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Asymptotics of weighted best-packing on rectifiable sets

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Abstract. We investigate the asymptotic behaviour, as N grows, of the largest minimal weighted pairwise distance between N points restricted to a rectifiable compact set embedded in Euclidean space, and we find the limit distribution of asymptotically optimal configurations.

Bibliography: 23 titles.

The classical best-packing problem is the problem of finding a configuration of N points on a given compact set A with the largest possible separation (largest minimal pairwise distance). Formulated for the Euclidean space $\mathbb{R}^d$ this problem becomes that of finding the largest density of a collection of non-overlapping equal balls in $\mathbb{R}^d$. Some of the significant results and reviews of the literature on this problem can be found in [1]–[6]. It is known, for example, that the solution to this problem for certain sets A in the plane coincides with asymptotically optimal nodes for cubature formulae (see, for example, [7]).

The best-packing problem is dual to the problem of ε-complexity of a compact set A which, for a given $\varepsilon > 0$, requires one to find the largest number of points on A that are at a distance at least ε from each other. The notion of ε-complexity was first introduced by Kolmogorov and Tikhomirov [8] and has, in particular, applications to the study of complexity of the behaviour of orbits in dynamical systems (see, for example, [9]).

The problems mentioned above have also been considered for different metric spaces. In this paper we study a weighted analogue of the best-packing problem which, in a sense, is similar to introducing a certain metric, or sequence of metrics. The solution to such problems can be applied to the construction of optimal weighted cubature formulae (see [10] and references therein for more information on such optimization problems) and computer aided geometric design when it is required to place points on a surface according to a prescribed non-uniform distribution (for example, to place more points on regions of the surface with higher curvature).

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§ 1. Definitions, notation and known results

Let $A \subset \mathbb{R}^p$ be an infinite set, where p is a positive integer. For a collection of distinct points $\omega_N := \{x_1, \dots, x_N\} \subset A$ and a function $w\colon A \times A \to [0, \infty]$ let

$$\delta^w(\omega_N) := \min_{1 \leqslant i \neq j \leqslant N} w(x_i, x_j)|x_i - x_j|,$$

where $|\cdot|$ is the Euclidean distance in $\mathbb{R}^p$. We also denote the *best weighted N-point packing distance on A* by

$$\delta_N^w(A) := \sup\{\delta^w(\omega_N) : \omega_N \subset A,\ \#\omega_N = N\}, \tag{1}$$

where $\#X$ stands for the cardinality of the set X. We are interested in the asymptotics (as $N \to \infty$) of the quantities in (1) as well as the asymptotic behaviour of the configurations that attain the supremum on the right-hand side of (1).

Without loss of generality we can assume that w is a symmetric function, since for $v(x, y) := \min\{w(x, y), w(y, x)\}$ we have $\delta^v(\omega_N) = \delta^w(\omega_N)$ for every configuration ω_N.

In the non-weighted case ($w \equiv 1$) we get the classical best-packing problem (we shall omit the superscript w in definition (1) and elsewhere when $w \equiv 1$). In this case the (not necessarily unique) configurations ω_N that achieve the supremum in (1) are called *best-packing configurations* on A. We remark that the exact solution to the best-packing problem when A is the sphere S^2 in $\mathbb{R}^3$ has been obtained for $N = 2, \dots, 12$ and $N = 24$ (see [4] or [5] for the list of references). The asymptotic behaviour as $N \to \infty$ of the best-packing distance on S^2 (up to the next order term) was obtained by Habicht and van der Waerden in [11] and [12].

Kolmogorov and Tikhomirov [8] obtained the main term of the quantity (1) as $N \to \infty$ when $w = 1$ and A is a compact set of positive Lebesgue measure in $\mathbb{R}^p$. In [13] the present authors analysed the asymptotic behaviour of the best-packing distance in the non-weighted case on d-rectifiable sets and their countable unions.

Throughout this paper, $\mathscr{H}_\alpha$ with $\alpha \leqslant p$ denotes α-dimensional Hausdorff measure in $\mathbb{R}^p$. When α is an integer d, then we choose a normalization so that an isometric copy of the cube $[0, 1]^d$ has $\mathscr{H}_d$-measure 1. A set $A \subset \mathbb{R}^p$ is *d-rectifiable*, $d \leqslant p$, if it is the image of a bounded subset T of $\mathbb{R}^d$ with respect to a Lipschitz mapping, that is a mapping $\varphi : T \to \mathbb{R}^p$ that satisfies for some constant λ

$$|\varphi(x) - \varphi(y)| \leqslant \lambda|x - y|, \qquad x, y \in T. \tag{2}$$

(See [14], [15].)

In what follows we assume that A is a compact set with $\mathscr{H}_d(A) < \infty$. In the first part of the paper we shall consider the weighted best-packing problem for a fixed weight on $A \times A$ satisfying certain conditions. A function $w\colon A \times A \to [0, \infty]$ is said to be a CBD-*weight function on $A \times A$* if

(a) w is continuous (as a function on $A \times A$) at $\mathscr{H}_d$-almost every point of the diagonal $D(A) := \{(x, x) : x \in A\}$,

(b) there is some neighbourhood G of $D(A)$ (relative to $A \times A$) such that $\sup_G w < \infty$, and

(c) $\inf_B w > 0$ on any closed subset $B \subset A \times A$ such that $B \cap D(A) = \varnothing$.

Here CBD stands for (almost) continuous and bounded on the diagonal. In particular, conditions (a)–(c) hold if w is greater than some positive number on $A \times A$ and continuous on the diagonal $D(A)$ (where continuity at a diagonal point (x_0, x_0) is meant in the sense of limits taken on $A \times A$).

Note that $w(x,y)|x-y|$ is, in general, not a metric. However, if a metric $\rho(x,y)$ is continuous with respect to the Euclidean distance in $\mathbb{R}^p$ and the limit

$$f(z) := \lim_{(x,y)\to(z,z)} \frac{\rho(x,y)}{|x-y|}, \qquad (x,y) \in (A \times A) \setminus D(A), \quad z \in A,$$

exists and is continuous, then we obtain best-packing results for the metric $\rho(x,y)$ by setting

$$w(x,y) := \begin{cases} \rho(x,y)/|x-y| & \text{if } x \neq y, \\ f(z) & \text{if } x = y = z. \end{cases}$$

For a review of results on packing in non-Euclidean spaces see, for example, [16], [17].

In this paper the asymptotic behaviour of the quantity (1) together with the weak-star limit distribution of asymptotically optimal configurations is obtained for compact d-rectifiable sets. We further extend these results in two ways: by considering varying weights and by allowing weights with singularities. Large values of the weight can significantly affect distances between optimal points, which leads to certain technical difficulties when the weight has singularities.

We set

$$C_{\infty,d} := \lim_{N\to\infty} \delta_N([0,1]^d) N^{1/d}.$$

Let Δ_d denote the largest density of packing equal non-overlapping balls in $\mathbb{R}^d$ (see [3] for the precise definition) and let β_d be the Lebesgue measure of the unit ball in $\mathbb{R}^d$. It follows from the definition that

$$C_{\infty,d} = 2\left(\frac{\Delta_d}{\beta_d}\right)^{1/d}. \tag{3}$$

The density Δ_d (and hence, constant $C_{\infty,d}$) has been obtained only for $d = 2$ (cf. [1]) and recently for $d = 3$ (cf. [6]). These results imply that

$$C_{\infty,2} = \sqrt{\frac{2}{\sqrt{3}}}, \qquad C_{\infty,3} = \sqrt[6]{2}$$

(clearly, $C_{\infty,1} = 1$).

The weighted best-packing problem represents the limiting case as $s \to \infty$ of the following weighted minimum energy problem

$$\mathscr{E}_s^w(A,N) = \inf_{x_1,\ldots,x_N \in A} \sum_{1 \leqslant i \neq j \leqslant N} \frac{w(x_i,x_j)}{|x_i - x_j|^s}.$$

This problem was considered by the present authors in [18] for a class of weights w that includes the reciprocals of CBD-weight functions and more general weights with a finite number of zeros on the diagonal. It was shown that if $A \subset \mathbb{R}^p$ is a closed d-rectifiable set and $s > d$, then

$$\lim_{N\to\infty} \frac{\mathscr{E}_s^w(A,N)}{N^{1+s/d}} = C_{s,d}\left(\int_A (w(x,x))^{-d/s}\, d\mathscr{H}_d(x)\right)^{-s/d}, \tag{4}$$

where $C_{s,d}$ is a positive constant independent of the set A. When $w \equiv 1$, we obtain the (non-weighted) minimal energy problem (see [19]–[23] and references therein).

The limit distribution of asymptotically optimal configurations is understood in the following sense. If $A \subset \mathbb{R}^p$ is compact and ν and $\{\nu_N\}_{N=1}^{\infty}$ are Borel probability measures on A, then the sequence $\{\nu_N\}_{N=1}^{\infty}$ is said to converge *weak-star* to ν (and we write $\nu_N \xrightarrow{*} \nu$ as $N \to \infty$) if for any function f continuous on A, we have

$$\lim_{N\to\infty} \int_A f\, d\nu_N = \int_A f\, d\nu.$$

We say that a sequence of configurations $\{\omega_N\}_{N=2}^{\infty}$, $\omega_N = \{x_{1,N},\dots,x_{N,N}\} \subset A$, $N = 2,3,\dots$, *has limit probability measure* ν if

$$\nu(\omega_N) := \frac{1}{N}\sum_{k=1}^{N} \delta_{x_{k,N}} \xrightarrow{*} \nu, \qquad N \to \infty, \tag{5}$$

where δ_x is the atomic probability measure in $\mathbb{R}^p$ centred at the point $x \in \mathbb{R}^p$.

To prove that (5) holds it is sufficient to show that for every subset $B \subset A$ whose boundary relative to A has ν-measure zero, we have

$$\lim_{N\to\infty} \frac{\#(\omega_N \cap B)}{N} = \nu(B). \tag{6}$$

In [22] and [23] the uniformity of the limit distribution of minimal s-energy configurations on d-rectifiable manifolds in $\mathbb{R}^p$ for $s \geqslant d$ is established. In [18] the present authors show that the density of the limit distribution of configurations asymptotically minimizing the weighted energy in (4) is proportional to $(w(x,x))^{-d/s}$, $s > d$. Afraimovich and Glebsky [9] study the properties of the limit distribution with respect to convergence along an ultrafilter of optimal ε-complexity configurations on compact sets in $\mathbb{R}^d$ endowed with a varying metric.

Our first goal is to establish a weighted analogue of the following theorem in [13].

Theorem 1.1. *Let $d \leqslant p$, where d, p are integers, and $A \subset \mathbb{R}^p$ an infinite closed d-rectifiable set. Then*

$$\lim_{N\to\infty} \delta_N(A) N^{1/d} = C_{\infty,d}\mathscr{H}_d(A)^{1/d} = 2\left(\frac{\Delta_d}{\beta_d}\right)^{1/d}\mathscr{H}_d(A)^{1/d}. \tag{7}$$

If $\mathscr{H}_d(A) > 0$, then for every sequence $\{\overline{\omega}_N\}_{N=2}^{\infty}$ of best-packing configurations on A such that $\#\overline{\omega}_N = N$, $N = 2,3,\dots$, we have

$$\nu(\overline{\omega}_N) \xrightarrow{*} \frac{\mathscr{H}_d\big|_A(\,\cdot\,)}{\mathscr{H}_d(A)}, \qquad N \to \infty.$$

Relation (7) for $d = p$ is the result of Kolomogorov and Tikhomirov obtained in [8].

§ 2. Main results

Let $\mathscr{H}_d^w$ and h_d^w be the Borel measures supported on A such that

$$\mathscr{H}_d^w(B) = \int_B w(x,x)^d \, d\mathscr{H}_d(x)$$

and

$$h_d^w(B) = \frac{\mathscr{H}_d^w(B)}{\mathscr{H}_d^w(A)}$$

for every Borel set $B \subset A$. A sequence of point configurations in A, $\{\omega_N\}_{N=2}^{\infty}$, $\#\omega_N = N$, $N = 2, 3, \dots$, is called *asymptotically w-best-packing* on A if

$$\delta^w(\omega_N) = \delta_N^w(A)(1 + o(1)), \qquad N \to \infty.$$

Theorem 2.1. *Let $d \leqslant p$, where d, p are integers. Suppose that $A \subset \mathbb{R}^p$ is an infinite closed d-rectifiable set and w is a* CBD*-weight function on $A \times A$. Then*

$$\lim_{N\to\infty} \delta_N^w(A) N^{1/d} = C_{\infty,d} \biggl(\int_A w(x,x)^d \, d\mathscr{H}_d(x) \biggr)^{1/d}, \tag{8}$$

where $C_{\infty,d}$ is as in (3).

Furthermore, if $\mathscr{H}_d(A) > 0$, then any asymptotically w-best-packing sequence of configurations $\widetilde{\omega}_N = \{x_1^N, \dots, x_N^N\}$, $N = 2, 3, \dots$, on A satisfies

$$\frac{1}{N} \sum_{k=1}^{N} \delta_{x_k^N} \xrightarrow{\ *\ } h_d^w, \qquad N \to \infty. \tag{9}$$

Example 2.2. Let $\mathbb{D}$ denote the open unit disc in the complex plane $\mathbb{C}$ and suppose that A is an infinite closed 1-rectifiable set in $\mathbb{D}$ (such as a rectifiable arc or curve) with positive length (that is, $\mathcal{H}_1(A) > 0$). Let $\omega_N^* = \{z_{1,N}^*, \dots, z_{N,N}^*\}$ maximize the minimum pseudohyperbolic metric distance $d(z,\zeta) = \bigl|(z-\zeta)/(1-z\bar{\zeta})\bigr|$ among all N-point subsets of A. Then from Theorem 2.1 with the weight $w(z,\zeta) = 1/|1 - z\bar{\zeta}|$, $z, \zeta \in \mathbb{D}$, it follows that the points ω_N^* are asymptotically uniformly distributed with respect to the infinitesimal Bergman (Poincaré) metric $|dz|/(1-|z|^2)$.

Similarly, if $A \subset \mathbb{D}$ is a closed set with positive area (that is, $\mathcal{H}_2(A) > 0$), then with the same weight as above, w-best-packing configurations have asymptotic density $1/(1-|z|^2)^2$ with respect to area measure (that is, $\mathcal{H}_2$).

Theorem 2.1 considers weights bounded on $D(A)$. Below, we study the case when the weight is allowed to have singularities on $D(A)$. Let $B(a,r)$ be the open ball in $\mathbb{R}^p$ centred at the point a with radius $r > 0$. For $t > 0$ we say that a function $w\colon A \times A \to [0,\infty]$ has a *singularity at $(a,a) \in D(A)$ of order at most t* if there are positive constants C and δ such that

$$w(x,y) \leqslant \frac{C}{|x-a|^t}\,, \qquad x, y \in A \cap B(a,\delta).$$

If w has a singularity $(a,a) \in D(A)$ whose order is too large, then it may act as an attracting 'sink' for optimal configurations, yielding $\delta_N^w(A) = \infty$. For example, let A be a closed ball in $\mathbb{R}^d$ centred at the origin and $w(x,y) = (|x|+|y|)^{-t}$, $x,y \in A$, $t>1$. If $\omega_N = \{x_1,\dots,x_N\}$ is a configuration of N distinct points in A, then $\delta^w(\gamma\omega_N) = \gamma^{1-t}\delta^w(\omega_N)$ for any $0<\gamma<1$. Taking $\gamma\to 0$ shows that $\delta_N^w(A)=\infty$.

A closed set $A\subset\mathbb{R}^p$ is called *α-regular at $a\in A$* if there are positive constants C_0 and δ_0 such that

$$C_0^{-1}r^\alpha \leqslant \mathscr{H}_\alpha(A\cap B(x,r)) \leqslant C_0 r^\alpha \tag{10}$$

for all $x\in A\cap B(a,\delta_0)$ and $0<r<\delta_0$.

Theorem 2.3. *Let $A\subset\mathbb{R}^p$ be an infinite closed d-rectifiable set, where $d\leqslant p$. Suppose that A is α_i-regular with $\alpha_i\leqslant d$ at a_i, $i=1,\dots,n$, for a finite collection of points $a_1,\dots,a_n\in A$ and that $w\colon A\times A\to[0,\infty]$ is a* CBD*-weight function on $K\times K$ for any compact set $K\subset A\setminus\{a_1,\dots,a_n\}$. If w has singularity of order at most $t<1$ at each (a_i,a_i), then the conclusions of Theorem 2.1 hold.*

Finally, we determine under suitable assumptions the asymptotic behaviour of weighted best-packing when the weight varies with N. Given a sequence $\{\upsilon_N\}_{N=2}^\infty$ of non-negative weight functions defined on $A\times A$, we say that a sequence of configurations $\{\omega_N\}_{N=2}^\infty$ on the set A, where $\#\omega_N=N$, $N=2,3,\dots$, is *asymptotically optimal* for this sequence of weights if

$$\delta^{\upsilon_N}(\omega_N) = \delta_N^{\upsilon_N}(A)(1+o(1)), \qquad N\to\infty.$$

Theorem 2.4. *Let $d\leqslant p$ be integers and let $A\subset\mathbb{R}^p$ be an arbitrary infinite closed d-rectifiable set. Suppose that either w is a* CBD*-weight function on $A\times A$ or that the set A and weight w satisfy the assumptions of Theorem 2.3. Let $\{\upsilon_N\}_{N=2}^\infty$ be a sequence of non-negative functions on $A\times A$ such that*

$$(1-\varepsilon_N)w(x,y) \leqslant \upsilon_N(x,y) \leqslant (1+\varepsilon_N)w(x,y), \qquad (x,y)\in A\times A, \tag{11}$$

where $\{\varepsilon_N\}_{N=2}^\infty\subset(0,1)$ is some sequence converging to zero. Then

$$\lim_{N\to\infty}\delta_N^{\upsilon_N}(A)N^{1/d} = C_{\infty,d}\biggl(\int_A w(x,x)^d\,d\mathscr{H}_d(x)\biggr)^{1/d}.$$

Moreover, if $\mathscr{H}_d(A)>0$, then for any sequence of configurations $\omega_N:=\{x_1^N,\dots,x_N^N\}$, $N=2,3,\dots$, that is asymptotically optimal for the sequence of weights $\{\upsilon_N\}_{N=2}^\infty$ we have

$$\frac{1}{N}\sum_{k=1}^N \delta_{x_k^N} \xrightarrow{\;*\;} h_d^w, \qquad N\to\infty. \tag{12}$$

§ 3. Auxiliary statements

In the following, we find it convenient to define $\delta_N^w(A)$ for finite sets to be 0 when $N>\#A$. Given positive integers $d\leqslant p$, let

$$\underline{g}_d^w(A) = \liminf_{N\to\infty}\delta_N^w(A)N^{1/d}, \qquad \overline{g}_d^w(A) = \limsup_{N\to\infty}\delta_N^w(A)N^{1/d}$$

and

$$g_d^w(A) = \lim_{N\to\infty} \delta_N^w(A)N^{1/d} \tag{13}$$

if this limit exists. When $w \equiv 1$ we shall denote the above limits by $\underline{g}_d(A)$, $\bar{g}_d(A)$ and $g_d(A)$, respectively. To prove Theorems 2.1 and 2.3 we first establish the following analogues of Lemmas 1 and 2 from [18].

Lemma 3.1. *Let B and D be two bounded sets in $\mathbb{R}^p$ and suppose that*

$$w\colon (B\cup D)\times(B\cup D)\to[0,\infty]$$

is an arbitrary weight function. Then

$$\bar{g}_d^w(B\cup D)^d \leqslant \bar{g}_d^w(B)^d + \bar{g}_d^w(D)^d. \tag{14}$$

Furthermore, if $\bar{g}_d^w(B), \bar{g}_d^w(D) < \infty$ and at least one of these quantities is positive, then

$$\lim_{\mathscr{N}\ni N\to\infty} \frac{\#(\widetilde{\omega}_N\cap B)}{N} = \frac{\bar{g}_d^w(B)^d}{\bar{g}_d^w(B)^d + \bar{g}_d^w(D)^d} \tag{15}$$

holds for every sequence $\{\widetilde{\omega}_N\}_{N\in\mathscr{N}}$ of N-point configurations in $B\cup D$ such that

$$\lim_{\mathscr{N}\ni N\to\infty} \delta^w(\widetilde{\omega}_N)N^{1/d} = \left(\bar{g}_d^w(B)^d + \bar{g}_d^w(D)^d\right)^{1/d}, \tag{16}$$

where $\mathscr{N}$ is some infinite subset of $\mathbb{N}$.

Proof. If $B\cup D$ is finite, then the result of the lemma holds trivially. Let $\mathscr{N}_1\subset\mathbb{N}$ be an infinite subset and let $\{\omega_N\}_{N\in\mathscr{N}_1}$ be a sequence of N-point configurations in $B\cup D$ such that the limit

$$\alpha := \lim_{\mathscr{N}_1\ni N\to\infty} \frac{\#(\omega_N\cap B)}{N}$$

exists. We shall show that

$$\limsup_{\mathscr{N}_1\ni N\to\infty} \delta^w(\omega_N)N^{1/d} \leqslant \left(\bar{g}_d^w(B)^d + \bar{g}_d^w(D)^d\right)^{1/d}. \tag{17}$$

Let $N_B = \#(\omega_N\cap B)$ and $N_D = \#(\omega_N\setminus B)$, $N\in\mathscr{N}_1$. Then

$$\begin{aligned}\delta^w(\omega_N) &= \min_{x\neq y\in\omega_N} w(x,y)|x-y| \\ &\leqslant \min\left\{\delta^w(\omega_N\cap B),\, \delta^w(\omega_N\setminus B)\right\} \leqslant \min\left\{\delta^w_{N_B}(B),\, \delta^w_{N_D}(D)\right\}.\end{aligned}$$

If $0<\alpha<1$, then we get

$$\begin{aligned}&\limsup_{\mathscr{N}_1\ni N\to\infty} \delta^w(\omega_N)N^{1/d} \\ &\quad\leqslant \min\left\{\limsup_{\mathscr{N}_1\ni N\to\infty} \delta^w_{N_B}(B)N_B^{1/d}\left(\frac{N}{N_B}\right)^{1/d},\ \limsup_{\mathscr{N}_1\ni N\to\infty} \delta^w_{N_D}(D)N_D^{1/d}\left(\frac{N}{N_D}\right)^{1/d}\right\} \\ &\quad\leqslant \min\left\{\bar{g}_d^w(B)\alpha^{-1/d},\, \bar{g}_d^w(D)(1-\alpha)^{-1/d}\right\} \leqslant \left(\bar{g}_d^w(B)^d + \bar{g}_d^w(D)^d\right)^{1/d}.\end{aligned}$$

If $\alpha = 0$, then

$$\begin{aligned}&\limsup_{\mathscr{N}_1 \ni N \to \infty} \delta^w(\omega_N) N^{1/d} \\ &\quad \leqslant \limsup_{\mathscr{N}_1 \ni N \to \infty} \delta^w(\omega_N \setminus B) N^{1/d} \leqslant \limsup_{\mathscr{N}_1 \ni N \to \infty} \delta^w_{N_D}(D) N_D^{1/d} \left(\frac{N}{N_D}\right)^{1/d} \\ &\quad \leqslant \bar{g}^w_d(D) \leqslant \left(\bar{g}^w_d(B)^d + \bar{g}^w_d(D)^d\right)^{1/d}.\end{aligned}$$

The case $\alpha = 1$ is handled analogously, which completes the justification of (17).

Now let $\{\overline{\omega}_N\}_{N=2}^{\infty}$ be a sequence of N-point configurations on $B \cup D$ such that for N sufficiently large

$$\delta^w(\overline{\omega}_N) > \delta^w_N(B \cup D) - \frac{1}{N^{2/d}}, \quad \text{if } \delta^w_N(B \cup D) < \infty$$

and

$$\delta^w(\overline{\omega}_N) > N, \quad \text{if } \delta^w_N(B \cup D) = \infty.$$

Since $\{\#(\overline{\omega}_N \cap B)/N\}_{N=2}^{\infty}$ is a bounded sequence, there exists an infinite subset $\mathscr{N}_2 \subset \mathbb{N}$ such that

$$\bar{g}^w_d(B \cup D) = \lim_{\mathscr{N}_2 \ni N \to \infty} \delta^w(\overline{\omega}_N) N^{1/d},$$

and the limit

$$\lim_{\mathscr{N}_2 \ni N \to \infty} \frac{\#(\overline{\omega}_N \cap B)}{N}$$

exists. Then, by (17) we have

$$\bar{g}^w_d(B \cup D) = \lim_{\mathscr{N}_2 \ni N \to \infty} \delta^w(\overline{\omega}_N) N^{1/d} \leqslant \left(\bar{g}^w_d(B)^d + \bar{g}^w_d(D)^d\right)^{1/d},$$

which establishes (14).

Now let $\{\widetilde{\omega}_N\}_{N \in \mathscr{N}}$ be any sequence of N-point configurations in $B \cup D$ such that (16) holds. Choose any subsequence $\mathscr{N}_3 \subset \mathscr{N}$ such that the limit

$$\beta := \lim_{\mathscr{N}_3 \ni N \to \infty} \frac{\#(\widetilde{\omega}_N \cap B)}{N}$$

exists. Assume that both $\bar{g}^w_d(B)$ and $\bar{g}^w_d(D)$ are positive. Then, using the above argument and (16), we have

$$\begin{aligned}\left(\bar{g}^w_d(B)^d + \bar{g}^w_d(D)^d\right)^{1/d} &= \lim_{\mathscr{N}_3 \ni N \to \infty} \delta^w(\widetilde{\omega}_N) N^{1/d} \\ &\leqslant F(\beta) := \min\left\{\bar{g}^w_d(B)\beta^{-1/d},\, \bar{g}^w_d(D)(1-\beta)^{-1/d}\right\}.\end{aligned} \tag{18}$$

The function $F(\beta)$ attains its maximum on $[0,1]$ only at the point

$$\alpha^* := \frac{\bar{g}^w_d(B)^d}{\bar{g}^w_d(B)^d + \bar{g}^w_d(D)^d}$$

and $F(\alpha^*) = \left(\bar{g}^w_d(B)^d + \bar{g}^w_d(D)^d\right)^{1/d}$. Then, in view of (18), we necessarily have $\beta = \alpha^*$.

Now suppose that $\bar{g}^w_d(B) = 0$, but $\bar{g}^w_d(D) > 0$. If $\beta > 0$, we can write

$$\begin{aligned}\bar{g}^w_d(D) = \big(\bar{g}^w_d(B)^d + \bar{g}^w_d(D)^d\big)^{1/d} &= \lim_{\mathscr{N}_3 \ni N \to \infty} \delta^w(\widetilde{\omega}_N) N^{1/d} \\ &\leqslant \lim_{\mathscr{N}_3 \ni N \to \infty} \delta^w(\widetilde{\omega}_N \cap B) N^{1/d} \leqslant \bar{g}^w_d(B)\beta^{-1/d} = 0,\end{aligned}$$

which contradicts the fact that $\bar{g}^w_d(D)$ must be positive. Hence, $\beta = 0 = \alpha^*$. If $\bar{g}^w_d(D) = 0$, we will get $\beta = 1 = \alpha^*$. Thus, we always have $\beta = \alpha^*$. In view of the arbitrariness of the subsequence $\mathscr{N}_3$, we obtain (15).

Lemma 3.2. *Let B and D be two sets in $\mathbb{R}^p$ such that* $\operatorname{dist}(B, D) > 0$. *Suppose that for some $h > 0$, $w(x, y) > h$ for $(x, y) \in B \times D$. Then*

$$\underline{g}^w_d(B \cup D)^d \geqslant \underline{g}^w_d(B)^d + \underline{g}^w_d(D)^d.$$

Proof. We can assume that $0 < \underline{g}^w_d(B), \underline{g}^w_d(D) < \infty$, since otherwise Lemma 3.2 holds trivially. Let

$$\alpha_* := \frac{\underline{g}^w_d(B)^d}{\underline{g}^w_d(B)^d + \underline{g}^w_d(D)^d}$$

and $a := \operatorname{dist}(B, D)$. Let $\{\omega^B_N\}_{N=2}^\infty$ and $\{\omega^D_N\}_{N=2}^\infty$ be sequences of N-point configurations in B and D, respectively, such that

$$\underline{g}^w_d(B) = \liminf_{N\to\infty} \delta^w(\omega^B_N) N^{1/d}, \qquad \underline{g}^w_d(D) = \liminf_{N\to\infty} \delta^w(\omega^D_N) N^{1/d}$$

(these sequences can be chosen in the same way as the sequence $\{\overline{\omega}_N\}$ in the proof of Lemma 3.1). For every $N \in \mathbb{N}$, let $N_B := \lfloor \alpha_* N \rfloor$ and $N_D := N - \lfloor \alpha_* N \rfloor$, where $\lfloor t \rfloor$ denotes the floor function of a number t. Then,

$$\delta^w_N(B \cup D) \geqslant \delta^w(\omega^B_{N_B} \cup \omega^D_{N_D}) \geqslant \min\big\{\delta^w(\omega^B_{N_B}),\ \delta^w(\omega^D_{N_D}),\ ha\big\}.$$

Hence,

$$\begin{aligned}\underline{g}^w_d(B \cup D) &= \liminf_{N\to\infty} \delta^w_N(B \cup D) N^{1/d} \\ &\geqslant \min\Big\{\liminf_{N\to\infty} \delta^w(\omega^B_{N_B}) N^{1/d},\ \liminf_{N\to\infty} \delta^w(\omega^D_{N_D}) N^{1/d},\ \lim_{N\to\infty} haN^{1/d}\Big\} \\ &= \min\left\{\liminf_{N\to\infty} \delta^w(\omega^B_{N_B}) N_B^{1/d} \left(\frac{N}{N_B}\right)^{1/d},\ \liminf_{N\to\infty} \delta^w(\omega^D_{N_D}) N_D^{1/d} \left(\frac{N}{N_D}\right)^{1/d}\right\} \\ &\geqslant \min\big\{\underline{g}^w_d(B)(\alpha^*)^{-1/d},\ \underline{g}^w_d(D)(1-\alpha_*)^{-1/d}\big\} = \big(\underline{g}^w_d(B)^d + \underline{g}^w_d(D)^d\big)^{1/d},\end{aligned}$$

which completes the proof of Lemma 3.2.

§ 4. The case of a weight bounded on the diagonal

Theorem 2.1 is a consequence of the following result.

Lemma 4.1. *Suppose that $A \subset \mathbb{R}^p$ is a compact set with $\mathscr{H}_d(A) < \infty$, and that $w\colon A \times A \to [0, \infty]$ is a* CBD*-weight function on $A \times A$. Furthermore, suppose that for any compact set $K \subset A$ the limit $g_d(K)$ exists and is given by*

$$g_d(K) = C_{\infty,d} \mathscr{H}_d(K)^{1/d}. \tag{19}$$

Then,

(a) $g_d^w(A)$ *exists and is given by*

$$g_d^w(A) = C_{\infty,d}\mathscr{H}_d^w(A)^{1/d}, \tag{20}$$

and

(b) *if a sequence* $\{\widetilde{\omega}_N\}_{N=2}^{\infty}$, *where* $\widetilde{\omega}_N = \{x_1^N, \dots, x_N^N\}$, *is asymptotically* w*-best-packing on the set* A *and* $\mathscr{H}_d(A) > 0$, *then*

$$\frac{1}{N}\sum_{k=1}^{N}\delta_{x_k^N} \xrightarrow{\ *\ } h_d^w, \qquad N \to \infty. \tag{21}$$

Proof. To prove the first part of the lemma (that is, relation (20)), we break A into disjoint 'pieces' of small diameter and estimate the w-best-packing radius of A by replacing w with its supremum or infimum on each of the 'pieces' and applying Lemmas 3.1 and 3.2.

For $\delta > 0$ suppose that $\mathscr{P}_\delta$ is a partition of A such that $\operatorname{diam} P \leqslant \delta$ and $\mathscr{H}_d(\overline{P}) = \mathscr{H}_d(P)$ for $P \in \mathscr{P}_\delta$, where $\overline{B}$ denotes the closure of a set B. For each $P \in \mathscr{P}_\delta$ choose a closed subset $Q_P \subset P$ so that $\mathscr{Q}_\delta := \{Q_P : P \in \mathscr{P}_\delta\}$ satisfies

$$\sum_{P\in\mathscr{P}_\delta} \mathscr{H}_d(Q_P) \geqslant \mathscr{H}_d(A) - \delta. \tag{22}$$

An example of such systems $\mathscr{P}_\delta$ and $\mathscr{Q}_\delta$ can be constructed as follows. Let $G_j[t]$ be the hyperplane in $\mathbb{R}^p$ consisting of all points whose jth coordinate equals t. If $(-a,a)^p$ is a cube containing A, then for $\mathbf{i} = (i_1, \dots, i_p) \in \{1, \dots, m\}^p$ let

$$R_{\mathbf{i}} := [t^1_{i_1-1}, t^1_{i_1}) \times \cdots \times [t^p_{i_p-1}, t^p_{i_p}),$$

where m and partitions $-a = t_0^j < t_1^j < \cdots < t_m^j = a$, $j = 1, \dots, p$, are chosen so that the diameter of any $R_{\mathbf{i}}$, $\mathbf{i} \in \{1, \dots, m\}^p$, is less than δ and $\mathscr{H}_d(G_j[t_i^j] \cap A) = 0$ for all i and j. (Since $\mathscr{H}_d(A) < \infty$, there are at most countably many values of t such that $\mathscr{H}_d(G_j[t] \cap A) > 0$.) Then, we may choose

$$\mathscr{P}_\delta = \bigl\{R_{\mathbf{i}} \cap A : \mathbf{i} \in \{1, \dots, m\}^p\bigr\}$$

and $\gamma \in (0,1)$ sufficiently close to 1 such that (22) holds for

$$\mathscr{Q}_\delta = \bigl\{Q_{\mathbf{i}} : \mathbf{i} \in \{1, \dots, m\}^p\bigr\},$$

where $Q_{\mathbf{i}} = (\gamma(\overline{R}_{\mathbf{i}} - c_{\mathbf{i}}) + c_{\mathbf{i}}) \cap A$ and $c_{\mathbf{i}}$ denotes the centre of $R_{\mathbf{i}}$.

For $B \subset A$, let

$$\overline{w}_B := \sup_{x,y\in B} w(x,y), \qquad \underline{w}_B := \inf_{x,y\in B} w(x,y)$$

and define the simple functions

$$\overline{w}_\delta(x) := \sum_{P\in\mathscr{P}_\delta} \overline{w}_P \chi_P(x), \qquad \underline{w}_\delta(x) := \sum_{P\in\mathscr{P}_\delta} \underline{w}_P \chi_P(x),$$

where $\chi_{_K}$ denotes the characteristic function of a set K. Since the distance between any two sets from $\mathscr{Q}_\delta$ is strictly positive, applying Lemma 3.2 and relation (19) we obtain

$$\begin{aligned}\underline{g}_d^w(A)^d &\geqslant \underline{g}_d^w\Big(\bigcup_{Q\in\mathscr{Q}_\delta} Q\Big)^d \geqslant \sum_{Q\in\mathscr{Q}_\delta} \underline{w}_Q^d \underline{g}_d(Q)^d \\ &\geqslant C_{\infty,d}^d \sum_{Q\in\mathscr{Q}_\delta} \underline{w}_Q^d \mathscr{H}_d(Q) \geqslant C_{\infty,d}^d \int_{\bigcup_{Q\in\mathscr{Q}_\delta} Q} \underline{w}_\delta(x)^d \, d\mathscr{H}_d(x). \end{aligned} \tag{23}$$

Here we assumed that $\delta_N^w(Q) = 0$ if $\#Q = 0$ or 1.

On the other hand, by Lemma 3.1, Theorem 1.1 and properties of the partition $\mathscr{P}_\delta$ we have

$$\begin{aligned}\overline{g}_d^w(A)^d &= \overline{g}_d^w\Big(\bigcup_{P\in\mathscr{P}_\delta} \overline{P}\Big)^d \leqslant \sum_{P\in\mathscr{P}_\delta} \overline{g}_d^w(\overline{P})^d \leqslant \sum_{P\in\mathscr{P}_\delta} \overline{w}_P^d \overline{g}_d(\overline{P})^d \\ &= C_{\infty,d}^d \sum_{P\in\mathscr{P}_\delta} \overline{w}_P^d \mathscr{H}_d(\overline{P}) = C_{\infty,d}^d \sum_{P\in\mathscr{P}_\delta} \overline{w}_P^d \mathscr{H}_d(P) \\ &= C_{\infty,d}^d \int_A \overline{w}_\delta(x)^d \, d\mathscr{H}_d(x). \end{aligned} \tag{24}$$

Since w is a CBD-weight function on $A\times A$, there is some neighbourhood G of $D(A)$ such that $\tau := \sup_G w < \infty$. For $\delta > 0$ sufficiently small, we have $P\times P \subset G$ for all $P \in \mathscr{P}_\delta$, and hence $\underline{w}_\delta(x) \leqslant w(x,x) \leqslant \overline{w}_\delta(x) \leqslant \tau$ for $x\in A$. Furthermore, w is continuous at $(x,x)\in D(A)$ for $\mathscr{H}_d$-almost all $x\in A$ and thus, for any such x, it follows that $\overline{w}_\delta(x)$ and $\underline{w}_\delta(x)$ converge to $w(x,x)$ as $\delta\to 0$. Therefore, by the Lebesgue Dominated Convergence Theorem, the integrals

$$\int_{\bigcup_{Q\in\mathscr{Q}_\delta} Q} \underline{w}_\delta(x)^d \, d\mathscr{H}_d(x), \qquad \int_A \overline{w}_\delta(x)^d \, d\mathscr{H}_d(x)$$

both converge to $\mathscr{H}_d^w(A)$ as $\delta \to 0$. Hence, using (23) and (24), we obtain (20).

We next prove relation (21). Let $\mathscr{H}_d(A) > 0$ and let $\widetilde{\omega}_N := \{x_1^N, \dots, x_N^N\}$, $N\in\mathbb{N}$, be an asymptotically w-best-packing sequence of configurations on A. Choose any set $B\subset A$ whose boundary relative to the set A (denote it by $\partial_A B$) has $\mathscr{H}_d$-measure zero. We shall show that

$$\lim_{N\to\infty} \frac{\#(\widetilde{\omega}_N \cap B)}{N} = h_d^w(B). \tag{25}$$

Note that $\overline{B}$ and $\overline{A\setminus B}$ are compact sets with finite $\mathscr{H}_d$-measure and the restriction of w on each of them is still a CBD-weight function. Every compact subset of these sets is a compact subset of A and hence, these two sets satisfy the assumptions of Lemma 4.1. Moreover, $\mathscr{H}_d^w(\partial_A \overline{B}) = \mathscr{H}_d^w(\partial_A \overline{A\setminus B}) = 0$. Thus,

$$g_d^w(\overline{B}) = C_{\infty,d}\mathscr{H}_d^w(\overline{B})^{1/d} \quad \text{and} \quad g_d^w(\overline{A\setminus B}) = C_{\infty,d}\mathscr{H}_d^w(\overline{A\setminus B})^{1/d},$$

and taking into account relation (20) we obtain that

$$\begin{aligned}\lim_{N\to\infty} \delta^w(\widetilde{\omega}_N)N^{1/d} &= \lim_{N\to\infty} \delta_N^w(A)N^{1/d} = g_d^w(A) = C_{\infty,d}\mathscr{H}_d^w(A)^{1/d} \\ &= C_{\infty,d}\big(\mathscr{H}_d^w(\overline{B}) + \mathscr{H}_d^w(\overline{A\setminus B})\big)^{1/d} = \big(g_d^w(\overline{B})^d + g_d^w(\overline{A\setminus B})^d\big)^{1/d}.\end{aligned}$$

By relation (16) of Lemma 3.1, we have

$$\lim_{N\to\infty}\frac{\#(\widetilde{\omega}_N\cap\overline{B})}{N}=\frac{g_d^w(\overline{B})^d}{g_d^w(\overline{B})^d+g_d^w(\overline{A\setminus B})^d}=h_d^w(\overline{B})=h_d^w(B). \tag{26}$$

Since relation (26) holds for any closed subset of A whose boundary relative to A has $\mathscr{H}_d$-measure zero, it follows that

$$\lim_{N\to\infty}\frac{\#(\widetilde{\omega}_N\cap(\overline{B}\setminus B))}{N}\leqslant\lim_{N\to\infty}\frac{\#(\widetilde{\omega}_N\cap\partial_A B)}{N}=h_d^w(\partial_A B)=0.$$

Consequently, (25) holds for all sets $B\subset A$ whose boundary relative to the set A has $\mathscr{H}_d$-measure zero. As mentioned in Section 1, this implies relation (21), which completes the proof.

Proof of Theorem 2.1. Since A is a compact d-rectifiable set, it follows that $\mathscr{H}_d(A)<\infty$ and any closed subset K of A is also a compact d-rectifiable set. By Theorem 1.1, relation (19) holds for K. Then Theorem 2.1 follows from Lemma 4.1.

§ 5. The case of the weight with singularities on the diagonal

Asymptotic behaviour of $\delta_N^w(A)$. The hypotheses of Theorem 2.3 imply that

$$\mathscr{H}_d^w(A)=\int_A w(x,x)^d\,d\mathscr{H}_d(x)$$

is finite and positive (see the proof below) and hence the same is true for $g_d^w(A)$. The essential ingredient in the proof of Theorem 2.3 is the following lemma which assumes lower regularity. Consistent with the definition in (10), we say that a set $K\subset\mathbb{R}^p$ is *lower α-regular at $a\in K$*, if there are positive constants C_0 and r_0 such that

$$C_0^{-1}r^\alpha\leqslant\mathscr{H}_\alpha(K\cap B(x,r)),\qquad x\in K\cap B(a,r_0),\quad 0<r<r_0.$$

Lemma 5.1. *Suppose that $\alpha>0$ and let $K\subset\mathbb{R}^p$ be a compact set that is lower α-regular at a point $a\in K$. Let $w\colon K\times K\to[0,\infty]$ be a weight function with a singularity at a of order at most $t<1$. Then there is a constant $C_1=C_1(w,K,t,\alpha)$ such that for any λ sufficiently small*

$$\bar{g}_\alpha^w(K\cap B(a,\lambda))\leqslant C_1\biggl(\int_{K\cap B(a,2\lambda)}\frac{1}{|x-a|^{t\alpha}}\,d\mathscr{H}_\alpha(x)\biggr)^{1/\alpha}. \tag{27}$$

Proof. Let r_0 and C_0 be as in the definition of lower α-regularity of the set K at a, and let C and δ be as in the definition of a being a singularity of w of order at most $t<1$.

Choose any $0<\lambda<\min\{r_0,\delta\}$. Let $\omega_N=\{x_1,\dots,x_N\}$ be an arbitrary configuration of N distinct points in $K\cap B(a,\lambda)$. For $i=1,\dots,N$, let $\rho_i=|x_i-a|$, $r_i=\min_{j:j\neq i}|x_j-x_i|$ and let y_i be a point in ω_N such that $|x_i-y_i|=r_i$.

Since $K\cap B(a,\lambda)$ is bounded, there are at most $L-1$ points $x_i\in\omega_N$ (for example, one could take $L=3^p+1$) such that $r_i\geqslant\lambda$. Reorder the points in ω_N so that $\rho_N\leqslant\rho_i$, $i=1,\dots,N-1$, and $r_i<\lambda$, $i=1,\dots,N-L$. Then,

$$\delta^w(\omega_N)=\min_{i\neq j}w(x_i,x_j)|x_i-x_j|\leqslant\min_{i=1,\dots,N-L}w(x_i,y_i)|x_i-y_i|.$$

For every $i = 1, \dots, N-L$, since $x_i, y_i \in K \cap B(a, \lambda) \subset K \cap B(a, \delta)$, we have

$$w(x_i, y_i) \leqslant \frac{C}{|x_i - a|^t}.$$

Hence,

$$\delta^w(\omega_N) \leqslant \min_{i=1,\dots,N-L} \frac{Cr_i}{|x_i - a|^t} = \min_{i=1,\dots,N-L} \frac{Cr_i}{\rho_i^t} \leqslant C\left(\frac{1}{N-L}\sum_{i=1}^{N-L} \frac{r_i^\alpha}{\rho_i^{t\alpha}}\right)^{1/\alpha}.$$

For $i = 1, \dots, N-1$,

$$r_i = \min_{j:j\neq i} |x_j - x_i| \leqslant |x_i - a| + \min_{j:j\neq i} |a - x_j| \leqslant \rho_i + \rho_N \leqslant 2\rho_i.$$

For every $x \in B(x_i, r_i/2)$, $i = 1, \dots, N-1$, we also have

$$|x - a| \leqslant |x - x_i| + |x_i - a| \leqslant \frac{r_i}{2} + \rho_i \leqslant 2\rho_i.$$

Taking into account the lower α-regularity of the set K at a, it is not difficult to see that

$$\begin{aligned}\frac{r_i^\alpha}{\rho_i^{t\alpha}} &\leqslant 2^\alpha C_0 \mathscr{H}_\alpha\left(K \cap B\left(x_i, \frac{r_i}{2}\right)\right)\frac{1}{\rho_i^{t\alpha}} \\ &\leqslant 2^\alpha C_0 \int_{K\cap B(x_i, r_i/2)} \frac{1}{\rho_i^{t\alpha}}\, d\mathscr{H}_\alpha(x) \leqslant 2^{\alpha(t+1)} C_0 \int_{K\cap B(x_i, r_i/2)} \frac{1}{|x-a|^{t\alpha}}\, d\mathscr{H}_\alpha(x).\end{aligned}$$

Consequently,

$$\delta^w(\omega_N) \leqslant 2^{t+1} C_0^{1/\alpha} C (N-L)^{-1/\alpha}\left(\sum_{i=1}^{N-L} \int_{K\cap B(x_i, r_i/2)} \frac{1}{|x-a|^{t\alpha}}\, d\mathscr{H}_\alpha(x)\right)^{1/\alpha}.$$

Since ω_N is an arbitrary N-point collection in $K \cap B(a, \lambda)$ and $B(x_i, r_i/2) \cap B(x_j, r_j/2) = \varnothing$, $i \neq j$, we can write

$$\delta_N^w(K \cap B(a, \lambda)) \leqslant 2^{t+1} C_0^{1/\alpha} C (N-L)^{-1/\alpha}\left(\int_{K\cap B(a, 2\lambda)} \frac{1}{|x-a|^{t\alpha}}\, d\mathscr{H}_\alpha(x)\right)^{1/\alpha}.$$

Hence,

$$\begin{aligned}\bar{g}_\alpha^w(K \cap B(a, \lambda)) &= \limsup_{N\to\infty} \delta_N^w(K \cap B(a, \lambda))(N-L)^{1/\alpha} \\ &\leqslant C_1\left(\int_{K\cap B(a, 2\lambda)} \frac{1}{|x-a|^{t\alpha}}\, d\mathscr{H}_\alpha(x)\right)^{1/\alpha},\end{aligned}$$

where $C_1 = 2^{t+1} C_0^{1/\alpha} C$, which completes the proof of Lemma 5.1.

Proof of Theorem 2.3. If K is α-regular, $0<\alpha\leqslant d$, as opposed to only lower α-regular at a, then the integral

$$\int_K \frac{1}{|x-a|^{t\alpha}}\ \ d\mathscr{H}_\alpha(x)$$

is finite (cf. [15], p. 109) and by absolute continuity of the Lebesgue integral we have

$$\lim_{\lambda\to 0}\int_{K\cap B(a,2\lambda)}\frac{1}{|x-a|^{t\alpha}}\, d\mathscr{H}_\alpha(x)=0.$$

By Lemma 5.1 we then have $\lim_{\lambda\to 0}\bar{g}^w_\alpha(K\cap B(a,\lambda))=0$ for any $0<\alpha\leqslant d$. Hence $\lim_{\lambda\to 0}\bar{g}^w_d(K\cap B(a,\lambda))=0$.

The α_i-regularity of A at a_i and the fact that w has a singularity of order at most $t<1$ at a_i, $i=1,\dots,n$, imply that

$$\int_A w(x,x)^d\, d\mathscr{H}_d(x)<\infty.$$

Suppose that $\varepsilon>0$. By Lemmas 5.1 and 3.1 we can find $\delta>0$ such that $B_\delta:=\bigcup_{i=1}^n A\cap B(a_i,\delta)$ satisfies $\bar{g}^w_d(B_\delta)<\varepsilon$ and for $A_\delta:=A\setminus B_\delta$,

$$\mathscr{H}^w_d(A_\delta)=\int_{A_\delta} w(x,x)^d\, d\mathscr{H}_d(x)\geqslant(1-\varepsilon)\mathscr{H}^w_d(A).$$

Since w is a CBD-weight function on $A_\delta\times A_\delta$, it follows from Theorem 2.1 that $g^w_d(A_\delta)$ exists and is given by $g^w_d(A_\delta)=C_{\infty,d}\mathscr{H}^w_d(A_\delta)^{1/d}$. By Lemma 3.1 we then get

$$\begin{aligned}\bar{g}^w_d(A)&\leqslant\left(\bar{g}^w_d(A_\delta)^d+\bar{g}^w_d(B_\delta)^d\right)^{1/d}\\&\leqslant\left(C^d_{\infty,d}\mathscr{H}^w_d(A_\delta)+\varepsilon^d\right)^{1/d}\leqslant\left(C^d_{\infty,d}\mathscr{H}^w_d(A)+\varepsilon^d\right)^{1/d}.\end{aligned}\tag{28}$$

We also have

$$\underline{g}^w_d(A)\geqslant\underline{g}^w_d(A_\delta)=C_{\infty,d}\mathscr{H}^w_d(A_\delta)^{1/d}\geqslant C_{\infty,d}(1-\varepsilon)^{1/d}\mathscr{H}^w_d(A)^{1/d}.\tag{29}$$

Taking $\varepsilon\to 0$ in (28) and (29) shows that $g^w_d(A)$ exists and that (8) also holds under the assumptions of Theorem 2.3.

We next show that (9) holds for the case of a weight with singularities. Let $B\subset A$ be any set such that $\mathscr{H}_d(\partial_A B)=0$. For every $\delta>0$ such that

$$\mathscr{H}_d(A\cap\partial_{\mathbb{R}^p}B(a_i,\delta))=0,\qquad i=1,\dots,n$$

(there are at most countably many δ's for which this does not hold) consider sets $B_1=B\cup\left(\bigcup_{i=1}^n A\cap B(a_i,\delta)\right)$ and $A\setminus B_1$. Note that

$$\mathscr{H}^w_d(\partial_A B_1)=\mathscr{H}^w_d(\partial_A A\setminus B_1)=0.$$

Both sets are d-rectifiable. Since $\overline{A\setminus B_1}$ is also a compact subset of $A\setminus\{a_1,\dots,a_n\}$, w is a CBD-weight function on $\overline{A\setminus B_1}\times\overline{A\setminus B_1}$. Then, by Theorem 2.1,

$$g^w_d(\overline{A\setminus B_1})=C_{\infty,d}\mathscr{H}^w_d(A\setminus B_1)^{1/d}.$$

The set $\overline{B}_1$ is compact and, for $\delta > 0$ sufficiently small, is α_i-regular at a_i, $i = 1, \ldots, n$. Restriction of w on $\overline{B}_1 \times \overline{B}_1$ will be a weight with singularities of order at most $t < 1$ at $a_1, \ldots, a_n$. Then, by (8), we have

$$g_d^w(\overline{B}_1) = C_{\infty,d}\mathscr{H}_d^w(B_1)^{1/d}.$$

Let $\{\overline{\omega}_N\}_{N=2}^{\infty}$, $\#\overline{\omega}_N = N$, be any asymptotically w-best-packing sequence. Then

$$\begin{aligned}\lim_{N\to\infty} \delta^w(\overline{\omega}_N)N^{1/d} &= g_d^w(A) = C_{\infty,d}\mathscr{H}_d^w(A)^{1/d} \\ &= C_{\infty,d}\big(\mathscr{H}_d^w(B_1) + \mathscr{H}_d^w(A \setminus B_1)\big)^{1/d} = \big(g_d^w(\overline{B}_1)^d + g_d^w(\overline{A \setminus B_1})^d\big)^{1/d}.\end{aligned}$$

Since $\mathscr{H}_d^w(A) > 0$, one of the quantities $\mathscr{H}_d^w(\overline{B}_1)$ or $\mathscr{H}_d^w(\overline{A \setminus B_1})$ has to be positive. Using relation (15) from Lemma 3.1, we have

$$\limsup_{N\to\infty} \frac{\#(\overline{\omega}_N \cap B)}{N} \leqslant \lim_{N\to\infty} \frac{\#(\overline{\omega}_N \cap \overline{B}_1)}{N} = \frac{g_d^w(\overline{B}_1)^d}{g_d^w(\overline{B}_1)^d + g_d^w(\overline{A \setminus B_1})^d} = h_d^w(B_1).$$

Since $\lim_{\delta\to 0} h_d^w(B_1) = h_d^w(B)$, we get that

$$\limsup_{N\to\infty} \frac{\#(\overline{\omega}_N \cap B)}{N} \leqslant h_d^w(B). \tag{30}$$

Repeating the same argument for the set $B_1 = (A \setminus B) \cup \big(\bigcup_{i=1}^n A \cap B(a_i, \delta)\big)$, we deduce that

$$\limsup_{N\to\infty} \frac{\#(\overline{\omega}_N \cap (A \setminus B))}{N} \leqslant h_d^w(A \setminus B).$$

Hence,

$$\liminf_{N\to\infty} \frac{\#(\overline{\omega}_N \cap B)}{N} \geqslant h_d^w(B).$$

Combining this relation with (30) yields

$$\lim_{N\to\infty} \frac{\#(\overline{\omega}_N \cap B)}{N} = h_d^w(B),$$

which implies that (9) also holds under the assumptions of Theorem 2.3.

Proof of Theorem 2.4. From relation (11), for any configuration

$$\omega_N := \{x_1, \ldots, x_N\} \subset A$$

we have

$$\begin{aligned}\delta^{v_N}(\omega_N) &= \min_{i\neq j} v_N(x_i, x_j)|x_i - x_j| \\ &\geqslant (1 - \varepsilon_N) \min_{i\neq j} w(x_i, x_j)|x_i - x_j| = (1 - \varepsilon_N)\delta^w(\omega_N).\end{aligned} \tag{31}$$

Analogously,

$$\delta^{v_N}(\omega_N) \leqslant (1 + \varepsilon_N)\delta^w(\omega_N). \tag{32}$$

Hence,

$$(1-\varepsilon_N)\delta_N^w(A) \leqslant \delta_N^{v_N}(A) \leqslant (1+\varepsilon_N)\delta_N^w(A),$$

and using Theorem 2.1 or 2.3, we get that

$$\lim_{N\to\infty} \delta_N^{v_N}(A)N^{1/d} = \lim_{N\to\infty} \delta_N^w(A)N^{1/d} = C_{\infty,d}(\mathscr{H}_d^w(A))^{1/d}. \tag{33}$$

Now let $\{\omega_N\}_{N=2}^\infty$ be an asymptotically optimal sequence of configurations for the sequence of weights $\{v_N\}_{N=2}^\infty$. Then, taking into account (33), we get

$$\delta^{v_N}(\omega_N) = \delta_N^{v_N}(A)(1+o(1)) = \delta_N^w(A)(1+o(1)), \qquad N\to\infty.$$

Relations (31) and (32) also imply that

$$\delta^{v_N}(\omega_N) = \delta^w(\omega_N)(1+o(1)), \qquad N\to\infty.$$

Then

$$\delta^w(\omega_N) = \delta_N^w(A)(1+o(1)), \qquad N\to\infty,$$

that is, the sequence $\{\omega_N\}_{N=2}^\infty$ is asymptotically w-best-packing on the set A. Applying again Theorem 2.1 or Theorem 2.3, we get (12).

Bibliography

[1] L. Fejes Tóth, *Lagerungen in der Ebene, auf der Kugel und im Raum*, Springer-Verlag, Berlin–Göttingen–Heidelberg 1953.

[2] L. Fejes Tóth, *Regular figures*, Macmillan, New York; Pergamon Press, Oxford 1964.

[3] C. A. Rogers, *Packing and covering*, Cambridge Tracts in Mathematics and Mathematical Physics, vol. 54, Cambridge Univ. Press, New York 1964.

[4] J. H. Conway and N. J. A. Sloane, *Sphere packings, lattices and groups*, Grundlehren Math. Wiss., vol. 290, Springer-Verlag, New York 1999.

[5] K. Böröczky, jr., *Finite packing and covering*, Cambridge Tracts in Math., vol. 154, Cambridge Univ. Press, Cambridge 2004.

[6] Th. C. Hales, "A proof of the Kepler conjecture", *Ann. of Math.* (2) **162**:3 (2005), 1065–1185.

[7] V. F. Babenko, "On the optimal error bound for cubature formulae on certain classes of continuous functions", *Anal. Math.* **3**:1 (1977), 3–9.

[8] A. N. Kolmogorov and V. M. Tikhomirov, "ε-entropy and ε-capacity of sets in functional spaces", *Uspekhi Mat. Nauk* **14**:2 (1959), 3–86; English transl. in *Amer. Math. Soc. Transl.* (2) **17** (1961), 227–364.

[9] V. Afraimovich and L. Glebsky, "Measures of ε-complexity", *Taiwanese J. Math.* **9**:3 (2005), 397–409.

[10] E. V. Chornaya, "On the optimization of weighted cubature formulae on certain classes of continuous functions", *East J. Approx.* **1**:1 (1995), 47–60.

[11] W. Habicht and B. L. van der Waerden, "Lagerung von Punkten auf der Kugel", *Math. Ann.* **123**:1 (1951), 223–234.

[12] B. L. van der Waerden, "Punkte auf der Kugel. Drei Zusätze", *Math. Ann.* **125**:1 (1952), 213–222.

[13] S. V. Borodachov, D. P. Hardin and E. B. Saff, "Asymptotics of best-packing on rectifiable sets", *Proc. Amer. Math. Soc.* **135**:8 (2007), 2369–2380.

[14] H. Federer, *Geometric measure theory*, Grundlehren Math. Wiss., vol. 153, Springer-Verlag, Berlin–Heidelberg–New York 1969.

[15] P. Mattila, *Geometry of sets and measures in Euclidean spaces. Fractals and rectifiability*, Cambridge Stud. Adv. Math., vol. 44, Cambridge Univ. Press, Cambridge 1995.

[16] K. Böröczky, "Packing of spheres in spaces of constant curvature", *Acta Math. Acad. Sci. Hungar.* **32**:3–4 (1978), 243–261.

[17] L. Bowen and Ch. Radin, "Densest packing of equal spheres in hyperbolic space", *Discrete Comput. Geom.* **29**:1 (2003), 23–39.

[18] S. V. Borodachov, D. P. Hardin and E. B. Saff, "Asymptotics for discrete weighted minimal Riesz energy problems on rectifiable sets", *Trans. Amer. Math. Soc.* **360**:3 (2008), 1559–1580.

[19] V. A. Yudin, "The minimum of potential energy of a system of point charges", *Diskret. Mat.* **4**:2 (1992), 115–121; English transl. in *Discrete Math. Appl.* **3**:1 (1993), 75–81.

[20] N. N. Andreev, "Disposition points on a sphere with minimum of energy", *Approximation theory. Harmonic analysis*, Tr. Mat. Inst. Steklova, vol. 219, Nauka, Moscow 1997, pp. 27–31; English transl. in *Proc. Steklov Inst. Math.* **219** (1997), 20–24.

[21] A. V. Kolushov and V. A. Yudin, "Extremal dispositions of points on the sphere", *Anal. Math.* **23**:1 (1997), 25–34.

[22] D. P. Hardin and E. B. Saff, "Discretizing manifolds via minimum energy points", *Notices Amer. Math. Soc.* **51**:10 (2004), 1186–1194.

[23] D. P. Hardin and E. B. Saff, "Minimal Riesz energy point configurations for rectifiable d-dimensional manifolds", *Adv. Math.* **193**:1 (2005), 174–204.

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