}{p} \right\rfloor \right) \delta(\mathcal{X}_{n_k + \lfloor n_k/p \rfloor}) = \liminf_{k \to \infty} (p+1) \left\lfloor \frac{n_k}{p} \right\rfloor \delta(\mathcal{X}_{n_k + \lfloor n_k/p \rfloor}).$$

Since $\liminf_{n \to \infty} n \, \delta(\mathcal{X}_n) \leq \liminf_{k \to \infty} (n_k + \lfloor n_k/p \rfloor) \, \delta(\mathcal{X}_{n_k + \lfloor n_k/p \rfloor})$, we obtain from (63)-(65) that

$$\frac{2}{p+1} \liminf_{n \to \infty} n \, \delta(\mathcal{X}_n) \leq L - \frac{p-1}{p} c.$$

Therefore

$$\liminf_{n \to \infty} n \, \delta(\mathcal{X}_n) \leq g(p) := \left(1 + \frac{1}{p} \right) \frac{p(L-c) + c}{2}.$$

If $c = L$ we get immediately that $\liminf_{n \to \infty} n \, \delta(\mathcal{X}_n) \leq L/2$. The function g attains a minimum for $p = \sqrt{c/(L-c)}$ and takes the value $L/2 + \sqrt{c(L-c)}$ at this point. This proves (30). $\square$

Proof of Proposition 2.13. Consider the sequence $\{a_n\}_{n=0}^{\infty} \subset [0,1]$ defined as follows:

- $a_0 := 1, a_1 := 0, a_2 := 1/2$.
- Assuming that the first $2^n + 1$ points have been selected, let $a_{2^n + i} := (2i - 1)/2^{n+1}$, $1 \leq i \leq 2^n$.

Obviously $\{a_n\}_{n=0}^{\infty}$ is a greedy best-packing sequence on $[0,1]$. However, the sequence of configurations $S_N := \{a_n\}_{n=0}^{N}$ is not uniformly distributed since

$$\lim_{n \to \infty} \frac{\operatorname{card}(S_{3 \cdot 2^{n-1}} \cap [0, 1/2])}{3 \cdot 2^{n-1} + 1} = \lim_{n \to \infty} \frac{2^n + 1}{3 \cdot 2^{n-1} + 1} = \frac{2}{3} \neq \frac{1}{2}.$$

Now we consider the sequence $\{b_n\}_{n=1}^{\infty} \subset [0,1]^2$ formed in the following way:

1) $b_1 := (1,1), b_2 := (0,0), b_3 := (0,1), b_4 := (1,0)$.

2) Assume that the first $(2^{n-1}+1)^2$, $n \geq 1$, points have been selected.

    2.1) We define the next $2^{2(n-1)}$ points as the centers of the $2^{2(n-1)}$ squares of area $2^{-2(n-1)}$ whose vertices are the first $(2^{n-1}+1)^2$ points $b_1, \ldots, b_{(2^{n-1}+1)^2}$. These $2^{2(n-1)}$ points are chosen in an arbitrary order.

    2.2) Now we select the next $2^n(2^{n-1}+1)$ points to be the middle points of the edges of the $2^{2(n-1)}$ squares mentioned above. The first group of points that we add consists of those points with abscissa equal to 0. The second group is formed by those with abscissa equal to 2^{-n}. In general, the points from the i-th group have abscissa $(i-1)/2^n$. We add exactly 2^n+1 groups, and in each one of them, the points are selected in an arbitrary order.

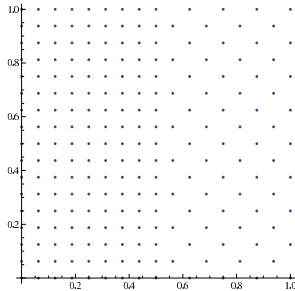Figure 5 illustrates the first 221 points of the sequence $\{b_n\}_{n=1}^{\infty}$.

FIGURE 5. Greedy best-packing points for square: a counterexample to a conjecture of Bos.

Using Voronoi cell decompositions one can show that $\{b_n\}_{n=1}^{\infty}$ is a greedy best-packing sequence on $[0,1]^2$. Indeed if we consider this Voronoi decomposition of $[0,1]^2$ corresponding to the points $\{b_i\}_1^N$, that is, $[0,1]^2 = \cup_{i=1}^N V_i$ where

$$V_i = \{x \in [0,1]^2 : |x - b_i| \leq |x - b_j| \text{ for all } j = 1, \ldots, N\},$$

then it is easy to see that each V_i is a convex polygon with 3, 4 or 5 sides and that b_{N+1} corresponds to a vertex of the V_i's that is of maximal distance from the points $\{b_i\}_{i=1}^N$.

To show that the sequence of configurations $T_N := \{b_i\}_{i=1}^N$ is not asymptotically uniformly distributed, we consider the subsequence of sets consisting of $N(n) = 3 \cdot 2^{2(n-1)} + 7 \cdot 2^{n-2} + 1$ points. We have that

$$\lim_{n \to \infty} \frac{\text{card}(T_{N(n)} \cap [0, 1/2] \times [0,1])}{N(n)} = \lim_{n \to \infty} \frac{(2^{n-1}+1)(2^n+1)}{N(n)} = \frac{2}{3} \neq \frac{1}{2}.$$

□

Using a similar argument it is possible to construct a greedy best-packing sequence on $[0,1]^p \subset \mathbb{R}^p$ that is not asymptotically uniformly distributed.

We remark that it is still plausible that for any infinite compact $A \subset \mathbb{R}^p$ there exists *at least one* greedy best-packing sequence that is asymptotically uniformly distributed on A.

6. Proofs of results from subsection 2.2.3

Proof of Theorem 2.18. Given a point $x \in S^d$, we define $C(x,r) := \{y \in S^d : |y - x| \leq r\}$. If σ_d denotes the normalized Lebesgue measure on S^d, then the following estimates hold (see formulas (3.7) and (3.4) in [21]):

$$(66) \qquad \int_{S^d \setminus C(x,r)} \frac{1}{|x-y|^d} d\sigma_d(y) = \gamma_d \log\left(\frac{1}{r}\right) + \mathcal{O}(1), \quad r \to 0,$$

$$(67) \qquad \sigma_d(C(x,r)) \leq \frac{1}{d}\gamma_d\, r^d, \quad d \geq 2,$$

where

$$(68) \qquad \gamma_d := \frac{\Gamma((d+1)/2)}{\Gamma(1/2)\Gamma(d/2)}.$$

If $d = 1$, inequality (67) is not valid since $\sigma_1(C(x,r)) = \frac{2}{\pi}\arcsin(\frac{r}{2})$, but instead we have

$$(69) \qquad \sigma_1(C(x,r)) = \gamma_1 r + \mathcal{O}(r^3), \quad r \to 0.$$

For $x \in S^d$ and $r > 0$,

$$\mathcal{H}_d^{d,w}(C(x,r)) = \int_{C(x,r)} w(y,y)^{-1} d\mathcal{H}_d(y) = \mathcal{H}_d(S^d)\int_{C(x,r)} w(y,y)^{-1} d\sigma_d(y).$$

Thus

$$(70) \qquad \mathcal{H}_d^{d,w}(C(x,r)) \leq \frac{M\mathcal{H}_d(S^d)\,\gamma_d\, r^d}{d}, \quad d \geq 2,$$

$$(71) \qquad \mathcal{H}_1^{1,w}(C(x,r)) \leq M\mathcal{H}_1(S^1)\gamma_1\, r + \mathcal{O}(r^3), \quad r \to 0,$$

where $M := \sup\{w(y,y)^{-1} : y \in S^d\}$.

Let $r \in (0,1)$ be fixed and set

$$D_i(r) := S^d \setminus C(a_i, rN^{-\frac{1}{d}}), \qquad D^N(r) := \bigcap_{i=1}^N D_i(r),$$

where a_i is the i-th element of the greedy (w,d)-energy sequence. From (70) and (71) we obtain that

$$(72) \qquad \mathcal{H}_d^{d,w}(D^N(r)) \geq \mathcal{H}_d^{d,w}(S^d) - \frac{M\mathcal{H}_d(S^d)\gamma_d\, r^d}{d}, \quad d \geq 2,$$

$$(73) \qquad \mathcal{H}_1^{1,w}(D^N(r)) \geq \mathcal{H}_1^{1,w}(S^1) - M\mathcal{H}_1(S^1)\gamma_1\, r + \mathcal{O}\left(\frac{r^3}{N^2}\right), \quad N \to \infty.$$

We may assume that the expressions in the right-hand side of the above inequalities are positive since we can take r sufficiently close to 0 and N sufficiently large (we will eventually let $r \to 0$ and $N \to \infty$).

Let $\epsilon > 0$. Since the function $w(x,y)/w(x,x)$ is uniformly continuous on $S^d \times S^d$, there exists $\delta > 0$ such that

$$\left| \frac{w(x,y)}{w(x,x)} - 1 \right| < \epsilon, \qquad \text{for} \quad |x-y| < \delta .$$

Consider the function

$$(74) \qquad U_{n,d}^w(x) := \sum_{i=1}^{n-1} \frac{w(x,a_i)}{|x-a_i|^d}, \quad x \in S^d, \quad n \geq 2 .$$

From the definition of a greedy (w,d)-energy sequence we know that $U_{n,d}^w(a_n) \leq U_{n,d}^w(x)$ for all $x \in S^d$. Let $2 \leq n \leq N$ and assume that $r < \delta$. Then $C(a_i, rN^{-\frac{1}{d}}) \subset C(a_i, \delta)$ for all $1 \leq i \leq n-1$ and so

$$\int_{D^N(r)} U_{n,d}^w(x) \, d\mathcal{H}_d^{d,w}(x) \leq \sum_{i=1}^{n-1} \int_{D_i(r)} \frac{w(x,a_i)}{w(x,x)} \frac{d\mathcal{H}_d(x)}{|x-a_i|^d}$$

$$\leq \sum_{i=1}^{n-1} \left(\int_{C(a_i,\delta) \setminus C(a_i, rN^{-\frac{1}{d}})} \frac{1+\epsilon}{|x-a_i|^d} \, d\mathcal{H}_d(x) + \int_{S^d \setminus C(a_i,\delta)} \frac{w(x,a_i)}{w(x,x)} \frac{d\mathcal{H}_d(x)}{|x-a_i|^d} \right)$$

$$\leq (n-1) \left((1+\epsilon)\mathcal{H}_d(S^d) \int_{S^d \setminus C(a_i, rN^{-\frac{1}{d}})} \frac{1}{|x-a_i|^d} d\sigma_d(x) + C(w,\delta) \right),$$

where $C(w,\delta)$ is some constant depending on δ and w. Using (66) it follows that

$$(75)$$
$$\int_{D^N(r)} U_{n,d}^w(x) \, d\mathcal{H}_d^{d,w}(x) \leq (n-1)(1+\epsilon)\mathcal{H}_d(S^d) \left(\frac{\gamma_d}{d} \log N - \gamma_d \log r + \mathcal{O}(1) \right).$$

Therefore,

$$E_d^w(\alpha_{N,d}^w) = 2 \sum_{n=2}^{N} U_{n,d}^w(a_n) \leq 2 \sum_{n=2}^{N} \frac{1}{\mathcal{H}_d^{d,w}(D^N(r))} \int_{D^N(r)} U_{n,d}^w(x) \, d\mathcal{H}_d^{d,w}(x)$$

$$\leq \frac{N(N-1)}{\mathcal{H}_d^{d,w}(D^N(r))} (1+\epsilon)\mathcal{H}_d(S^d) \left(\frac{\gamma_d}{d} \log N - \gamma_d \log r + \mathcal{O}(1) \right).$$

Consequently, from (72) and (73) we get that for $d \geq 1$,

$$\limsup_{N \to \infty} \frac{E_d^w(\alpha_{N,d}^w)}{N^2 \log N} \leq \frac{1}{\mathcal{H}_d^{d,w}(S^d) - \frac{M\mathcal{H}_d(S^d)\gamma_d r^d}{d}} (1+\epsilon)\mathcal{H}_d(S^d)\frac{\gamma_d}{d} .$$

After letting $r \to 0$ and $\epsilon \to 0$ we obtain that

$$\limsup_{N \to \infty} \frac{E_d^w(\alpha_{N,d}^w)}{N^2 \log N} \leq \frac{\mathcal{H}_d(S^d)\,\gamma_d}{\mathcal{H}_d^{d,w}(S^d)\,d} = \frac{\text{Vol}(B^d)}{\mathcal{H}_d^{d,w}(S^d)} .$$

Finally, since $\mathcal{E}_d^w(S^d, N) \leq E_d^w(\alpha_{N,d}^w)$ for all N, applying (31) it follows that

$$\lim_{N\to\infty} \frac{E_d^w(\alpha_{N,d}^w)}{N^2 \log N} = \frac{\text{Vol}(B^d)}{\mathcal{H}_d^{d,w}(S^d)}\,.$$

The statement about the weak-star convergence of the normalized counting measure associated with $\alpha_{N,d}^w$ is also an application of Theorem 2.16. $\qquad\square$

Remark: It is not difficult to see that greedy k_s-energy sequences on $S^d \subset \mathbb{R}^{d+1}$ satisfy the following property for any $s \in [0, \infty)$. If $\{a_n\}_{n=1}^\infty$ denotes such a sequence, then for each integer $m \geq 1$, the choice of a_{2m} is unique and $a_{2m} = -a_{2m-1}$.

It is also easily seen that on S^2 the configuration formed by the first six points of any greedy k_s-energy sequence does not depend on s and is a rotation of the configuration $\{(1,0,0), (-1,0,0), (0,1,0), (0,-1,0), (0,0,1), (0,0,-1)\}$ (cf. [24]).

Proof of Theorem 2.19. If $R := \text{diam}(A)$ is the diameter of A, $r < R$ and $x \in A$, then
(76)
$$\int_{A\setminus B(x,r)} \frac{1}{|x-y|^p}\, dy \leq \int_{B(x,R)\setminus B(x,r)} \frac{1}{|x-y|^p}\, dy = \mathcal{H}_{p-1}(S^{p-1}) \log(R/r)\,.$$

Defining

$$D_i(r) := A \setminus B(a_i, rN^{-\frac{1}{p}})\,, \qquad D^N(r) := \bigcap_{i=1}^N D_i(r)\,,$$

where a_i is the i-th element of the greedy (w, p)-energy sequence, the proof of Theorem 2.18 is applicable here and yields the result. For instance, using (76) the expression similar to (75) is
(77)
$$\int_{D^N(r)} U_{n,p}^w(x)\, d\mathcal{H}_p^{p,w}(x) \leq (n-1)(1+\epsilon)\mathcal{H}_{p-1}(S^{p-1})\Big(\frac{1}{p}\log N - \log r + \mathcal{O}(1)\Big)\,.$$

Since $\text{Vol}(B^p) = p^{-1}\mathcal{H}_{p-1}(S^{p-1})$, (35) follows from (77) and Theorem 2.16. The limit (36) is a consequence of (35) and Theorem 2.16. $\qquad\square$

Proof of Theorem 2.20. We follow closely the argument on page 20 of [4]. The following result is known as Frostman's lemma (see [26]).

Lemma 6.1. *Let $\delta > 0$ and A be a Borel set in $\mathbb{R}^p$. Then $\mathcal{H}_\delta(A) > 0$ if and only if there exists $\mu \in \mathcal{M}^+(A)$ such that $\mu(A) > 0$ and*

(78)
$$\mu(B(x,r)) \leq r^\delta, \qquad x \in \mathbb{R}^p, \quad r > 0\,,$$

where $B(x,r)$ denotes the open ball centered at x and radius r. Furthermore, one can select μ so that $\mu(A) \geq c_{p,\delta}\, \mathcal{H}_\delta^\infty(A)$, where $c_{p,\delta}$ is independent of A.

Let μ be a measure from Lemma 6.1, and set $r_0 := (\mu(A)/2N)^{1/\delta}$. Define the sets

$$D_j := B(a_j, r_0), \qquad \mathcal{D}_N := A \setminus \bigcup_{j=1}^{N-1} D_j,$$

where a_j denotes the j-th element of the greedy (w, s)-energy sequence. Then, using (78),

$$(79) \qquad \mu(\mathcal{D}_N) \geq \mu(A) - \sum_{j=1}^{N-1} \mu(D_j) \geq \mu(A) - (N-1)r_0^\delta > \frac{\mu(A)}{2} > 0.$$

Consider the function $U_{N,s}^w$ defined in (74). From (79) we obtain

$$U_{N,s}^w(a_N) \leq \frac{1}{\mu(\mathcal{D}_N)} \int_{\mathcal{D}_N} U_{N,s}^w(x) d\mu(x) \leq \frac{2}{\mu(A)} \sum_{j=1}^{N-1} \int_{\mathcal{D}_N} \frac{w(x, a_j)}{|x - a_j|^s} d\mu(x)$$

$$\leq \frac{2\|w\|}{\mu(A)} \sum_{j=1}^{N-1} \int_{A \setminus D_j} \frac{1}{|x - a_j|^s} d\mu(x),$$

where $\|w\| := \sup\{w(x, y) : x, y \in A\}$. Set $R := \mathrm{diam}(A)$. Then $\mu(A) \leq R^\delta$ by (78). If $y \in A$ and $r \in (0, R]$, then

$$\int_{A \setminus B(y,r)} \frac{1}{|x - y|^s} d\mu(x) \leq \int_0^{r^{-s}} \mu(\{x \in A : \frac{1}{|x - y|^s} > t\}) dt$$

$$\leq \frac{\mu(A)}{R^s} + \int_{R^{-s}}^{r^{-s}} \mu(B(y, t^{-1/s})) dt \leq R^{\delta - s} + \int_{R^{-s}}^{r^{-s}} t^{-\delta/s} dt$$

$$\leq \begin{cases} R^{\delta - s} + \frac{s}{s - \delta} r^{\delta - s}, & \text{if } s > \delta, \\[2ex] 1 + \delta \log\left(\frac{R}{r}\right), & \text{if } s = \delta. \end{cases}$$

Therefore, for $s > \delta$ we obtain

$$(80) \quad U_{N,s}^w(a_N) \leq \frac{2\|w\|}{\mu(A)}(N-1)\left(R^{\delta - s} + \frac{s}{s - \delta} r_0^{1 - s/\delta}\right) \leq C_1 \|w\|\left(\frac{N}{\mu(A)}\right)^{s/\delta},$$

where $C_1 > 0$ is a constant independent of N and w. If $s = \delta$, then

$$(81) \qquad U_{N,\delta}^w(a_N) \leq \frac{2\|w\|}{\mu(A)}(N-1)\left(1 + \delta \log\left(\frac{R}{r_0}\right)\right) \leq C_2 \|w\|\left(\frac{N \log N}{\mu(A)}\right),$$

where $C_2 > 0$ is also independent of N and w. The sequence $\{U_{i,s}^w(a_i)\}_N$ is non-decreasing since

$$U_{i+1,s}^w(a_{i+1}) \geq U_{i,s}^w(a_i) + \frac{w(a_{i+1}, a_i)}{|a_{i+1} - a_i|^s}, \qquad i \geq 1.$$

Therefore, applying $\mu(A) \geq c_{p,\delta} \, \mathcal{H}_\delta^\infty(A)$ and (80)-(81), Theorem 2.20 readily follows from

$$E_s^w(\alpha_{N,s}^w) = 2 \sum_{i=2}^N U_{i,s}^w(a_i) \, .$$

□

Proof of Corollary 2.21. Since $E_s^w(\alpha_{N,s}^w) \geq \mathcal{E}_s^w(A, N)$ for every N and $s \geq d$, the lower bounds in (37) and (38) follow from (33) and (31), respectively. The upper bounds follow from Theorem 2.20. □

Proof of Corollary 2.22. Assume the existence of a point $a \in A$ and $\epsilon > 0$ such that $\{a_n\}_{n=1}^\infty \cap B(a, \epsilon) = \emptyset$. Let $\alpha_{N,s}^w = \{a_1, \ldots, a_N\}$. Then

$$E_s^w(\alpha_{N,s}^w) = 2 \sum_{1 \leq i < j \leq N} \frac{w(a_i, a_j)}{|a_i - a_j|^s} \leq 2 \sum_{j=2}^N \sum_{i=1}^{j-1} \frac{w(a_i, x)}{|a_i - x|^s} \, ,$$

where the last inequality is valid for any $x \in A$. In particular, taking $x = a$ we get

$$E_s^w(\alpha_{N,s}^w) \leq \frac{\|w\|}{\epsilon^s} N(N-1) \, ,$$

where $\|w\| = \sup\{w(x, y) : x, y \in A\}$. This inequality contradicts the first inequalities in (37) and (38). □

Proof of Proposition 2.23. Assume that there exists a subinterval $I = [c, d] \subset [a, b]$ for which (39) is not satisfied. Let N_l be a subsequence such that

$$\lim_{l \to \infty} \frac{(\text{card}\{1 \leq n \leq N_l : a_n \in I\})^{1+\frac{1}{s}}}{N_l} = 0 \, .$$

Select $\epsilon > 0$ sufficiently small so that $J = [c + \epsilon/2, d - \epsilon/2] \subset I$ is not empty. If we define $\nu_l := \text{card}\{1 \leq n \leq N_l : a_n \in J\}$, then there exists a subinterval of J of length at least $(d - c - \epsilon)/(\nu_l + 1)$ not containing any point from $\{a_n \in J : 1 \leq n \leq N_l\}$. Let x_l be the center of such a subinterval. We have, for $\alpha_{N_l,s}^w = \{a_1, \ldots, a_{N_l}\}$,

$$(82) \qquad E_s^w(\alpha_{N_l,s}^w) = 2 \sum_{n=2}^{N_l} U_{n,s}^w(a_n) \leq 2 \sum_{n=2}^{N_l} U_{n,s}^w(x_l) = 2 \sum_{n=2}^{N_l} \sum_{i=1}^{n-1} \frac{w(x_l, a_i)}{|x_l - a_i|^s}$$

$$\leq 2\|w\| \left[\frac{N_l - 1}{|x_l - a_1|^s} + \frac{N_l - 2}{|x_l - a_2|^s} + \cdots + \frac{1}{|x_l - a_{N_l-1}|^s} \right] = 2\|w\|(S_{I,l} + T_{I,l}) \, ,$$

where $\|w\| = \sup\{w(x, y) : x, y \in [a, b]\}$ and

$$S_{I,l} := \sum_{a_i \in I, 1 \leq i \leq N_l - 1} \frac{N_l - i}{|x_l - a_i|^s} \, , \qquad T_{I,l} := \sum_{a_i \notin I, 1 \leq i \leq N_l - 1} \frac{N_l - i}{|x_l - a_i|^s} \, .$$

For each $a_i \notin I$, $|a_i - x_l| \geq \epsilon/2$; hence

$$(83) \qquad\qquad\qquad 2\,T_{I,l} \leq (2/\epsilon)^s \, N_l^2 \, .$$

If $a_i \in I, 1 \leq i \leq N_l - 1$, then $|a_i - x_l| \geq (d - c - \epsilon)/2(\nu_l + 1)$. Therefore, if we define $\tau_l := \text{card}\{1 \leq i \leq N_l - 1 : a_i \in I\}$, it follows that

$$(84) \qquad 2\,S_{I,l} \leq \frac{2^{s+1}}{(d - c - \epsilon)^s}\,(\nu_l + 1)^s\,\tau_l\,N_l\,.$$

By hypothesis, $\tau_l^{1+s}/N_l^s \to 0$ as $l \to \infty$. We deduce from (82)-(84) that

$$\lim_{l \to \infty} \frac{E_s^w(\alpha_{N_l,s}^w)}{N_l^{1+s}} = 0\,,$$

which contradicts the fact that

$$\liminf_{N \to \infty} \frac{E_s^w(\alpha_{N,s}^w)}{N^{1+s}} \geq \lim_{N \to \infty} \frac{\mathcal{E}_s^w([a,b],N)}{N^{1+s}} = \frac{2\zeta(s)}{\mathcal{H}_1^{s,w}([a,b])^s} > 0\,.$$

$\square$

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