# HIGHER-ORDER THREE-TERM RECURRENCES AND ASYMPTOTICS OF MULTIPLE ORTHOGONAL POLYNOMIALS

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Abstract. The asymptotic theory is developed for polynomial sequences that are generated by the three-term higher-order recurrence

$$Q_{n+1} = zQ_n - a_{n-p+1}Q_{n-p},\ p \in \mathbb{N},\ n \geq p,$$

where $z$ is a complex variable and the coefficients $a_k$ are positive and satisfy the perturbation condition $\sum_{n=1}^{\infty} |a_n - a| < \infty$. Our results generalize known results for $p = 1$, that is, for orthogonal polynomial sequences on the real line that belong to the Blumenthal-Nevai class. As is known, for $p \geq 2$, the role of the interval is replaced by a starlike set $S$ of $p+1$ rays emanating from the origin on which the $Q_n$ satisfy a multiple orthogonality condition involving $p$ measures. Here we obtain strong asymptotics for the $Q_n$ in the complex plane outside the common support of these measures as well as on the (finite) open rays of their support. In so doing, we obtain an extension of Weyl's famous theorem dealing with compact perturbations of bounded self-adjoint operators. Furthermore, we derive generalizations of the classical Szegő functions, and we show that there is an underlying Nikishin system hierarchy for the orthogonality measures that is related to the Weyl functions. Our results also have application to Hermite-Padé approximants as well as to vector continued fractions.

## 1. Introduction

Let $\{Q_n\}$ be the sequence of algebraic polynomials in the complex variable $z$ defined by the higher-order recurrence relation

$$Q_{n+1} = zQ_n - a_{n-p+1}Q_{n-p}, \quad n \geq p, \quad z \in \mathbb{C}, \quad p \in \mathbb{N}, \tag{1}$$

with initial conditions

$$Q_j(z) = z^j, \quad j = 0, 1, \ldots, p\,. \tag{2}$$

Such recurrences provide examples of simple *difference operators* of order $p+1$ and the study of the associated polynomials $\{Q_n\}$ facilitates an understanding of the spectral properties of such operators. Another application is to *Hermite-Padé* rational approximants of a vector of analytic functions. The polynomial numerators and denominators of these rational approximants satisfy recurrence relations of higher order. We will present the main definitions and elaborate on these applications below.

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For the constant coefficients case,

$$a_n := a > 0, \qquad n = 1, 2, \ldots, \tag{3}$$

the recurrence (1) generalizes the recurrence relation defining the classical Tchebyshev polynomials. (For the initial conditions (2), the generalization is that of second-kind Tchebyshev polynomials.) The constant coefficients case also can be interpreted as the recurrence for the *Faber polynomials* of a hypocycloidal region of $p+1$ cusps, for which the first-kind Tchebyshev polynomials correspond to the case $p = 1$ (with the 2-cusp hypocycloid interpreted as the interval $[-2\sqrt{a},\ 2\sqrt{a}]$). Such Faber polynomials have been studied in [10], [15], where many interesting properties were discovered. For example, it was shown in [15] that the zeros of these polynomials have some interlacing properties and lie on the starlike set

$$S_0 := [0, \alpha] \cup [0, \alpha_1] \cup [0, \alpha_2] \ldots \cup [0, \alpha_p] \tag{4}$$

with end points

$$\mathcal{A} := \{\alpha,\ \alpha_1 := \varepsilon^{(1)}_{(p+1)}\alpha,\ \alpha_2 := \varepsilon^{(2)}_{(p+1)}\alpha, \quad \ldots, \alpha_p := \varepsilon^{(p)}_{(p+1)}\alpha\}, \tag{5}$$

where $\alpha := [(p+1)/p^{p/(p+1)}]a^{1/(p+1)}$ and $\varepsilon^{(k)}_{(p+1)} := \exp(k2\pi i/(p+1)),\ k = 1, \ldots, p,$ are roots of unity.

The orthogonality property of the polynomials defined by (1) - (2) was investigated in [3]. In particular, for the case of positive coefficients in (1) :

$$a_n > 0, \qquad n \geq 1, \tag{6}$$

the following *Favard-type* theorem was proved.

**Theorem 1.1** (cf. [3])**.** *If $a_n > 0$ for $n \geq 1$ in (1), then there exists a system of positive measures $\{\mu_j\}_{j=1}^p$ such that the polynomials $Q_n$ defined by (1)-(2) satisfy the following multiple-orthogonal (non-Hermitian) relations:*

$$\begin{aligned} &\textstyle\int_S Q_n(t)t^r d\mu_j(t) = 0, \quad r = 0, 1, \ldots, k, \quad j = 1, 2, \ldots, d\ , \\ &\textstyle\int_S Q_n(t)t^r d\mu_j(t) = 0, \quad r = 0, 1, \ldots, k-1, \quad j = d+1, d+2, \ldots, p\ , \end{aligned} \tag{7}$$

*where $n = kp + d$, $0 \leq d \leq (p-1)$, and $S$ is the starlike set*

$$S := \bigcup_{k=0}^{p} \exp(2\pi i k/(p+1)) \times [0, \infty). \tag{8}$$

*Moreover, the measures $\mu_j$, $j = 1, \ldots, p,$ have common support which is a subset of $S$ and they are invariant under rotations in the angles $2\pi k/(p+1)$, $k = 1, 2, \ldots, p$.*

As remarked in [3], the condition (6) is a sufficient but not a necessary condition for the existence of such a system of positive measures $\{\mu_j\}_{j=1}^p$ . One corollary of this theorem (cf. [3] and also [26]) is the existence of the global solution of the Cauchy problem with positive, bounded initial data for a discrete nonlinear dynamical system, namely the so-called *Bogoyavlenskii lattice*, (cf. [4]) :

$$\dot{a}_n = a_n(\sum_{k=1}^{p} a_{n+k} - \sum_{k=1}^{p} a_{n-k}), \qquad n \geq 1, \quad a_{-n} = 0, \qquad n \geq 0,$$

which, for $p = 1$, reduces to the special case of the Toda lattice, also called *Langmuire* or *Volterra* equations (cf. [16], [20], [24]).

In this paper we study the multiple orthogonal polynomials $Q_n$ defined by (1)-(2), where the coefficients $a_n$ satisfy (6) and the limiting relation

$$\lim_{n\to\infty} a_n = a(>0)\,. \tag{9}$$

Condition (9) generalizes the *Blumenthal-Nevai* class in the theory of orthogonal polynomials (cf. [6], [7], [22], [28] ). We prove here a *Weyl-type* theorem characterizing the support of the measures of orthogonality $\{\mu_j\}_{j=1}^p$. By imposing an extra condition on the speed of convergence in (9); namely,

$$\sum_{n=1}^{\infty} |a_n - a| < \infty, \tag{10}$$

we investigate further the analytic properties of the measures $\{\mu_j\}_{j=1}^p$ and obtain strong asymptotics for the multiple orthogonal polynomials $Q_n$. In other words, we deal with *direct spectral problems* in the $l^1$ *perturbation* class (6) - (10).

## 2. Outline and main results

The following two sections (Sections 3 and 4) have an introductory character. There we present some notions related to difference operators and Hermite-Padé rational approximants. It is known (cf. for example [2]) that a difference operator of order $p+1$ can be restored from its *spectral data* consisting of $p$ *resolvent functions* also called *Weyl functions.* A procedure for the solution of this *inverse spectral problem* is based on the expansion, by means of the *Jacobi-Perron algorithm*, of the vector of these $p$ resolvent functions to the *vector continued fraction.* The vector of *convergents* of this continued fraction is the Hermite-Padé rational approximant of that vector of the resolvent functions.

In Section 5 we obtain some preliminary results regarding the constant coefficients case (3). These results will be used latter to develop the asymptotic theory for the perturbed polynomials (1) - (2) under conditions (6) - (10). We also perform an analysis of the characteristic algebraic function $w(z) := \{w_j(z)\}_{j=0}^p$ of the difference equation (1) :

$$w^{p+1} - zw^p + a = 0. \tag{11}$$

The branch points of this function are points of the set (5). Also this function has a branch point of order $(p-1)$ at infinity. There is a unique branch $w_0(z)$ of this function meromorphic at infinity that has a holomorphic continuation in the domain outside $S_0$:

$$w_0 \in \mathrm{H}(\Omega_0), \quad \Omega_0 := \mathbb{C}\backslash S_0,$$

where $S_0$ is defined in (4). The function $w_0$ plays an essential role in the theory since it describes the main term of the asymptotics of the polynomials $Q_n$. Then we prove that the polynomials

$$U_n(z) := Q_n(z)$$

defined by the recurrence relations (1) - (2) with constant coefficients $a_n = a$ form the vector of the rational Hermite-Padé approximants

$$\left(\frac{U_{n-1}}{U_n}, \frac{U_{n-2}}{U_n}, \dots, \frac{U_{n-p}}{U_n}\right)$$

for the vector of functions (which are the Weyl functions of the associated difference operator)

$$\left(\frac{1}{w_0}, \frac{1}{w_0^2}, \dots, \frac{1}{w_0^p}\right),$$

which are holomorphic in $\bar{\mathbb{C}}\setminus S_0$. Furthermore, we obtain some explicit relations and bounds that will be used for the asymptotic analysis of the perturbed case.

In Section 6 we prove an analogue (Theorem 6.1) of Weyl's theorem concerning the support of the measures $\mu_j$ appearing in (7) for the perturbed case (9). This theorem states that the support is the starlike set $S_0$ (defined in (4)) plus a countable set of mass points with only accumulation at the end points of $S_0$. We emphasize that this result is not a direct corollary of the Weyl perturbation theorem (as it is for $p = 1$), because in the case $p \geq 2$ the spectrum of the corresponding difference operator is not equal to the common support of the spectral measures (i.e. the support of the measures in the multiple orthogonality relations (7)).

In Section 7 we consider the $l^1$ perturbation class (10). In the case $p = 1$, i.e. for standard orthogonal polynomials on an interval (and their associated Jacobi operator), it is known (see [12], [21], [19], [28]) that

**(i)** uniform asymptotics of associated orthogonal polynomials holds on any compact subset outside of $[-2, 2]$;

**(ii)** uniform asymptotics of associated orthogonal polynomials holds on any compact subset of $(-2, 2)$;

**(iii)** the spectral measure of the Jacobi operator on $(-2, 2)$ is absolutely continuous and the Radon-Nikodym derivative of the spectral measure is continuous on $(-2, 2)$.

We shall prove analogues of these results for $p \geq 2$ for the perturbation class (10). The techniques we use are similar to the methods of perturbation theory developed for the classical case $p = 1$ in the papers cited above. The starting point is a *comparison equation* (Theorem 7.1) which connects the perturbed polynomials $Q_n$ defined by (1) - (2) with the polynomials $U_n$ defined by the recurrence with constant coefficients $a_n = a$ (see, for example, [28] for the case $p = 1$). Namely, we show that

$$Q_n = U_n + \sum_{k=0}^{n-1} (1 - a_{k+1}) U_{n-p-1-k} Q_k \,, \qquad U_{-1} = U_{-2} = \ldots = U_{-p} = 0. \tag{12}$$

Using bounds for the polynomials $U_n$, this equation allows us to obtain corresponding bounds for the polynomials $Q_n$ and to introduce the series

$$\Phi_0(z) := 1 + \sum_{k=0}^{\infty} \frac{1 - a_{k+1}}{w_0^{p+1}} \cdot \frac{Q_k(z)}{w_0^k(z)} \,,$$

which converges on compact subsets of $\Omega = \mathbb{C}\backslash S_0$ as well as on "both sides" of $S_0$, except for the set of end points $\mathcal{A}$. We further show that the function

$$F_0(z) := \frac{w_0(z)^p \; \Phi_0(z)}{\prod_{k=1}^{p}(w_0 - w_k)} \tag{13}$$

plays the same role as the Szegő function for the classical case $p = 1$. Using this function we derive (Theorem 7.2) the strong uniform asymptotics of $Q_n$ on compact subsets $K$ of the domain $\Omega$ :

$$\lim_{n\to\infty} \frac{Q_n(z)}{w_0^n(z)} = F_0(z) \;, \quad z \in K \subset \Omega \,,$$

and we further show (Theorem 7.5) that, uniformly on compact subsets $\widetilde{K}$ of $S_0 \setminus \mathcal{A}$, there holds

$$\frac{Q_n(t)}{|w_0(t)|^n} = \left(\frac{w_{0,+}(t)}{|w_0(t)|}\right)^n F_{0,+}(t) + \left(\frac{w_{0,-}(t)}{|w_0(t)|}\right)^n F_{0,-}(t) + o(1) \,, \qquad t \in \widetilde{K} \subset S_0 \setminus \mathcal{A}. \tag{14}$$

(We remark that the derivation of the asymptotics (14) on the starlike set requires a new approach in comparison with the classical case $p = 1$.) The proven asymptotics imply the convergence of the Hermite-Padé approximants to the vector of resolvent (Weyl) functions $\{f_j\}_{j=1}^p$, and the jumps of these functions give us the measures $\{\mu_j\}_{j=1}^p$ in (7), which are absolutely continuous on $S_0$ :

$$d\mu_j(t) = \rho_j(t)|dt|, \qquad t \in S_0, \quad j = 1, \dots, p. \tag{15}$$

The final two sections (Sections 8 and 9) are devoted to the study of the analytical properties of the measures $\{\mu_j\}_{j=1}^p$ and the Szegő function $F_0$ (see (13) ). Section 8 deals with the constant recurrence coefficients case (3), and, in Section 9, we consider the $l^1$ perturbation class (10). Here an important role is played by the notion of a *Nikishin system* - the canonical system of functions appearing in the asymptotic theory of Hermite-Padé approximants (see [23], [24], [13], [8], [9], [5], [1])). Investigating the analytic continuation of the weight functions in (15) we prove (Theorems 8.1 and 9.1) that they form a Nikishin system. This allows us to obtain a system of *boundary value problems* on the *Riemann surface* of the algebraic function (11) that characterizes the Szegő function $F_0$ (cf. Theorems 8.4 and 9.2).

## 3. Difference operators and spectral data

Consider the following nonsymmetric $(p + 2)$-banded (lower Hessenberg) infinite matrix

$$A := \begin{pmatrix} a_{0,0} & a_{0,1} & 0 & 0 & 0 & \dots & \dots \\ a_{1,0} & a_{1,1} & a_{1,2} & 0 & 0 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{p,0} & a_{p,1} & a_{p,2} & \dots & a_{p,p+1} & 0 & \dots \\ 0 & a_{p+1,1} & a_{p+1,2} & \dots & a_{p+1,p+1} & a_{p+1,p+2} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}. \tag{16}$$

The matrix $A$ defines in the space $l^2$ an operator that we will denote by the same symbol $A$. Let $\{e_n\}_0^\infty$ be the standard basis of $l^2$. At first the operator $A$ is defined on the linear space span$(e_j)$ by

$$\begin{aligned} Ae_0 &= a_{0,0}e_0 + a_{1,0}e_1 + \cdots + a_{p,0}e_p, \\ Ae_k &= a_{k-1,k}e_{k-1} + a_{k,k}e_k + \cdots + a_{k+p,k}e_{k+p}, \ \ k \geq 1 \end{aligned}$$

and then we take its closure, which always exists. The following functions are called *resolvent* or *Weyl functions* of the operator $A$ :

$$f_j(z) := (R_z e_{j-1}, e_0), \quad j = 1, 2, ..., p\,, \tag{17}$$

where $R_z := (zI - A)^{-1}$ is the resolvent operator of $A$, and $(\cdot,\cdot)$ denotes the inner product in the space $l^2$ of complex sequences . We remark that the operator $A$ is a bounded operator if and only if

$$\sup_{i,j} |a_{i,j}| < \infty\,.$$

In this case, the Weyl functions (17) admit the following power (Neumannn) series expansion at infinity:

$$f_j(z) = \sum_{k=0}^{\infty} \frac{(A^k e_{j-1}, e_0)}{z^{k+1}}, \quad |z| > ||A||\,. \tag{18}$$

For a fixed $j = 1, 2, ..., p$ the quantities

$$f_k^{(j)} := (A^k e_{j-1}, e_0)$$

are called the *moments of operator* $A$ associated with the Weyl function $f_j(z)$.

The recurrence (1) is associated with the following nonsymmetric difference operator with only 2 nonzero diagonals :

$$L := \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \ldots & \ldots \\ 0 & 0 & 1 & 0 & 0 & \ldots & \ldots \\ 0 & 0 & 0 & 1 & 0 & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_1 & 0 & 0 & 0 & \ldots & \ldots & \ldots \\ 0 & a_2 & 0 & 0 & 0 & \ldots & \ldots \\ 0 & 0 & a_3 & 0 & 0 & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}. \tag{19}$$

The special structure of the matrix (19) implies that the moments of this operator have the following property: $f_n^{(j)} = (L^n e_{j-1}, e_0) = 0$, $n \neq (p+1)k + j - 1$, $k \geq 0$. Theorem 1.1 states that there exists a system of positive measures $\mu_j$ with common support on the starlike set $S$ and invariant under rotations such that

$$f_k^{(j)} = \int t^k d\mu_j(t), \quad k \geq 0, \quad j = 1, 2, \ldots, p \tag{20}$$

or, in the terms of Weyl functions,

$$f_j(z) = ((zI - L)^{-1} e_{j-1}, e_0) = \int_S \frac{d\mu_j(t)}{z - t}, \quad j = 1, 2, \ldots, p. \tag{21}$$

The representations of the moments (20) and of the Weyl functions (21) can be considered as *spectral* representations and the system of measures $\{\mu_j\}$ can be considered as the *system of spectral measures* of the operator $L$. In the more general setting of banded Hessenberg operators, the spectral properties and connections with multiple orthogonal polynomials and Hermite-Padé approximants were investigated in [2], [17], [18].

If $\lim_{n\to\infty} a_n = a$, then the operator $L$ is a *compact perturbation* of the background operator with constant entries on the diagonals. In the case of a tridiagonal Jacobi operator it corresponds to Blumenthal-Nevai class.

## 4. Hermite-Padé approximants and vector-continued fractions

Here we briefly summarize some notions from the theory of Hermite-Padé approximants, multiple orthogonal polynoimials and vector continued fractions. ( For the details, see [24], [2] and [14]).

Let $\vec{g} := (g_1, g_2, \dots, g_p)$ be a system of formal power series:

$$g_j(z) := \sum_{k=0}^{\infty} \frac{s_k^{(j)}}{z^{k+1}} \,. \tag{22}$$

For any vector index $\vec{n} = (n_1, n_2, \dots, n_p)$, $n_j \in \mathbb{N}$, $j = 1, 2, \dots, p$, the numerators $P^{(1)}$, $P^{(2)}$, ..., $P^{(p)}$ and denominator $Q$ of the (simultaneous) Hermite-Padé approximant to $\vec{g}$ associated with $\vec{n}$ are defined by the following relations:

$$Q(z)g_j(z) - P^{(j)}(z) = \frac{c_j}{z^{n_j+1}} + \cdots \,, \qquad j = 1, 2, \dots, p \,,$$

where $Q \not\equiv 0$, $\deg Q \le n$, and $n = n_1 + n_2 + \cdots + n_p$. In this case the vector of rational functions

$$\vec{\pi}_{\vec{n}} := \left( \frac{P^{(1)}}{Q}, \frac{P^{(2)}}{Q}, \dots, \frac{P^{(p)}}{Q} \right)$$

is called the *Hermite-Padé (H-P) approximant* of the system $\vec{g}$.

The spectral problem for the operator $A$ defined by the matrix (16) leads to the following difference equation of order $(p+1)$:

$$a_{n,n-p}y_{n-p} + a_{n,n-p+1}y_{n-p+1} + \cdots + a_{n,n}y_n + a_{n,n+1}y_{n+1} = zy_n. \tag{23}$$

Let $q_n(z)$, $p_n^{(j)}(z)$, $j = 1, 2, \dots, p$, be the $(p+1)$ linearly independent solutions of (23) defined by the following initial conditions:

$$q_0 = 1, \; q_n = 0, \; n < 0 \,; \quad p_j^{(j)} = 1/a_{j-1,j}; \quad p_n^{(j)} = 0, \; n < j, \quad j = 1, 2, \dots, p \,,$$

where we assume that $a_{j-1,j} \neq 0$. Then $q_n(z)$ is a polynomial of degree exactly $n$ and $p_n^{(j)}(z)$ is a polynomial of degree exactly $(n-j)$, $j = 1, 2, \dots, p$. The connection between the spectral problem and Hermite-Padé approximants of the system of the Weyl functions (17) of the operator (16) is given by the following known result.

**Theorem 4.1** (cf. [18]). *For $n = kp + s$, the vector of rational functions*

$$\vec{\pi}_{\vec{n}} := \left( \frac{p_n^{(1)}(z)}{q_n(z)}, \frac{p_n^{(2)}(z)}{q_n(z)}, \dots, \frac{p_n^{(p)}(z)}{q_n(z)} \right)$$

*is the Hermite-Padé approximant of the system of the Weyl functions* (17) *of the operator* (16) *corresponding to the index $\vec{n} = (k+1, k+1, \dots, k+1, k, \dots, k)$.*

In particular, for the operator $L$ defined by (19), the denominators $Q_n$ and numerators $P_n^{(j)}$ of the Hermite-Padé approximants are defined by the recurrences

$$Y_{n+1} = zY_n - a_{n-p+1}Y_{n-p},\ n \geq p\ ,$$

with initial conditions

$$\begin{array}{llllllllllll}
Q_0 & = & 1, & Q_1 & = & z, & Q_2 & = & z^2, & \dots & Q_p & = z^p \\
P_0^{(1)} & = & 0, & P_1^{(1)} & = & 1, & P_2^{(1)} & = & z, & \dots & P_p^{(1)} & = z^{p-1} \\
P_0^{(2)} & = & 0, & P_1^{(2)} & = & 0, & P_2^{(2)} & = & 1, & \dots & P_p^{(2)} & = z^{p-2} \\
\dots & \dots & \dots & \dots & \dots & & & & & & & \\
P_0^{(p)} & = & 0, & P_1^{(p)} & = & 0, & P_2^{(p)} & = & 0, & \dots & P_p^{(p)} & = 1\ .
\end{array}$$

Thus the polynomials defined by the recurrence (1) are exactly the common denominators of the Hermite-Padé approximants of the system of Weyl functions for the operator $L$.

Hermite-Padé approximants are connected with vector continued fractions (cf. [24]). According to the Jacobi-Perron rule, the quotient and product of two vectors are defined by

$$\frac{(1, 1, \dots, 1)}{(y_1, y_2, \dots, y_p)} := \left( \frac{1}{y_p}, \frac{y_1}{y_p}, \dots, \frac{y_{p-1}}{y_p} \right),$$

$$(x_1, x_2, \dots, x_p)(y_1, y_2, \dots, y_p) := (x_1y_1, x_2y_2, \dots, x_py_p).$$

Let $\vec{g}$ be a system of formal power series (22). Then it is possible to write

$$\vec{g} = \frac{\vec{1}}{(\frac{g_2}{g_1}, \frac{g_3}{g_1}, \dots, \frac{g_p}{g_1}, \frac{1}{g_1})} = \frac{\vec{c_1}}{\vec{p_1} + \vec{r_1}}\ ,$$

where $\vec{p_1}$ is a vector of polynomials (the polynomial parts of power series for $g_j/g_1$), $\vec{c_1}$ is a constant vector $\vec{c_1}=(c_1, 1, 1, ..., 1)$ chosen so that the last component of $\vec{p_1}$ is a monic polynomial, and $\vec{r_1}$ is a quotient of the same type as $\vec{g}$. This development may be repeated and we can associate with the system $\vec{g}$ the following infinite vector continued fraction:

$$\vec{g} \sim \frac{\vec{c_1}|}{|\vec{p_1}} + \frac{\vec{c_2}|}{|\vec{p_2}} + \dots + \frac{\vec{c_n}|}{|\vec{p_n}} + \dots\ .$$

This algorithm is called the (modified) *Jacobi-Perron algorithm.* The connection with the system of Weyl functions of the operator $A$ is given by the following.

**Theorem 4.2** (cf. [17])**.** *The (modified) Jacobi-Perron algorithm applied to the system of the Weyl functions of the operator* (16) *gives the following vector continued fraction (VCF)*

$$\frac{(1/h_0, 1, \dots, 1)|}{|(0, 0, \dots, 0, z + b_{0,0})} + \frac{(1/h_1, 1, \dots, 1)|}{|(0, 0, \dots, b_{1,0}, z + b_{1,1})} + \dots + \tag{24}$$
$$+ \frac{(1/h_{p-1}, 1, \dots, 1)|}{|(b_{p-1,0}, b_{p-1,1}, \dots, z + b_{p-1,p-1})} + \dots + \frac{(b_{n,n-p}, 1, \dots, 1)|}{|(b_{n,n-p+1}, b_{n,n-p+2}, \dots, z + b_{n,n})} + \dots\ ,$$

*where* $b_{i,j} = -(h_j/h_i)a_{i,j}$, $i \geq 0$, $j \geq 0$ *and* $h_k = 1/(a_{0,1}a_{1,2} \cdots a_{k-1,k})$, $h_0 = 1$.

In particular, for the vector of Weyl functions (21) of the operator $L$ defined by (19) we have the following VCF

$$(f_1(z),\dots,f_p(z)) = \frac{(1,1,\dots,1)|}{|(0,0,\dots,0,z)}+\frac{(1,1,\dots,1)|}{|(0,0,\dots,0,z)}+\dots+\frac{(1,1,\dots,1)|}{|(0,0,\dots,z)}+ \\ \frac{(-a_1,1,\dots,1)|}{|(0,0,\dots,z)}+\frac{(-a_2,1,\dots,1)|}{|(0,0,\dots,z)}+\dots\ . \tag{25}$$

We note that the continued fraction (25) solves the inverse spectral problem; namely, from the spectral data (21), it recovers the coefficients of the operator $L$ in (19). As in the scalar case $p = 1$ (cf . [27], [30] ), the Jacobi-Perron VCF (25) can be transformed to a Stieltjes VCF (for details, see [3] ).

We will make later use of the following observation.

**Remark 4.1:** If we transform the system of the Weyl functions $\vec{f} := (f_1, f_2, \dots, f_p)$ into a new system of resolvent functions $\vec{\phi} = \vec{f}\cdot X$, where $X$ is upper triangular matrix, then the essential part of the VCF for the new system of resolvent functions $\vec{\phi}$ is the same, only the first $p$ floors change. More precisely, one has the following VCF for the system $\vec{\phi}$

$$\frac{(b_{0,-p},1,\dots,1)|}{|(b_{0,-p+1},b_{0,-p+2},\dots,b_{0,-1},z+b_{0,0})}+\frac{(b_{1,-p+1},1,\dots,1)|}{|(b_{1,-p+2},b_{1,-p+3},\dots,b_{1,0},z+b_{1,1})}+ \\ +\dots+\frac{(b_{n,n-p},1,\dots,1)|}{|(b_{n,n-p+1},b_{n,n-p+2},\dots,b_{n,n-1},z+b_{n,n})}+\dots,$$

where $b_{i,j} = -(h_j/h_i)a_{i,j}$, $i \ge 0$, $j \ge 0$, and $b_{i,j}$ for $j < 0$ are given by the matrix equation $B = \mathrm{diag}(1/h_0, 1/h_1, \dots, 1/h_{p-1})\cdot X$, where we put

$$B := \begin{pmatrix} b_{0,-p} & b_{0,-p+1} & \dots & b_{0,-1} \\ 0 & b_{1,-p+1} & \dots & b_{1,-1} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & b_{p-1,-1} \end{pmatrix}.$$

We also mention that there exists transformation connecting general band operator (16) with "generic" operator (19) (for details see [3], [29]).

## 5. Constant recurrence coefficients

5.1. **Algebraic function.** Let $w(z)$ be the multiple-valued algebraic function defined by the equation (11), where without lost of generality we set $a = 1$:

$$w^{p+1} - zw^p + 1 = 0. \tag{26}$$

The inverse function with respect to $w(z)$ is the rational function

$$z = w + \frac{1}{w^p}. \tag{27}$$

Thus $w(z)$ is an algebraic function of order $p+1$ and genus 0 and the rational function $z(w)$ gives the composition of a conformal map of the sphere to the Riemann surface $\mathfrak{R}$ of the function $w(z)$ and the projection of $\mathfrak{R}$ to the complex plane. The finite branch points of this function are the points of $\mathcal{A}$ in (5), i.e.

$$\{\alpha,\ \alpha_1 := \varepsilon^{(1)}_{(p+1)}\alpha,\ \alpha_2 := \varepsilon^{(2)}_{(p+1)}\alpha, \quad \dots,\ \alpha_p := \varepsilon^{(p)}_{(p+1)}\alpha\}\,,$$

with $\alpha = (p+1)/p^{p/(p+1)}$ and roots of unity $\varepsilon^{(k)}_{(p+1)} := \exp(k2\pi i/(p+1)),\ k = 1,\dots,p$ . Also $w(z)$ has a branch point of order $p-1$ at infinity and, as $z \to \infty$, its branches $w_j$ have the following behavior:

$$\begin{aligned} w_0(z) &= z + \cdots \,, \\ w_j(z) &= \mathcal{O}(1/z^{1/p})\ ,\ j = 1,\dots,p. \end{aligned} \tag{28}$$

Clearly the function $w(z)$ has no finite poles and no zeros.

Following Nuttall [25], we shall fix global piecewise holomorphic branches of $w(z)$ prescribed by the conditions

$$|w_0(z)| \geq |w_1(z)| \geq \ldots \geq |w_p(z)|, \quad \forall z \in \mathbb{C}. \tag{29}$$

To describe the domains of holomorphicity of such branches it is convenient, in addition to the sets $S_0$ in (4) and $S$ in (8), to introduce the set (see Figure 1)

$$s := \{z : z = \xi \exp(\frac{i\pi}{p+1}),\ \xi \in S\}. \tag{30}$$

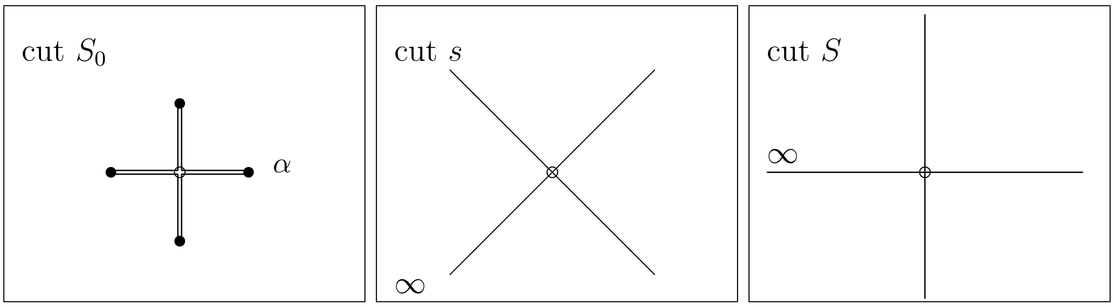


FIGURE 1. Cuts of the complex plane forming Riemann surface of $w(z)$ ($p = 4$)

**Proposition 1.** *There exist global branches of $w(z)$ satisfying (29) in $\mathbb{C}$, with strict inequality holding in $\mathbb{C}\setminus\{S \cup s\}$, such that*

$$w_j \in \mathrm{H}(\Omega_j), \quad j = 0,1,\dots,p,$$

*where*

$$\begin{aligned} &\Omega_0 := \mathbb{C}\setminus S_0, && \Omega_1 := \begin{cases} \mathbb{C}\setminus S_0, & p = 1 \\ \mathbb{C}\setminus\{S_0 \cup s\}, & p > 1 \end{cases}, \\ &\Omega_k := \mathbb{C}\setminus\{s \cup S\}, \quad (k < p), && \Omega_p := \begin{cases} \mathbb{C}\setminus s, & p = 2l \\ \mathbb{C}\setminus\ S, & p = 2l+1 \end{cases}. \end{aligned} \tag{31}$$

*Proof.* **1.** We first consider $w(z)$ for $z \in \mathbb{R}^+ := [0, \infty]$. The inverse function (27) (for $w \in \mathbb{R}^+$) is decreasing on the interval $(0, w_\alpha)$ and increasing on $(w_\alpha, \infty)$, where $w_\alpha := p^{1/(p+1)}$. Consequently $w(z)$ has two positive branches on $(\alpha, +\infty)$, which we denote as $w_0$ and $w_1$, such that

$$w_0(x) > w_\alpha > w_1(x), \qquad x \in (\alpha, +\infty).$$

We also remark that for even $p$ there also exists a negative branch of $w(z)$ on $\mathbb{R}^+$ which we denote as $w_p$. Moreover, it is possible to prove (introducing notation for other branches) that

$$|w_0| \geq |w_1| > |w_2| = |w_3| > \ldots > \begin{cases} |w_{p-1}| = |w_p| \,, & p = 2k+1 \\ -w_p \,, & p = 2k \end{cases} , \qquad \text{on } \mathbb{R}^+. \tag{32}$$

**2.** To prolong the branches of $w(z)$ from $\mathbb{R}^+$ to $\mathbb{C}$ we make use of some basic symmetry relations. From (26) we have

$$(e^{\frac{2\pi i}{p+1}} w)^{p+1} - z\, e^{\frac{2\pi i}{p+1}} (e^{\frac{2\pi i}{p+1}} w)^p + 1 = 0;$$

hence, for any branch $w_j$, there exists a choice of branch $w_k$, $j, k \in \{0, 1, \ldots, p\}$ such that

$$w_j\,(z\, e^{\frac{2\pi i}{p+1}}) = e^{\frac{2\pi i}{p+1}}\, w_k(z), \qquad z \in \mathbb{C}\,. \tag{33}$$

Another symmetry relation follows from the Schwarz reflection principle (because $\Im w_1 = 0$ on $[\alpha, \infty]$); namely,

$$w_1(z) = \overline{w_1(\bar{z})}\,, \qquad z \in \mathbb{C}\,. \tag{34}$$

We shall use (34) also for other branches of $w(z)$ when we consider them as the analytical continuation of $w_1$.

**3.** Now we define branches of $w(z)$ globally in $\mathbb{C}$.

Thanks to the Monodromy Theorem, the branch $w_0$ can be prolonged from a neighborhood of infinity to the domain $\Omega_0$ defined in (31), i.e. $w_0 \in H(\Omega_0)$. Furthermore in a neighborhood of infinity we must have (33) for $j = k = 0$, and therefore this relation holds in $\Omega_0$:

$$w_0\,(z\, e^{\frac{2\pi i}{p+1}}) = e^{\frac{2\pi i}{p+1}}\, w_0(z), \qquad z \in \Omega_0\,.$$

The branch $w_1$ (due to the order of the infinity branch point (28)) has a holomorphic continuation (from $\mathbb{R}^+ \setminus [0, \alpha]$) to the simply connected domain $\tilde{A}$ (see Figure 2) defined by

$$\tilde{A} := \{\, z \,:\, |\arg z| < \frac{\pi}{p+1} \} \setminus [0, \alpha]\,.$$

Next we define the branch $w_1$ in $\Omega_1$ using (33) (starting from $\tilde{A}$) :

$$w_1\,(z\, e^{\frac{2\pi i}{p+1}}) = e^{\frac{2\pi i}{p+1}}\, w_1(z), \qquad z \in \Omega_1\,.$$

Note that, due to (34) and (33), the branch $w_1$ has a jump on $s$ : $\overline{w_{1+}} = e^{\frac{2\pi i}{p+1}}\, w_{1-}$. Thus $w_1 \in H(\Omega_1)$.

In a similar way we define the remaining branches of $w(z)$. First, we define them in the fundamental sector

$$A := \{\, z \,:\, |\arg z| < \frac{\pi}{p+1} \}\,,$$

and then extend them to $\mathbb{C} \setminus A$ by means of the symmetry relation (33) :

$$w_j\,(z\, e^{\frac{2\pi i}{p+1}}) = e^{\frac{2\pi i}{p+1}}\, w_j(z), \quad j = 2, \ldots, p\,, \quad z \in \Omega_j\,.$$

A branch with even index, $w_{2k}$, is defined in $A$ as a direct analytic continuation of the branch $w_{2k-1}$ to the domain $A^+ := \{ z : 0 < \arg z < \frac{\pi}{p+1}\}$ from the upper part of the boundary of $A^+$, and to the domain $A^- := \{ z : -\frac{\pi}{p+1} < \arg z < 0\}$ from the lower part of the boundary of $A^-$ (see Figure 2).

A branch with odd index, $w_{2k+1}$, is defined in $A$ as a direct analytic continuation of the branch $w_{2k}$ to the domain $A^+$ from the lower part of the boundary of $A^+$, and to the domain $A^-$ from the upper part of the boundary of $A^-$ (see Figure 2).

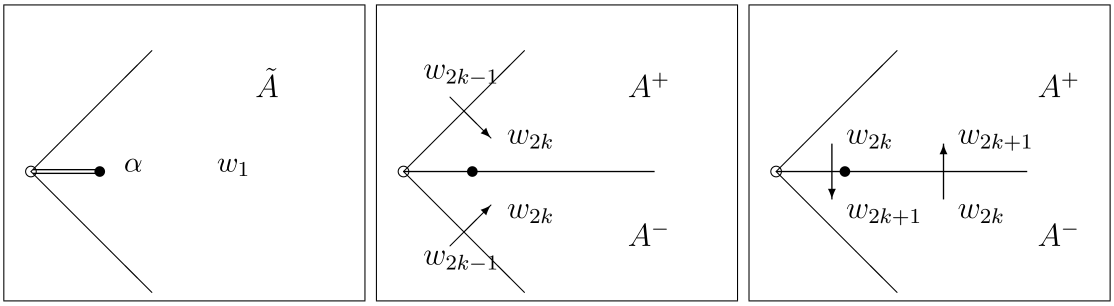


FIGURE 2. Fundamental domains for branches of $w(z)$ $(p = 4)$

**4.** It remains to prove that the branches of $w(z)$ as defined above satisfy the relations (29), with strict inequality holding in $\mathbb{C}\backslash\{S \cup s\}$. To prove that

$$|w_0(z)| > |w_1(z)| \qquad z \in \overline{\mathbb{C}} \setminus S_0 \ , \tag{35}$$

we define in the domain

$$B := \{ z : 0 < \arg z < \frac{2\pi}{p+1}\} \ ,$$

the holomorphic function $w_{12} \in H(B)$ :

$$w_{12}(z) := \begin{cases} w_1(z) \ , & \frac{\pi}{p+1} < \arg z < \frac{2\pi}{p+1} \\ w_2(z) \ , & 0 < \arg z \le \frac{\pi}{p+1} \end{cases} .$$

Applying the Phragmen-Lindelöf maximum principle to the function $w_0/w_{12} \in H(B)$ we deduce (due to (32) and (33)) the validity of (35) in $B$, and therefore (because of (33)) inequality (35) holds throughout $\mathbb{C}$.

The remaining relations in (29) can be verified in a similar way, using (32), (33) and the definition of the branches $\{w_j\}_{j=1}^p$. □

We now define a Riemann surface $\mathfrak{R}$ for the function $w(z)$ with sheet structure corresponding to the choice of the branches as in (29), i.e. the projection of a sheet gives the domain of meromorphicity (single-valuedness) of a branch of the algebraic function (29). Using the notation $\pi$ for the projection of $\mathfrak{R}$ on $\mathbb{C}$ and $\pi_k^{-1}$ for its inverse branches, we have

$$\mathfrak{R} = \overline{\bigcup_{j=0}^{p} \mathfrak{R}_j}, \quad \mathfrak{R}_0 := \pi_0^{-1}(\Omega_0 \cup \{\infty\}), \ \mathfrak{R}_j := \pi_j^{-1}(\Omega_j), \ j = 1, \dots, p \ . \tag{36}$$

We denote by $\partial\mathfrak{R}_{j,k}$ a closed contour in $\mathbb{C}^2$ separating sheets $\mathfrak{R}_j$ and $\mathfrak{R}_k$. We assume that the contour $\partial\mathfrak{R}_{j,k}$ is orientated such that sheet $\mathfrak{R}_j$ lies to the left (+) side. We have (see

Figures 1 and 3)

$$\begin{aligned} \pi(\partial\mathfrak{R}_{0,1}) &= \partial\Omega_0 = S_{0+} \cup S_{0-}, \\ \pi(\partial\mathfrak{R}_{j,j+1}) &= \begin{cases} s_+ \cup s_-, & j = 2l-1 \\ S_+ \cup S_-, & j = 2l \end{cases} , \quad j = 1, \dots, p-1 \ . \end{aligned} \tag{37}$$

Thus

$$\mathfrak{R} = \left\{ \bigcup_{j=0}^{p} \mathfrak{R}_j \right\} \bigcup \left\{ \bigcup_{j=0}^{p-1} \mathfrak{R}_{j,j+1} \right\} .$$

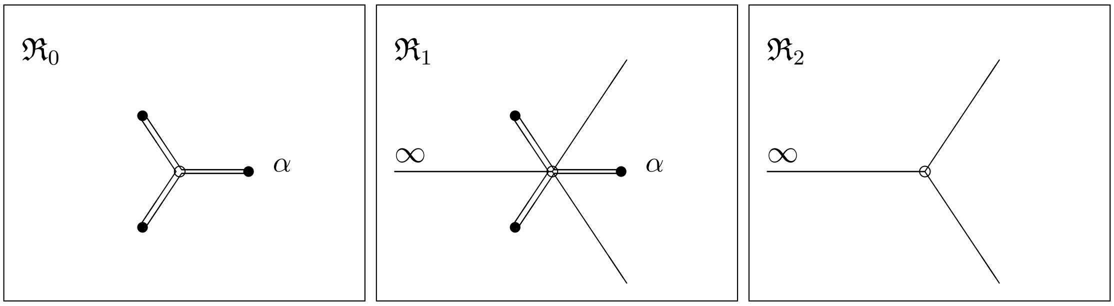


FIGURE 3. Nuttall's sheet structure for the Riemann surface for function $w(z)$ $(p = 2)$

It is useful to recall the Vieta relations for (26) which hold at any point $z$ of the complex plane:

$$\prod_{j=0}^{p} w_j = (-1)^{p+1}, \quad \sum_{j=0}^{p} w_j = z, \tag{38}$$

and the remaining elementary symmetric functions of the branches (roots) $\{w_j\}_{j=0}^{p}$ must be identically zero in $\mathbb{C}$.

5.2. **Polynomials with the constant recurrence coefficients.** Here we consider generalizations of Tchebyshev polynomials of the second kind, which are defined by the recurrences

$$U_{n+1} = zU_n - U_{n-p}, \quad n \geq p, \tag{39}$$

with initial conditions

$$U_0 = 1, \quad U_1 = z, \quad U_2 = z^2, \ \dots, U_p = z^p \ .$$

The associated operator takes the form

$$L_0 = \begin{pmatrix} 0 & 1 & 0 & 0 & \ddots \\ 0 & 0 & 1 & 0 & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ 1 & 0 & 0 & 0 & \ddots \\ 0 & 1 & 0 & 0 & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} . \tag{40}$$

The spectral equation for the operator is equivalent to the recurrence equation : $L_0 U = zU$, where $U^T = (U_0, U_1, U_2, \ldots)$ .

We present here an explicit form for these polynomials. With the help of the functions $w_j(z)$ we have

$$U_n = A_0 w_0^n + A_1 w_1^n + \cdots + A_p w_p^n \ ,$$

where the coefficients $A_0(z)$, $A_1(z)$,..., $A_p(z)$ are defined (from initial conditions) by the equations

$$\begin{cases} A_0 + A_1 + A_2 + \cdots + A_p = 1 \\ A_0 w_0 + A_1 w_1 + A_2 w_2 + \cdots + A_p w_p = z \\ A_0 w_0^2 + A_1 w_1^2 + A_2 w_2^2 + \cdots + A_p w_p^2 = z^2 \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ A_0 w_0^p + A_1 w_1^p + A_2 w_2^p + \cdots + A_p w_p^p = z^p \end{cases} .$$

From Cramer's rule we find

$$A_i = \prod_{k=0, k\neq i}^{p} (z - w_k) / \prod_{k=0, k\neq i}^{p} (w_i - w_k).$$

It is easy to check from the Vieta formulas (38) that

$$\prod_{k=0}^{p} (z - w_k) = z^{p+1} - \sum_{j=0}^{p} w_j z^p + \cdots + (-1)^{p+1} \prod_{j=0}^{p} w_j = 1 \ ,$$

and taking into account that

$$w_i^{p+1} - z w_i^p + 1 = 0 \implies \frac{1}{z - w_i} = w_i^p \ ,$$

one has

$$\prod_{k=0, k\neq i}^{p} (z - w_k) = \frac{1}{z - w_i} = w_i^p \ .$$

Thus

$$U_n = \sum_{i=0}^{p} \frac{w_i^{n+p}}{\prod_{k=0, k\neq i}^{p} (w_i - w_k)} \ . \tag{41}$$

5.3. **Hermite-Padé approximants for the Weyl functions.** We can calculate explicitly the Hermite-Padé approximants for the system of Weyl functions for the background operator $L_0$ using Theorem 4.1. In fact, the common denominator $Q_n$ and numerators $P_n^{(j)}$, $j = 1, 2, \ldots, p$ are defined in this case by the recurrences

$$Y_{n+1} = z Y_n - Y_{n-p}, \ n \geq p \ ,$$

with initial conditions $Q_n = z^n$, $0 \leq n \leq p$, $P_n^{(j)} = 0$, $0 \leq n < j$, $P_n^{(j)} = z^{n-j}$, $j \leq n \leq p$. Consequently, $Q_n(z) = U_n(z)$ and

$$P_n^{(j)} = U_{n-j}, \quad j = 1, 2, \ldots, p \ .$$

Thus the Hermite-Padé approximants are of the form

$$\left( \frac{U_{n-1}}{U_n}, \frac{U_{n-2}}{U_n}, \ldots, \frac{U_{n-p}}{U_n} \right) . \tag{42}$$

5.4. **Weyl functions of the operator and orthogonality measures (spectral measures of operator).** The vector of Weyl functions for the operator $L_0$ defined by (40) has the continued fraction representation (cf.(25)):

$$(f_1, f_2, \ldots, f_p) = \frac{(1,1,\ldots,1)|}{|(0,0,\ldots,0,z)} + \ldots + \frac{(1,1,\ldots,1)|}{|(0,0,\ldots,z)} + \frac{(-1,1,\ldots,1)|}{|(0,0,\ldots,z)} + \frac{(-1,1,\ldots,1)|}{|(0,0,\ldots,z)} + \ldots .$$

Consider the system of functions $(g_1, g_2, \ldots, g_p)$ associated with the tail of this VCF. Then one has

$$(g_1, g_2, \ldots, g_p) := \frac{(-1,1,\ldots,1)|}{|(0,0,\ldots,z)} + \frac{(-1,1,\ldots,1)|}{|(0,0,\ldots,z)} + \ldots = \frac{(-1,1,\ldots,1)|}{|(0,0,\ldots,z)} + (g_1, g_2, \ldots, g_p).$$

This implies the following relations for the functions $g_j(z)$:

$$g_2 = -g_1^2, \quad g_3 = g_1^3, \quad g_4 = -g_1^4, \ldots, g_p = (-1)^{p+1} g_1^p, \qquad (-1)^{p+1} g_1^{p+1} + z g_1 + 1 = 0 .$$

Using these relations in the representation

$$(f_1, f_2, \ldots, f_p) = \frac{(1,1,\ldots,1)|}{|(0,0,\ldots,0,z)} + \ldots + \frac{(1,1,\ldots,1)|}{|(0,0,\ldots,z)} + (g_1, g_2, \ldots, g_p),$$

where the same floor is repeated $p$ times, we get for the Weyl functions of the operator $L_0$:

$$f_1 = -g_1, \quad f_2 = g_1^2, \quad \ldots, \quad f_p = (-1)^p g_1^p,$$

with $f_1^{p+1} - z f_1 + 1 = 0$. In general, $f_j = f_1^j$, $j = 1, 2, \ldots, p$, where the function $f_1(z)$ is the solution of the algebraic equation

$$f_1^{p+1} - z f_1 + 1 = 0,$$

with $f_1 \to 0$ as $z \to \infty$. Hence $f_1 = 1/w_0(z)$, where $w_0$ is the holomorphic branch of $w$ defined in (28)-(29). Finally one has

$$f_1(z) = \frac{1}{w_0}, \quad f_2(z) = \frac{1}{w_0^2(z)}, \quad \ldots \quad , f_p(z) = \frac{1}{w_0^p(z)}. \tag{43}$$

From the explicit representation (43) of the Weyl functions we can easily deduce properties of the associated measures of orthogonality.

**Lemma 5.1.** *The common support of the measures $\mu_j^0$, $j = 1, \ldots, p$ (spectral measures of $L_0$) is the set $S_0$ defined in (4). All measures $\mu_j^0$ are absolutely continuous on $S_0$.*

*Proof.* We have for the Weyl functions of the operator $L_0$:

$$f_k(z) = \frac{1}{[w_0(z)]^k}, \quad k = 1, 2, \ldots, p ,$$

where $w_0(z)$ is as above. This representation implies immediately that all measures have a common support on the set $S_0$ and they are absolutely continuous with respect to Lebesgue measure on this set. Moreover, if we put $d\mu_j(t) = \rho_j(t)|dt|$, $t \in S_0$, then $\rho_j(t)$ is continuous and positive on $S_0$, and the following symmetry property holds: $\rho_j(\exp(2\pi i/(p+1))t) = \rho_j(t)$, $t \in S_0$. $\square$

Here we illustrate the connection between the two weight functions for the case $p = 2$. Considering the boundary values of $f_j^{\pm}, j = 1,2$ on the both sides of the intervals that comprise the set $S_0$, we have from the explicit representation of the Weyl functions that

$$\rho_1(t) = f_1^+ - f_1^- = \frac{1}{w_0^+} - \frac{1}{w_0^-},$$

and

$$\rho_2(t) = (f_1^2)^+ - (f_1^2)^- = \rho_1(t)(\frac{1}{w_0^+} + \frac{1}{w_0^-}).$$

It remains to notice ( due to the sheet structure (31)-(37) and (38)) that on $S_0$ we have

$$\frac{1}{w_0^+} + \frac{1}{w_0^-} = \frac{1}{w_0} + \frac{1}{w_1} = -\frac{1}{w_2} \Longrightarrow \rho_2(t) = -\frac{1}{w_2(t)}\rho_1(t), \quad t \in S_0.$$

5.5. **Bounds for polynomials with the constant recurrence coefficients.** To study the asymptotics of perturbed polynomials, we need some special properties of the polynomials $U_n$ generated from recurrences with constant coefficients (cf. (39)). The lemmas of this section provide useful bounds for these polynomials.

We set

$$\Delta(z) := \prod_{0 \le i < j} (w_i(z) - w_j(z)) \ . \tag{44}$$

**Lemma 5.2.** *The following bound for $U_n(z)$ holds in $\Omega_0 = \mathbb{C} \setminus S_0$ :*

$$|\Delta(z)| \cdot |U_n(z)| \cdot |w_0(z)|^{-n-p} \le C, \quad z \in \Omega_0, \quad n = 0, 1, \dots,$$

*for some constant $C$.*

*Proof.* We have from (41)

$$\Delta \cdot U_n \cdot w_0^{-n-p} = \sum_{i=0}^{p} \frac{\Delta}{\prod_{k=0,k\neq i}^{p}(w_i - w_k)} \cdot \frac{w_i^{p-1}}{w_0^{p-1}} \cdot \left(\frac{w_i}{w_0}\right)^{n+1}.$$

Recall that $|w_0(z)| > \max\{|w_j(z)|,\, j = 1, 2, \dots, p\}$ in $\Omega_0$, and that $|w_j(z)| \le K$ for $j = 1, 2, \dots, p,$ for some constant $K$. Consequently, for $z \in \Omega_0$,

$$\left| \frac{\Delta}{\prod_{k=0,k\neq 0}^{p}(w_0 - w_k)} \cdot \frac{w_0^{p-1}}{w_0^{p-1}} \cdot \left(\frac{w_0}{w_0}\right)^{n+1} \right| = \left| \prod_{0<i<j} (w_i - w_j) \right| \le C_0,$$

and for $i = 1, 2, \dots, p$ we deduce that

$$\left| \frac{\Delta}{\prod_{k=0,k\neq i}^{p}(w_i - w_k)} \cdot \frac{w_i^{p-1}}{w_0^{p-1}} \cdot \left(\frac{w_i}{w_0}\right)^{n+1} \right| \le \left| \prod_{s=1,s\neq i}^{p} \left(1 - \frac{w_s}{w_0}\right) \cdot \prod_{0<s<j,\, s,j\neq i}^{p} (w_i - w_j) \right| \le C_i.$$

The lemma now follows with $C = C_0 + C_1 + \dots + C_p$. □

**Lemma 5.3.** *The following bound holds on the open intervals of $S_0$ :*

$$|U_n(t)| \cdot |w_0(t)|^{-n} \le M(n+1), \quad t \in S_0 \setminus \mathcal{A}, \ n = 0, 1, \dots,$$

*where $\mathcal{A}$ is given in (5).*

*Proof.* By symmetry, it is sufficient to prove the lemma for the interval $[0, \alpha)$. To fix the branches of the algebraic function we will use their values on $[0, \alpha)$ obtained by approaching the interval from above. We have

$$w_0(t) = |w_0(t)| \exp(i\theta),\ w_1(t) = |w_0(t)| \exp(-i\theta), \quad 0 \le \theta \le \frac{\pi}{p+1},$$

and

$$|w_i(t) - w_j(t)| \ge c > 0,\ i \ne j,\ i, j \ne 0, 1, \quad [w_1(t) - w_0(t)] = -2i|w_0(t)| \sin(\theta).$$

From formula (41) for $U_n(t)$ follows

$$U_n = \sum_{i=0}^{p} \frac{w_i^{n+p}}{\prod_{k=0,k\ne i}^{p}(w_i - w_k)} = \frac{w_0^{n+p}}{\prod_{k=0,k\ne 0}^{p}(w_0 - w_k)} + \frac{w_1^{n+p}}{\prod_{k=0,k\ne 1}^{p}(w_1 - w_k)} + O(|w_0(t)|^n).$$

Furthermore,

$$\frac{w_0^{n+p}}{\prod_{k=0,k\ne 0}^{p}(w_0 - w_k)} + \frac{w_1^{n+p}}{\prod_{k=0,k\ne 1}^{p}(w_1 - w_k)} = \frac{w_0^{n+p}\prod_{k=2}^{p}(w_1 - w_k) - w_1^{n+p}\prod_{k=2}^{p}(w_0 - w_k)}{(w_0 - w_1)\prod_{k=2}^{p}(w_1 - w_k)(w_0 - w_k)}.$$

On writing $\prod_{k=2}^{p}(w - w_k) = \sum_{k=2}^{p} b_k(t) w^{p-k}$, and taking into account that

$$w_0^{n+p}\prod_{k=2}^{p}(w_1 - w_k) - w_1^{n+p}\prod_{k=2}^{p}(w_0 - w_k) = \sum_{k=2}^{p} b_k(t)[w_1^{p-k}w_0^{n+p} - w_0^{p-k}w_1^{n+p}] =$$

$$= \sum_{k=2}^{p} b_k(t)|w_0(t)|^{n+2p-k}(2i)\sin((n+k)\theta),$$

we get

$$|U_n(t)| \cdot |w_0(t)|^{-n} \le M_1 \sum_{k=2}^{p} \left| \frac{\sin((n+k)\theta)}{\sin(\theta)} \right| + M_2 \le M(n+1),$$

and the lemma follows. □

The maximum modulus principle applied to the preceding lemma yields

**Corollary 5.1.** *There holds*

$$|U_n(z)| \cdot |w_0(z)^{-n}| \le M(n+1)\,, \quad z \in \Omega_0\ .$$

## 6. A Weyl-type theorem

**Theorem 6.1.** *Suppose that $a_n > 0$ for all $n \ge 1$ and*

$$\lim_{n\to\infty} a_n = 1.$$

*Then the measures $\mu_j$, $j = 1, 2, \ldots, p$, of multiple orthogonality (7) for the polynomials (1) - (2) have a common support consisting of the set $S_0$ in (4) plus a countable set of mass points that lie on the set*

$$S = \bigcup_{k=0}^{p} \exp(2\pi i k/(p+1)) \times [0, \infty)$$

*with only accumulation points at the ends of the intervals of $S_0$.*

*Proof.* We first note that any operator $L$ of the form (19) whose entries satisfy condition (9) is a compact perturbation of the operator $L_0$ in (40). By the Weyl perturbation theorem for bounded operators, the essential spectrum of the operator $L$ is the same as for $L_0$. For a symmetric bounded Jacobi operator, the spectrum of $L_0$ coincidess with the support of the spectral measure $\mu^0$ and the theorem follows for the case $p = 1$. For the case $p > 1$ the situation is different; namely, the spectrum of the operator $L_0$ can be a set that is larger than the common support of the measures $\mu_j^0$, $j = 1, 2, \dots, p$. On the other hand, it is clear that the support of measures $\mu_j^0$ is connected with the properties of the Weyl functions $f_j(z)$ (see the integral representation above). First we note that, by Lemma 5.1, the common support of the measures $\mu_j^0$ is the set $S_0$.

To continue with the proof of the theorem, it is convenient to introduce the following family of operators:

$$L_0(c,b) := \begin{pmatrix} 0 & b & 0 & 0 & 0 & \dots & \dots \\ 0 & 0 & b & 0 & 0 & \dots & \dots \\ 0 & 0 & 0 & b & 0 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ c & 0 & 0 & 0 & \dots & \dots & \dots \\ 0 & c & 0 & 0 & 0 & \dots & \dots \\ 0 & 0 & c & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \tag{45}$$

and find their spectrums.

**Lemma 6.1.** *The resolvent set $G_0(c,b)$ of the operator $L_0(c,b)$ is described by the following condition*

$$G_0(c,b) = \{\lambda \in \mathbb{C} \ : \ |v_0(\lambda)| < 1 < |v_j(\lambda)|, \quad j = 1, 2, \dots, p \ \},$$

*where $v_0(\lambda)$, $v_1(\lambda)$, ..., $v_p(\lambda)$ are the branches of the algebraic function*

$$cv^{p+1} - \lambda v + b = 0,$$

*with $v_0(\lambda)$ having the smallest modulus.*

*Proof.* To find the spectrum of the operator $L_0(c,b)$ we transform it to the operator in the Hardy space $H^2$ of analytic functions in the unit disk. For any sequence $\{x_k\}_{k=0}^{\infty} \in l^2$ we associate the function $x(v) := \sum x_k v^k \in H^2$ and define the operator $K_0(c,b)$ by

$$K_0(c,b)x = y, \quad y(v) = b\frac{x(v) - x(0)}{v} + cv^p x(v).$$

The operator $K_0(c,b)$ is isomorphic to $L_0(c,b)$ and therefore has the same spectrum. To describe the resolvent set of $K_0(c,b)$ we consider the operator $[K_0(c,b) - \lambda I]$ and find all $\lambda$ such that this operator has a bounded inverse. By the Banach-Schauder theorem, a necessary and sufficient for $\lambda$ to be in the resolvent set is that the equation $(K_0(c,b) - \lambda I)x = y$ has a unique solution in $H^2$ for any function $y(w) \in H^2$. Clearly, if a solution of this equation exists, it must have the form

$$x(v) = \frac{vy(v) + bx(0)}{cv^{p+1} - \lambda v + b}.$$

The following cases are possible

(1) The denominator $cv^{p+1} - \lambda v + b$ has no zeros in the disk $\{|v| \le 1\}$. In this case function $x(v)$ is in the space $H^2$ but the solution of the equation $(K_0(c,b) - \lambda I)x = y$ is not unique because we can choose $x(0)$ arbitrarily.
(2) The denominator $cv^{p+1} - \lambda v + b$ has only one zero in the disk $\{|v| < 1\}$ (remember that $\lambda$ is fixed). Denote this zero by $v_0$. Then by choosing $x(0) = -v_0 y(v_0)/b$ we get $x(v) \in H^2$ for any $y \in H^2$ and this solution of the equation $(K_0(c,b) - \lambda I)x = y$ is unique.
(3) The denominator $cv^{p+1} - \lambda v + b$ has only one zero $v_0$ in the disk $\{|v| \le 1\}$ and this zero has modulus one. In this case the equation $(K_0(c,b) - \lambda I)x = y$ has no solution in $H^2$ for some $y(v) \in H^2$ (it's not possible to choose the appropriate value for $x(0)$).
(4) The denominator $cv^{p+1} - \lambda v + b$ has more than one zero in the disk $\{|v| \le 1\}$. In this case, for $y(v) \equiv 1$, there is no solution of the equation $(K_0(c,b) - \lambda I)x = y$ analytic in the unit disk.

The above analysis shows that the point $\lambda$ is in the resolvent set of the operator $K_0(c,b)$ if and only if one zero of the equation $cv^{p+1} - \lambda v + b = 0$ is inside the unit circle and all other zeros are outside the unit circle. The lemma is proved. □

The next step in the proof of Theorem 6.1 is to describe the intersection of spectrums of the family of operators $L_0(c,b)$ (see Figure 4).

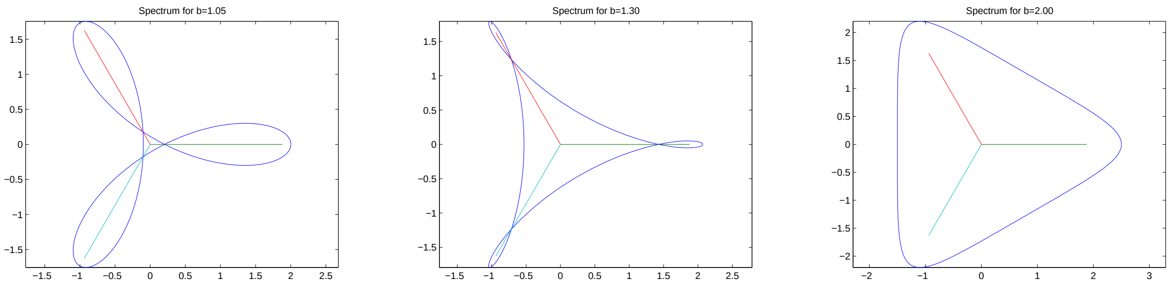


FIGURE 4. Boundary of the spectrum of operator $L_0(c,b)$ for $p = 2$, $cb^p = 1$ , and $b = 1.05,\ 1.30,\ 2.00$

**Lemma 6.2.** *If the parameters $c$ and $b$ of the operator* (45) *satisfy the relation $cb^p = 1$, then the intersection of the spectrums of the operators $L_0(c,b)$ for $b > 1$ is the set $S_0$.*

*Proof.* Under the condition $cb^p = 1$ the equation $cv^{p+1} - \lambda v + b = 0$ becomes $(v/b)^{p+1} - \lambda(v/b) + 1 = 0$. Consider the algebraic function $V$ defined by the equation

$$V^{p+1} - \lambda V + 1 = 0.$$

The branches of $V$ are given by $V_j(\lambda) = 1/w_j(\lambda)$, where the $w_j$'s are given in Proposition 1 of Section 5.1. Thus, $V_0(\lambda) \to 0$ as $\lambda \to \infty$, $V_0$ is analytic in the domain $\Omega_0 := \mathbb{C} \backslash S_0$, $|V_0(\lambda)| < 1$ for $\lambda \in \Omega_0$ and $|V_0(\lambda)| < |V_j(\lambda)|$, $\lambda \in \Omega_0$ for $j = 1, 2, \ldots, p$.

For fixed $c$ and $b > 0$ with $cb^p = 1$ we have the following relation for the algebraic functions $v(\lambda)$ and $V(\lambda)$: $v_j(\lambda) = bV_j(\lambda)$. Thus the part of the resolvent set of the operator $L_0(c,b)$ that lies in $\Omega_0$ is given by

$$G_0(c,b) = \{\lambda \in \Omega_0 \ : \ |V_0(\lambda)| < \frac{1}{b} < |V_j(\lambda)|,\ j = 1, \ldots, p\}.$$

The union of all these sets for $b > 1$ is exactly the set where $|V_0| < |V_j|$, $j = 1, 2, \ldots, p$, that is, $\Omega_0$. □

**Conclusion of Proof of Theorem 6.1.** For the operator $L$ we consider the family of operators

$$L(c,b) := \begin{pmatrix} 0 & b & 0 & 0 & 0 & \ldots & \ldots \\ 0 & 0 & b & 0 & 0 & \ldots & \ldots \\ 0 & 0 & 0 & b & 0 & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_1 \cdot c & 0 & 0 & 0 & \ldots & \ldots & \ldots \\ 0 & a_2 \cdot c & 0 & 0 & 0 & \ldots & \ldots \\ 0 & 0 & a_3 \cdot c & 0 & 0 & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}$$

with additional condition $cb^p = 1$. The operator $L(c, b)$ is then a compact perturbation of the operator $L_0(c, b)$ for the same parameters $c$, $b$. The Weyl functions of the operator $L(c, b)$ are closely related with the Weyl functions of the operator $L$ in (19). Indeed, the VCF for the system of Weyl functions of the operator $L(c, b)$ is the following (see (24)):

$$\frac{(1,1,\ldots,1)|}{|(0,0,\ldots,0,z)} + \frac{(b,1,\ldots,1)|}{|(0,0,\ldots,0,z)} + \ldots + \frac{(b^{p-1},1,\ldots,1)|}{|(0,0,\ldots,z)} +$$

$$+ \frac{(-a_1,1,\ldots,1)|}{|(0,0,\ldots,z)} + \frac{(-a_2,1,\ldots,1)|}{|(0,0,\ldots,z)} + \ldots .$$

Here we used the relations $b_{n,n-p} = -(h_{n-p}/h_n)a_{n,n-p} = -b^p c a_{n-p+1} = -a_{n-p+1}$. By Remark 4.1 we deduce the following relations between the Weyl functions

$$f_j[L(c,b)] = b^{j-1} f_j[L], \quad j = 1, 2, \ldots, p \,.$$

The same relations are true for the Weyl functions of the operators $L_0(c, b)$ and $L_0$. The support of measures $\mu_j$ is situated on the starlike set $S_0$ and all Weyl functions are analytic outside the support. Consequently, the resolvent sets of the operators $L(c, b)$ do not include the support of Weyl functions. By the Weyl perturbation theorem for bounded operators we conclude that the essential spectrum of the operator $L(c, b)$ is the same as for $L_0(c, b)$. Thus the intersections of essential spectrums of $L(c, b)$ for $b > 1$ is the same as for $L_0(c, b)$. This implies (cf. Lemmas 5.1, 6.1 - 6.2 ) that the intersection is $S_0$. The support of the measures $\mu_j$ is then the set $S_0$ plus a countable number of mass points. These points can be situated (due to Theorem 1.1) only on the starlike set $S$, and the theorem follows. □

## 7. $l^1$ PERTURBATION CLASS

7.1. **Comparison equation.** Our goal here is to establish a connection between the polynomials $U_n$ defined by (39) and the perturbed polynomials $Q_n$ defined by (1)-(2) (see [28] for the case $p = 1$). As before, $L$ denotes the operator associated with the $Q_n$'s.

**Theorem 7.1.** *The following comparison equation holds:*

$$Q_n = U_n + \sum_{k=0}^{n-1} (1 - a_{k+1}) U_{n-p-1-k} Q_k, \tag{46}$$

*where* $U_{-1} = U_{-2} = U_{-3} = \cdots = U_{-p} = 0$.

*Proof.* To simplify the calculations we put $Q_{-j} = a_{-j} = 0$, $j = 1, 2, \ldots, p$. First we have

$$zQ_k = Q_{k+1} + a_{k-p+1}Q_{k-p}, \quad k \geq 0\,,$$

$$zU_{n-p-k} = U_{n-p-k+1} + U_{n-2p-k}, \quad k \geq 0\,.$$

Then we multiply the first equation by $U_{n-p-k}$, the second by $Q_k$ and take the difference. We get as the result

$$U_{n-p+1-k}Q_k + U_{n-2p-k}Q_k = U_{n-p-k}Q_{k+1} + a_{k-p+1}U_{n-p-k}Q_{k-p}, \quad k \geq 0.$$

Writing this relation for $k = 0, 1, 2, \ldots, n-p$ we have

$$\begin{array}{rcl}
U_{n-p+1}Q_0 + U_{n-2p}Q_0 & = & U_{n-p}Q_1 + a_{-p+1}U_{n-p}Q_{-p} \\
U_{n-p}Q_1 + U_{n-2p-1}Q_1 & = & U_{n-p-1}Q_2 + a_{-p+2}U_{n-p-1}Q_{-p+1} \\
U_{n-p-1}Q_2 + U_{n-2p-2}Q_2 & = & U_{n-p-2}Q_3 + a_{-p+3}U_{n-p-2}Q_{-p+2} \\
\cdots\cdots\cdots\cdots & \cdots & \cdots\cdots\cdots\cdots \\
U_{n-2p+1}Q_p + U_{n-3p}Q_p & = & U_{n-2p}Q_{p+1} + a_1U_{n-2p}Q_0 \\
\cdots\cdots\cdots\cdots & \cdots & \cdots\cdots\cdots\cdots \\
U_1Q_{n-p} + U_{-p}Q_{n-p} & = & U_0Q_{n-p+1} + a_{n-2p+1}U_0Q_{n-2p}
\end{array}.$$

Summing these relations we get (taking into account cancellations)

$$Q_{n-p+1} = U_{n-p+1} + \sum_{k=0}^{n-2p}(1 - a_{k+1})U_{n-2p-k}Q_k.$$

Writing $n$ in place of $n-p+1$ we get

$$Q_n = U_n + \sum_{k=0}^{n-p-1}(1 - a_{k+1})U_{n-p-1-k}Q_k = U_n + \sum_{k=0}^{n-1}(1 - a_{k+1})U_{n-p-1-k}Q_k.$$

and the theorem follows. □

7.2. **Bounds for polynomials $Q_n$.** In this subsection we find bounds for $Q_n(z)$ in $\Omega_0$ and on $S_0$.

**Lemma 7.1.** *The following bounds for* $Q_n(z)$ *hold on* $\Omega_0$ :

$$|\Delta(z)| \cdot |Q_n(z)| \cdot |w_0(z)|^{-n-p} \leq C \cdot \exp\left(\frac{C}{|\Delta(z)||w_0(z)|}\sum_{k=0}^{n-1}|1 - a_{k+1}|\right), \quad z \in \Omega_0,$$

*where* $\Delta(z)$ *is defined in* (44) *and* $C$ *is the constant from the Lemma 5.2.*

*Proof.* From the comparison equation (46), we have

$$\frac{\Delta Q_n w_0^{-n-p}}{C} = \frac{\Delta U_n w_0^{-n-p}}{C} + \sum_{k=0}^{n-1}\frac{C}{\Delta w_0}(1 - a_{k+1})\frac{\Delta U_{n-p-k-1}w_0^{-n+k+1}}{C}\frac{\Delta Q_k w_0^{-k-p}}{C}.$$

For given $z \in \Omega_0$ put

$$r_k := \frac{1}{C}|\Delta(z)| \cdot |Q_k(z)| \cdot |w_0(z)|^{-n-p}, \quad d_k := \frac{C}{|\Delta(z)||w_0(z)|}|1 - a_{k+1}|, \quad k \geq 0.$$

Then we get from Lemma 5.2 that

$$r_n \leq 1 + \sum_{k=0}^{n-1} d_k \cdot r_k.$$

Applying Gronwall's inequality yields

$$r_n \leq 1 \cdot \exp(\sum_{k=0}^{n-1} d_k),$$

and the lemma follows. □

In precisely the same way we deduce

**Lemma 7.2.** *There holds on $S_0 \setminus \mathcal{A}$*

$$|Q_n(t)| \cdot |w_0(t)|^{-n} \leq M(n+1) \exp\left( \frac{M}{|w_0(t)|^{p+1}} \sum_{k=0}^{n-1} (k+1)|1 - a_{k+1}| \right),$$

*where $M$ is the constant from Lemma 5.3.*

7.3. $l^1$ **perturbation. Asymptotics of $Q_n$.** Suppose now that

$$\sum_{k=0}^{\infty} |1 - a_{k+1}| < +\infty \tag{47}$$

and for $z \in \Omega_0$ define the function

$$\Phi_0(z) := 1 + \sum_{k=0}^{\infty} \frac{1 - a_{k+1}}{w_0^{p+1}} \cdot \frac{Q_k(z)}{w_0^k(z)}. \tag{48}$$

This series converges (Lemma 7.1) on a compact subsets of $\Omega_0 = \mathbb{C} \setminus S_0$ and on the both sides of $S_0 \setminus \mathcal{A}$ (Lemma 7.2). We now describe the asymptotics of $Q_n$ in the domain $\Omega_0$.

**Theorem 7.2.** *For the polynomials $Q_n(z)$ defined by* (1)-(2) *with coefficients satisfying* (47)*, we have*

$$\lim_{n\to\infty} \frac{Q_n(z)}{w_0^n(z)} = A_0(z)\Phi_0(z),$$

*uniformly on compact sets in $\Omega_0$, where*

$$A_0(z) := \frac{w_0(z)^p}{\prod_{k=1}^{p}(w_0 - w_k)}. \tag{49}$$

*Proof.* From the comparison equation (46) we have

$$\frac{Q_n}{w_0^n} = \frac{U_n}{w_0^n} + \sum_{k=0}^{n-1} \frac{1 - a_{k+1}}{w_0^{p+1}} \cdot \frac{U_{n-p-k-1}}{w_0^{n-p-k-1}} \cdot \frac{Q_k}{w_0^k}.$$

Since (cf. (41))

$$U_n(z)/w_0(z)^n \to A_0(z), \quad n \to \infty,$$

uniformly on a compact subsets of $\Omega_0$, the Lebesgue dominated convergence theorem implies that

$$\frac{Q_n(z)}{w_0^n(z)} \to A_0(z)\Phi_0(z),$$

uniformly on compact subsets of $\Omega_0$. □

7.4. $l^1$ **perturbation. Asymptotics of Hermite-Padé polynomials.** Again we assume that condition (47) is satisfied and we consider the Hermite-Padé approximants to the vector of Weyl functions of the operator $L$. Recall that $Q_n$ is the common denominator of the Hermite-Padé approximants and the numerators of the approximants are $P_n^{(j)}$, $j = 1, 2, \ldots, p$ which are defined by the same recurrences

$$Y_{n+1} = zY_n - a_{n-p+1}Y_{n-p}, \quad n \geq p \ ,$$

with initial conditions

$$P_n^{(j)} = 0, \ 0 \leq n < j; \quad P_n^{(j)} = z^{n-j}, \ j \leq n \leq p.$$

The comparison equation for $Q_n$ holds also for the polynomials $P_{n+j}^{(j)}$ but with $a_{k+1}$ replaced by $a_{k+j+1}$ :

$$P_{n+j}^{(j)} = U_n + \sum_{k=0}^{n-1}(1 - a_{k+j+1})U_{n-p-1-k}P_{k+j}^{(j)},$$

or what is the same (taking into account initial conditions)

$$P_n^{(j)} = U_{n-j} + \sum_{k=0}^{n-1}(1 - a_{k+1})U_{n-p-1-k}P_k^{(j)}.$$

Proceeding in the same way as we obtained bounds for the polynomials $Q_n$ we can obtain estimates for the polynomials $P_n^{(j)}$.

**Lemma 7.3.** *The following bounds for $P_n^{(j)}(z)$ hold on $\Omega_0$ :*

$$|\Delta(z)| \cdot |P_n^{(j)}(z)| \cdot |w_0(z)|^{-n+j-p} \leq C \cdot \exp\left(\frac{C}{|\Delta(z)||w_0(z)|}\sum_{k=0}^{n-1}|1 - a_{k+1}|\right), \quad z \in \Omega_0,$$

*and, on $S_0 \setminus \mathcal{A}$,*

$$|P_n^{(j)}(t)| \cdot |w_0(t)|^{-n+j} \leq M(n+1)\exp\left(\frac{M}{|w_0(t)|^{p+1}}\sum_{k=0}^{n-1}(k+1)|1 - a_{k+1}|\right).$$

These bounds enable us to determine the asymptotic behavior of the Hermite-Padé polynomials $P_n^{(j)}$. For $z \in \Omega_0$ we define the function

$$\Phi_j(z) := 1 + \sum_{k=0}^{\infty}\frac{1 - a_{k+1}}{w_0^{p+1}} \cdot \frac{P_k^{(j)}(z)}{w_0^{k-j}(z)}, \quad j = 1, 2, \ldots, p \ . \tag{50}$$

All series converge (Lemma 7.3) on a compact subsets of $\Omega_0$. In the same way as for $Q_n(z)$ we get

**Theorem 7.3.** *For the numerators of the Hermite-Padé approximants* $P_n^{(j)}$, $j = 1, 2, \ldots, p$, *of the Weyl functions of the operator* $L$ *satisfying condition* (47) *we have*

$$\lim_{n\to\infty} \frac{P_n^{(j)}(z)}{w_0^{n-j}(z)} = A_0(z)\Phi_j(z),$$

*uniformly on compact sets in* $\Omega_0$, *where* $A_0(z)$ *is given in* (49).

7.5. **Weyl functions and spectral measures.** For the $l^1$ perturbation class we now are able to find the Weyl functions $f_j(z)$ of (17) for the associated operator $L$ given by the matrix (19) and use them to determine the measures of orthogonality $\mu_j$, $j = 1, 2, \ldots, p$ (see (7)).

**Theorem 7.4.** *Suppose* (47) *is satisfied. Then for the Weyl functions of the operator* $L$ *the following representation holds for* $z \in \Omega_0$ :

$$f_j(z) = \lim_{n\to\infty} \frac{P_n^{(j)}(z)}{Q_n(z)} = \frac{1}{w_0^j} \cdot \frac{\Phi_j}{\Phi_0}, \quad j = 1, 2, \ldots, p. \tag{51}$$

*Proof.* The theorem follows immediately from the asymptotic formulas for the Hermite-Padé polynomials $Q_n$ and $P_n^{(j)}$ (see Theorems 7.2 and 7.3). $\square$

To find the asymptotics of polynomials $Q_n$ on $S_0$, we first establish the following important relation.

**Lemma 7.4.** *For* $n \geq 0$ *there holds*

$$U_n - (\sum_{i=1}^{p} w_i)U_{n-1} + (\sum_{0<i<j}^{p} w_i w_j)U_{n-2} + \cdots + (-1)^p(\prod_{i=1}^{p} w_i)U_{n-p} = w_0^n. \tag{52}$$

*Proof.* Putting $\mathbf{V} := (-1)^{p(p+1)/2} \prod_{0<i<j}(w_i - w_j)$, we have from (41) and (49)

$$\mathbf{V}[U_n - (\sum_{i=1}^{p} w_i)U_{n-1} + \cdots + (-1)^p(\prod_{i=1}^{p} w_i)U_{n-p}] =$$

$$\begin{vmatrix} U_{n-p} & 1 & \ldots & 1 \\ U_{n-p+1} & w_1 & \ldots & w_p \\ \ldots & \ldots & \ldots & \ldots \\ U_n & w_1^p & \ldots & w_p^p \end{vmatrix} = \begin{vmatrix} A_0 w_0^{n-p} & 1 & \ldots & 1 \\ A_0 w_0^{n-p+1} & w_1 & \ldots & w_p \\ \ldots & \ldots & \ldots & \ldots \\ A_0 w_0^n & w_1^p & \ldots & w_p^p \end{vmatrix} = A_0 w_0^{n-p} \begin{vmatrix} 1 & 1 & \ldots & 1 \\ w_0 & w_1 & \ldots & w_p \\ \ldots & \ldots & \ldots & \ldots \\ w_0^p & w_1^p & \ldots & w_p^p \end{vmatrix}$$

$$= (-1)^{p(p+1)/2} w_0^{n-p} \frac{w_0^p}{\prod_{k=1}^p (w_0 - w_k)} \prod_{0\leq i<j} (w_i - w_j) = w_0^n \mathbf{V}.$$

This implies the relation (52). $\square$

Formula (52) implies the following lemma which we will use to obtain asymptotics for the polynomials $Q_n$ on $S_0$.

**Lemma 7.5.** *Suppose* (47) *is satisfied. Then the following limit holds uniformly on the open subintervals of $S_0$ (on the both sides of $S_0$):*

$$\lim_{n\to\infty}\frac{1}{w_0(t)^n}\left[Q_n(t)-(\sum_{i=1}^{p}w_i)Q_{n-1}(t)\cdots+(-1)^p(\prod_{i=1}^{p}w_i)Q_{n-p}(t)\right]=\Phi_0(t). \tag{53}$$

*Proof.* We substitute the comparison equation

$$Q_m=U_m+\sum_{k=0}^{m-1}(1-a_{k+1})U_{m-p-1-k}Q_k,\quad m\geq 0$$

into the left-hand side of (53). Using (52) we get

$$\frac{1}{w_0(t)^n}\left[Q_n(t)-(\sum_{i=1}^{p}w_i)Q_{n-1}(t)\cdots+(-1)^p(\prod_{i=1}^{p}w_i)Q_{n-p}(t)\right]=$$
$$\frac{1}{w_0(t)^n}\left[w_0^n+\sum_{k=0}^{n-1}(1-a_{k+1})w_0^{n-p-1-k}Q_k\right]=1+\sum_{k=0}^{n-1}\frac{1-a_{k+1}}{w_0^{p+1}}\frac{Q_k}{w_0^k}.$$

This sequence converges to $\Phi_0(t)$ on the both sides of $S_0$ (see (48)) and the lemma is proved. ◻

**Theorem 7.5.** *If* (47) *is satisfied, then as $n\to\infty$,*

$$\frac{Q_n(t)}{|w_0(t)|^n}=\left(\frac{w_{0,+}(t)}{|w_0(t)|}\right)^n F_{0,+}(t)+\left(\frac{w_{0,-}(t)}{|w_0(t)|}\right)^n F_{0,-}(t)+o(1),$$

*uniformly on a compact subsets of $S_0\setminus\mathcal{A}$, where*

$$F_0(z):=A_0(z)\Phi_0(z).$$

*Proof.* : Introducing the notation $\Phi_{0,n}$ :

$$Q_n-\left(\sum_{i=1}^{p}w_i\right)Q_{n-1}+\cdots+(-1)^p\left(\prod_{i=1}^{p}w_i\right)Q_{n-p}=:w_0^n\Phi_{0,n},$$

we have from (53)

$$\Phi_{0,n}\to\Phi_0,$$

uniformly on a compact subsets of both sides of $S_0$ (except extreme points).
Setting

$$V_n:=Q_n-\left(\sum_{i=2}^{p}w_i\right)Q_{n-1}+\cdots+(-1)^{p-1}\left(\prod_{i=2}^{p}w_i\right)Q_{n-p+1},$$

we have the identity

$$V_n-w_1V_{n-1}=w_0^n\Phi_{0,n}.$$

From the definition we note an important relation:

$$V_{n,+}=V_{n,-}\quad\text{on } S_0,$$

which implies that on $S_0$

$$\begin{cases} V_n-w_{1,+}V_{n-1}=w_{0,+}^n\Phi_{0,n,+} \\ V_n-w_{1,-}V_{n-1}=w_{0,-}^n\Phi_{0,n,-}\end{cases}.$$

From this and taking into account that

$$w_{0,-} = w_{1,+}; \quad w_{1,+} = w_{0,-}; \quad w_{0,+} = w_{1,-}; \quad w_{1,-} = w_{0,+} \quad \text{on } S_0,$$

we get

$$V_n = w_{0,+} \frac{w_{0,+}^n \Phi_{0,n,+}}{w_{0,+} - w_{1,+}} + w_{0,-} \frac{w_{0,-}^n \Phi_{0,n,-}}{w_{0,-} - w_{1,-}} .$$

Put $w_n = \dfrac{V_n}{|w_0|^n}$, $u_n = \dfrac{Q_n}{|w_0|^n}$. Then

$$w_n = u_n - \left( \sum_{i=2}^{p} \frac{w_i}{|w_0|} \right) u_{n-1} + \cdots + (-1)^{p-1} \prod_{i=2}^{p} \left( \frac{w_i}{|w_0|} \right) u_{n-p+1} =$$

$$= \left( \frac{w_{0,+}}{|w_0|} \right)^n \left( \frac{w_0 \Phi_0}{w_0 - w_1} \right)_+ + \left( \frac{w_{0,-}}{|w_0|} \right)^n \left( \frac{w_0 \Phi_0}{w_0 - w_1} \right)_- + \gamma_n(x) ,$$

where $\gamma_n(x) \to 0$ uniformly on a compact subsets of $S_0$ (except extreme points). To complete the proof we need the following.

**Lemma 7.6.** *Consider the linear recurrence equation with constant coefficients*

$$y_n + d_1 y_{n-1} + \cdots + d_s y_{n-s} = c \cdot b^n + \gamma_n , \tag{54}$$

*where* $d_i, c, b$ *are constants,* $|b| = 1$, $c \neq 0$, $\gamma_n \to 0$, *as* $n \to \infty$ *and all roots of the characteristic equation*

$$\lambda^s + d_1 \lambda^{s-1} + \cdots + d_s = 0$$

*are simple and satisfy* $|\lambda_j| < 1$, $j = 1, 2, \ldots, s$. *Then every solution to ((54)) satisfies*

$$y_n = \frac{c \cdot b^s}{\prod_{j=1}^{s} (b - \lambda_j)} \cdot b^n + o(1) , \quad \text{as } n \to \infty .$$

*Proof.* Any solution of the recurrence equation can be written in the form

$$y_n = c_1(n) \lambda_1^n + c_2(n) \lambda_2^n + \cdots + c_s(n) \lambda_s^n , \quad n \geqslant 0 ,$$

where $c_1(n)$, $c_2(n), \ldots, c_s(n)$ are defined by the system

$$\begin{cases} \lambda_1^{n+1} \triangle c_1 + \lambda_2^{n+1} \triangle c_2 + \cdots + \lambda_s^{n+1} \triangle c_s = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \lambda_1^{n+s-1} \triangle c_1 + \lambda_2^{n+s-1} \triangle c_2 + \cdots + \lambda_s^{n+s-1} \triangle c_s = 0 \\ \lambda_1^{n+s} \triangle c_1 + \lambda_2^{n+s} \triangle c_2 + \cdots + \lambda_s^{n+s} \triangle c_s = c \cdot b^{n+s} + \gamma_n \end{cases} ,$$

where we have introduced the notation $\triangle c(n) := c(n+1) - c(n)$. The solution of this system of equations for $\triangle c_j(n)$ is

$$\triangle c_j(n) = \frac{1}{\lambda_j^{n+1}} (-1)^{s+j} \frac{\mathbf{V}_j(\lambda_1, \lambda_2, \ldots, \lambda_s)}{\mathbf{V}(\lambda_1, \lambda_2, \ldots, \lambda_s)} \left( c b^{n+s} + \gamma_n \right) , \quad j = 1, 2, \ldots, s ,$$

with

$$\mathbf{V}(\lambda_1,\, \lambda_2, \dots, \lambda_s) := \begin{vmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \dots & \lambda_s \\ \dots & \dots & \dots & \dots \\ \lambda_1^{s-1} & \lambda_2^{s-1} & \dots & \lambda_s^{s-1} \end{vmatrix},$$

and $\mathbf{V}_j(\lambda_1,\, \lambda_2, \dots, \lambda_s)$ is the same determinant but with the last row and column $j$ deleted. Summing we have

$$c_j(n) = c_j(0) + \sum_{k=0}^{n-1} \big[c_j(k+1) - c_j(k)\big] =$$

$$= c_j(0) + (-1)^{s+j} \frac{\mathbf{V}_j(\lambda_1,\, \lambda_2, \dots, \lambda_s)}{\mathbf{V}(\lambda_1,\, \lambda_2, \dots, \lambda_s)} \left[ \frac{cb^s}{\lambda_j} \cdot \sum_{k=0}^{n-1} \left(\frac{b}{\lambda_j}\right)^k + \sum_{k=0}^{n-1} \frac{\gamma_k}{\lambda_j^{k+1}} \right] .$$

Thus, as $n \to \infty$,

$$y_n = \sum_{j=1}^{s} \frac{\mathbf{V}_j(\lambda_1,\, \lambda_2, \dots, \lambda_s)}{\mathbf{V}(\lambda_1,\, \lambda_2, \dots, \lambda_s)} (-1)^{s+j} \left[ \frac{cb^s}{b-\lambda_j} \cdot \left( \frac{b^n}{\lambda_j^n} - 1 \right) + \sum_{k=0}^{n-1} \frac{\gamma_k}{\lambda_j^{k+1}} \right] \lambda_j^n + \sum_{j=1}^{s} c_j(0) \lambda_j^n =$$

$$= \sum_{j=1}^{s} \frac{1}{\prod\limits_{\substack{k=1\\k\neq j}}^{s} (\lambda_j - \lambda_k)} \cdot \frac{cb^s}{(b-\lambda_j)} \cdot b^n + o(1) ,$$

because $c_j(0)\lambda_j^n \to 0$ and (since $\gamma_k \to 0$ and $|\lambda_j| < 1$)

$$\sum_{k=1}^{n} \gamma_k \lambda_j^{n-k} \to 0.$$

Now consider the sum

$$\sum_{j=1}^{s} \frac{1}{\prod\limits_{\substack{k=1\\k\neq j}}^{s} (\lambda_j - \lambda_k)} \cdot \frac{1}{(b-\lambda_j)} ,$$

which is a rational function in the variable $b$ with poles at the $\lambda_j$'s. The numerator of this function is polynomial in $b$ of degree $(s-1)$ and one can easy check that it has the same value at any point $\lambda_j$, $j = 1, 2, \dots, s$. Thus

$$\sum_{j=1}^{s} \frac{1}{\prod\limits_{\substack{k=1\\k\neq j}}^{s} (\lambda_j - \lambda_k)} \cdot \frac{1}{(b-\lambda_j)} = \frac{1}{\prod_{j=1}^{s} (b-\lambda_j)},$$

and the lemma follows. □

To complete the proof of Theorem 7.5, we simply apply Lemma 7.6 with

$$y_n := u_n,\ s := p-1,\ c := \frac{w_0 \Phi_0}{w_0 - w_1},\ b := \frac{w_0}{|w_0|},\ \gamma_n := \gamma_n(x),\ \lambda_j := \frac{w_{j+1}}{|w_0|},\ j = \overline{1, p-1} .$$

□

We conclude this section with a description of the measures of orthogonality.

**Corollary 7.1.** *Suppose (47) is satisfied. Then, on $S_0 \setminus \mathcal{A}$, all measures $\mu_j$ from (7) are absolutely continuous $d\mu_j(t) = \rho_j(t)|dt|$, $t \in S_0 \setminus \mathcal{A}$, where the weight functions $\rho_j(t)$ are given by*

$$\rho_j(t) = \left[\frac{1}{w_0(t)^j}\frac{\Phi_j(t)}{\Phi_0(t)}\right]^+ - \left[\frac{1}{w_0(t)^j}\frac{\Phi_j(t)}{\Phi_0(t)}\right]^- , \quad j = 1, 2, \ldots, p\ .$$

## 8. Analytic structure of the Weyl functions and their Hermite-Padé approximants for the operator with constant coefficients.

In this section we show that vector of Weyl functions (43) for the operator with constant coefficients (40) has features of the so-called *Nikishin system.* The Nikishin system is one of the model systems of functions with well understood asymptotic behavior of their Hermite-Padé approximants (see [23], [24], [13], [8], [9], [5], [1]).

8.1. **Analytic continuation of the jumps of the Weyl functions for the operator with constant coefficients.** We take the Riemann surface $\mathfrak{R}$ defined in (36) and using the notation (37) we define star-like contours

$$S_{j-1} := \pi(\partial\mathfrak{R}_{j-1,j})\ , \quad j = 1, \ldots, p. \tag{55}$$

Let $\{\sigma_j\}_{j=1}^p$ be a system of locally integrable functions on $\{S_{j-1}\}_{j=1}^p$, respectively, which are non-vanishing and have constant argument, so that

$$\operatorname{supp}\sigma_j = S_{j-1}\ , \quad j = 1, \ldots, p.$$

We say that the system of functions

$$\{f_j\}\ , \quad f_j \in H(\overline{\mathbb{C}}\backslash S_0)\ , \quad j = 1, \ldots, p\ , \tag{56}$$

forms a *Nikishin system with respect to the system of weights $\{\sigma_k\}_{k=1}^p$ on $\{S_{k-1}\}_{k=1}^p$*, if $\{f_j\}$ have locally integrable boundary values on $S_0$, such that

$$f_{j+} - f_{j-}\Big|_{S_0} = \sigma_1 \cdot f_j^{(2)}\ , \quad j = 1, \ldots, p\ ,$$

where

$$f_1^{(2)} \equiv 1\ , \quad f_j^{(2)} \in H(\overline{\mathbb{C}}\backslash S_1)\ , \quad j = 2, \ldots, p\ ,$$

and the functions $\{f_j^{(2)}\}_{j=2}^p$ in their turn form the Nikishin system with respect to $\{\sigma_k\}_{k=2}^p$ on $\{S_{k-1}\}_{k=2}^p$.

We see that the Nikishin system starting from $f_j^{(1)} := f_j\ ,\ j = 1, \ldots, p$ , defines inductively a hierarchy of analytic functions

$$\{f_j^{(k)}\}\ , \quad k = 1, \ldots, p\ , \quad j = k, \ldots, p\ , \tag{57}$$

by means of analytic continuation of their jumps

$$f_{j+}^{(k)} - f_{j-}^{(k)}\Big|_{S_{k-1}} = \sigma_k \cdot f_j^{(k+1)}\ , \tag{58}$$

from $S_{k-1}$ to the whole domain of their holomorphicity :

$$f_k^{(k+1)} \equiv 1\ , \quad f_j^{(k+1)} \in H(\overline{\mathbb{C}}\backslash S_k)\ , \quad k = 1, \ldots, p\ , \quad j = k, \ldots, p\ . \tag{59}$$

We remark that we use the notion of Nikishin system with respect to the system of sets $\{S_{k-1}\}_{k=1}^p$, which includes the unbounded sets, and weight functions $\{\sigma_k\}_{k=2}^p$ that are not necessarily globally integrable on $\{S_{k-1}\}_{k=2}^{p}$. This differs from the usual definition of the Nikishin system by means of Cauchy integrals on a system of intervals of the real axis (see [23], [24], [13]).

**Theorem 8.1.** *The system of the Weyl functions for the operator with constant coefficients (see* (43)*)*

$$f_j = \frac{1}{w_0^j}\ , \quad j = 1, \ldots, p\,,$$

*forms a Nikishin system* (57) - (59) *with respect to the system of weights*

$$\sigma_k := \frac{1}{w_{k-1,+}} - \frac{1}{w_{k-1,-}} \quad on \quad S_{k-1}\ , \quad k = 1, \ldots, p\ . \tag{60}$$

*Moreover, the hierarchy of the Nikishin functions* (57) *has the form*

$$\begin{aligned}
&f_j^{(1)} = \frac{1}{w_0^j}\ , \quad f_j^{(1)} \in H(\overline{\mathbb{C}}\backslash S_0)\ , \quad j = 1, \ldots, p\,,\\
&f_j^{(2)} = \sum_{\nu_0=0}^{j-1} \frac{1}{w_0^{j-1-\nu_0} w_1^{\nu_0}}\ , \quad f_j^{(2)} \in H(\overline{\mathbb{C}}\backslash S_1)\ , \quad j = 2, \ldots, p\,,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&f_j^{(k+1)} = \sum_{\nu_0=k-1}^{j-1} \frac{1}{w_0^{j-1-\nu_0}} \sum_{\nu_1=k-2}^{\nu_0-1} \frac{1}{w_1^{\nu_0-1-\nu_1}} \cdots \sum_{\nu_{k-1}=0}^{\nu_{k-2}-1} \frac{1}{w_{k-1}^{\nu_{k-2}-1-\nu_{k-1}} w_k^{\nu_{k-1}}}\ ,\\
&f_j^{(k+1)} \in H(\overline{\mathbb{C}}\backslash S_k)\ , \quad j = k+1, \ldots, p\,,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&f_p^{(p)} = \left(\frac{1}{w_0} + \frac{1}{w_1} + \ldots + \frac{1}{w_{p-1}}\right) = -\frac{1}{w_p}\ , \quad f_p^{(p)} \in H(\overline{\mathbb{C}}\backslash S_{p-1})\ .
\end{aligned} \tag{61}$$

*Proof.* We proceed by induction. We have

$$f_{j+}^{(1)} - f_{j-}^{(1)}\Big|_{S_0} = \left(\frac{1}{w_{0+}^j} - \frac{1}{w_{0-}^j}\right) = \left(\frac{1}{w_{0+}} - \frac{1}{w_{0-}}\right) \sum_{\nu_0=0}^{j-1} \frac{1}{w_{0+}^{j-1-\nu_0} w_{1+}^{\nu_0}}\ .$$

Setting

$$\sigma_1 := \left(\frac{1}{w_{0+}} - \frac{1}{w_{0-}}\right)\ , \quad f_j^{(2)} := \sum_{\nu_0=0}^{j-1} \frac{1}{w_0^{j-1-\nu_0} w_1^{\nu_0}}\ , \quad j = 2, 3, \ldots, p\ ,$$

we see that the $f_j^{(2)}$'s are symmetric functions with respect to $w_0$ and $w_1$. Therefore the boundary values of $f_j^{(2)}$ from both sides of the cut $S_0$ are the same, and we have

$$f_j^{(2)} \in H(S_0)\ , \quad j = 2, \ldots, p\ .$$

So, we can analytically prolong the functions $f_j^{(2)}$ from $S_0$ to the complex plane wherever $w_1$ maintains its holomorphicity, i.e.

$$f_j^{(2)} \in H(\overline{\mathbb{C}}\backslash S_1)\ .$$

Thus the first step of the induction is proved.

Now suppose that (61), (58) and (60) hold for some $k$. Then

$$f_{j+}^{(k+1)} - f_{j-}^{(k+1)}\Big|_{S_k} = \sum_{\nu_0=k-1}^{j-1} \frac{1}{w_0^{j-1-\nu_0}} \cdots \sum_{\nu_{k-1}=0}^{\nu_{k-2}-1} \frac{1}{w_{k-1}^{\nu_{k-2}-1-\nu_{k-1}}} \left[\frac{1}{w_{k+}^{\nu_{k-1}}} - \frac{1}{w_{k-}^{\nu_{k-1}}}\right] . \tag{62}$$

Here we used the fact that $f_j^{(k+1)}$ is also a symmetric function with respect to the consecutive branches $w_p$ and $w_{p+1}$, $p = 0, 1, \ldots, k-2$. Moreover, we notice that the term with index $\nu_{k-1} = 0$ does not contribute to the sums of (62), which permits a shift of lower indices in all sums of (62) :

$$f_{j+}^{(k+1)} - f_{j-}^{(k+1)}\Big|_{S_k} = \sum_{\nu_0=k}^{j-1} \frac{1}{w_0^{j-1-\nu_0}} \cdots \sum_{\nu_{k-1}=1}^{\nu_{k-2}-1} \frac{1}{w_{k-1}^{\nu_{k-2}-1-\nu_{k-1}}} \left[\frac{1}{w_{k+}^{\nu_{k-1}}} - \frac{1}{w_{k-}^{\nu_{k-1}}}\right] ,$$

and using the boundary condition

$$w_{k,\pm} = w_{k+1,\mp} \quad \text{on} \quad S_k ,$$

we continue

$$f_{j+}^{(k+1)} - f_{j-}^{(k+1)}\Big|_{S_k} = \left(\frac{1}{w_{k+}} - \frac{1}{w_{k-}}\right) \sum_{\nu_0=k}^{j-1} \frac{1}{w_0^{j-1-\nu_0}} \cdots \sum_{\nu_{k-1}=1}^{\nu_{k-2}-1} \frac{1}{w_{k-1}^{\nu_{k-2}-1-\nu_{k-1}}} \sum_{\nu_k=0}^{\nu_{k-1}-1} \frac{1}{w_{k,+}^{\nu_{k-1}-1-\nu_k} w_{k+1,+}^{\nu_k}} .$$

Thus for the index $k+1$ we obtained the boundary condition (58) and the expressions for $\sigma_{k+1}$ as in (60) and for $f_j^{(k+2)}$, $j = k+2, \ldots, p$ as in (61). Analyticity of $f_j^{(k+2)}$ on $S_k$ and the domain of holomorphicity for $f_j^{(k+2)} \in H(\overline{\mathbb{C}} \setminus S_{k+1})$ is checked in the same way as in the first step of the induction, i.e. using the symmetry of $f_j^{(k+2)}$ with respect to consecutive branches of $w$. □

8.2. **Hierarchy of the functions of the second kind for H-P approximants.** Following the asymptotic theory of Hermite-Padé approximants for Nikishin systems (see, for example, [24], [13]), we inductively define (for each fixed $n \in \mathbb{N}$) a system of functions $\{\Psi_n^{(k)}\}_{k=0}^p$ by means of the following Riemann-Hilbert BVP problems :

$$\begin{cases} \Psi_n^{(k)} \in H(\overline{\mathbb{C}} \setminus S_{k-1}) \\ \Psi_{n+}^{(k)} - \Psi_{n-}^{(k)}\Big|_{S_{k-1}} = \sigma_k \Psi_n^{(k-1)} \\ \Psi_n^{(k)}(z) = O\left(z^{-\frac{n+p-k+1}{p}}\right) , \quad z \to \infty \end{cases} , \quad k = 1, \ldots, p . \tag{63}$$

Here $\{\sigma_k\}_{k=1}^p$ are the weight functions (60) on $\{S_{k-1}\}_{k=1}^p$, forming the Nikishin system (56) of the Weyl functions (43) for the operator with the constant coefficients. The initial function for the system $\{\Psi_n^{(k)}\}$ is chosen as

$$\Psi_n^{(0)} = Q_n , \tag{64}$$

where $Q_n$ is the common denominator of the H-P approximants.
We note that, for $k = 1$,

$$\begin{aligned} &\sigma_1 = \rho_1 \quad \text{on} \quad S_0 \\ &\Psi_n^{(1)} = R_n^{(1)} = Q_n f_1 - P_n^{(1)} \ . \end{aligned} \tag{65}$$

In our case, since the polynomials

$$Q_n = U_n$$

are defined by means of the recurrence relations (39) with constant coefficients, we can express the system $\{\Psi_n^{(k)}\}$ explicitly. We have

**Theorem 8.2.** *For the system* $\{\sigma_k\}_{k=1}^p$ *of the weight functions* (60)*, the solutions of the Riemann-Hilbert problems* (63) - (64) *have a form:*

$$\begin{aligned} &\Psi_n^{(0)} := U_n = \sum_{i=0}^{p} w_i^{n+p} \frac{1}{\prod\limits_{\substack{k=0\\k\neq i}}^{p} (w_i - w_k)} \ , \\ &\Psi_n^{(1)} := \frac{1}{w_0} \sum_{i=1}^{p} w_i^{n-p+1} \frac{1}{\prod\limits_{\substack{k=1\\k\neq i}}^{p} (w_i - w_k)} \ , \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\Psi_n^{(l)} := \frac{1}{\prod\limits_{j=0}^{l-1} w_j} \sum_{i=l}^{p} w_i^{n+p-l} \frac{1}{\prod\limits_{\substack{k=l\\k\neq i}}^{p} (w_i - w_k)} \ , \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\Psi_n^{(p)} = \frac{1}{w_0 \ldots w_{p-1}} w_p^n = (-1)^{p+1} w_p^{n+1}. \end{aligned} \tag{66}$$

*Proof.* We have (see (65) and (42))

$$\Psi_n^{(1)} = \Psi_n^{(0)} \frac{1}{w_0} - \Psi_{n-1}^{(0)} \ .$$

We define $\Psi_n^{(l)}$, $l = 1, 2, \ldots, p$, inductively by the same formula, just shifting indices :

$$\Psi_n^{(l)} := \Psi_n^{(l-1)} \frac{1}{w_{l-1}} - \Psi_{n-1}^{(l-1)} \ , \tag{67}$$

and first we prove that for $\Psi_n^{(l)}$ defined by (67) we get the expressions (66). We proceed by induction. For $l = 1$ we have

$$\Psi_n^{(1)} := \frac{1}{w_0}\sum_{i=0}^{p} w_i^{n+p}\frac{1}{\prod\limits_{\substack{k=0\\k\neq i}}^{p}(w_i - w_k)} - \sum_{i=0}^{p} w_i^{n+p-1}\frac{1}{\prod\limits_{\substack{k=0\\k\neq i}}^{p}(w_i - w_k)} =$$

$$= \frac{1}{w_0}\sum_{i=1}^{p} w_i^{n+p}\frac{1}{\prod\limits_{\substack{k=0\\k\neq i}}^{p}(w_i - w_k)} - \frac{1}{w_0}\sum_{i=1}^{p} w_0 w_i^{n+p-1}\frac{1}{\prod\limits_{\substack{k=0\\k\neq i}}^{p}(w_i - w_k)} =$$

$$= \frac{1}{w_0}\sum_{i=1}^{p} w_i^{n+p-1}\frac{1}{\prod\limits_{\substack{k=1\\k\neq i}}^{p}(w_i - w_k)} .$$

Similarly we establish the induction step :

$$\Psi_n^{(l+1)} := \frac{1}{w_l\prod\limits_{j=0}^{l-1} w_j}\sum_{i=l}^{p}\frac{w_j^{n+p-l}}{\prod\limits_{\substack{k=l\\k\neq i}}^{p}(w_i - w_k)} - \frac{1}{\prod\limits_{j=0}^{l-1} w_j}\sum_{i=l}^{p}\frac{w_j^{n-1+p-l}}{\prod\limits_{\substack{k=l\\k\neq i}}^{p}(w_i - w_k)} =$$

$$= \frac{1}{\prod\limits_{j=0}^{l} w_j}\sum_{i=l+1}^{p}\frac{w_j^{n-1+p-l}}{\prod\limits_{\substack{k=l+1\\k\neq i}}^{p}(w_i - w_k)} .$$

Thus the formulas (66) are proven for $\{\Psi_n^{(l)}\}_{l=0}^{p}$ defined by (67). Now we check that these functions satisfy the system of Riemann-Hilbert problems (63). The first and the third relations in (63) follow from the representation (66) and the analytic properties of the branches $\{w_j\}_{j=0}^{p}$ of the algebraic function (11). The second relation in (63) follows from the representation (67) and from the definition (60) of the weights $\{\sigma_k\}$. □

8.3. **Asymptotics of H-P approximants and BVP for the Szegő functions.** A peculiarity of the H-P approximants for Nikishin systems is that analysis of their asymptotics (see [13]) involves not only asymptotics of polynomials denominators $Q_n$ and functions of the second kind $R_n^{(j)} = f_j Q_n - P_n^{(j)}$, $j = 1, \ldots, p$, but also the whole hierarchy of the functions of the second kind $\{\Psi_n^{(k)}\}_{k=0}^{p}$ is involved (we recall, that $\Psi_n^{(0)} = Q_n$, $\Psi_n^{(2)} = R_n^{(1)}$, but $\Psi_n^{(k)}$ for $k > 2$ have no direct meaning in terms of approximation or orthogonality). However, from the explicit formulas (66) for the system $\{\Psi_n^{(k)}\}$ we can derive asymptotic formulas

**Theorem 8.3.** *The solutions of the Riemann-Hilbert problems* (63) - (64) *for the system of the weight functions* $\{\sigma_k\}_{k=1}^{p}$ *defined in* (60) *have the following uniform asymptotic behavior* *as* $n \to \infty$ *:*

$$1)\Psi_n^{(l)} = \frac{w_l^n\ F_l}{\prod\limits_{m=0}^{l-1}(w_m - w_l)}\ (1 + O(q^n))\ , \quad \textit{on compacta of} \quad \overline{\mathbb{C}}\backslash\{S_{l-1}\cup S_l\}\ ;$$

$$2)\frac{\Psi_n^{(l)}}{|w_l|^n} = \frac{1}{|w_l|^n}\left(\frac{w_l^n\ F_l}{\prod\limits_{m=0}^{l-1}(w_m - w_l)}\right)_+ + \frac{1}{|w_l|^n}\left(\frac{w_l^n\ F_l}{\prod\limits_{m=0}^{l-1}(w_m - w_l)}\right)_- + O(q^n) \ \ \textit{on compacta of}\, S_l,$$

$$3)\Psi_{n\pm}^{(l)} = \left(\frac{w_l^n\ F_l}{\prod\limits_{m=0}^{l-1}(w_m - w_l)}\right)_\pm\ (1 + O(q^n))\ , \quad \textit{on compacta of} \quad S_{l-1}\ , \tag{68}$$

*Here* $q \in (0,1),\ l = 0,1,2,\ldots,p,$ *and the functions* $F_l$ *are given by*

$$F_l := \frac{1}{\prod\limits_{m=0}^{l-1} w_m}\frac{w_l^{p-l}\prod\limits_{m=0}^{l-1}(w_m - w_l)}{\prod\limits_{k=l+1}^{p}(w_l - w_k)}, \tag{69}$$

*and satisfy*

$$F_l \in H\left(\mathbb{C}\backslash\{S_l \cup S_{l-1}\}\right), \quad F_l(z) \underset{z\to\infty}{\to} F_l(\infty) \neq 0,\infty. \tag{70}$$

*Proof.* The asymptotic formulas 1) and 3) are direct corollaries of (66) and the ordering of the branches of the algebraic function $w$ in (29). For the formula 2) we use the fact that

$$\left(\frac{w_l}{\prod\limits_{k=l+1}^{p}(w_l - w_k)}\right)_\pm = \left(\frac{w_{l+1}}{(w_{l+1} - w_l)\prod\limits_{k=l+2}^{p}(w_{l+1} - w_k)}\right)_\mp \qquad \text{on} \quad S_l\ .$$

□

We remark that Theorem 7.3 gives asymptotics of the H-P denominators

$$U_n = \Psi_n^{(0)} = w_0^n F_0(1 + O(q^n)) \quad \text{in} \quad \overline{\mathbb{C}}\backslash S_0\ ,$$

$$\frac{U_n}{|w_0|^n} = \left(\frac{w_0}{|w_0|}\right)_+^n F_{0+} + \left(\frac{w_0}{|w_0|}\right)_-^n F_{0-} + O(q^n) \quad \text{on} \quad S_0\ ,$$

(we use in (68) the convention that a product over an empty set of indices is equal to 1), and also it gives asymptotics of the remainder function of the H-P approximants for the first function $f_1 = 1/w_0$ :

$$R_n^{(1)} = Q_n f_1 - P_n^{(1)} = U_n\frac{1}{w_0} - U_{n-1} = \Psi_n^{(1)}\ .$$

Thus we have

$$R_n^{(1)} = \frac{w_1^n}{w_0 - w_1} F_1 (1 + O(q^n)) \quad \text{in} \quad \overline{\mathbb{C}} \setminus \{S_0 \cup S_1\} \ ,$$

$$\frac{R_n^{(1)}}{|w_1|^n} = \left(\frac{w_1}{|w_1|}\right)_+^n \left(\frac{F_1}{w_0 - w_1}\right)_+ + \left(\frac{w_1}{|w_1|}\right)_-^n \left(\frac{F_1}{w_0 - w_1}\right)_- \quad \text{on} \quad S_1 \ , \tag{71}$$

$$R_{n\pm}^{(1)} = \left(\frac{w_1^n}{w_0 - w_1} F_1\right)_\pm (1 + O(q^n)) \quad \text{on} \quad S_0 \ .$$

We see from the second formula in (71) that the Hermite-Padé approximants to $f_1 = 1/w_0$ not only interpolate the function at the point infinity, but they also interpolate this function in some points of the set $S_1$. This phenomena is typical for the H-P approximants of Nikishin systems (see [13]).

The functions $\{F_l\}_{l=0}^p$ in the asymptotical formula (68) are called *Szegő functions* (analogous to the asymptotics for classical orthogonal polynomials). The main feature of such functions is that they can be characterized by means of a boundary value problem which we describe below.

Given a set of locally integrable weight functions

$$\sigma_l \quad \text{on} \quad S_{l-1} \ , \quad l = 1, 2, \ldots, p \ ,$$

we set

$$\omega_l := (w_{l-1,+} - w_{l-1,-})\Big|_{S_{l-1}} \ , \quad l = 1, 2, \ldots, p \ . \tag{72}$$

Consider the Riemann surface $\mathfrak{R}$ (see (36)) and contours $\partial\mathfrak{R}_{l-1,l}$ (see (55)) that separate the consecutive sheets. We define on each such contour the function

$$\overset{\circ}{\sigma}_l := \begin{cases} \sigma_l \omega_l & \text{on} \quad S_{l-1,-} \\ \\ \sigma_l \omega_l & \text{on} \quad S_{l-1,+} \end{cases} \ , \tag{73}$$

lift the values of $\overset{\circ}{\sigma}_l$ to $\partial\mathfrak{R}_{l-1,l}$ in accordance with (55), (37), and formulate the following BVP on $\mathfrak{R}$.

Find a piecewise holomorphic function $\mathcal{F}$ on $\mathfrak{R}$ with locally integrable boundary values such that

$$\begin{aligned} &1) \ \mathcal{F} \in H\left(\mathfrak{R} \setminus \bigcup_{l=1}^p \partial\mathfrak{R}_{l-1,l}\right) \ , \ \exists\, \mathcal{F}_\pm \in L^1_{loc}\left(\bigcup_{l=1}^p \partial\mathfrak{R}_{l-1,l}\right) \ , \\ &2) \ \mathcal{F}_+ = \mathcal{F}_- \frac{1}{\overset{\circ}{\sigma}_l} \quad \text{on} \quad \partial\mathfrak{R}_{l-1,l} \ , \ l = 1, \ldots, p \ , \\ &3) \ \mathcal{F}(\infty^{(0)})\, \mathcal{F}(\infty^{(1)}) \cdots \mathcal{F}(\infty^{(p)}) = 1 \ . \end{aligned} \tag{74}$$

**Proposition 2.** *If $\overset{\circ}{\sigma}_l$ is defined on $\partial\mathfrak{R}_{l-1,l}$ as in* (73)*, by means of a continuous, non-vanishing function $\sigma_l \omega_l$, then the solution of the BVP* (74) *on the Riemann surface* (37) *exists and unique. Moreover, for all $z \in \mathbb{C}$ we have*

$$\mathcal{F}(z^{(0)})\, \mathcal{F}(z^{(1)}) \cdots \mathcal{F}(z^{(p)}) \ = \ 1 \ , \tag{75}$$

*where* $z^{(j)} := \pi_j^{-1}(z), \quad j = 0, 1, \dots, p.$

*Proof.* Existence. The Riemann surface (37) has genus 0. Therefore our BVP is equivalent to the BVP on the complex plane with boundary conditions on the union of contours. However, due to (73) and the fact that $\overset{\circ}{\sigma}_l$ is continuous and non-vanishing, the index of this BVP is equal to zero. Such a BVP always has a solution (see [11]).
Uniqueness. Suppose we have two solutions $\mathcal{F}_1$ and $\mathcal{F}_2$. Then the function $\mathcal{F}_1/\mathcal{F}_2$ is holomorphic on the whole of $\mathfrak{R}$ and therefore this function is a constant which must equal one, because of condition 3) in (74). Analogously, (75) follows from the fact that the function of variable $z \in \mathbb{C}$, on the left-hand side of (75) is holomorphic on the whole plane $\mathbb{C}$. □

Now we obtain a characterization of the Szegő functions from the asymptotics (68) by means of a BVP on $\mathfrak{R}$.

**Theorem 8.4.** *The Szegő functions* $\{F_l\}_{l=0}^p$*, which satisfy the asymptotics* (68)*, give the solution of the BVP* (74) *on* $\mathfrak{R}$*; namely*

$$F_l = \mathcal{F}\Big|_{\mathfrak{R}_l}, \quad l = 0, 1, \dots, p\,. \tag{76}$$

*Proof.* The first condition in (74) is fulfilled because of (70). To check the second condition in (74) we consider (69) on $S_l$ , $l = 0, 1, \dots, p-1$ :

$$\frac{F_{l,+}}{F_{l+1,-}} = \frac{w_{l,-}\, w_{l,+}^{p-l}}{w_{l+1,-}^{p-l-1}(w_{l,+} - w_{l+1,+})(w_{l,-} - w_{l+1,-})} = \frac{w_{l,-}\, w_{l,+}}{(w_{l,+} - w_{l+1,+})(w_{l,-} - w_{l+1,-})} =$$

$$= \frac{F_{l,-}}{F_{l+1,+}} = \frac{1}{\omega_{l+1}\sigma_{l+1}}\,.$$

Here we used (72) and (60). Thus (73) holds, which implies the second condition in (74). Finally, the third condition in (74) follows directly from (69) :

$$\prod_{l=0}^{p} F_l = \frac{w_0^p}{\prod_{k=1}^{p}(w_0 - w_k)}\; \frac{w_1^{p-1}(w_0 - w_1)}{w_0 \prod_{k=2}^{p}(w_1 - w_k)} \cdots \frac{w_l^{p-l} \prod_{m=0}^{l-1}(w_m - w_l)}{\prod_{m=0}^{l-1} w_m \prod_{k=l+1}^{p}(w_l - w_k)} \cdots \frac{\prod_{m=0}^{p-1}(w_m - w_{p-1})}{\prod_{m=0}^{p-1} w_m} = 1\,.$$

By uniqueness of the solution (Proposition 2), the proof is complete. □

## 9. Analytic structure of the Weyl and of the Szegő functions for the perturbed operator

The goal of this section is to show that after a perturbation satisfying (10), the vector of Weyl functions remains to be a Nikishin system (56)-(57). Using this link we then obtain a characterization of the Szegő functions for the strong asymptotics of the polynomials $Q_n$ in (1) (see Theorems 7.2 and 7.5) by means of the BVP (74) on the Riemann surface $\mathfrak{R}$.

9.1. **Analysis of the system $\{f_j\}_{j=1}^p$. Determination of the jumps $\{\sigma_j\}_{j=1}^p$ on the stars $\{S_{j-1}\}_{j=1}^p$.** For the vector of the Weyl functions (18) for the perturbed operator (19) we have the formula (51) of the Theorem 7.4 :

$$f_j = \frac{1}{w_0^j}\,\frac{\Phi_j}{\Phi_0}\,, \quad j = 1, \ldots, p\,,$$

where $\Phi_l \in H(\mathbb{C}\backslash S_0)\,,\ l = 0, 1, \ldots, p$ are defined by (48) and (50). We also use the following representations of $\{\Phi_l\}_{l=0}^p$ (for $l = 0$ it becomes (53) of Lemma 7.5 ; for $l > 1$ these representations can be proved analogously) :

$$\Phi_j = \lim_{n\to\infty} \frac{1}{w_0^{n-j}} \left[ P_n^{(j)} - \left(\sum_{i=1}^p w_i\right) P_{n-1}^{(j)} + \cdots + (-1)^p \left(\prod_{i=1}^p w_i\right) P_{n-p}^{(j)} \right], \quad P_n^{(0)} := Q_n\,, \quad (77)$$

where convergence is uniform, not only on compact subsets of $\overline{\mathbb{C}}\backslash S_0$, but also on the compacta containing the boundary points of the domain $\overline{\mathbb{C}}\backslash S_0$, i.e. on both sides $\pm$ of the starlike set $S_0 \setminus \mathcal{A}$.

Using the identity

$$\left[ P_n^{(j)} - \left(\sum_{i=1}^p w_i\right) P_{n-1}^{(j)} + \cdots + (-1)^p \left(\prod_{i=1}^p w_i\right) P_{n-p}^{(j)} \right] = \frac{\triangle_n^{(j)}}{\mathbf{V}}\,, \quad (78)$$

where

$$\triangle_n^{(j)} := \begin{vmatrix} 1 & \ldots & 1 & P_{n-p}^{(j)} \\ w_1 & \ldots & w_p & P_{n-p+1}^{(j)} \\ \vdots & & \vdots & \vdots \\ w_1^p & \ldots & w_p^p & P_n^{(j)} \end{vmatrix}\,, \quad \mathbf{V} = \begin{vmatrix} 1 & \ldots & 1 \\ w_1 & \ldots & w_p \\ \vdots & & \vdots \\ w_1^{p-1} & \ldots & w_p^{p-1} \end{vmatrix}\,, \quad (79)$$

we have for the Weyl functions

$$f_j = \lim_{n\to\infty} f_{j,n}\,, \quad f_{j,n} := \frac{\triangle_n^{(j)}}{\triangle_n^{(0)}}\,, \quad j = 1, 2, \ldots, p\,, \quad (80)$$

where convergence is uniform on compact subsets of the domain $\overline{\mathbb{C}}\backslash S_0$, including its boundary points.

The goal of this subsection is to prove that the system of the Weyl function (80) is a Nikishin system and to find the system of weights $\sigma_k$ on $S_{k-1},\ k = 1, \ldots, p$ , forming $\{f_j^{(k)}\}$, the hierarchy of the Nikishin functions (57) - (59), as follows:

$$
\begin{aligned}
&f_j^{(1)} := f_j \;\in\; H(\overline{\mathbb{C}}\backslash S_0)\;, \quad j = 1, \dots, p \\
&\qquad\qquad f_{j+}^{(1)} - f_{j-}^{(1)}\Big|_{S_0} = \sigma_1 f_j^{(2)}\;, \;\; f_1^{(2)} \equiv 1\,, \\
&f_j^{(2)} \;\in\; H(\overline{\mathbb{C}}\backslash S_1)\;, \quad j = 2, \dots, p \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
&\qquad\qquad f_{j+}^{(l)} - f_{j-}^{(l)}\Big|_{S_{l-1}} = \sigma_l f_j^{(l+1)}\;, \;\; f_l^{(l+1)} \equiv 1\,, \\
&f_j^{(l+1)} \;\in\; H(\overline{\mathbb{C}}\backslash S_l)\;, \quad j = l+1, \dots, p \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
&f_p^{(p)} \;\in\; H(\overline{\mathbb{C}}\backslash S_{p-1})\;.
\end{aligned}
\tag{81}
$$

We have

**Theorem 9.1.** *The system of Weyl functions* (80) *for the operator* (19) *with coefficients satisfying* (10) *forms a Nikishin system* (56) *with respect to the system of weights*

$$
\sigma_l = -\lim_{n\to\infty} \sigma_{l,n}\,, \quad \sigma_{l,n} := \frac{\triangle_{n-}^{(0,1,\dots,l-2)}\triangle_{n+}^{(0,1,\dots,l)}}{\triangle_{n+}^{(0,1,\dots,l-1)}\triangle_{n-}^{(0,1,\dots,l-1)}} \quad \text{on} \quad S_{l-1}\,, \quad l = 1, \dots, p\,, \tag{82}
$$

*where*

$$
\triangle_n^{(j_0,j_1,\dots,j_l)} := \begin{vmatrix} 1 & \dots & 1 & P_{n-p}^{(j_0)} & P_{n-p}^{(j_1)} & \dots & P_{n-p}^{(j_l)} \\ w_{l+1} & \dots & w_p & P_{n-p+1}^{(j_0)} & P_{n-p+1}^{(j_1)} & \dots & P_{n-p+1}^{(j_l)} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ w_{l+1}^p & \dots & w_p^p & P_n^{(j_0)} & P_n^{(j_1)} & \dots & P_n^{(j_l)} \end{vmatrix}\,, \tag{83}
$$

$$
\triangle_n^{(0,1,\dots,l-2)}\Big|_{l=1} =: \triangle = \begin{vmatrix} 1 & 1 & \dots & 1 \\ w_0 & w_1 & \dots & w_p \\ \vdots & \vdots & & \vdots \\ w_0^p & w_1^p & \dots & w_p^p \end{vmatrix}\,, \tag{84}
$$

*and the convergence in* (82)*is uniform on compact subsets of* $S_{l-1}$. *The hierarchy of the Nikishin functions* (81) *has the form*

$$
f_j^{(l+1)} = \lim_{n\to\infty} f_{j,n}^{(l+1)}, \quad f_{j,n}^{(l+1)} := \frac{\triangle_n^{(0,1,\dots,l-1,j)}}{\triangle_n^{(0,1,\dots,l-1,l)}}\;, \quad j = l+1, \dots, p,\; l = 0, \dots, p-1\;, \tag{85}
$$

*where convergence is uniform on compact subsets of the domain* $\overline{\mathbb{C}}\backslash S_l$, *including its boundary points (except for points of* $\mathcal{A}$*).*

**Remark 9.1.** We note, that in passing from $S_{(l-1)+}$ to $S_{(l-1)-}$ we interchange neighboring columns of the determinants in (83). This produces a sign change in the product of determinants in (82) :

$$
\triangle_{n-}^{(0,1,\dots,l-2)}\triangle_{n+}^{(0,1,\dots,l)} = -\triangle_{n+}^{(0,1,\dots,l-2)}\triangle_{n-}^{(0,1,\dots,l)}.
$$

*Proof.* We first prove (by induction) that for each $n \in \mathbb{N}$, the system (see (80))

$$f_{j,n}\ , \quad j = 1, \dots, p \tag{86}$$

is a Nikishin system with respect to the weight functions $\{\sigma_{l,n}\}_{l=1}^{p}$ from (82), and the corresponding hierarchy $\{f_{j,n}^{(l)}\}$ is given by (85). Then convergence in (82) and (85) will follow from the convergence in (80).

To prove that (86) is a Nikishin system we use a determinant identity. We denote by $\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}, \overrightarrow{d}$ vector-columns of size $(p+1)$ and by $\mathbf{X}$ a matrix with $(p-1)$ columns and $(p+1)$ rows. Let $\left[\overrightarrow{a} \bigsqcup \overrightarrow{b} \bigsqcup \mathbf{X}\right]$ be the $(p+1)\times(p+1)$ matrix that is obtained by adjoining (as first two columns) $\overrightarrow{a}$ and $\overrightarrow{b}$ to the matrix $\mathbf{X}$. Then the following identity holds:

$$\begin{aligned} \det\,[\overrightarrow{a} \bigsqcup \mathbf{X} \bigsqcup \overrightarrow{c}] * \det\left[\overrightarrow{b} \bigsqcup \mathbf{X} \bigsqcup \overrightarrow{d}\right] - \det\left[\overrightarrow{a} \bigsqcup \mathbf{X} \bigsqcup \overrightarrow{d}\right] * \det\left[\overrightarrow{b} \bigsqcup \mathbf{X} \bigsqcup \overrightarrow{d}\right] = \\ = \det\left[\overrightarrow{a} \bigsqcup \overrightarrow{b} \bigsqcup \mathbf{X}\right] * \det\left[\mathbf{X} \bigsqcup \overrightarrow{c} \bigsqcup \overrightarrow{d}\right] . \end{aligned} \tag{87}$$

Now we proceed with the induction. For the first step we have

$$f_{j,n}^{(1)} := f_{j,n} = \frac{\triangle_n^{(j)}}{\triangle_n^{(0)}}\ , \quad j = 1, 2, \dots, p\ ,$$

and, on $S_0$,

$$f_{j,n+}^{(1)} - f_{j,n-}^{(1)}\Big|_{S_0} = \frac{\triangle_{n+}^{(j)}\triangle_{n-}^{(0)} - \triangle_{n-}^{(j)}\triangle_{n+}^{(0)}}{\triangle_{n+}^{(0)}\triangle_{n-}^{(0)}}\ . \tag{88}$$

Note that the first column is changed in the boundary values of the determinants $\triangle_{n+}^{(j)}$ and $\triangle_{n-}^{(j)}$ (see (79)). The changes of the boundary values of the other columns (2nd and 3rd, 4th and 5th, etc.) produce a change in sign of the determinant (interchanging two columns). This change of sign does not affect (88) because there determinants occur in pairs. Thus we can apply the identity (87) to the numerator of (88) :

$$f_{j,n+}^{(1)} - f_{j,n-}^{(1)}\Big|_{S_0} = -\frac{\triangle_{-}\triangle_{n+}^{(0,j)}}{\triangle_{n+}^{(0)}\triangle_{n-}^{(0)}} = -\frac{\triangle_{-}\triangle_{n+}^{(0,1)}}{\triangle_{n+}^{(0)}\triangle_{n-}^{(0)}} \cdot \frac{\triangle_{n+}^{(0,j)}}{\triangle_{n+}^{(0,1)}}\ . \tag{89}$$

The same reasoning (see also Remark 9.1) implies that

$$\frac{\triangle_{n+}^{(0,j)}}{\triangle_{n+}^{(0,1)}} = \frac{\triangle_{n-}^{(0,j)}}{\triangle_{n-}^{(0,1)}} \quad \text{on} \quad S_0\ .$$

Thus the function $f_{j,n}^{(2)} := \triangle_n^{(0,j)} \,/\, \triangle_n^{(0,1)}\ ,\ j = 2, \dots, p,$ has no jump on $S_0$ and therefore

$$f_{j,n}^{(2)} \in H\left(\overline{\mathbb{C}} \backslash S_1\right)\ . \tag{90}$$

Setting

$$\sigma_{1,n} := -\frac{\triangle_{-}\triangle_{n+}^{(0,1)}}{\triangle_{n+}^{(0)}\triangle_{n-}^{(0)}} = \frac{\triangle_{+}\triangle_{n-}^{(0,1)}}{\triangle_{n-}^{(0)}\triangle_{n-}^{(0)}}\ ,$$

we have from (89), (90) that the theorem holds for $l = 1$.

Now suppose that

$$f_{j,n}^{(l)} := \frac{\triangle_n^{(0,1,\dots,l-2,j)}}{\triangle_n^{(0,1,\dots,l-2,l-1)}} \in H\left(\overline{\mathbb{C}}\backslash S_{l-1}\right) \ , \quad j = l, l+1, \dots, p \, ,$$

is true. For the jump of $f_{j,n}^{(l)}$ on $S_{l-1}$ we have

$$f_{j,n+}^{(l)} - f_{j,n-}^{(l)}\Big|_{S_{l-1}} = \frac{\triangle_{n+}^{(0,1,\dots,l-2,j)}\triangle_{n-}^{(0,1,\dots,l-2,l-1)} - \triangle_{n-}^{(0,1,\dots,l-2,j)}\triangle_{n+}^{(0,1,\dots,l-2,l-1)}}{\triangle_{n+}^{(0,1,\dots,l-2,l-1)}\triangle_{n-}^{(0,1,\dots,l-2,l-1)}} \, .$$

Again using (87) we obtain

$$f_{j,n+}^{(l)} - f_{j,n-}^{(l)}\Big|_{S_{l-1}} = -\frac{\triangle_{n-}^{(0,1,\dots,l-2)}\triangle_{n+}^{(0,1,\dots,l-1,j)}}{\triangle_{n+}^{(0,1,\dots,l-1)}\triangle_{n-}^{(0,1,\dots,l-1)}} = \sigma_l f_{j,n}^{(l+1)} \, ,$$

in accordance with (82), (85) and (81). This completes the induction. □

9.2. **BVP on $\mathfrak{R}$ for the Szegő functions related to the $Q_n$.** We have from Theorem 7.2 the asymptotic formula

$$\lim_{n\to\infty} \frac{Q_n}{w_0^n} = A_0\Phi_0 =: F_0 \ ,$$

uniformly on compact subsets of $\mathbb{C}\backslash S_0$. Substituting the expressions for $A_0$ and $\Phi_0$ (see (77), (78))

$$A_0 = \frac{w_0^p}{\prod_{k=1}^{p}(w_0 - w_k)} \ ; \quad \Phi_0 = \lim_{n\to\infty}\frac{1}{w_0^n}\frac{\begin{vmatrix} 1 & \dots & 1 & P_{n-p}^{(0)} \\ w_1 & \dots & w_p & P_{n-p+1}^{(0)} \\ \vdots & & \vdots & \vdots \\ w_1^p & \dots & w_p^p & P_n^{(0)} \end{vmatrix}}{\begin{vmatrix} 1 & \dots & 1 \\ w_1 & \dots & w_p \\ \vdots & & \vdots \\ w_1^{p-1} & \dots & w_p^{p-1} \end{vmatrix}},$$

we have (uniformly on compact subsets of $\mathbb{C}\backslash S_0$ including the boundary points of $S_{0\pm} \setminus \mathcal{A}$)

$$F_0 = \lim_{n\to\infty} F_{0,n} \ , \quad F_{0,n} = \frac{1}{w_0^{n-p}}\frac{\triangle_n^{(0)}}{\triangle}, \tag{91}$$

where we use the notation of (79) and (84).

Next we prove that the function $F_0$ is the solution (76) to the BVP (74)-(73)-(72) on the Riemann surface $\mathfrak{R}$, where the jumps in the BVP are formed by the weight functions $\{\sigma_l\}_{l=1}^p$ defined in (82).

**Theorem 9.2.** *Let $\mathcal{F}$ be the solution of the BVP* (74) *on $\mathfrak{R}$, with jumps $\{\sigma_l\}_{l=1}^p$ in* (73) *defined by* (82). *Let $\{F_l\}_{l=0}^p$ be the values of the solution taken from the different sheets of $\mathfrak{R}$ :*

$$F_l = \mathcal{F}\Big|_{\mathfrak{R}_l} \ , \qquad l = 0, 1, \dots, p \, .$$

*Then $F_0$ is the Szegő function* (91)*, and for the remaining of $F_l$'s we have*

$$F_l = \lim_{n\to\infty} F_{l,n} \ , \quad F_{l,n} = \frac{(w_l - w_{l-1})\ldots(w_l - w_0)}{w_l^{n-p}} \frac{\triangle_n^{(0,1,\ldots,l)}}{\triangle_n^{(0,1,\ldots,l-1)}} \ , \tag{92}$$

*uniformly on compact subsets of $\mathbb{C}\backslash S_0$, including the boundary points of $S_{0\pm}$ (cf.* (83) *for notation).*

*Proof.* Again, as in the proof of the Theorem 9.1, it is enough to prove that $\{F_{l,n}\}$ forms the solution of the BVP (74) with jumps $\{\sigma_{l,n}\}$, and then the convergence in (91) yields the result. Thus, for $\{F_{l,n}\}$ in (92), we have to check the conditions 1), 2) and 3) from (74).

1) Evidently

$$F_{l,n} \in H\left(\overline{\mathbb{C}}\backslash\{S_{l-1}\cup S_l\}\right) \ , \quad l = 1,\ldots,p-1 \ ,$$

$$F_{0,n} \in H\left(\overline{\mathbb{C}}\backslash S_0\right) \ , \quad F_{p,n} \in H\left(\overline{\mathbb{C}}\backslash S_{p-1}\right) \ .$$

2) We need to prove that on $S_l$, $l = 0,1,\ldots,p-1$, there holds

$$F_{l+1,n\pm} = F_{l,n\mp}\,\sigma_{l+1}\,\omega_{l+1} \ . \tag{93}$$

We begin with the case $l = 0$. From (91), (82) and (72) we have on $S_0$

$$F_{0,n\mp}\,\sigma_1\,\omega_1 = -\frac{1}{w_{0\mp}^{n-p}} \frac{\triangle_{n\mp}^{(0)}}{\triangle_\mp} \frac{\triangle_-\,\triangle_n^{(0,1)}}{\triangle_{n+}^{(0)}\,\triangle_{n-}^{(0)}} (w_{0+} - w_{0-}) \ .$$

Taking into account Remark 9.1 we can continue :

$$F_{0,n\mp}\,\sigma_1\,\omega_1 = -\frac{1}{w_{0\mp}^{n-p}} \frac{\triangle_{n\mp}^{(0)}}{\triangle_\mp} \frac{\triangle_\mp\,\triangle_{n\pm}^{(0,1)}}{\triangle_{n+}^{(0)}\,\triangle_{n-}^{(0)}} (w_{0\pm} - w_{0\mp}) = \frac{1}{w_{1\pm}^{n-p}} \frac{\triangle_{n\pm}^{(0,1)}}{\triangle_{n\pm}^{(0)}} (w_1 - w_0)_\pm = F_{1,n\pm} \text{ on } S_0 \ .$$

Thus for $l = 0$ the boundary condition (93) is verified. Now we consider $l = 1,2,\ldots,p-1$. We have on $S_l$

$$F_{l,n\mp}\,\sigma_{l+1}\,\omega_{l+1} =$$

$$= -\frac{(w_l - w_{l-1})_\mp\ldots(w_l - w_0)_\mp}{w_{l\mp}^{n-p}} \frac{\triangle_{n\mp}^{(0,1,\ldots,l)}}{\triangle_{n\mp}^{(0,1,\ldots,l-1)}} \frac{\triangle_{n\mp}^{(0,1,\ldots,l-1)}\triangle_{n\pm}^{(0,1,\ldots,l+1)}}{\triangle_{n+}^{(0,1,\ldots,l)}\triangle_{n-}^{(0,1,\ldots,l)}} (w_{l\pm} - w_{l\mp}) =$$

$$= -\frac{(w_{l+1} - w_{l-1})_\pm(w_{l+1} - w_{l-2})_\pm\ldots(w_{l+1} - w_0)_\pm}{w_{l+1,\pm}^{n-p}} \frac{\triangle_{n\pm}^{(0,1,\ldots,l+1)}}{\triangle_{n\pm}^{(0,1,\ldots,l)}} (w_l - w_{l+1})_\pm = F_{l+1,n\pm} \ .$$

Thus condition (93) holds.

3) Finally, taking the product of the $F_{l,n}$, we have

$$\prod_{l=0}^{p} F_{l,n} = \frac{\triangle_n^{(0,1,\ldots,p)}\triangle}{\triangle\prod_{l=0}^{p} w_l^{n-p}} = 1 \ ,$$

where we used the fact that

$$\triangle_n^{(0,1,\dots,p)} = \begin{vmatrix} Q_{n-p} & P_{n-p}^{(1)} & \dots & P_{n-p}^{(p)} \\ Q_{n-p+1} & P_{n-p+1}^{(1)} & \dots & P_{n-p+1}^{(p)} \\ \vdots & \vdots & & \vdots \\ Q_n & P_n^{(1)} & \dots & P_{n-p+1}^{(p)} \end{vmatrix} = \prod_{l=0}^{p} w_l^{n-p} .$$

□

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