Riesz Energy and Sets of Revolution in $\mathbb{R}^3$

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Dedicated to V. Zaharyuta on the occasion of his 70th birthday

Abstract. Let $A \subseteq \mathbb{R}^2$ be a compact set in the right-half plane and $\Gamma(A)$ the set in $\mathbb{R}^3$ obtained by rotating A about the vertical axis. We review recent results concerning the support of the equilibrium measure on $\Gamma(A)$ for the Riesz kernel $k_s(\mathbf{x},\mathbf{y}) := 1/|\mathbf{x}-\mathbf{y}|^s$ $(0 < s < 1)$ and the logarithmic kernel $k_0(\mathbf{x},\mathbf{y}) := \log(1/|\mathbf{x}-\mathbf{y}|)$ (limit case $s \to 0$). Here $|\cdot|$ denotes the Euclidean distance. The main tool is to reduce the minimum energy problem on $\Gamma(A)$ in $\mathbb{R}^3$ for the singular kernel k_s to a related problem on A in $\mathbb{R}^2$ for a continuous kernel $\mathcal{K}_s$. Some open problems are posed.

1. Introduction

Let K be an infinite compact set in $\mathbb{R}^p$ whose d-dimensional Hausdorff measure, $\mathcal{H}_d(K)$, is finite and positive (hence, d is the Hausdorff dimension of K). [We normalize the Hausdorff measure $\mathcal{H}_d$ so that the d-dimensional unit cube in $\mathbb{R}^p$ has measure 1.] For a collection of $N(\geq 2)$ distinct points $X_N := \{\mathbf{x}_1, \dots, \mathbf{x}_N\} \subseteq K$, and $s > 0$, the *discrete Riesz s-energy of X_N* is defined by

$$E_s(X_N) := \sum_{j \neq k} \frac{1}{|\mathbf{x}_j - \mathbf{x}_k|^s} = \sum_{j=1}^{N} \sum_{\substack{k=1,\\ k\neq j}}^{N} \frac{1}{|\mathbf{x}_j - \mathbf{x}_k|^s},$$

while the *N-point Riesz s-energy of K* is defined by

$$\mathcal{E}_s(K,N) := \inf\{E_s(X_N) : X_N \subseteq K, |X_N| = N\}, \tag{1.1}$$

where $|X|$ denotes the cardinality of the set X. Since K is compact, there must be at least one N-point set $X_{s,N} \subseteq K$ such that $\mathcal{E}_s(K,N) = E_s(X_{s,N})$.

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This class of minimal discrete s-energy problems can be considered as a bridge between logarithmic energy problems and best-packing ones. Indeed, when $s \to 0$ and N is fixed, the minimal energy problem turns into the problem for the logarithmic potential energy

$$E_0(X_N) := \sum_{j \neq k} \log \frac{1}{|\mathbf{x}_j - \mathbf{x}_k|},$$

which is minimized over all N-point configurations $\{\mathbf{x}_1, \dots, \mathbf{x}_N\} \subseteq K$.

On the other hand, when $s \to \infty$, and N is fixed, we get the best-packing problem (cf. [**FT64**], [**CS99**]); that is, the problem of finding N-point configurations $X_N \subseteq K$ with the largest separation radius:

$$\delta(X_N) := \min_{j \neq k} |\mathbf{x}_j - \mathbf{x}_k|. \tag{1.2}$$

We are interested in the geometrical properties of optimal s-energy N-point configurations for a set K; that is, sets X_N for which the infimum in (1.1) is attained. Indeed, these configurations are useful in statistical sampling, weighted quadrature, and computer-aided geometric design where the selection of a "good" finite (but possibly large) collection of points is required to represent a set or manifold K. Since the exact determination of optimal configurations seems, except in a handful of cases, beyond the realm of possibility, our focus is on the asymptotics of such configurations. Specifically, we consider the following questions.

(i) What is the asymptotic behavior of the quantity $\mathcal{E}_s(K, N)$ as N gets large?
(ii) How are optimal point configurations $X_{s,N}$ distributed as $N \to \infty$?

In the case $0 \leq s < \dim K$ (the Hausdorff dimension of K), answers to questions (i) and (ii) are determined by the *equilibrium measure* $\mu_{s,K}$ that minimizes the continuous energy integral

$$\mathcal{I}_s[\mu] := \int \int k_s(\mathbf{x}, \mathbf{y}) \, d\mu(\mathbf{x}) \, d\mu(\mathbf{y})$$

over the class $\mathcal{M}(K)$ of (Radon) probability measures μ supported on K. Let $V_s(K) := \inf_{\mu \in \mathcal{M}(K)} \mathcal{I}_s[\mu]$. Specifically (cf. [**Lan72**, Ch. II no. 12]), we have

$$\lim_{N \to \infty} \mathcal{E}_s(K, N)/N^2 = V_s(K) = \mathcal{I}_s[\mu_{s,K}]$$

and (in the weak-star sense)

$$\frac{1}{N} \sum_{\mathbf{x} \in X_{s,N}} \delta_{\mathbf{x}} \xrightarrow{*} \mu_{s,K},$$

where $\delta_{\mathbf{x}}$ denotes the atomic measure centered at $\mathbf{x}$. In the case when $K = \mathbb{S}^d$, the unit sphere in $\mathbb{R}^{d+1}$, the equilibrium measure is simply the normalized surface area measure and it follows that optimal energy points on the sphere are uniformly distributed in this sense.

The *hypersingular* case when $s \geq d$ was studied by the second and third authors together with S. Borodachov in [**HS04, HS05, BHS08**]. In this case, $I_s(\mu) = \infty$ for any probability measure supported on K and, hence, K has s-capacity 0 and no equilibrium measure for the continuous energy integral problem. However, for any *d-rectifiable* set K, the following holds:

$$\lim_{N \to \infty} \mathcal{E}_s(K, N)/N^{1+s/d} = C_{s,d}/(\mathcal{H}_d(K))^{s/d},$$

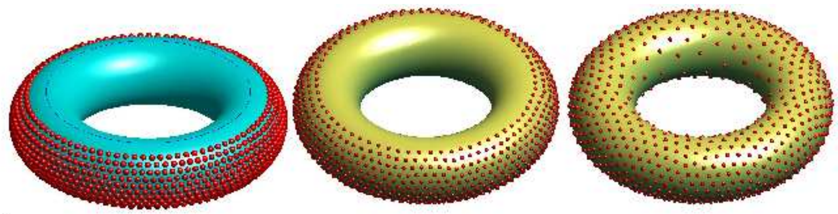

FIGURE 1. Near minimum Riesz s-energy configurations ($N = 1000$ points) on a torus in $\mathbb{R}^3$ for $s = 0$, 0.2, and 1.

where $C_{s,d}$ is a positive constant independent of K. Furthermore, if $\mathcal{H}_d(K) > 0$, then

$$\frac{1}{N} \sum_{\mathbf{x} \in X_{s,N}} \delta_{\mathbf{x}} \stackrel{*}{\longrightarrow} \mathcal{H}_d(\cdot)/\mathcal{H}_d(K). \tag{1.3}$$

For the critical index $s = d$, we have (under some smoothness conditions)

$$\lim_{N\to\infty} \mathcal{E}_s(K, N)/(N^2 \log N) = \mathrm{Vol}(\mathcal{B}_d)/\mathcal{H}_d(K),$$

where $\mathcal{B}_d$ is the unit ball in $\mathbb{R}^d$, and, if $\mathcal{H}_d(K) > 0$, then again (1.3) holds.

Numerical experiments, conducted by Rob Womersley [**Wom05**], suggest that minimum s-energy configurations on a torus are confined to the "outer-most" part with positive curvature (Figure 1) for $s \geq 0$ sufficiently small, which, if true, implies that the support of the k_s-equilibrium measure on this torus would also be contained in this set. Conversely, if the k_s-equilibrium measure is concentrated on the "outer-most" part, the fraction of points of a minimum s-energy N-point system not in this set tends to zero as N goes to infinity. In [**HSS07**] the last two authors together with Herbert Stahl showed that, indeed, the support of the k_s-equilibrium measure on a compact set of revolution K with no points on the axis of rotation is a subset of the "outer-most" part of K in the logarithmic case ($s = 0$). In [**BHS07b**] we studied the case $0 < s < 1$.

In this paper we review results from these two papers concerning the support of equilibrium measures $\mu_{s,K}$ on sets of revolution K in $\mathbb{R}^3$ and pose some open problems.

2. The energy problem on sets of revolution

Let A be a non-empty compact set in the right-half plane $\mathbb{H}^+$ and $K = \Gamma(A)$ the set of revolution in $\mathbb{R}^3$ obtained by rotating A about the vertical axis. Classical potential theory yields that for $0 \leq s \leq 1$ the equilibrium measure $\mu_{s,\Gamma(A)}$ on $\Gamma(A)$ is supported on the *outer boundary* $\partial\Gamma(A)_\infty$ *of* $\Gamma(A)$ which is the boundary of the unbounded component of the complement of $\Gamma(A)$. (In the Coulomb case $s = 1$ the support of $\mu_{s,\Gamma(A)}$ is essentially the outer boundary of $\Gamma(A)$.) In the next sections we will review results from [**HSS07**] and [**BHS07b**] which will give us more insight into the nature of $\operatorname{supp} \mu_{s,\Gamma(A)}$.

On a set of revolution it is sufficient to consider rotational symmetric measures. A Borel measure $\tilde{\mu} \in \mathcal{M}(\mathbb{R}^3)$ is *rotationally symmetric about the vertical axis* if

$$\tilde{\mu}(\mathbf{R}_\phi B) = \tilde{\mu}(B)$$

for all Borel sets $B \subseteq \mathbb{R}^3$ and for all rotations $\mathbf{R}_\phi$ about the vertical axis. Thus, the energy problem on $\Gamma(A)$ in $\mathbb{R}^3$ for the **singular** Riesz kernel k_s can be reduced to the energy problem on A in $\mathbb{R}^2$ for a new kernel $\mathcal{K}_s$ (which is **continuous** if $0 \leq s < 1$) by rewriting the energy integral

$$\mathcal{I}_s[\tilde{\mu}] = \int\int \mathcal{K}_s(z,w) d\mu(z) d\mu(w) =: \mathcal{J}_{\mathcal{K}_s}[\mu], \tag{2.1}$$

where the compactly supported *rotational symmetric* measure $\tilde{\mu} \in \mathcal{M}(\mathbb{R}^3)$, admits a decomposition

$$d\tilde{\mu} = \frac{d\phi}{2\pi}\, d\mu, \qquad \mu = \tilde{\mu} \circ \Gamma \in \mathcal{M}(\mathbb{H}^+),$$

into the normalized Lebesgue measure on the half-open interval $[0, 2\pi)$ and a measure μ on the right-half plane $\mathbb{H}^+$. For convenience, we identify $\mathbb{H}^+$ with the complex right-half plane $\{z : \mathrm{Re}[z] \geq 0\}$. As mentioned in [**HSS07**], the kernel $\mathcal{K}_s(z,w)$ is given by the integral

$$\mathcal{K}_s(z,w) = \frac{1}{2\pi} \int_0^{2\pi} k_s(\mathbf{R}_\phi z, w) d\phi. \tag{2.2}$$

The $\mathcal{K}_s$-energy $V_{\mathcal{K}_s}$ of A is given by

$$V_{\mathcal{K}_s}(A) := \inf \left\{ \mathcal{J}_{\mathcal{K}_s}[\nu] : \nu \in \mathcal{M}(A) \right\}. \tag{2.3}$$

For $\nu \in \mathcal{M}(A)$, we define the $\mathcal{K}_s$-potential W_s^ν by

$$W_s^\nu(z) := \int \mathcal{K}_s(z,w) d\nu(w), \qquad z \in \mathbb{H}^+. \tag{2.4}$$

The existence and uniqueness of the equilibrium measure on A and a Frostman-type result follow from the properties of the equilibrium measure on $\Gamma(A)$.

PROPOSITION 2.1. *Suppose A is a non-empty compact set in $\mathbb{H}^+$ with positive logarithmic capacity ($s = 0$) or positive s-capacity ($0 < s < 1$). Then $\lambda_{s,A} := \mu_{s,\Gamma(A)} \circ \Gamma$ is the unique measure in $\mathcal{M}(A)$ that minimizes $\mathcal{J}_{\mathcal{K}_s}[\nu]$ over all measures $\nu \in \mathcal{M}(A)$. The equilibrium measure $\lambda_{s,A}$ on A for the kernel $\mathcal{K}_s$ is supported on the outer boundary of A. Furthermore:*

$$W_s^{\lambda_{s,A}} \geq V_{\mathcal{K}_s}(A) \qquad \text{everywhere on } A, \tag{2.5}$$

$$W_s^{\lambda_{s,A}} \leq V_{\mathcal{K}_s}(A) \qquad \text{everywhere on } \operatorname{supp} \lambda_{s,A}, \tag{2.6}$$

and

$$V_{\mathcal{K}_s}(A) = \mathcal{J}_{\mathcal{K}_s}[\lambda_{s,A}] = \mathcal{I}_s[\mu_{s,\Gamma(A)}] = V_s(\Gamma(A)).$$

By studying the $\mathcal{K}_s$-equilibrium measure on sets obtained by translating a given set $A \subseteq \mathbb{H}^+$ a distance R units to the right and taking the limit $R \to \infty$, one can obtain further information. Specifically, for $0 < s < 1$ and $z, w \in \mathbb{H}^+$, the asymptotic expansion of $\mathcal{K}_s(R+z, R+w)$ for large R is (cf. [**BHS07b**, Lemma 3 of section IV]) of the form

$$\mathcal{K}_s(R+z, R+w) = V_s(\mathbb{T}) R^{-s} - B_2(s) \frac{\left|z-w\right|^{1-s}}{2R} - B_3(s) \frac{\mathrm{Re}[z - w_*]}{2R} R^{-s} + \mathcal{O}(\frac{s}{R^2}),$$

where $V_s(\mathbb{T}) = \Gamma(1-s)/[\Gamma(1-s/2)]^2$ is the s-energy of the unit circle $\mathbb{T}$, $B_2(s) = 2^{-s}[s/(1-s)] V_{-s}(\mathbb{T})$, and $B_3(s) = s V_s(\mathbb{T})$.

This motivates the introduction of the "finite R" kernel

$$\mathcal{K}_s^{(R)}(z,w) := 2R \left[\mathcal{K}_s(R+z, R+w) - V_s(\mathbb{T}) R^{-s} \right] \tag{2.7}$$

and the "infinity" kernel

$$\mathcal{K}_s^{(\infty)}(z,w) := -B_2(s)\,|z-w|^{1-s}\,.$$

These kernels are then related by

$$\mathcal{K}_s^{(R)}(z,w) = \mathcal{K}_s^{(\infty)}(z,w) + \mathcal{O}(1/R), \qquad R\to\infty,$$

which holds uniformly on compact subsets in the interior of $\mathbb{H}^+\times\mathbb{H}^+$, and are connected to the logarithmic case (see Section 3) by means of

$$\lim_{s\to 0}\mathcal{K}_s^{(R)}(z,w)/s = \mathcal{K}_0^{(\infty)}(z,w) + \mathcal{O}(1/R), \qquad R\to\infty.$$

It follows from (2.7) that the equilibrium measure $\lambda_{s,A}^R$ on A for $\mathcal{K}_s^{(R)}$ is equal to $\lambda_{s,A+R}(\cdot+R)$, that is, $\lambda_{s,A}^R(B) = \lambda_{s,A+R}(B+R)$ for any measurable set $B\subseteq\mathbb{H}^+$, where $B+R$ denotes the translate $\{b+R : b\in B\}$. The kernel $\mathcal{K}_s^{(\infty)}$ falls into a class of kernels studied by Björck [**Bjoe56**]. From his results we obtain the following proposition.

Proposition 2.2. *Let $0\leq s<1$. If A is a non-empty compact set in the interior of $\mathbb{H}^+$, then there is a unique equilibrium measure $\lambda_{s,A}^\infty$ minimizing $\mathcal{J}_{K_s^{(\infty)}}[\mu]$ over all $\mu\in\mathcal{M}(A)$. Moreover, $\lambda_{s,A}^R$ converges weak-star to $\lambda_{s,A}^\infty$ as $R\to\infty$.*

Throughout this paper we will use the notation $z=x+iy$, $w=u+iv$ with $x,y,u,v\in\mathbb{R}$. Then $w_* := -u+iv$ denotes the reflection of w in the imaginary axis. One also has $|\mathbf{R}_\phi z - w|^2 = x^2+u^2-2xu\cos\phi+(y-v)^2$.

3. The Logarithmic Case $s=0$

The logarithmic case ($s=0$) has been investigated by Hardin, Saff, and Stahl in [**HSS07**]. The kernel in (2.2) has the representation

$$\mathcal{K}_0(z,w) = \log\frac{2}{|z-w_*|+|z-w|}, \qquad z,w\in\mathbb{H}^+.$$

The level sets of $\mathcal{K}_0(\cdot,w)$ are ellipses with foci w and w_*. The kernel is symmetric, that is, $\mathcal{K}_0(w,z)=\mathcal{K}_0(z,w)$. Furthermore, $\mathcal{K}_0$ is continuous at any $(z,w)\in\mathbb{H}^+\times\mathbb{H}^+$ unless $z=w=iy$ for some $y\in\mathbb{R}$. The "infinity" kernel is given by

$$\mathcal{K}_0^{(\infty)}(z,w) = -\mathrm{Re}[z-w_*] - |z-w|, \qquad z,w\in\mathbb{H}^+. \tag{3.1}$$

If $A\subseteq\mathbb{H}^+$ is compact, let $\operatorname{proj}A$ denote the projection of the set A onto the imaginary axis and for $y\in\operatorname{proj}A$, define $x_A(y) := \max\{x : (x,y)\in A\}$. We denote by A_+ the "right-most" portion of A, that is,

$$A_+ := \{(x_A(y),y) : y\in\operatorname{proj}A\}\,.$$

The following main result is proved in [**HSS07**].

Theorem 3.1. *Suppose A is a non-empty compact set in $\mathbb{H}^+$ such that A_+ is contained in the interior of $\mathbb{H}^+$. Then the support of the equilibrium measure $\lambda_{0,A}\in\mathcal{M}(A)$ is contained in A_+. (The same holds for $\lambda_{0,A}^\infty$.)*

Corollary 3.2 (horizontal line-segment). *Let A be a non-empty compact subset of the line-segment $[a+ib, c+ib]$, $0<a<c$ and $b\in\mathbb{R}$. Then $\lambda_{0,A}$ ($\lambda_{0,A}^\infty$) is the unit point charge concentrated at the "right-most" point of A.*

More can be said if the "right-most" part A_+ is contained in the graph of a simple smooth curve $\gamma : [a,b] \to \mathbb{H}^+$; that is, $A_+ \subseteq \gamma^* := \{\gamma(t) : a \leq t \leq b\}$. Strict convexity of $\mathcal{K}_0(\gamma(\cdot), \gamma(t))$ on the intervals $[a,t]$ and $[t,b]$ for each fixed $t \in [a,b]$ implies the existence of some closed interval $I \subseteq [a,b]$ such that $\operatorname{supp} \lambda_{0,A} = \gamma(I) \cap A_+$. Note that A_+ is only required to be a compact subset of γ^*. For example, A_+ may be a Cantor subset of γ^*.

Corollary 3.3 (vertical line-segment). *Suppose A is a non-empty compact subset in the interior of H^+ such that A_+ is contained in a vertical line segment $[R+ic, R+id]$ for some $R > 0$. Then* $\operatorname{supp} \lambda_{0,A} = A_+$.

For large R, $\operatorname{supp} \lambda_{0,A}$ thins out in the "middle". In fact, it is shown in [**HSS07**] that $\operatorname{supp} \lambda_{0,A}^{\infty}$ consists of the two "endpoints" of A_+. (The behavior for $s > 0$ is different, cf. Corollary 4.5(b) below.)

Corollary 3.4 (circle). *Suppose $C \subset \mathbb{C}$ is a circle of radius $r > 0$ and center a with $\operatorname{Re}[a] > 0$ and suppose A is a compact set in $\mathbb{H}^+$ such that $A_+ \subset C_+$. Then* $\operatorname{supp} \lambda_{0,A} = A_+^{\theta} := A_+ \cap \{a + re^{it} : |t| \leq \theta\}$ *for some $\theta \in [0, \pi/2]$. In particular, if A_+ is a circular arc contained in C_+, then so is* $\operatorname{supp} \lambda_{0,A}$*; consequently,* $\operatorname{supp} \mu_{0,\Gamma(A)}$ *is connected.*

Moreover, the following can be shown for a torus.

Corollary 3.5 (torus). *Let A be a circle with center $(R,0)$ and radius r with $0 < r < R$. Then, for each $\epsilon > 0$ there is some $R > 0$ such that the support of $\lambda_{0,A}^{R}$ is contained in $A_+^{\pi/3+\epsilon}$. Consequently, for each $\epsilon > 0$ and R/r sufficiently large, the support of the equilibrium measure $\mu_{0,\Gamma(A)}$ on the torus $\Gamma(A)$ is a proper subset of $\Gamma(A_+^{\pi/3+\epsilon})$.*

4. The Case $0 < s < 1$

In the case $0 < s < 1$ the kernel in (2.2) can be represented in terms of a hypergeometric ${}_2\mathrm{F}_1$-function

$$\mathcal{K}_s(z,w) = |z - w_*|^{-s} \, {}_2\mathrm{F}_1\left(\begin{matrix} s/2, 1/2 \\ 1 \end{matrix}; 1 - \frac{|z-w|^2}{|z-w_*|^2}\right), \qquad z, w \in \mathbb{H}^+.$$

The level sets of $\mathcal{K}_s(\cdot, w)$, $w \in \mathbb{H}^+$ fixed, look like Cassinian ovals as shown in Figure 2. For $0 < s < 1$, the kernel $\mathcal{K}_s$ is clearly continuous at any $(z,w) \in \mathbb{H}^+ \times \mathbb{H}^+$ unless $z = w = iy$ for some $y \in \mathbb{R}$. For $s \geq 1$ the kernel $\mathcal{K}_s$ is singular on the diagonal (w,w). As $s \to 0^+$, we recover the logarithmic kernel $\mathcal{K}_0$ discussed in the last section

$$\lim_{s \to 0^+} \frac{\mathcal{K}_s(z,w) - 1}{s} = \mathcal{K}_0(z,w), \qquad z, w \in \mathbb{H}^+.$$

The "infinity" kernel is given by

$$\mathcal{K}_s^{(\infty)}(z,w) = -\frac{2}{1-s} \frac{\Gamma((1+s)/2)}{\sqrt{\pi}\,\Gamma(s/2)} |z-w|^{1-s}, \qquad z, w \in \mathbb{H}^+.$$

The existence of compact sets A for which $\operatorname{supp}\lambda_{s,A}$ is not all of the outer boundary of A is shown in the next result. We define

$$\mathcal{K}_s^*(z,w) := \left[\mathcal{K}_s(z,w) + \mathcal{K}_s(z,\overline{w})\right]/2. \tag{4.1}$$

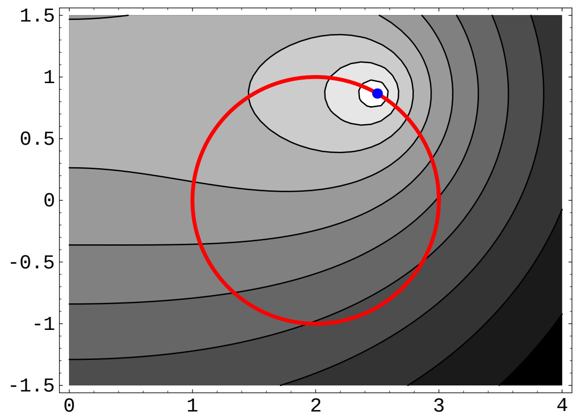

FIGURE 2. Level curves for $\mathcal{K}_{1/2}(z,w)$ for w a fixed point on the unit circle centered at $(2,0)$.

THEOREM 4.1 (3-point Theorem). *Let $0<s<1$. Let $x>0$ and z' be in the interior of $\mathbb{H}^+$. Let A be a non-empty compact subset of $\{w\in\mathbb{H}^+ : \mathcal{K}_s(x,w)\geq \mathcal{K}_s(x,z')\}$ in the interior of $\mathbb{H}^+$ with $x, z', \overline{z'}\in A$.*

(a) *If $\Delta_s := \mathcal{K}_s(x,z') - \mathcal{K}_s^*(z',z') > 0$, then $x\notin \operatorname{supp}\lambda_{s,A}$.*
(b) *If $z' = 1+i\gamma$, $\gamma>0$, and*

$$4\left(\gamma+\sqrt{1+\gamma^2}\right) > \left(\sqrt{(1+x)^2+\gamma^2}+\sqrt{(1-x)^2+\gamma^2}\right)^2, \tag{4.2}$$

then $\Delta_s>0$ (and hence, by (a), $x\notin\operatorname{supp}\lambda_{s,A}$) for $s>0$ sufficiently small.
(c) *If $x=1/2$ and $z'=1+i/2$, then $\Delta_s>0$ (and hence, by (a), $x\notin\operatorname{supp}\lambda_{s,A}$) for all $0<s<1/3$. (The graph of Δ_s is shown in Figure 3.)*

The difference $\Delta_s = \mathcal{K}_s(x,z') - \mathcal{K}_s^*(z',z')$ compares the potential due to a unit point charge at x with the potential due to half unit charges at z' and its complex conjugate $\overline{z'}$. Positivity of Δ_s implies $\mathcal{K}_s(x,\cdot) > \mathcal{K}_s^*(z',\cdot)$ on A, which in turn implies $W_s^{\lambda_{s,A}}(x) > V_{\mathcal{K}_s}(A)$; hence $x\notin\operatorname{supp}\lambda_{s,A}$ by variational inequality (2.6).

In Theorem 4.1.(c) we give a range for s. Based upon numerical experimention we state

CONJECTURE 4.2. *To every $0<s<1$ there exists a compact set $A\neq\emptyset$ in the interior of $\mathbb{H}^+$ such that $\operatorname{supp}\lambda_{s,A_s}$ is a proper subset of the outer boundary of A.*

EXAMPLE 4.3. Let A be the rectangle with lower left corner $1/2-i/2$ and upper right corner $1+i/2$. From Theorem 4.1 it follows that $1/2\notin\operatorname{supp}\lambda_{s,A}$ for $0<s<1/3$. Alternatively, if A is the left-half circle with radius $1/2$ centered at 1, it again follows from Theorem 4.1 that $1/2\notin\operatorname{supp}\lambda_{s,A}$ for $0<s<1/3$. (See Figure 3.) In contrast, as A is moved to the right R units and $R\to\infty$, we get $\operatorname{supp}\lambda_{s,A}^{\infty} = A$ for every $0<s<1$.

The converse, the existence of sets A for which $\operatorname{supp}\lambda_{s,A}$ equals the outer boundary S_A of A for all $0<s<1$, can be shown by using a convexity argument utilized in the following result.

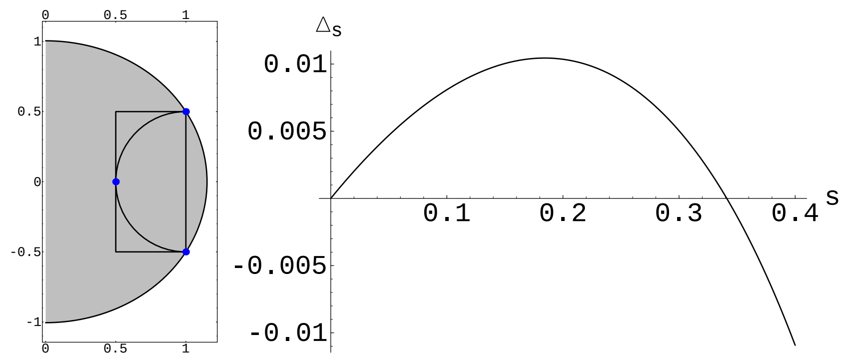

FIGURE 3. Examples of sets A satisfying Theorem 4.1.(c). Δ_s for $x = \gamma = 1/2$.

LEMMA 4.4. *Let $0 < s < 1$ and A be a compact set in the interior of $\mathbb{H}^+$.*

(i) *If $\gamma : [a, b] \to \mathbb{H}^+$, $a < b$, is a simple continuous* **non-closed** *curve with $S_A \subseteq \gamma^* := \{\gamma(t) : a \le t \le b\}$, and $\mathcal{K}_s(\gamma(\cdot), \gamma(t))$ is a strictly convex function on the intervals $[a, t]$ and $[t, b]$ for each fixed $t \in [a, b]$, then there is some closed interval $I \subseteq [a, b]$ such that* $\operatorname{supp} \lambda_{s,A} = \gamma(I) \cap S_A$.

(ii) *If $\gamma : [0, b] \to \mathbb{H}^+$ is a simple continuous* **closed** *curve, that is $\gamma(0) = \gamma(b)$, with $S_A \subseteq \gamma^*$ and extended periodically by $\gamma(t) = \gamma(t + b)$, and $\mathcal{K}_s(\gamma(\cdot), \gamma(t))$ is a strictly convex function on the interval $[t, t+b]$ for each fixed $t \in [0, b]$, then* $\operatorname{supp} \lambda_{s,A} = S_A$.

Using Lemma 4.4 it follows that any compact subset A of a horizontal or vertical line-segment satisfies $\operatorname{supp} \lambda_{s,A} = A$ for every $0 < s < 1$. We compare and contrast this with the logarithmic case, where it is still true that $\operatorname{supp} \lambda_{0,A} = A$ in case of a vertical line-segment (Corollary 3.3). However, in case of a horizontal line-segment one has that $\lambda_{0,A}$ is a unit point charge at the "right-most" point of A (Corollary 3.2).

COROLLARY 4.5 (horizontal and vertical line-segment). *Suppose A is a non-empty compact subset of either* (a) *the horizontal line-segment $[a + ic, b + ic]$, $0 < a < b$, or* (b) *the vertical line-segment $[R + ic, R + id]$, $R > 0$, $c < d$. Then* $\operatorname{supp} \lambda_{s,A} = \operatorname{supp} \lambda_{s,A}^{\infty} = A$ *for every $0 < s < 1$.*

CONJECTURE 4.6. *Suppose C is a circle with radius 1 centered at $R + i0$ in $\mathbb{H}^+$ (so that $\Gamma(A)$ is a torus in $\mathbb{R}^3$). Based on several numerical experiments for the discrete energy (see Figure 4), we conjecture that the support of the equilibrium measure $\lambda_{s,C}$ is connected and is increasing with respect to growing s. Furthermore, there seems to be a critical value $s_0 < 1$ (at least for R sufficiently large) such that* $\operatorname{supp} \lambda_{s,C} = C$ *for $s \ge s_0$.*

5. Transfinite Diameter and Limit Distribution

Let $X_{s,N}$, $N \ge 2$, be a sequence of minimum s-energy N-point systems on a compact set K in $\mathbb{R}^p$. Since the class $\mathcal{M}(K)$ of all Radon probability measures supported on K is weak-star compact, the sequence of discrete probability measures

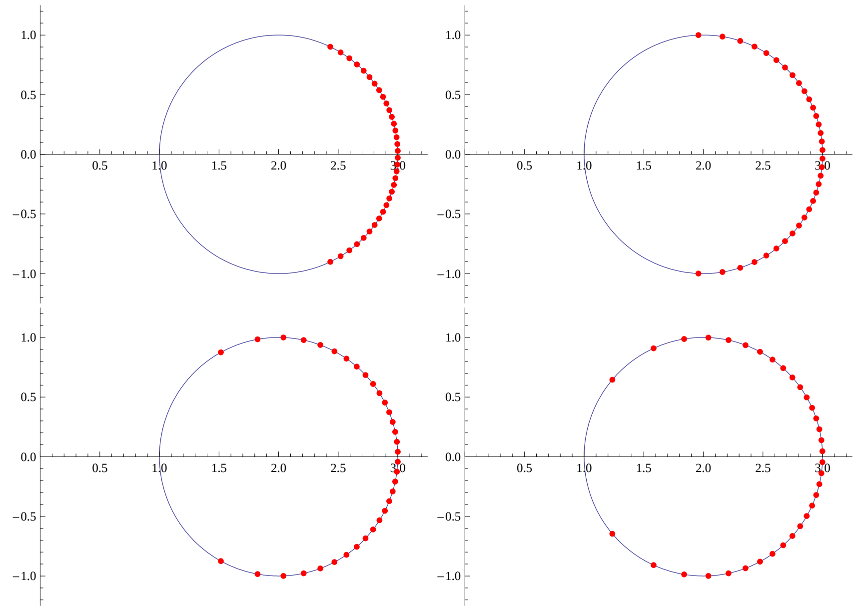

FIGURE 4. Minimum $\mathcal{K}_s$-energy configurations ($N = 34$ points) for $s = 0$, 0.5 (top circles), 0.75, 1 (bottom circles).

$\mu(X_{s,N})$, $N \geq 2$, induced by $X_{s,N}$, always has a cluster point $\mu^*_{s,K}$ in $\mathcal{M}(K)$. But one can say more. Define the quantities

$$D^s(X_N) := N(N-1) \Big/ \sum_{j \neq k} |\mathbf{x}_j - \mathbf{x}_k|^{-s} \qquad (s > 0)$$

called the *N-th generalized transfinite diameter*. Note, $[D^s(X_N)]^{1/s}$ can be seen as a generalization of the harmonic mean ($s = 1$) of the $N(N-1)$ distances $|\mathbf{x}_j - \mathbf{x}_k|$, $j \neq k$. Clearly, $[D^s(X_N)]^{1/s} \geq \delta(X_N)$, where $\delta(X_N)$ is defined in (1.2). The limit $\lim_{s\to\infty}[D^s(X_N)]^{1/s}$ for N fixed is studied in [**BHS07a**]. It is known [**Lan72**, Ch. II no. 12] that the positive quantities $D^s(X_{s,N})$, $N \geq 2$, satisfy

$$D^s(X_{s,2}) \geq D^s(X_{s,3}) \geq \cdots \geq D^s(X_{s,N}) \geq \cdots .$$

This implies the existence of the non-negative limit

$$D^s(K) := \lim_{N\to\infty} D^s(X_{s,N}),$$

which is called the *generalized transfinite diameter of order s* of the compact set K. The generalized transfinite diameter $D^s(K)$ was introduced by Pólya and Szegő in [**PS31**]. It is related to the N-point Riesz s-energy $\mathcal{E}_s(N,K)$ of K defined in (1.1) and the s-capacity of K (the reciprocal of $V_s(K)$ [$s > 0$] or the exponential $\exp\{-V_0(K)\}$ [$s = 0$]; $V_s(K)$ is defined in the Introduction) in the following way

$$D^s(K) = \lim_{N\to\infty} \frac{N(N-1)}{\mathcal{E}_s(N,K)} = \mathrm{cap}_s K. \qquad (s > 0) \tag{5.1}$$

(In the logarithmic case $s = 0$ we define $D^0(X_N)$ to be the $N(N-1)$-root of the product of all mutual distances $\prod_{j\neq k} |\mathbf{x}_j - \mathbf{x}_k|$. Then (5.1) holds except its middle part is replaced by $\lim_{N\to\infty} \exp\{-\mathcal{E}_0(N,K)/[N(N-1)]\}$.) By (5.1), the positivity of the s-capacity ($s > 0$) of K implies the weak-star convergence of the sequence of measures $\mu(X_{s,N})$, $N \geq 2$, to the limit distribution $\mu^*_{s,K}$. Moreover, by uniqueness of the equilibrium measure $\mu_{s,K}$ on K, one has $\mu^*_{s,K} = \mu_{s,K}$.

For sets K with s-capacity zero the situation is more complicated. For example, for K a compact set in $\mathbb{R}^d$ one gets from [**HS05**] that

$$\lim_{N\to\infty} (\log N) D^d(X_{d,N}) = \frac{\mathcal{H}_d(K)}{\beta_d}, \tag{5.2}$$

$$\lim_{N\to\infty} N^{s/d-1} D^s(X_{s,N}) = \frac{[\mathcal{H}_d(K)]^{s/d}}{C_{s,d}} \quad (s > d), \tag{5.3}$$

where β_d is the volume of the d-dimensional unit ball and $C_{s,d}$ is a positive constant independent of K. (The value of $C_{s,d}$ is known explicitly only for $d = 1$.)

Similarly, one can redefine the "N-th generalized transfinite diameter" $D_{\mathcal{K}_s}(Z_N) := N(N-1)/\sum_{n\neq k} \mathcal{K}_s(z_j, z_k)$ to be related to the kernel $\mathcal{K}_s$ $(0 < s < 1)$. Since $\mathcal{K}_s$ is symmetric, positive, and continuous, one can easily show that the sequence $D_{\mathcal{K}_s}(Z_{s,N})$, $N \geq 2$, induced by minimum $\mathcal{K}_s$-energy N-point systems $Z_{s,N}$, $N \geq 2$, is a non-increasing sequence bounded from below whose limit is $D_{\mathcal{K}_s}(A)$, the *generalized $\mathcal{K}_s$-transfinite diameter of A.* Furthermore, one has

$$D_{\mathcal{K}_s}(A) = \lim_{N\to\infty} \frac{N(N-1)}{\mathcal{E}_{\mathcal{K}_s}(N,A)} = \operatorname{cap}_{\mathcal{K}_s} A = \operatorname{cap}_s \Gamma(A). \quad (0 < s < 1) \tag{5.4}$$

The last equality holds by (2.1). By (5.4), the positivity of the s-capacity ($s > 0$) of $\Gamma(A)$ implies the weak-star convergence of the sequence of measures $\lambda(Z_{s,N}) = (1/N)\sum_{z\in Z_{s,N}} \delta_z$, $N \geq 2$, to the limit distribution $\lambda^*_{s,A}$. Moreover, by uniqueness of the equilibrium measure $\lambda_{s,A}$ on A, one has $\lambda^*_{s,A} = \lambda_{s,A}$.

An interesting question is whether the minimum s-energy N-point systems $X_{s,N}$ $(0 < s < 1)$ are always contained in the support of $\mu_{s,K}$ for every N.

Open Problem: *For what sets of revolution $\Gamma(A)$ and values of $0 \leq s < 1$ is it true that the points of minimum s-energy configurations are always contained in the support of the k_s-equilibrium measure on $\Gamma(A)$? Same question for the kernel $\mathcal{K}_s$ and A.*

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