MENKE POINTS ON THE REAL LINE AND THEIR CONNECTION TO CLASSICAL ORTHOGONAL POLYNOMIALS

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Abstract. We investigate the properties of extremal point systems on the real line consisting of two interlaced sets of points solving a modified minimum energy problem. We show that these extremal points for the intervals $[-1,1]$, $[0,\infty)$ and $(-\infty,\infty)$, which are analogues of Menke points for a closed curve, are related to the zeros and extrema of classical orthogonal polynomials. Use of external fields in the form of suitable weight functions instead of constraints motivates the study of "weighted Menke points" on $[0,\infty)$ and $(-\infty,\infty)$. We also discuss the asymptotic behavior of the Lebesgue constant for the Menke points on $[-1,1]$.

Dedicated to Jesus Dehesa on the occasion of his 60^{th} *birthday.*

1. Introduction

Let q and p be two positive numbers representing charges at the left endpoint and right endpoint, respectively, of the interval $[-1,1]$. The problem of finding n points $x_1^{(n)},\dots,x_n^{(n)}$, the locations of unit point charges, in the interior of $[-1,1]$ such that the expression

$$T_n(x_1,\dots,x_n) := \prod_{i=1}^{n}(1-x_i)^p \prod_{j<k}|x_j-x_k| \prod_{\ell=1}^{n}(1+x_\ell)^q \tag{1.1}$$

is maximized, or equivalently, $\log(1/T_n)$ is minimized over all n-point systems $x_1,\dots,x_n$ in $[-1,1]$, is a classical problem that owes its solution to Stieltjes [12]. The quantity $\log(1/T_n)$ can be interpreted as the potential energy of the point charges at $x_1,\dots,x_n$ in an external field

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exerted by the charge p at $x = 1$ and the charge q at $x = -1$, where the "points" interact according to a logarithmic potential. Stieltjes showed that the points $x_1^{(n)}, \cdots, x_n^{(n)}$ of minimal logarithmic energy are, in fact, the zeros of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$, where $\alpha = 2p - 1$ and $\beta = 2q - 1$. The zeros of Laguerre and Hermite polynomials admit a similar interpretation (see, for example, [13, Theorems 6.7.2 and 6.7.3]). Additional constraints are needed to prevent these zeros from escaping to infinity. A more modern approach is to have external fields in form of appropriate weight functions instead of constraints. (See, for example, [1] for a discussion of this model.) We also refer the interested reader to the short survey article [4].

In this note we investigate the properties of extremal point systems on the real intervals $[-1, 1]$, $[0, \infty)$, and $(-\infty, \infty)$, that consist of two interlaced sets of points solving a modified minimum energy problem. We will see that these extremal points are related to the zeros and extrema of classical orthogonal polynomials. Moreover, each of these two interlaced sets solves a separate extremal problem. In the unbounded case an additional constraint is needed that prevents points from escaping to infinity. We show that this constraint can be lifted by introducing an external field in form of a suitable weight function. We will also discuss the effectiveness of a certain class of extremal points on the interval $[-1, 1]$ by considering the associated Lebesgue constants.

Our study was motivated by the work of K. Menke [5], [6] who introduced certain interlaced optimal point sets on closed analytic Jordan curves $\mathcal{C}$ in the complex plane. To describe such points, we first provide $\mathcal{C}$ with a positive orientation (denoted by $\prec$) and let $w_1, \dots, w_n$ and $z_1, \dots, z_n$ be two sets of points on $\mathcal{C}$ interlaced in the following way:

$$z_1 \prec w_1 \prec \cdots \prec w_{n-1} \prec z_n \prec w_n \prec z_1. \tag{1.2}$$

Then points that maximize the resultant

$$R_n(z_1, \dots, z_n, w_1, \dots, w_n) := \prod_{j=1}^{n} \prod_{k=1}^{n} |z_j - w_k| \tag{1.3}$$

are called *Menke points*.

Recall that N points $\zeta_1^{(N)}, \cdots, \zeta_N^{(N)}$ of a compact set K of the complex plane that maximize the product $\prod_{j \neq k} |\zeta_j - \zeta_k|$ over all N-point subsets of K are known as *Fekete points* for K. In particular, according to the previously mentioned result of Stieltjes, the zeros of $(1 - x^2) P_{N-2}^{(1,1)}(x)$ form an N-point Fekete set for the interval $[-1, 1]$. One intriguing result in [5] is the comparison of Menke and Fekete points for a closed analytic Jordan curves $\mathcal{C}$. If Ψ is a conformal mapping of the exterior of the unit disk onto the exterior of $\mathcal{C}$ with $\Psi(\infty) = \infty$, then the pre-images under Ψ of $2n$ Menke points are more nearly equally spaced on the unit circle (error decays geometrically) than the pre-images of $2n$ Fekete points (error decays with $1/n$) as n becomes large. (See also [8], [9], and [11].)

To define the Menke points for a closed infinite subset A of the real line $\mathbb{R}$, we consider two finite sets of points $X = \{x_j\}$ and $Y = \{y_k\}$ such that $X \cup Y \subset A$ with X and Y interlaced. In contrast to the case of a closed curve, we need to distinguish two cases according to the parity of the total number of points $|X \cup Y|$.

2. Menke points on the interval $[-1, 1]$

Let q and p be two positive numbers. Analogous to the Stieltjes problem mentioned above, for the case of an odd number (say $2n + 1$) of interlaced points on the interval $A = [-1, 1]$,

$$-1 = y_0 < x_1 < y_1 < \cdots < y_{n-1} < x_n < y_n = 1, \tag{2.1}$$

we maximize the function

$$\mathcal{T}_n(x_1, \ldots, x_n, y_1, \ldots, y_{n-1}) := \prod_{i=1}^{n} (1 - x_i)^p \prod_{j=1}^{n} \prod_{k=1}^{n-1} |x_j - y_k| \prod_{\ell=1}^{n} (1 + x_\ell)^q \tag{2.2}$$

over all configurations satisfying (2.1), while for an even number of points (say $2n$), we assume that

$$-1 = x_1 < y_1 < \cdots < y_{n-1} < x_n < y_n = 1 \tag{2.3}$$

and maximize the function

$$\tau_n(x_2, \ldots, x_n, y_1, \ldots, y_{n-1}) := \prod_{i=2}^{n} (1 - x_i)^p \prod_{j=2}^{n} \prod_{k=1}^{n-1} |x_j - y_k| \prod_{\ell=2}^{n-1} (1 + y_\ell)^q \tag{2.4}$$

over all configurations satisfying (2.3).

A system $X = \{x_j\}$, $Y = \{y_k\}$ for which the maximum is attained in (2.2) or (2.4) is called a (p, q)-*Menke system for* $[-1, 1]$. In our definition, we always regard the endpoints of the interval $[-1, 1]$ as Menke points; there is no loss of generality in such an assumption since if, say, y_0 and y_n were regarded as variable points with $-1 \leq y_0 < x_1$ and $x_n < y_n \leq 1$, then maximizing the quantity

$$\mathcal{T}_n^*(x_1, \ldots, x_n, y_0, \ldots, y_n) := \prod_{i=1}^{n} (y_n - x_i)^p \prod_{j=1}^{n} \prod_{k=1}^{n-1} |x_j - y_k| \prod_{\ell=1}^{n} (x_\ell - y_0)^q$$

would clearly imply that $y_0 = -1$ and $y_n = 1$.

It turns out that Menke points are related to the zeros and extrema of Jacobi polynomials. We shall prove in Section 7 the following:

Theorem 1. *Let $p > 0$, $q > 0$, and let the points in (2.1) form a (p, q)-Menke system for $[-1, 1]$ maximizing (2.2). Then the points $x_1, \dots, x_n$ are the zeros and the points $y_0, \dots, y_n$ are the extrema of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ on $[-1, 1]$, where $\alpha = p - 1$, $\beta = q - 1$.*

In particular, for $p = q$, the Menke points satisfy

$$x_{n-k+1} = -x_k, \quad k = 1, \dots, n, \qquad y_{n-k} = -y_k, \quad k = 0, \dots, n.$$

Theorem 2. *Let $p > 0$, $q > 0$, and let the points in (2.3) form a (p, q)-Menke system for $[-1, 1]$ maximizing (2.4). Then the points $x_2, \dots, x_n$ and the points $y_1, \dots, y_{n-1}$ are the zeros of the Jacobi polynomial $P_{n-1}^{(p-1,q)}(x)$ and $P_{n-1}^{(p,q-1)}(x)$, respectively.*

In particular, for $p = q$, the Menke points satisfy

$$y_{n-k+1} = -x_k, \qquad k = 1, \dots, n.$$

Theorems 1 and 2 should be compared with the previously mentioned result of Stieltjes concerning the minimization of (1.1). In fact, on combining that result with Theorem 1, we deduce that the x-points of a (p, q)-Menke system on $[-1, 1]$ with $(2n + 1)$ points solve a separate extremal problem. The same holds true for the y-points.

Corollary 3. *Under the hypotheses of Theorem 1 the points $x_1, \dots, x_n$ maximize the product*

$$\prod_{i=1}^{n} (1 - x_i)^{p/2} \prod_{j<k} |x_j - x_k| \prod_{\ell=1}^{n} (1 + x_\ell)^{q/2}, \tag{2.5}$$

and the points $y_1, \dots, y_{n-1}$ maximize the product

$$\prod_{i=1}^{n-1} (1 - y_i)^{(p+1)/2} \prod_{j<k} |y_j - y_k| \prod_{\ell=1}^{n-1} (1 + y_\ell)^{(q+1)/2}. \tag{2.6}$$

Proof. Note that the critical points of $P_n^{(\alpha,\beta)}$ are the zeros of $P_{n-1}^{(\alpha+1,\beta+1)}$. □

A similar corollary, whose statement is left to the reader, follows from Theorem 2 and aforementioned result of Stieltjes.

In Section 6 we will discuss the effectiveness of Menke points for polynomial interpolation by considering the associated Lebesgue constants for the case $p = q = 1$.

3. Menke points on the interval $[0,\infty)$

For an unbounded closed set, the existence of Menke points requires that an additional constraint be imposed. For the interval $[0,\infty)$, the setting for the Laguerre polynomials, we impose this constraint on the centroid of the x-points.

Given a positive charge p and an even number (say $2n$) of interlaced points

$$0 = y_0 < x_1 < y_1 < \cdots < x_{n-1} < y_{n-1} < x_n, \tag{3.1}$$

we seek to maximize the function

$$U_n(x_1,\ldots,x_n,y_1,\ldots,y_{n-1}) := \prod_{k=1}^{n} x_k^p \prod_{j=1}^{n}\prod_{k=1}^{n-1} |x_j - y_k| \tag{3.2}$$

subject to the additional condition that the centroid of the x-points satisfies

$$\frac{1}{n}(x_1 + \cdots + x_n) = K, \tag{3.3}$$

where K is a pre-assigned positive real number.

In the case of an odd number of points we maximize the function

$$U_n(x_0,\ldots,x_n,y_1,\ldots,y_n) := \prod_{k=1}^{n} y_k^p \prod_{j=1}^{n}\prod_{k=1}^{n} |x_j - y_k| \tag{3.4}$$

subject to the condition that the x-points and y-points are interlaced,

$$0 = x_0 < y_1 < x_1 < \cdots < y_{n-1} < x_{n-1} < y_n < x_n, \tag{3.5}$$

and, again, the x-centroid satisfies (3.3).

A solution of either of these optimization problems will be called a *p-Menke system for $[0,\infty)$ with x-centroid at K*. Notice that the left endpoint zero is regarded as a Menke point in this setting (again without loss of generality).

We shall prove the following results (see Section 7).

Theorem 4. *Given $p > 0$ and $K > 0$, let (3.1) form a $(2n)$-point p-Menke system for $[0,\infty)$ with x-centroid at K. Then the points $x_1,\ldots,x_n$ are the zeros and the points $y_0,\ldots,y_{n-1}$ are the extrema of the generalized Laguerre polynomial $L_n^{(\alpha)}(ct)$ on $[0,\infty)$, where $\alpha = p-1$ and $c = (n+\alpha)/K$.*

Recall that the generalized Laguerre polynomials $L_n^{(\alpha)}(t)$, where $\alpha > -1$, are orthogonal on the interval $[0,\infty)$ with respect to the weight function $t^{\alpha}e^{-t}$.

Theorem 5. *Given $p > 0$ and $K > 0$, let (3.5) form a $(2n+1)$-point p-Menke system for $[0,\infty)$ with x-centroid at K. Then the points $x_1, \dots, x_n$ and the points $y_1, \dots, y_n$ are, respectively, the zeros of the Laguerre polynomials $L_n^{(p)}(ct)$ and $L_n^{(p-1)}(ct)$, where $c = (n+p)/K$.*

Theorems 4 and 5 should be compared with the following classical result.

Proposition 6 ([13, Thm. 6.7.2])**.** *For a positive charge p at the fixed point $x = 0$ and unit point charges at the variable points $x_1, \dots, x_n$ in the interval $[0,\infty)$ such that the x-centroid satisfies*

$$\frac{1}{n}\left(x_1 + \cdots + x_n\right) \leq K, \tag{3.6}$$

where K is a pre-assigned positive real number, the maximum of

$$\prod_{i=1}^{n} x_i^p \prod_{j<k} |x_j - x_k| \tag{3.7}$$

is attained if and only if the $x_1, \dots, x_n$ are the zeros of the Laguerre polynomial $L_n^{(\alpha)}(ct)$, where $\alpha = 2p-1$ and $c = (n+\alpha)/K$.

From Theorem 4 and Proposition 6 it follows that the x-points of a p-Menke system for $[0,\infty)$ with centroid K solve an extremal problem. The same holds for the y-points.

Corollary 7. *Under the hypotheses of Theorem 4, the points $x_1, \dots, x_n$ maximize the product*

$$\prod_{i=1}^{n} x_i^{p/2} \prod_{j<k} |x_j - x_k| \tag{3.8}$$

subject to the constraint

$$\frac{1}{n}\left(x_1 + \cdots + x_n\right) \leq K, \tag{3.9}$$

and the points $y_1, \dots, y_{n-1}$ maximize the product

$$\prod_{i=1}^{n-1} y_i^{1+p/2} \prod_{j<k} |y_j - y_k| \tag{3.10}$$

subject to the constraint

$$y_1 + \cdots + y_{n-1} \leq n\,(K-1) + p. \tag{3.11}$$

An analogous corollary, whose statement is left to the reader, follows from Theorem 5 and Proposition 6.

4. Menke points on the interval $(-\infty, \infty)$

For the real line, the setting of the classical Hermite polynomials, we define the Menke points by imposing a constraint on the moment of inertia of the points. First we consider Menke systems with an odd number of points. For unit charges at the points $x_1, \dots, x_n, y_1, \dots, y_{n-1}$ in $(-\infty, \infty)$ we want to maximize the function

$$V_n(x_1, \dots, x_n, y_1, \dots, y_{n-1}) := \prod_{j=1}^{n} \prod_{k=1}^{n-1} |x_j - y_k| \tag{4.1}$$

subject to the conditions that

$$x_1 < y_1 < x_2 < \dots < y_{n-1} < x_n \tag{4.2}$$

and the x-moment of inertia satisfies

$$\frac{1}{n}\left(x_1^2 + \dots + x_n^2\right) = L, \tag{4.3}$$

where L is a pre-assigned positive real number. A solution of this optimization problem will be called *Menke system for* $(-\infty, \infty)$ *with* x*-moment of inertia* L.

Theorem 8. *Let* (4.2) *form a* $(2n-1)$*-point Menke system for* $(-\infty, \infty)$ *with* x*-moment of inertia* $L(> 0)$*. Then the points* $x_1, \dots, x_n$ *are the zeros and the points* $y_1, \dots, y_{n-1}$ *are the extrema of the Hermite polynomial* $H_n(ct)$ *with* $c = \sqrt{(n-1)/(2L)}$*. Furthermore, the Menke points satisfy*

$$x_{n-k+1} = -x_k, \quad k = 1, \dots, n, \qquad y_{n-k} = -y_k, \quad k = 1, \dots, n-1.$$

Recall that the Hermite polynomials $H_n(t)$ are orthogonal on the interval $(-\infty, \infty)$ with respect to the weight function e^{-t^2}.

Theorem 8 should be compared with the following classical result.

Proposition 9 ([13, Thm. 6.7.3])**.** *For unit point masses at each of the variable points* $x_1, \dots, x_n$ *in* $(-\infty, \infty)$ *such that the "moment of inertia" satisfies*

$$\frac{1}{n}\left(x_1^2 + \cdots x_n^2\right) \le L, \tag{4.4}$$

where L *is a pre-assigned positive real number, the maximum of*

$$\prod_{j<k} |x_j - x_k| \tag{4.5}$$

is attained if and only if the points $x_1, \dots, x_n$ *are the zeros of the Hermite polynomial* $H_n(ct)$*,* $c = \sqrt{(n-1)/(2L)}$*.*

From Theorem 8 and Proposition 9 it follows that both the x-points and the y-points of a $(2n+1)$-point Menke system solve an extremal problem.

Corollary 10. *Under the hypotheses of Theorem 8, the points $x_1, \dots, x_n$ maximize the product*

$$\prod_{j<k} |x_j - x_k| \tag{4.6}$$

subject to the constraint

$$\frac{1}{n}\left(x_1^2 + \cdots + x_n^2\right) \le L, \tag{4.7}$$

and the points $y_1, \dots, y_{n-1}$ maximize the product

$$\prod_{j<k} |y_j - y_k| \tag{4.8}$$

subject to the constraint

$$y_1^2 + \cdots + y_{n-1}^2 \le n\,(L-1) + 1. \tag{4.9}$$

We next consider the optimization problem for an even number of points. A *Menke system of $2n$ points for $(-\infty, \infty)$ with total moment of inertia L* is a collection of points $x_1, \dots, x_n, y_1, \dots, y_n$ maximizing the function

$$V_n(x_1, \dots, x_n, y_1, \dots, y_n) := \prod_{j=1}^{n} \prod_{k=1}^{n} |x_j - y_k| \tag{4.10}$$

subject to the conditions that

$$x_1 < y_1 < x_2 < \cdots < y_{n-1} < x_n < y_n \tag{4.11}$$

and the total moment of inertia satisfies

$$\frac{1}{2n}\left(x_1^2 + \cdots + x_n^2 + y_1^2 + \cdots + y_n^2\right) = L, \tag{4.12}$$

where L is a pre-assigned positive real number.

We shall prove in Section 7 the following result.

Theorem 11. *Let the $2n$ points, $n \ge 1$, in (4.11) form a Menke system for $(-\infty, \infty)$ with total moment of inertia $L (> 0)$. Then the points $x_1, \dots, x_n$ are the zeros of the polynomial*

$$H_n(ct) + \sqrt{2n} H_{n-1}(ct) \tag{4.13}$$

and the points $y_1, \dots, y_n$ are the zeros of the polynomial

$$H_n(ct) - \sqrt{2n} H_{n-1}(ct), \tag{4.14}$$

where $c = \sqrt{(n+1)/(4L)}$. *Furthermore*

$$y_{n-k+1} = -x_k, \qquad k = 1, \ldots, n,$$

and

$$y_1 + \cdots + y_n = -(x_1 + \cdots + x_n) = \sqrt{\frac{2nL}{n+1}}.$$

5. Weighted Menke points on unbounded intervals

Additional constraints are needed to prevent points from escaping to infinity when solving the corresponding modified energy problems for Menke points on unbounded intervals. A more modern approach is to replace constraints with external fields in the form of suitably chosen weight functions. The monograph [10] deals in detail with the logarithmic minimum energy problem when an external field is applied. Ismail [1] gave an electrostatic model for zeros for general orthogonal polynomials subject to certain integrability conditions on their weight function $w(x)$. The modified energy problem for Menke points will, however, differ in at least one significant detail: the points (charges) are decomposed into two interlaced configurations each of which may be subject to its own external field. That is, we seek to maximize a function of the type

$$\prod_j \prod_k w_x(x_j) w_y(y_k) \left| x_j - y_k \right|$$

subject to interlacing constraints on the x and y points in the presence of given weight functions w_x and w_y.

5.1. **Weighted p-Menke Points on $[0, \infty)$.** Let p be a positive charge placed at zero. Let $w_x(t) = t^{s_x} \exp\{-\lambda_x t\}$, $s_x \geq 0$ and $\lambda_x > 0$. We similarly define w_y but with constants s_y and and λ_y instead of s_x and λ_x. For an odd number of points (counting $x_0 = 0$) we seek to maximize the function

(5.1)

$$F_n^o(x_1, \ldots, x_n, y_1, \ldots, y_n) := \prod_{k=1}^{n} \left[w_y(y_k) y_k \right]^p \prod_{j=1}^{n} \prod_{k=1}^{n} w_x(x_j) w_y(y_k) \left| x_j - y_k \right|$$

subject to the conditions (3.5), and for an even number (counting $y_0 = 0$) of points we maximize the function

(5.2)

$$F_n^e(x_1, \ldots, x_n, y_1, \ldots, y_{n-1}) := \prod_{j=1}^{n} \left[w_x(x_j) x_j \right]^p \prod_{j=1}^{n} \prod_{k=1}^{n-1} w_x(x_j) w_y(y_k) \left| x_j - y_k \right|$$

subject to the constraints (3.1).

A solution of either of these optimization problems will be called a *weighted p-Menke system for* $[0,\infty)$ *with weight functions* w_x *and* w_y. (Without loss of generality we regard the point at zero as a Menke point.)

We shall prove the following results (see Section 7).

Theorem 12. *Given* $p > 0$. *Let the* $2n+1$ *points in* (3.5) *form a weighted p-Menke system for* $[0,\infty)$ *with weight functions* $w_x(t) = t^{s_x}\exp\{-\lambda_x t\}$, $s_x \geq 0$, $\lambda_x > 0$, *and* $w_y(t) = t^{s_y}\exp\{-\lambda_y t\}$, $s_y \geq 0$, $\lambda_y > 0$. *Then the points* $x_1,\dots,x_n$ *are the zeros of the polynomial*

$$nL_n^{(\alpha-1)}(\beta t) + b\Delta L_{n-1}^{(\alpha-1)}(\beta t) \tag{5.3}$$

and the points $y_1,\dots,y_n$ *are the zeros of the polynomial*

$$nL_n^{(\alpha-1)}(\beta t) - d\Delta L_{n-1}^{(\alpha-1)}(\beta t), \tag{5.4}$$

where $a = p + (p+n)s_y$, $b = (p+n)\lambda_y$, $c = ns_x$, $d = n\lambda_x$, *and* $\alpha = 1+a+c$, $\beta = b+d$. *The quantity* Δ *is the positive solution of*

$$(n+a+b\Delta)(n+c-d\Delta) = ac. \tag{5.5}$$

Moreover, $\Delta = x_1+\cdots+x_n-y_1-\cdots-y_n$.

Remark 13. Theorem 12 is the analogue of Theorem 5. For "switched off" y-field (that is $s_y = 0$ and $\lambda_y \to 0$) and $s_x = 0$ and $n\lambda_x K = p+n$ both theorems give the same Menke points.

Theorem 14. *Given* $p > 0$. *Let the* $2n(\geq 4)$ *points in* (3.1) *form a weighted p-Menke system for* $[0,\infty)$ *with weight functions* w_x *and* w_y *as in Theorem 12. Then the points* $x_1,\dots,x_n$ *are the zeros of the polynomial*

$$L_n^{(\alpha-2)}(\beta t) - (s_y - \lambda_y t)\, L_{n-1}^{(\alpha-1)}(\beta t) \tag{5.6}$$

and the points $y_1,\dots,y_{n-1}$ *are the zeros of the generalized Laguerre polynomial*

$$L_{n-1}^{(\alpha-1)}(\beta t), \tag{5.7}$$

where $\alpha = 1+p+(p+n-1)s_x+ns_y$ *and* $\beta = (p+n-1)\lambda_x + n\lambda_y$. *Furthermore,* $\lambda_x(\bar{x}-\bar{y}) = 1+s_x$ *and* $\beta\bar{y} = (n-1)(n+\alpha-2)$, *where* $\bar{x} = x_1+\cdots+x_n$ *and* $\bar{y} = y_1+\cdots+y_{n-1}$. *The* x*-centroid* $\bar{x}/n$ *satisfies*

$$\beta\,(\bar{x}/n) = (p+n-1+\lambda_y/\lambda_x)\,(1+s_x) + (n-1)\,s_y.$$

Remark 15. Theorem 14 is the analogue of Theorem 4 but the former has a much wider scope. For "switched off" external y-field (that is $s_y = 0$ and letting $\lambda_y \to 0$) and $s_x = 0$ and $\lambda_x K = 1$ (K is the prescribed x-centroid in Theorem 4) both theorems give the same results.

5.2. **Weighted Menke Points on** $(-\infty, \infty)$**.** We consider the weight functions $w_x(t) = \exp\{-\lambda_x t^2\}$, $\lambda_x > 0$, and $w_y(t) = \exp\{-\lambda_y t^2\}$, $\lambda_y > 0$. For an even number of points we seek to maximize the function

$$G_n^e(x_1, \ldots, x_n, y_1 \ldots, y_n) := \prod_{j=1}^{n} \prod_{k=1}^{n} w_x(x_j) w_y(y_k) \, |x_j - y_k| \tag{5.8}$$

subject to the conditions (4.11), and for an odd number of points we maximize the function

$$G_n^o(x_1, \ldots, x_n, y_1, \ldots, y_{n-1}) := \prod_{j=1}^{n} \prod_{k=1}^{n-1} w_x(x_j) w_y(y_k) \, |x_j - y_k| \tag{5.9}$$

subject to the conditions (4.2).

A solution of either of these optimization problems will be called a *weighted Menke system for* $(-\infty, \infty)$ *with weight functions* w_x *and* w_y.

We shall prove the following results (see Section 7).

Theorem 16. *Let the* $2n(\geq 2)$ *points in* (4.11) *form a weighted Menke system for* $(-\infty, \infty)$ *with weight functions* w_x *and* w_y *as given above. Then the points* $x_1, \ldots, x_n$ *are the zeros of the polynomial*

$$H_n(\beta t) + \sqrt{2 \, (\lambda_y/\lambda_x) \, n} H_{n-1}(\beta t) \tag{5.10}$$

and the points $y_1, \ldots, y_n$ *are the zeros of the polynomial*

$$H_n(\beta t) - \sqrt{2 \, (\lambda_x/\lambda_y) \, n} H_{n-1}(\beta t), \tag{5.11}$$

where $\beta = \sqrt{(\lambda_x + \lambda_y) n}$.

Remark 17. Theorem 16 is the analogue of Theorem 11. If $\lambda_x = \lambda_y = \lambda$, that is x-points and y-points are subject to the same external field, then the parameter λ characterizing the external field in Theorem 16 and the total moment of inertia L (pre-assigned in Theorem 11) given by relation (4.12) are connected via formula $8n\lambda L = n + 1$.

Theorem 18. *Let the* $2n - 1(\geq 3)$ *points in* (4.2) *form a weighted Menke system for* $(-\infty, \infty)$ *with weight functions* w_x *and* w_y *as above. Then the points* $x_1, \ldots, x_n$ *are the zeros of the polynomial*

$$H_n(\beta t) - 2 \, (\lambda_y/\lambda_x) \, n H_{n-2}(\beta t) \tag{5.12}$$

and the points $y_1, \ldots, y_{n-1}$ *are the zeros of the polynomial*

$$H_{n-1}(\beta t), \tag{5.13}$$

where $\beta = \sqrt{(n-1)\lambda_x + n\lambda_y}$. *Furthermore,*

$$x_{n-j+1} = -x_j, \qquad j = 1, \ldots, n,$$
$$y_{n-k} = -y_k, \qquad k = 1, \ldots, n-1.$$

Remark 19. Theorem 18 is the analogue of Theorem 8. If $\lambda_x = \lambda_y = \lambda$ and λ (characterizing the field in Theorem 18) and L (the total moment of inertia in Theorem 8) satisfy the relation $n - 1 = 2L(2n-1)\lambda$, then both theorems give the same Menke points.

6. Lebesgue constants for Menke Points on [-1,1]

In this section we consider the Lebesgue functions and Lebesgue constants for the Menke points on $[-1,1]$ for the case $p = q = 1$.

It follows from Theorem 1 that, for such p and q, the $(2n-1)$ Menke points on $[-1,1]$ are the zeros of $(1-x^2)P_{n-1}(x)P'_{n-1}(x)$, where $P_n(x)$ is the *n-th* Legendre polynomial. We denote this set of points by

$$\mathcal{M}(2n-1) := \{t_1^{(2n-1)}, t_2^{(2n-1)}, \cdots, t_{2n-1}^{(2n-1)}\} \tag{6.1}$$

and let $\ell_j(t), j = 1, \cdots, 2n-1$, denote the fundamental Lagrange polynomials for this set of points, that is $\ell_j(t)$ is a polynomial of degree $2n-2$ satisfying

$$\ell_j(t_i^{(2n-1)}) = \delta_{ij}, \qquad i, j = 1, \cdots, 2n-1. \tag{6.2}$$

The *Lebesgue function* for the set of points $\mathcal{M}(2n-1)$ is given by

$$\Lambda_{\mathcal{M}}(t, 2n-1) := \sum_{j=1}^{2n-1} |\ell_j(t)| \tag{6.3}$$

and the corresponding *Lebesgue constant* is

$$\Lambda_{\mathcal{M}}(2n-1) := \max_{-1 \le t \le 1} \Lambda_{\mathcal{M}}(t, 2n-1). \tag{6.4}$$

For an even number of points, we know from Theorem 2 that for $p = q = 1$ the $2n$ Menke points are the zeros of $(1-x^2)P_{n-1}^{(0,1)}(x)P_{n-1}^{(1,0)}(x)$. Denoting this set of points by $\mathcal{M}(2n)$, we similarly define the associated Lebesgue function $\Lambda_{\mathcal{M}}(t, 2n)$ and the Lebesgue constant $\Lambda_{\mathcal{M}}(2n)$.

Recall that the Lebesgue constant $\Lambda_{\mathcal{M}}(N)$ is the norm of the associated interpolation operator from $C[-1,1]$ to the space of polynomials of degree at most $N-1$, defined via interpolation in the points of $\mathcal{M}(N)$. As such it provides a measure of how close polynomial interpolants approximate a continuous function. More precisely, if $f \in C[-1,1]$ and $L_{\mathcal{M}(N)}(t)$ is the unique polynomial of degree $N-1$ that interpolates f

in the N points of $\mathcal{M}(N)$, then for the uniform norm on $[-1, 1]$ there holds

$$\left\| f - L_{\mathcal{M}(N)} \right\| \leq (1 + \Lambda_{\mathcal{M}}(N)) E_{N-1}(f),$$

where $E_{N-1}(f)$ denotes the error in best uniform approximation to f by polynomials of degree at most $N - 1$.

It is well-known that the Lebesgue constants $\Lambda_{\mathcal{T}}(N)$ for *any* triangular scheme $\mathcal{T}$ of the interpolation points on $[-1, 1]$ grow with order at least $\log N$ as $n \to \infty$ (see [3] for the historical discussion and the characterization of optimal schemes). As we shall prove in a later paper, this optimal growth rate is also achieved for the Menke points.

Theorem 20. *The Lebesgue constants for the* $(1, 1)$*-Menke points* $\mathcal{M}(N)$ *on* $[-1, 1]$ *satisfy*

$$\Lambda_{\mathcal{M}}(N) = \mathcal{O}(\log N) \qquad \text{as } N \to \infty.$$

As a numerical example of the effectiveness of the interpolation in the Menke points we computed the maximum (uniform) error in the Lagrange interpolation for the simple function $f(x) = |x|$, $-1 \leq x \leq 1$. For comparison purposes, Table 1 lists the maximum error over $[-1, 1]$ when the interpolation points are chosen to be

(1) zeros of the Chebyshev polynomial $T_N(x)$ rescaled so that the first and last zeros coincide, respectively, with -1 and 1;
(2) the $(1, 1)$-Menke points for the interval $[-1, 1]$;
(3) the Fekete points for the interval $[-1, 1]$.

We see from Table 1 on the next page that the maximum absolute error of interpolation is least for the listed values of N if the interpolation points (even or odd) are chosen to be the Menke points in comparison to the Chebyshev points or the Fekete points. A plot of the corresponding Lebesgue functions over $[-1, 1]$ reveals that the Menke Lebesgue function is smaller than the other Lebesgue functions if we stay away from the endpoints. Thus, it is reasonable that the interpolation in the Menke points might better approximate functions such as $|x|$ that have singularities only in the interior of the interval.

7. Proofs

7.1. **Proof of Theorem 1.** We give an argument similar to that in [13]. For a (p, q)-Menke system of $2n+1$ points, we have $\partial \log \mathcal{T}_n / (\partial x_\ell) =$

TABLE 1. Maximum error of Lagrange Interpolation in N points for $f(x) = |x|$ on $[-1, 1]$

N	Chebyshev	Menke	Fekete
8	0.129946	0.122238	0.135684
9	0.068008	0.066984	0.070580
12	0.084777	0.081689	0.087608
13	0.046472	0.046136	0.047867
16	0.063106	0.061455	0.064782
17	0.035361	0.035211	0.036227
20	0.050309	0.049285	0.051416
21	0.028556	0.028477	0.029144
24	0.041845	0.041148	0.042629
25	0.023955	0.023908	0.024379
35	0.017083	0.017080	0.017308
36	0.027831	0.027531	0.028191
44	0.022756	0.022558	0.023001
45	0.013278	0.013270	0.013416
55	0.010860	0.010859	0.010954
56	0.017871	0.017750	0.018024
64	0.015634	0.015542	0.015751
65	0.009188	0.009185	0.009255
75	0.007962	0.007961	0.008013
76	0.013163	0.013098	0.0132473

0 ($\ell = 1, \dots, n$) and $\partial \log \mathcal{T}_n / (\partial y_\ell) = 0$ ($\ell = 1, \dots, n-1$), or

$$\frac{p}{x_\ell - 1} + \sum_{k=1}^{n-1} \frac{1}{x_\ell - y_k} + \frac{q}{x_\ell + 1} = 0, \qquad \ell = 1, \dots, n, \tag{7.1}$$

$$\sum_{k=1}^{n} \frac{1}{y_\ell - x_k} = 0, \qquad \ell = 1, \dots, n-1. \tag{7.2}$$

Introducing the polynomials $f(x) = (x - x_1)(x - x_2) \cdots (x - x_n)$ and $g(y) = (y - y_1)(y - y_2) \cdots (y - y_{n-1})$, we observe that, by (7.2),

$$\frac{f'(y_\ell)}{f(y_\ell)} = \sum_{k=1}^{n} \frac{1}{y_\ell - x_k} = 0, \qquad \ell = 1, \dots, n-1. \tag{7.3}$$

Since f has no zeros at $y_1, \dots, y_{n-1}$, it follows that the polynomial $f'(x)$ of degree $(n-1)$ vanish at $(n-1)$ points $x = y_1, \dots, y_{n-1}$. Thus, $f'(x)$

is a multiple of $g(x)$. The latter and (7.1) yield

$$\frac{f''(x_\ell)}{f'(x_\ell)} = \frac{g'(x_\ell)}{g(x_\ell)} = \sum_{k=1}^{n-1} \frac{1}{x_\ell - y_k} = -\frac{q}{x_\ell + 1} + \frac{p}{1 - x_\ell} \tag{7.4}$$

for all $\ell = 1, \ldots, n$, or equivalently,

$$\left(1 - x_\ell^2\right) f''(x_\ell) + [q - p - (p+q)x_\ell]\, f'(x_\ell) = 0, \quad \ell = 1, \ldots, n. \tag{7.5}$$

With $\alpha = p - 1$ and $\beta = q - 1$, the last relation means that

$$(1 - x^2) f''(x) + [\beta - \alpha - (\alpha + \beta + 2)x] f'(x) \tag{7.6}$$

is a polynomial of degree n which vanish for all zeros of $f(x)$. Hence, it is a multiple of $f(x)$. By comparing the terms in x^n, we get for the constant factor the expression $-n(n + \alpha + \beta + 1)$. The polynomial solutions of the resulting linear homogeneous differential equation of the second order,

$$\begin{aligned} \left(1 - x^2\right) f''(x) + [\beta - \alpha - (\alpha + \beta + 2)x]\, f'(x) \\ + n\,(n + \alpha + \beta + 1)\, f(x) = 0, \end{aligned} \tag{7.7}$$

are the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, and therefore $f(x)$ is a constant multiple of $P_n^{(\alpha,\beta)}(x)$ and $g(y)$ is a constant multiple of $[P_n^{(\alpha,\beta)}]'(y) = (1/2)(n+\alpha+\beta+1)P_{n-1}^{(\alpha+1,\beta+1)}(y)$. The last assertion of the theorem follows immediately from symmetry properties of the Jacobi polynomials when $p = q$. □

7.2. **Proof of Theorem 2.** For a (p, q)-point Menke system of $2n$ points, we have the conditions $\partial \log \tau_n/(\partial x_\ell) = 0$ ($\ell = 2, \ldots, n$) and $\partial \log \tau_n/(\partial y_\ell) = 0$ ($\ell = 1, \ldots, n - 1$), or

$$\frac{p}{x_\ell - 1} + \sum_{j=1}^{n-1} \frac{1}{x_\ell - y_j} = 0, \qquad \ell = 2, \ldots, n, \tag{7.8}$$

$$\frac{q}{y_\ell + 1} + \sum_{j=2}^{n} \frac{1}{y_\ell - x_j} = 0, \qquad \ell = 1, \ldots, n - 1. \tag{7.9}$$

Setting $f(x) := (x - x_2) \cdots (x - x_n)$ and $g(x) = (x - y_1) \cdots (x - y_{n-1})$, we have by equation (7.9) we have

$$\frac{f'(y_\ell)}{f(y_\ell)} = \sum_{j=2}^{n} \frac{1}{y_\ell - x_j} = -\frac{q}{1 + y_\ell}, \qquad \ell = 1, \ldots, n - 1. \tag{7.10}$$

Since $f(x)$ has no zeros at $y_1, \ldots, y_{n-1}$, it follows that the polynomial $(1+x)f'(x) + pf(x)$ of degree $(n-1)$ vanish at $x = y_1, \ldots, y_{n-1}$. Thus

$$(1 + x)\, f'(x) + qf(x) = (n - 1 + q)\, g(x). \tag{7.11}$$

Similarly, we obtain

$$(1-x)\,g'(x)-pg(x)=(1-n-p)\,f(x) \tag{7.12}$$

Eliminating $g(x)$ from (7.11) and (7.12), we get

$$\begin{aligned}\left(1-x^2\right)f''(x)+\left[(1-p+q)-(1+p+q)\,x\right]f'(x)\\ +(n-1)\,(p+q+n-1)\,f(x)=0.\end{aligned} \tag{7.13}$$

For $\alpha=p-1$ and $\beta=q$ the above equation represents the differential equation of $(n-1)$-st Jacobi polynomial $P_{n-1}^{(\alpha,\beta)}(x)$. Thus, $f(x)$ is a constant multiple of $P_{n-1}^{(\alpha,\beta)}(x)$. Similarly, by eliminating $f(x)$ from (7.11) and (7.12) and proceeding as before, we arrive at the differential equation

$$\begin{aligned}\left(1-x^2\right)g''(x)+\left[(-1-p+q)-(1+p+q)\,x\right]g'(x)\\ +(n-1)\,(p+q+n-1)\,g(x)=0.\end{aligned} \tag{7.14}$$

By taking $\alpha=p$ and $\beta=q-1$ it follows that $g(x)$ is a constant multiple of $P_{n-1}^{(\alpha,\beta)}(x)$. □

7.3. **Proof of Theorem 4.** Clearly, an extremal system exists and, via convexity argument, one can show it is unique. Maximizing the function U_n is equivalent with minimizing $F_n:=\log(1/U_n)$. Defining $h(x_1,\dots,x_n):=(x_1+\dots+x_n)/n-K$, we have the following necessary conditions for optimality:

$$\nabla_{\mathbf{x}}F_n=-\lambda\nabla_{\mathbf{x}}h,\qquad \nabla_{\mathbf{y}}F_n=-\lambda\nabla_{\mathbf{y}}h, \tag{7.15}$$

or equivalently,

$$-\frac{p}{x_\ell}-\sum_{k=1}^{n-1}\frac{1}{x_\ell-y_k}=-\frac{\lambda}{n},\qquad \ell=1,\dots,n, \tag{7.16}$$

$$-\sum_{j=1}^{n}\frac{1}{y_\ell-x_j}=0,\qquad \ell=1,\dots,n-1. \tag{7.17}$$

Setting $f(x)=(x-x_1)(x-x_2)\cdots(x-x_n)$ and $g(y)=(y-y_1)(y-y_2)\cdots(y-y_{n-1})$, we get from (7.17) that

$$\frac{f'(y_\ell)}{f(y_\ell)}=\sum_{j=1}^{n}\frac{1}{y_\ell-x_j}=0,\qquad \ell=1,\dots,n-1. \tag{7.18}$$

Since $f(y_\ell)\neq 0$ for all $\ell=1,\dots,n-1$, we have that f', a polynomial of degree $n-1$, has the same zeros as g. Thus, $f'(x)$ is a constant multiple of $g(x)$. The latter and (7.16) yield

$$\frac{f''(x_\ell)}{f'(x_\ell)} = \frac{g'(x_\ell)}{g(x_\ell)} = \sum_{k=1}^{n-1} \frac{1}{x_\ell - y_k} = \frac{\lambda}{n} - \frac{p}{x_\ell}, \qquad \ell = 1, \dots, n, \tag{7.19}$$

or equivalently,

$$x_\ell f''(x_\ell) + \left(p - \frac{\lambda}{n} x_\ell\right) f'(x_\ell) = 0, \qquad \ell = 1, \dots, n. \tag{7.20}$$

The left-hand side of (7.20) is a polynomial of degree n in x that vanishes at n points $x_1, \dots, x_n$; hence this polynomial is a constant multiple of $f(x)$. We get

$$x f''(x) + \left(p - \frac{\lambda}{n} x\right) f'(x) + \lambda f(x) = 0. \tag{7.21}$$

A change of variables $u = cx$ with $nc = \lambda$ and the substitution $\alpha + 1 = p$ lead to the associated Laguerre differential equation

$$u w'' + (\alpha + 1 - u)\, w' + n w = 0, \tag{7.22}$$

whose polynomial solutions are the associated Laguerre polynomials $w = L_n^{(\alpha)}(u)$. Thus, $f(x)$ is a constant multiple of $L_n^{(\alpha)}(cx)$ and $g(y)$ is a constant multiple of the derivative $f'(y)$. The constant c can be obtained from the relation

$$cnK = cx_1 + \cdots + cx_n = u_1 + \cdots + u_n \tag{7.23}$$

and the fact that the sum of zeros $u_1 + \cdots + u_n$ equals $n(n + \alpha)$ which follows from

$$L_n^{(\alpha)}(u) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \binom{n+\alpha}{n-k} u^k = \frac{(-1)^n}{n!} (u - u_1) \cdots (u - u_n).$$

□

The proof of Theorem 5 is similar to the preceding argument and is therefore omitted.

7.4. **Proof of Theorem 11.** Since the proof of the Theorem 8 is more straightforward, we leave it to the reader and proceed with the proof of Theorem 11.

It is easily seen that an extremal system exists and is unique. Let $h(x_1, \dots, x_n, y_1, \dots, y_n)$ be defined as the difference between the left-hand side and the right-hand side of (4.12). Maximizing the function V_n is equivalent with minimizing $F_n := \log(1/V_n)$. We have the following necessary conditions for optimality:

$$\nabla_{\mathbf{x}} F_n = -\lambda \nabla_{\mathbf{x}} h, \qquad \nabla_{\mathbf{y}} F_n = -\lambda \nabla_{\mathbf{y}} h, \tag{7.24}$$

or equivalently,

$$-\sum_{k=1}^{n} \frac{1}{x_\ell - y_k} = -\frac{\lambda x_\ell}{n}, \qquad \ell = 1, \dots, n, \tag{7.25}$$

$$-\sum_{j=1}^{n} \frac{1}{y_\ell - x_j} = -\frac{\lambda y_\ell}{n}, \qquad \ell = 1, \dots, n. \tag{7.26}$$

We introduce the polynomials $f(x) := (x - x_1)(x - x_2) \cdots (x - x_n)$ and $g(y) := (y - y_1)(y - y_2) \cdots (y - y_n)$. Using (7.25) and (7.26), we get

$$\frac{f'(y_\ell)}{f(y_\ell)} = \sum_{j=1}^{n} \frac{1}{y_\ell - x_j} = \frac{\lambda y_\ell}{n}, \qquad \ell = 1, \dots, n, \tag{7.27}$$

$$\frac{g'(x_\ell)}{g(x_\ell)} = \sum_{k=1}^{n} \frac{1}{x_\ell - y_k} = \frac{\lambda x_\ell}{n}, \qquad \ell = 1, \dots, n. \tag{7.28}$$

Since the expressions are symmetric (that is, can be obtained from each other by substituting g for f and x for y), it is sufficient to consider one relation. The expression $f'(x) - (\lambda/n) x f(x)$ is a polynomial of degree $n + 1$ which vanishes at $x = y_1, \dots, y_n$. Hence

$$f'(x) - \frac{\lambda}{n} x f(x) = c\,(x + \Delta)\, g(x) \tag{7.29}$$

for some non-negative constant c and a real zero $-\Delta$. The constant c and the zero $-\Delta$ can be obtained from comparing the coefficients of the following expansions

$$\begin{aligned} -\frac{\lambda}{n} x f(x) &= -\frac{\lambda}{n} x^{n+1} + \frac{\lambda}{n} (x_1 + \cdots + x_n)\, x^n + \cdots, \\ c\,(x + \Delta)\, g(x) &= c x^{n+1} - c\,(-\Delta + y_1 + \cdots + y_n)\, x^n + \cdots. \end{aligned}$$

We obtain $c = -\lambda/n$ and $\Delta = \bar{y} - \bar{x}$, where we defined $\bar{x}{:=}x_1 + \cdots + x_n$ and $\bar{y}{:=}y_1 + \cdots + y_n$. Combining these facts we deduce that

$$f'(x) - \frac{\lambda}{n} x f(x) + \frac{\lambda}{n} (x + \Delta)\, g(x) = 0, \tag{7.30}$$

$$g'(x) - \frac{\lambda}{n} x g(x) + \frac{\lambda}{n} (x - \Delta)\, f(x) = 0. \tag{7.31}$$

Adding and subtracting these equations yield

$$\begin{aligned} (f + g)' - \frac{\lambda}{n} \Delta\, (f - g) &= 0, \\ (f - g)' - 2\frac{\lambda}{n} x\, (f - g) + \frac{\lambda}{n} \Delta\, (f + g) &= 0. \end{aligned}$$

Setting $2F = f + g$ and $2G = f - g$, we have

$$F'(x) - \frac{\lambda\Delta}{n} G(x) = 0, \tag{7.32}$$

$$G'(x) - \frac{2\lambda}{n} x G(x) + \frac{\lambda\Delta}{n} F(x) = 0. \tag{7.33}$$

Next, we take the derivative of both sides of (7.32) and use (7.33) to derive the second order differential equation

$$F''(x) - \frac{2\lambda}{n} x F'(x) + \left(\frac{\lambda\Delta}{n}\right)^2 F(x) = 0. \tag{7.34}$$

A change of variables $u = \sqrt{\lambda/n}x$ leads to the Hermite differential equation

$$w'' - 2uw' + \frac{\lambda}{n}\Delta^2 w = 0, \tag{7.35}$$

which has a polynomial solution if and only if $\lambda\Delta^2/n$ is an even positive integer. In the case $\lambda\Delta^2 = 2n^2$ its solution is a constant multiple of the Hermite polynomial $H_n(u)$ of degree n. Thus,

$$F(x) = H_n(\frac{\sqrt{2n}}{\Delta} x). \tag{7.36}$$

From (7.32) we obtain

$$G(x) = \frac{\Delta}{2n} F'(x). \tag{7.37}$$

Consequently, $f = F + G$ is a multiple of

$$H_n(\frac{\sqrt{2n}}{\Delta} x) + \sqrt{2n} H_{n-1}(\frac{\sqrt{2n}}{\Delta} x), \qquad n = 1, 2, \ldots, \tag{7.38}$$

and $g = F - G$ is a multiple of

$$H_n(\frac{\sqrt{2n}}{\Delta} x) - \sqrt{2n} H_{n-1}(\frac{\sqrt{2n}}{\Delta} x), \qquad n = 1, 2, \ldots, \tag{7.39}$$

which also justifies the identity $g(x) = (-1)^n f(-x)$. The last identity implies that $\bar{y} = -\bar{x}$, $\Delta = 2\bar{y} = -2\bar{x}$ ($\bar{x} < \bar{y}$ follows from the ordering of the points $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$), and $\sum_{j=1}^n x_j^2 = \sum_{k=1}^n y_k^2$. This allows us to relate the total moment of inertia L of the given Menke system and the difference $\Delta = \bar{y} - \bar{x}$. Using

$$H_n(z) = n! \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k}{k!(n-2k)!} (2z)^{n-2k}, \tag{7.40}$$

we compare the coefficient of x^{n-2} in (7.38) and in

$$f(x) = x^n - \bar{x}x^{n-1} + \left(\bar{x}^2 - \sum_{j=1} x_j^2\right) x^{n-2} + \cdots . \tag{7.41}$$

Only the first Hermite polynomial in (7.38) needs to be considered. That is

$$\left(\frac{\sqrt{8n}}{\Delta}\right)^n x^n + n! \frac{-1}{(n-2)!} \left(\frac{\sqrt{8n}}{\Delta}\right)^{n-2} x^{n-2} + \cdots = cf(x). \tag{7.42}$$

We get

$$c = \left(\frac{\sqrt{8n}}{\Delta}\right)^n, \qquad -\frac{n-1}{8}\Delta^2 = \bar{x}^2 - \sum_{j=1}^{n} x_j^2, \tag{7.43}$$

and it follows that

$$\sum_{k=1}^{n} y_k^2 = \sum_{j=1}^{n} x_j^2 = \frac{\Delta^2}{4} + \frac{n-1}{2}\frac{\Delta^2}{4} = \frac{n+1}{8}\Delta^2.$$

From (4.12), we have

$$2nL = \sum_{k=1}^{n} x_k^2 + \sum_{j=1}^{n} x_j^2 = \frac{n+1}{4}\Delta^2, \tag{7.44}$$

or equivalently,

$$\Delta = \sqrt{\frac{8n}{n+1}L}, \qquad n = 1, 2, \dots . \tag{7.45}$$

□

7.5. **Proof of Theorem 12.** It is easy to see that an extremal system exists. Let

$$f(x) := (x - x_1)(x - x_2) \cdots (x - x_n) = x^n - f_1 x^{n-1} + \cdots ,$$
$$g(y) := (y - y_1)(y - y_2) \cdots (y - y_n) = y^n - g_1 y^{n-1} + \cdots .$$

Maximizing the function F_n^o is equivalent to minimizing the function $V_n^o := \log(1/F_n^o)$. The corresponding optimality conditions give

$$\frac{f'(y_\ell)}{f(y_\ell)} = \sum_{j=1}^{n} \frac{1}{y_\ell - x_j} = -(p+n)\left(\frac{s_y}{y_\ell} - \lambda_y\right) - \frac{p}{y_\ell}, \qquad \ell = 1, \dots, n,$$

$$\frac{g'(x_\ell)}{g(x_\ell)} = \sum_{k=1}^{n} \frac{1}{x_\ell - y_k} = -n\left(\frac{s_x}{x_\ell} - \lambda_x\right), \qquad \ell = 1, \dots, n.$$

Since $f(y_\ell) \neq 0$ for all $\ell = 1, \ldots, n$, we have for some constant $A = A_n$

$$xf'(x)+[p + (p + n)\, s_y - (p + n)\, \lambda_y x]\, f(x) = -\,(p + n)\, \lambda_y\, (x - A)\, g(x). \tag{7.46}$$

Similarly, for some constant $B = B_n$

$$xg'(x) + n\,(s_x - \lambda_x x)\, g(x) = -n\lambda_x\,(x - B)\, f(x). \tag{7.47}$$

To simplify notation we write

$$xf'(x) + (a - bx)\, f(x) + b\,(x - A)\, g(x) = 0, \tag{7.48}$$

$$xg'(x) + (c - dx)\, g(x) + d\,(x - B)\, f(x) = 0, \tag{7.49}$$

where we define the quantities

$$a := p + (p + n)\, s_y, \quad b := (p + n)\, \lambda_y, \quad c := ns_x, \quad d := n\lambda_x. \tag{7.50}$$

It is easy to see that

$$b\,(A + g_1 - f_1) = n + a, \qquad d\,(B + f_1 - g_1) = n + c. \tag{7.51}$$

In particular, it follows that

$$bd\,(A + B) = (b + d)\, n + ad + bc. \tag{7.52}$$

Evaluating (7.48) and (7.49) at $x = 0$ and using that f and g do not vanish at $x = 0$, we derive

$$ac - bdAB = 0. \tag{7.53}$$

It is convenient to introduce new functions F and G via $f = F + G$ and $g = F - G$. Thus, (7.48) and (7.49) are transformed to

$$xF'(x) + xG'(x) + (a + bA - 2bx)\, G(x) + (a - bA)\, F(x) = 0, \tag{7.54}$$

$$xF'(x) - xG'(x) - (c + dB - 2dx)\, G(x) + (c - dB)\, F(x) = 0. \tag{7.55}$$

Adding and subtracting these two differential equations yields

$$\begin{aligned} &2xF'(x) + (a + c - bA - dB)\, F(x) \\ &\qquad + [a - c + bA - dB + 2\,(d - b)\, x]\, G(x) = 0, \end{aligned} \tag{7.56}$$

$$\begin{aligned} &2xG'(x) + [a + c + bA + dB - 2\,(b + d)\, x]\, G(x) \\ &\qquad + (a - c - bA + dB)\, F(x) = 0. \end{aligned} \tag{7.57}$$

By eliminating $F(x)$ from (7.56) and (7.57), we obtain

$$x^2G''(x) + (\alpha - \beta x)\, xG'(x) + (\gamma - \delta x)\, G(x) = 0,$$

where

$$\begin{aligned} \alpha &:= 1 + a + c, & \beta &:= b + d, \\ \gamma &:= ac - bdAB, & \delta &:= b + d + ad + bc - bd\,(A + B)\,. \end{aligned}$$

Taking into account both relations (7.52) and (7.53), we derive

$$xG''(x) + (\alpha - \beta x)\, G'(x) + (n-1)\, \beta G(x) = 0.$$

A change of variables $u = \beta x$ leads to the Laguerre differential equation

$$uh''(u) + (\alpha - u)\, h'(u) + (n-1)\, h(u) = 0, \tag{7.58}$$

whose polynomial solution is a constant multiple of the generalized Laguerre polynomial $L_{n-1}^{(\alpha-1)}(u)$. Consequently, $G(x) = C_1 L_{n-1}^{(\alpha-1)}(\beta x)$ for some $C_1 \neq 0$. By relations (7.57), $f = F + G$ and $g = F - G$, and using properties of Laguerre polynomials, we obtain

$$\begin{aligned}
&-(a - c - bA + dB)\, f(x) \\
&\qquad = C_1 \left\{ 2\beta x \left\{ L_{n-1}^{(\alpha-1)} \right\}' (\beta x) + [2c + 2bA - 2\beta x]\, L_{n-1}^{(\alpha-1)}(\beta x) \right\} \\
&\qquad = 2C_1 \left\{ n L_n^{(\alpha-1)}(\beta x) - (n + a - bA)\, L_{n-1}^{(\alpha-1)}(\beta x) \right\}, \\
&-(a - c - bA + dB)\, g(x) \\
&\qquad = C_1 \left\{ 2\beta x \left\{ L_{n-1}^{(\alpha-1)} \right\}' (\beta x) + [2a + 2dB - 2\beta x]\, L_{n-1}^{(\alpha-1)}(\beta x) \right\} \\
&\qquad = 2C_1 \left\{ n L_n^{(\alpha-1)}(\beta x) - (n + c - dB)\, L_{n-1}^{(\alpha-1)}(\beta x) \right\}.
\end{aligned}$$

Set $\Delta := f_1 - g_1$ (which is positive by (3.5)). By eliminating A and B from Equations (7.51), (7.52), and (7.53), we derive

$$(n + a + b\Delta)\,(n + c - d\Delta) = ac. \tag{7.59}$$

□

7.6. **Proof of Theorem 14.** Clearly, an extremal system exists. Proceeding as in the proof of Theorem 12, we define

$$f(x) := \prod_{j=1}^{n} (x - x_j) = x^n - f_1 x^{n-1} + f_2 x^{n-2} - \cdots + (-1)^n x_1 \cdots x_n,$$
$$g(y) := \prod_{k=1}^{n-1} (y - y_k) = y^{n-1} - g_1 y^{n-2} + g_2 y^{n-3} - \cdots + (-1)^{n-1} y_1 \cdots y_{n-1}.$$

Maximizing the function F_n^e is equivalent to minimizing the function $V_n^e := \log(1/F_n^e)$. The corresponding optimality conditions give

$$\frac{f'(y_\ell)}{f(y_\ell)} = \sum_{j=1}^{n} \frac{1}{y_\ell - x_j} = -n\left(\frac{s_y}{y_\ell} - \lambda_y\right), \quad \ell = 1, \dots, n-1,$$

$$\frac{g'(x_\ell)}{g(x_\ell)} = \sum_{k=1}^{n-1} \frac{1}{x_\ell - y_k} = -(n-1+p)\left(\frac{s_x}{x_\ell} - \lambda_x\right) - \frac{p}{x_\ell}, \quad \ell = 1, \dots, n.$$

Since $f(y_\ell) \neq 0$ for all $\ell = 1, \dots, n-1$ and $g(x_\ell) \neq 0$ for all $\ell = 1, \dots, n$, we have

$$xf'(x) + (a - bx) f(x) + b(x-A)(x-B) g(x) = 0, \tag{7.60}$$

$$xg'(x) + (c - dx) g(x) + df(x) = 0, \tag{7.61}$$

where

$$a := ns_y, \quad b := n\lambda_y, \quad c := p + (n-1+p) s_x, \quad d := (n-1+p)\lambda_x.$$

The constants A and B follow from comparison of coefficients. We find that

$$b(A+B) = n + a + b(f_1 - g_1),$$
$$bAB = (n+a) g_1 - (n-1+a) f_1 + b[f_2 - g_2 + (f_1 - g_1) g_1].$$

By eliminating $f(x)$ from (7.60) and (7.61), we get

$$x^2 g''(x) + (\alpha - \beta x) xg'(x) + (\gamma + \delta x) g(x) = 0, \tag{7.62}$$

where

$$\alpha := 1 + a + c, \qquad \beta := b + d,$$
$$\gamma := ac - bdAB, \qquad -\delta := d + ad + bc - bd(A+B).$$

Since g is a polynomial of degree $n-1$ we get from (7.62) the conditions

$$(n-1)\beta = \delta, \qquad \gamma = 0. \tag{7.63}$$

(These relations follow by equating the constant terms and equating the x^{n-1}-terms.) Thus, for $x > 0$ it is sufficient to consider

$$xg''(x) + (\alpha - \beta x) g'(x) + (n-1)\beta g(x) = 0.$$

A change of variables $u = \beta x$ and $g(x) = h(u)$ leads to the Laguerre differential equation

$$ug''(x) + (\alpha - u) h'(x) + (n-1) h(x) = 0, \tag{7.64}$$

whose polynomial solution is given by a constant multiple of the generalized Laguerre polynomial $L_{n-1}^{(\alpha-1)}(u)$. Hence $g(x) = C_1 L_{n-1}^{(\alpha-1)}(\beta x)$

for some constant $C_1 \neq 0$. By (7.61) and a differentiation formula for Laguerre polynomials

$$-\left(d/C_1\right) f(x) = \left(c - dx\right) L_{n-1}^{(\alpha-1)}(\beta x) - \beta x L_{n-2}^{(\alpha)}(\beta x).$$

An alternative representation, obtained by using a second differentiation formula and a three-term recurrence relation, is given by

$$-\frac{d}{C_1 n} f(x) = L_n^{(\alpha-1)}(\beta x) - \left(1 + \frac{a}{n} - \frac{b}{n} x\right) L_{n-1}^{(\alpha-1)}(\beta x). \tag{7.65}$$

The centroids can be obtained by using explicit representations of the Laguerre polynomials. However, it is easier to use the relations between AB, $A+B$, and the quantities f_1 and g_1. Clearly, $d(f_1 - g_1) = n - 1 + c$ and it is well-known that $u_1 + \cdots + u_{n-1} = (n-1)(n+\alpha-2)$, where $u_1, \dots, u_{n-1}$ are the zeros of $L_{n-1}^{(\alpha-1)}(\beta x)$. Thus, $\beta g_1 = (n-1)(n+\alpha-2)$. □

7.7. **Proof of Theorem 16.** Maximizing the function G_n^e is the same as minimizing $V_n^e := \log(1/G_n^e)$. Defining

$$f(x) := \prod_{j=1}^{n} (x - x_j) = x^n - f_1 x^{n-1} + \cdots,$$
$$g(y) := \prod_{k=1}^{n} (y - y_j) = y^n - g_1 y^{n-1} + \cdots,$$

by the optimality conditions, we have that

$$\frac{g'(x_\ell)}{g(x_\ell)} = \sum_{k=1}^{n} \frac{1}{x_\ell - y_k} = -n \frac{w'(x_\ell)}{w(x_\ell)} = 2n\lambda_x x_\ell, \qquad \ell = 1, \dots, n, \tag{7.66}$$

$$\frac{f'(y_\ell)}{f(y_\ell)} = \sum_{j=1}^{n} \frac{1}{y_\ell - x_j} = -n \frac{w'(y_\ell)}{w(y_\ell)} = 2n\lambda_y y_\ell, \qquad \ell = 1, \dots, n. \tag{7.67}$$

Since both $f(y_\ell) \neq 0$ and $g(x_\ell) \neq 0$ for all $\ell = 1, \dots, n$ and proceeding as usual, we obtain

$$f'(x) - 2axf(x) + 2a\left(x - A\right) g(x) = 0,$$
$$g'(x) - 2bxg(x) + 2b\left(x - B\right) f(x) = 0,$$

where $a := n\lambda_y$ and $b := n\lambda_x$. Comparing the coefficients of x^n, we get

$$B = -A = \Delta := g_1 - f_1 (> 0). \tag{7.68}$$

Introducing the functions F and G via $f = F + G$ and $g = F - G$, we have

$$(F+G)'(x) + 2a\Delta F(x) - 2a(\Delta + 2x)G(x) = 0,$$
$$(F-G)'(x) - 2b\Delta F(x) - 2b(\Delta - 2x)G(x) = 0.$$

Adding and subtracting these equations, we obtain

$$F'(x) + (a-b)\Delta F(x) - [(a+b)\Delta + 2(a-b)x]G(x) = 0,$$
$$G'(x) - [(a-b)\Delta + 2(a+b)x]G(x) + (a+b)\Delta F(x) = 0. \tag{7.69}$$

Eliminating $G(x)$ from the above equations we get

$$G''(x) - 2(a+b)xG'(x) - 2\left(a + b - 2ab\Delta^2\right)G(x) = 0. \tag{7.70}$$

A change of variables $u = \beta x$, $\beta = \sqrt{a+b}$, leads to the Hermite differential equation

$$h''(u) - 2uh'(u) + 2\left[2ab\Delta^2/\beta^2 - 1\right]h(u) = 0,$$

which has a polynomial solution if and only if the square bracketed expression is equal to an integer ≥ 0. Since we know that G is a polynomial of degree $n-1$, we derive the relation

$$2ab\Delta^2 = (a+b)\,n, \tag{7.71}$$

and thereby deduce that h is a constant multiple of the Hermite polynomial H_{n-1}. Hence, $G(x) = C_1 H_{n-1}(\beta x)$ for some $C_1 \neq 0$. By (7.69) and expressing the derivative of a Hermite polynomial in terms of Hermite polynomials, we get

$$F(x) = C_1\left\{\frac{1}{\beta\Delta}H_n(\beta x) + \frac{a-b}{a+b}H_{n-1}(\beta x)\right\}.$$

Consequently, $f = F + G$ is a constant multiple of

$$H_n(\beta x) + 2a(\Delta/\beta)H_{n-1}(\beta x) \tag{7.72}$$

and $g = F - G$ is a constant multiple of

$$H_n(\beta x) - 2b(\Delta/\beta)H_{n-1}(\beta x). \tag{7.73}$$

Furthermore,

$$2a(\Delta/\beta) = \sqrt{2(a/b)\,n}, \qquad 2b(\Delta/\beta) = \sqrt{2(b/a)\,n}.$$

The symmetry of the weighted Menke points follows from the properties of the Hermite polynomials. □

7.8. **Proof of the Theorem 18.** Proceeding as in the proof of Theorem 16, we define

$$f(x) := \prod_{j=1}^{n} (x - x_j) = x^n - f_1 x^{n-1} + f_2 x^{n-2} - \cdots ,$$

$$g(y) := \prod_{k=1}^{n-1} (y - y_k) = y^{n-1} - g_1 y^{n-2} + g_2 y^{n-3} - \cdots .$$

By the optimality conditions, we have

$$\frac{f'(y_\ell)}{f(y_\ell)} = \sum_{j=1}^{n} \frac{1}{y_\ell - x_j} = 2n\lambda_y y_\ell, \quad \ell = 1, \dots, n-1,$$

$$\frac{g'(x_\ell)}{g(x_\ell)} = \sum_{k=1}^{n-1} \frac{1}{x_\ell - y_k} = 2\,(n-1)\,\lambda_x x_\ell, \quad \ell = 1, \dots, n.$$

Consequently,

$$f'(x) - 2axf(x) + 2a\,(x-A)\,(x-B)\,g(x) = 0, \tag{7.74}$$

$$g'(x) - 2bxg(x) + 2bf(x) = 0, \tag{7.75}$$

where we define $a := n\lambda_y$ and $b := (n-1)\lambda_x$. By eliminating $f(x)$ from (7.74) and (7.75), we get

$$\begin{aligned} &g''(x) - 2\,(a+b)\,xg'(x) \\ &\qquad - 2\,[b + 2abAB - 2ab\,(A+B)\,x]\,g(x) = 0, \end{aligned}$$

which, on comparing highest order terms, implies the condition

$$A + B = 0. \tag{7.76}$$

A change of variables $u = \beta x$ with $\beta = \sqrt{a+b}$ now leads to the Hermite differential equation

$$h'' - 2uh' + 2\frac{b + 2abAB}{a+b} h = 0. \tag{7.77}$$

It has a polynomial solution of degree $n-1$ if and only if

$$\frac{b + 2abAB}{a+b} = n - 1. \tag{7.78}$$

Thus, $g(x) = C_1 H_{n-1}(\beta x)$ for some $C_1 \neq 0$. By (7.75) and properties of Hermite polynomials

$$\begin{aligned} 2b\beta f(x) &= C_1 \left\{ 2b\beta x H_{n-1}(\beta x) - 2\,(n-1)\,\beta^2 H_{n-1}(\beta x) \right\} \\ &= C_1 \left\{ bH_n(\beta x) - 2\,(n-1)\,aH_{n-2}(\beta x) \right\}. \end{aligned}$$

The symmetry relations for the weighted Menke points follows from properties of the Hermite polynomials. □

References

[1] M.E.H. Ismail, *An electrostatics model for zeros of general orthogonal polynomials*, Pacific J. Math. **193** (2000), no. 2, 355–369. MR MR1755821 (2001i:33009)

[2] ______, *More on electrostatic models for zeros of orthogonal polynomials*, Proceedings of the International Conference on Fourier Analysis and Applications (Kuwait, 1998), vol. 21, 2000, pp. 191–204. MR MR1759996 (2001h:33002)

[3] T.A. Kilgore, *A characterization of the Lagrange interpolating projection with minimal Tchebycheff norm*, J. Approx. Theory **24** (1978), 273-288. MR MR523977 (80d:41002)

[4] F. Marcellán, A. Martínez-Finkelshtein, and P. Martínez-González, *Electrostatic models for zeros of polynomials: old, new, and some open problems*, J. Comput. Appl. Math. **207** (2007), no. 2, 258–272. MR MR2345246

[5] K. Menke, *Extremalpunkte und konforme Abbildung*, Math. Ann. **195** (1972), 292–308. MR MR0291427 (45 #520)

[6] ______, *Zur Approximation des transfiniten Durchmessers bei bis auf Ecken analytischen geschlossenen Jordankurven*, Israel J. Math. **17** (1974), 136–141. MR MR0355029 (50 #7506)

[7] A.D. Polyanin and V.F. Zaitsev, *Handbook of exact solutions for ordinary differential equations*, second ed., Chapman & Hall/CRC, Boca Raton, FL, 2003. MR MR2001201 (2004g:34001)

[8] Ch. Pommerenke, *Über die Verteilung der Fekete-Punkte*, Math. Ann. **168** (1967), 111–127. MR MR0206238 (34 #6057)

[9] ______, *Über die Verteilung der Fekete-Punkte. II*, Math. Ann. **179** (1969), 212–218. MR MR0247108 (40 #377)

[10] E.B. Saff and V. Totik, *Logarithmic Potentials with External Field*, Springer-Verlag Berlin Heidelberg, 1997. MR MR1485778 (99h:31001)

[11] M. Stiemer, *Effective discretization of the energy integral and Grunsky coefficients in annuli*, Constr. Approx. **22** (2005), no. 1, 133–147. MR MR2132771 (2006b:30050)

[12] T.J. Stieltjes, *Sur certains polynômes qui vérifient une équation différentielle linéaire du second ordre et sur la theorie des fonctions de lamé*, Acta Math. **6** (1885), no. 1, 321–326. MR MR1554669

[13] G. Szegő, *Orthogonal Polynomials*, American Mathematical Society, New York, 1939, American Mathematical Society Colloquium Publications, Vol. 23. MR MR0000077 (1,14b)

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