ASYMPTOTICS OF BEST-PACKING ON RECTIFIABLE SETS

S.V. BORODACHOV, D.P. HARDIN, AND E.B. SAFF

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ABSTRACT. We investigate the asymptotic behavior, as N grows, of the largest minimal pairwise distance of N points restricted to an arbitrary compact rectifiable set embedded in Euclidean space, and we find the limit distribution of such optimal configurations. For this purpose, we compare best-packing configurations with minimal Riesz s-energy configurations and determine the s-th root asymptotic behavior (as $s \to \infty$) of the minimal energy constants.

We show that the upper and the lower dimension of a set defined through the Riesz energy or best-packing coincides with the upper and lower Minkowski dimension, respectively.

For certain sets in $\mathbf{R}^d$ of integer Hausdorff dimension, we show that the limiting behavior of the best-packing distance as well as the minimal s-energy for large s is different for different subsequences of the cardinalities of the configurations.

1. PRELIMINARIES.

The problem of finding a configuration of N points on the sphere with the minimal pairwise distance between the points being as large as possible is classical and is known as *Tammes's problem* or the *hard spheres problem*. When formulated for the whole Euclidean space, the analogous problem is that of finding a collection (or packing) of non-overlapping equal balls with the largest density. More information on this problem and its generalizations can be found in [2], [4], [7], [18]. In the present paper we investigate the best-packing problem on certain classes of "non-smooth" sets.

1.1. **Best-packing problem.** We denote by $\mathbf{R}^{d'}$ the embedding space, reserving the symbol d for the dimension of the set being considered. For a collection of N distinct points $\omega_N = \{y_1, \ldots, y_N\} \subset \mathbf{R}^{d'}$ we set

$$\delta(\omega_N) := \min_{1 \le i \ne j \le N} |y_i - y_j|,$$

and for an infinite set $A \subset \mathbf{R}^{d'}$, we let

$$(1.1) \qquad \delta_N(A) := \sup\{\delta(\omega_N) \ : \ \omega_N \subset A, \ \#\omega_N = N\}$$

be the *best-packing distance* of N-point configurations on A, where $\#X$ denotes the cardinality of the set X. We can consider only compact infinite sets A, since quantity (1.1) is infinite for unbounded sets and is the same for a set and its closure.

For the case when A is the unit sphere $S^2 \subset \mathbf{R}^3$ exact values of $\delta_N(A)$ have been determined for $2 \le N \le 12$ and $N = 24$ (see [2] for references). For arbitrarily large values of N, the precise determination of best-packing distances is, in general, an intractable problem. Thus we focus on their asymptotic behavior. For this purpose we introduce the following notation. Let $0 < \alpha \le d'$ and set

$$(1.2) \quad \underline{g}_{\infty,\alpha}(A) := \liminf_{N \to \infty} \ \delta_N(A) \cdot N^{1/\alpha}, \qquad \overline{g}_{\infty,\alpha}(A) := \limsup_{N \to \infty} \ \delta_N(A) \cdot N^{1/\alpha}.$$

We further put

$$g_{\infty,\alpha}(A) := \lim_{N \to \infty} \delta_N(A) \cdot N^{1/\alpha},$$

if this limit exists. On relating these quantities to the *largest sphere packing density* in $\mathbf{R}^d$, $d \in \mathbf{N}$, which we denote by Δ_d (see (1.4) below), it can be shown that $g_{\infty,d}([0,1]^d)$ exists and is given by

$$(1.3) \qquad C_{\infty,d} := g_{\infty,d}([0,1]^d) = 2 \left(\Delta_d/\beta_d\right)^{1/d},$$

where β_d is the Lebesgue measure (volume) of the unit ball in $\mathbf{R}^d$. It is not difficult to show that $g_{\infty,d}(A)$ exists for d-dimensional smooth manifolds and domains. Here we shall establish the existence of $g_{\infty,d}(A)$ for a class of rectifiable sets and provide a formula for it in terms of the largest sphere packing density in $\mathbf{R}^d$; we also describe the limiting distribution of best-packing points (see Theorem 2.2).

Recall that the definition of Δ_d is as follows (cf. [6, Chapter 3] or [18, Chapter 1]). Let $\mathcal{L}_d$ stand for the Lebesgue measure in $\mathbf{R}^d$. Denote by Λ_d the set of collections $\mathcal{P}$ of non-overlapping unit balls in $\mathbf{R}^d$ for which the density

$$\rho(\mathcal{P}) := \lim_{r \to \infty} (2r)^{-d} \cdot \mathcal{L}_d \left(\cup_{B \in \mathcal{P}} B \cap [-r,r]^d\right)$$

exists. Then

$$(1.4) \qquad \Delta_d := \sup_{\mathcal{P} \in \Lambda_d} \rho(\mathcal{P}).$$

It is known that $\Delta_1 = 1$, $\Delta_2 = \pi/\sqrt{12}$ (Thue in 1892 and Fejes-Toth [6]), and $\Delta_3 = \pi/\sqrt{18}$ (Hales [9]). The exact value of Δ_d for $d > 3$ is unknown; so far, only upper and lower estimates, which differ by an exponential factor as $d \to \infty$, have been obtained for Δ_d (see [4] for this and other references).

Concerning the case when $A = S^2$, the papers [12], [20] prove that $g_{\infty,2}(S^2) = \left(8\pi/\sqrt{3}\right)^{1/2}$. Furthermore, the results of [9] imply that for the unit sphere $S^3 \subset \mathbf{R}^4$ we have $g_{\infty,3}(S^3) = \sqrt{2}\pi^{2/3}$.

1.2. **Minimum energy problem.** The best-packing problem can be viewed as the limiting case of the problem of minimization of the discrete Riesz s-energy as $s \to \infty$. The setting of this latter problem is as follows: for $s > 0$ and a collection $\omega_N = \{y_1, \ldots, y_N\} \subset \mathbf{R}^{d'}$, let

$$E_s(\omega_N) := \sum_{i=1}^N \sum_{j:j \ne i} \frac{1}{|y_i - y_j|^s}.$$

The *minimal discrete N-point Riesz s-energy* of a set A is defined to be

$$(1.5) \qquad \mathcal{E}_s(A, N) := \inf\{E_s(\omega_N) : \omega_N \subset A, \ \#\omega_N = N\}.$$

It is not difficult to see that, for N fixed, $\lim_{s\to\infty} \mathcal{E}_s(A, N)^{1/s} = 1/\delta_N(A)$. We remark that for the case when $s = 1$ and $A = S^2$, the determination of $\mathcal{E}_s(A, N)$ corresponds to the famous Thomson problem for the Coulomb potential (cf. [19]). Exact solutions to this problem are known for $N = 2, 3, 4, 6, 12$ and for some interesting cases in higher dimensions (see [21], [14], [1] and references therein).

When $0 < s < n := \dim_H A$ (the Hausdorff dimension of the set A), potential theory implies that $\mathcal{E}_s(A, N) \asymp N^2$, $N \to \infty$ (cf. e.g. [15]). When $s > n$, local interactions dominate, and as described in Theorem 2.1 below, $\mathcal{E}_s(A, N) \asymp N^{1+s/n}$, $N \to \infty$, for compact rectifiable sets A of positive n-dimensional Hausdorff measure.

1.3. Notation and definitions.

To describe the precise rate of growth of $\mathcal{E}_s(A, N)$, where A may have non-integer dimension, let $0 < \alpha \le d'$ and for $s > \alpha$ define

$$(1.6) \qquad \underline{g}_{s,\alpha}(A) := \liminf_{N\to\infty} \frac{\mathcal{E}_s(A, N)}{N^{1+s/\alpha}}, \qquad \overline{g}_{s,\alpha}(A) := \limsup_{N\to\infty} \frac{\mathcal{E}_s(A, N)}{N^{1+s/\alpha}}$$

and

$$(1.7) \qquad g_{s,\alpha}(A) := \lim_{N\to\infty} \frac{\mathcal{E}_s(A, N)}{N^{1+s/\alpha}},$$

if this limit exists. Given a positive integer $d \le d'$, denote by $\mathcal{H}_d$ the d-dimensional Hausdorff measure in $\mathbf{R}^{d'}$ normalized so that an isometric image of $[0, 1]^d$ has measure 1. Following [5] a set $A \subset \mathbf{R}^{d'}$ is called *d-rectifiable* if it is an image of a bounded set from $\mathbf{R}^d$ with respect to a Lipschitz mapping. A set $A \subset \mathbf{R}^{d'}$ is called *$(\mathcal{H}_d, d)$-rectifiable*, if $\mathcal{H}_d(A) < \infty$ and A is a union of at most a countable collection of d-rectifiable sets and a set of $\mathcal{H}_d$-measure zero.

Given $\alpha \ge 0$, let $\beta_\alpha = \frac{\pi^{\alpha/2}}{\Gamma(1+\alpha/2)}$. When α is an integer, this formula agrees with the above definition of β_d. Let also $A(\epsilon)$, $\epsilon > 0$, be the ϵ-neighborhood of the set $A \subset \mathbf{R}^{d'}$. *The lower and the upper Minkowski content* of A are defined by

$$(1.8) \qquad \underline{\mathcal{M}}_\alpha(A) := \liminf_{\rho\to 0^+} \frac{\mathcal{L}_{d'}(A(\rho))}{\beta_{d'-\alpha}\rho^{d'-\alpha}} \quad \text{and} \quad \overline{\mathcal{M}}_\alpha(A) := \limsup_{\rho\to 0^+} \frac{\mathcal{L}_{d'}(A(\rho))}{\beta_{d'-\alpha}\rho^{d'-\alpha}},$$

respectively. If they coincide, then the quantity $\mathcal{M}_\alpha(A) := \underline{\mathcal{M}}_\alpha(A) = \overline{\mathcal{M}}_\alpha(A)$ is called the *α-dimensional Minkowski content* of the set A. It is known (cf. [5, Theorem 3.2.39]) that for every closed d-rectifiable set $A \subset \mathbf{R}^{d'}$ we have

$$(1.9) \qquad \mathcal{M}_d(A) = \mathcal{H}_d(A).$$

Let A be compact with $\mathcal{H}_d(A) > 0$ and $\{\omega_N\}_{N=2}^\infty$ be a sequence of point configurations on A such that $\#\omega_N = N$, $N \ge 2$. We say that $\{\omega_N\}_{N=2}^\infty$ is *asymptotically uniformly distributed on A with respect to $\mathcal{H}_d$* if, for every subset $B \subset A$ whose boundary relative to A has $\mathcal{H}_d$-measure zero, we have

$$(1.10) \qquad \frac{\#(\omega_N \cap B)}{N} \to \frac{\mathcal{H}_d(B)}{\mathcal{H}_d(A)}, \quad N \to \infty.$$

This definition can also be stated in terms of the weak* convergence of measures.

2. Main results.

First, we describe known results on the asymptotic behavior of the minimal s-energy on sets with $\dim_H A = d$ and $s > d$. Let

$$(2.1) \qquad C_{s,d} := \lim_{N \to \infty} \frac{\mathcal{E}_s \left([0,1]^d, N \right)}{N^{1+s/d}}, \quad s > d.$$

In [16] it is shown that $C_{s,1} = 2\zeta(s)$, $s > 1$, where $\zeta(s) = \sum_{i=1}^{\infty} i^{-s}$ is the classical zeta-function. The articles [10], [11] show that for $s > d$ the limit $C_{s,d}$ exists, is finite and positive. However, the value of $C_{s,d}$ is still unknown for $d > 1$.

The following result is proved in [11], [3] (for curves in $\mathbf{R}^{d'}$, it follows from [16]).

Theorem 2.1. *Let $s > d$ and $d' \geq d$, where d and d' are integers. For every infinite compact $(\mathcal{H}_d, d)$-rectifiable set A in $\mathbf{R}^{d'}$ with $\mathcal{M}_d(A) = \mathcal{H}_d(A)$, the limit $g_{s,d}(A)$ defined in (1.7) exists and is given by*

$$(2.2) \qquad g_{s,d}(A) = C_{s,d} \mathcal{H}_d(A)^{-s/d}.$$

Moreover, if A is d-rectifiable with $\mathcal{H}_d(A) > 0$, then any sequence $\{\omega_N^*\}_{N=2}^{\infty}$ of s-energy minimizing collections on A such that $\#\omega_N^* = N$ is asymptotically uniformly distributed on A with respect to $\mathcal{H}_d$.

In the case $d = d'$, Theorem 2.1 applies to any compact set $A \subset \mathbf{R}^{d'}$, since such a set is trivially d-rectifiable. In this case $\mathcal{H}_d = \mathcal{L}_{d'}$, but we use the notation $\mathcal{H}_d$ to handle both the cases $d < d'$ and $d = d'$.

In this paper we get an analogue of Theorem 2.1 for best-packing configurations.

Theorem 2.2. *Let $d \leq d'$, where d, d' are integers, and $A \subset \mathbf{R}^{d'}$ be an infinite compact $(\mathcal{H}_d, d)$-rectifiable set. If $\mathcal{M}_d(A) = \mathcal{H}_d(A)$, then $g_{\infty,d}(A)$ exists and is given by*

$$(2.3) \qquad g_{\infty,d}(A) = C_{\infty,d} \cdot \mathcal{H}_d(A)^{1/d} = 2(\Delta_d/\beta_d)^{1/d} \cdot \mathcal{H}_d(A)^{1/d}.$$

Moreover, if $\overline{\mathcal{M}}_d(A) > \mathcal{H}_d(A)$, then

$$(2.4) \qquad \overline{g}_{\infty,d}(A) > C_{\infty,d} \cdot \mathcal{H}_d(A)^{1/d}.$$

If A is d-rectifiable with $\mathcal{H}_d(A) > 0$, then every sequence $\{\overline{\omega}_N\}_{N=2}^{\infty}$ of best-packing configurations on A such that $\#\overline{\omega}_N = N$ is asymptotically uniformly distributed on A with respect to $\mathcal{H}_d$.

In view of relation (1.9), and the fact that any $(\mathcal{H}_d, d)$-rectifiable set can be approximated by its d-rectifiable subsets, we either have $\mathcal{M}_d(A) = \mathcal{H}_d(A)$ or $\overline{\mathcal{M}}_d(A) > \mathcal{H}_d(A)$, so that either (2.3) or (2.4) must hold.

For results similar to (2.3) for $d = d'$ that concern the covering radius, see [8].

We next relate the fundamental constants $C_{s,d}$ and $C_{\infty,d}$.

Theorem 2.3. *The limit $\lim_{s \to \infty} C_{s,d}^{1/s}$ exists for each integer $d > 1$ and*

$$\lim_{s \to \infty} C_{s,d}^{1/s} = \frac{1}{C_{\infty,d}} = \frac{1}{2} \left(\frac{\beta_d}{\Delta_d} \right)^{1/d}.$$

We next show that relation (2.2) can fail for certain $(\mathcal{H}_d, d)$-rectifiable sets.

Proposition 2.4. *If $A \subset \mathbf{R}^{d'}$ is a compact $(\mathcal{H}_d, d)$-rectifiable set with $\overline{\mathcal{M}}_d(A) > \mathcal{H}_d(A)$, then for s sufficiently large*

$$\underline{g}_{s,d}(A) < C_{s,d} \mathcal{H}_d(A)^{-s/d}.$$

As an example of a rectifiable set for which (1.9) does not hold, we mention a compact $(\mathcal{H}_2, 2)$-rectifiable set $B \subset \mathbf{R}^3$ with $0 < \mathcal{H}_2(B) < \infty = \mathcal{M}_2(B)$ given in [5, p. 276]. Proposition 2.5 will imply that $g_{s,2}(B) = 0$, $s > 3$, and $g_{\infty,2}(B) = \infty$.

Remarks for general sets. Let $\underline{\dim}_M A$ and $\overline{\dim}_M A$ denote the lower and the upper Minkowski dimension of a set $A \subset \mathbf{R}^{d'}$. One can also introduce the lower and the upper dimension of a set using s-energy or best-packing. Let $\underline{\dim}_\infty A :=$ $\inf(\{\alpha > 0 : \underline{g}_{\infty,\alpha}(A) = 0\} \cup \{d'\}) = \sup(\{\alpha \in (0, d'] : \underline{g}_{\infty,\alpha}(A) = \infty\} \cup \{0\})$ and for a fixed $s > d'$ denote $\underline{\dim}_s A := \inf(\{\alpha > 0 : \overline{g}_{s,\alpha}(A) = \infty\} \cup \{d'\}) =$ $\sup(\{\alpha \in (0, d'] : \overline{g}_{s,\alpha}(A) = 0\} \cup \{0\})$ with $\overline{\dim}_\infty A$ and $\overline{\dim}_s A$ being defined in an analogous way through $\overline{g}_{\infty,\alpha}$ or $\underline{g}_{s,\alpha}$. The following proposition implies that for any set $A \subset \mathbf{R}^{d'}$ we have $\underline{\dim}_s A = \underline{\dim}_\infty A = \underline{\dim}_M A$ and $\overline{\dim}_s A = \overline{\dim}_\infty A = \overline{\dim}_M A$, provided $s > d'$.

Proposition 2.5. *If $0 < \alpha \leq d' < s$, there are positive constants $c_1 = c_1(s, \alpha)$ and $c_2 = c_2(s, \alpha)$ such that for any infinite set $A \subset \mathbf{R}^{d'}$ we have*

$$(2.5) \qquad c_1 \underline{\mathcal{M}}_\alpha(A)^{-s/\alpha} \leq \overline{g}_{s,\alpha}(A) \leq c_2 \underline{\mathcal{M}}_\alpha(A)^{-s/\alpha},$$

$$(2.6) \qquad c_1 \overline{\mathcal{M}}_\alpha(A)^{-s/\alpha} \leq \underline{g}_{s,\alpha}(A) \leq c_2 \overline{\mathcal{M}}_\alpha(A)^{-s/\alpha}.$$

There are also positive constants $c_3 = c_3(\alpha)$ and $c_4 = c_4(\alpha)$ such that for every infinite set $A \subset \mathbf{R}^{d'}$

$$(2.7) \qquad c_3 \underline{\mathcal{M}}_\alpha(A)^{1/\alpha} \leq \underline{g}_{\infty,\alpha}(A) \leq c_4 \underline{\mathcal{M}}_\alpha(A)^{1/\alpha},$$

$$(2.8) \qquad c_3 \overline{\mathcal{M}}_\alpha(A)^{1/\alpha} \leq \overline{g}_{\infty,\alpha}(A) \leq c_4 \overline{\mathcal{M}}_\alpha(A)^{1/\alpha}.$$

It is known that $\underline{\dim}_M A \geq \dim_H A$ with a strict inequality possible for some compact sets (cf. e.g. [17, p. 77]). Hence, for such sets A and any $\dim_H A < \alpha < \alpha_1 < \underline{\dim}_M A$ we have $\mathcal{H}_\alpha(A) = 0$, but $g_{s,\alpha_1}(A) = 0$, $s > d'$, and $g_{\infty,\alpha_1}(A) = \infty$.

For every $s \in (d', \infty]$ and compact sets with sufficiently large gap between $\underline{\mathcal{M}}_\alpha(A)$ and $\overline{\mathcal{M}}_\alpha(A)$ we will have $\underline{g}_{s,\alpha}(A) < \overline{g}_{s,\alpha}(A)$. Moreover, if $\underline{\dim}_M A < \overline{\dim}_M A$ (cf. e.g. [17, p. 77] for examples), the order of the best-packing radius and the minimal s-energy for $s > d'$ will vary depending on the subsequence of cardinalities of configurations.

We also show that the condition of $(\mathcal{H}_d, d)$-rectifiability in Theorems 2.1 and 2.2 is crucial in the sense that there are non-rectifiable compact sets with $\dim_H A = d$ and $0 < \mathcal{H}_d(A) < \infty$ such that $g_{\infty,d}(A)$ and $g_{s,d}(A)$ (for sufficiently large s) do not exist. Indeed, we show that this is true for a class of Cantor-type sets which we will denote by $\mathcal{K}$.

We say that a non-empty compact set $K \subset \mathbf{R}^{d'}$ belongs to the class $\mathcal{K}$, if there are a finite number of distinct similitudes $S_1, \ldots, S_p : \mathbf{R}^{d'} \to \mathbf{R}^{d'}$ with the same contraction coefficient $\sigma \in (0, 1)$ such that

$$(2.9) \qquad \bigcup_{i=1}^{p} S_i(K) = K, \quad \text{and} \quad S_i(K) \cap S_j(K) = \emptyset, \ i \neq j.$$

According to [13], we have $\lambda := \dim_H K = -\log_\sigma p$ and $0 < \mathcal{H}_\lambda(K) < \infty$. This is a subclass of the class of self-similar sets constructed in [13] (this construction is also cited in [17, Section 4.13]).

Class $\mathcal{K}$ contains the classical Cantor subset of $[0, 1]$. Parameters p and σ can also be chosen so that $\dim_H K$ is any integer between 0 and d'. For example, if a_1, a_2 and a_3 are vertices of an equilateral triangle on the plane and S_i, $i = 1, 2, 3$, is the homothety of the plane with respect to a_i and the contraction coefficient $1/3$, then we get a set of Hausdorff dimension one, known as the Sierpinski gasket [17, p. 75].

Proposition 2.6. *Let K be a compact set from the class $\mathcal{K}$ with $\lambda = \dim_H K$. Then, for s sufficiently large we have $0 < \underline{g}_{s,\lambda}(K) < \overline{g}_{s,\lambda}(K) < \infty$.*

3. Proofs.

With regard to the extended real number limits in $[0, \infty]$, we agree that $1/0 = 0^{-s} = \infty^s = \infty$, $1/\infty = \infty^{-s} = 0$, $s > 0$.

Proposition 3.1. *For every infinite set $A \subset \mathbf{R}^{d'}$ and $0 < \alpha \leq d'$ we have*

$$(3.1) \qquad \lim_{s \to \infty} \left(\overline{g}_{s,\alpha}(A)\right)^{1/s} = \frac{1}{\underline{g}_{\infty,\alpha}(A)} \quad \text{and} \quad \lim_{s \to \infty} \left(\underline{g}_{s,\alpha}(A)\right)^{1/s} = \frac{1}{\overline{g}_{\infty,\alpha}(A)}.$$

Proposition 3.1 immediately yields the following statements.

Proposition 3.2. *Let $A \subset \mathbf{R}^{d'}$ be an infinite set and $0 < \alpha \leq d'$. If for every s sufficiently large $\underline{g}_{s,\alpha}(A) = \overline{g}_{s,\alpha}(A)$, then $g_{\infty,\alpha}(A)$ exists and*

$$\lim_{s \to \infty} (g_{s,\alpha}(A))^{1/s} = \frac{1}{g_{\infty,\alpha}(A)}.$$

Proposition 3.3. *Let $A \subset \mathbf{R}^{d'}$ be an infinite set such that $\underline{g}_{\infty,\alpha}(A) < \overline{g}_{\infty,\alpha}(A)$ for some $0 < \alpha \leq d'$. Then for sufficiently large s we have $\underline{g}_{s,\alpha}(A) < \overline{g}_{s,\alpha}(A)$.*

Proof of Proposition 3.1. Lower estimates. We can assume $A \subset \mathbf{R}^{d'}$ to be compact, since on unbounded sets $g_{s,\alpha}(A) = 0$ and $g_{\infty,\alpha}(A) = \infty$ and the minimal s-energy (as well as the best-packing radius) is the same for A and its closure.

Choose an arbitrary $\epsilon \in (0, 1)$ and let $s > \alpha$. Let N be sufficiently large and $\omega_N^* := \{x_{1,N}, \ldots, x_{N,N}\}$ be an s-energy minimizing N-point collection on A. Set $N_\epsilon := \lfloor (1 - \epsilon)N \rfloor$, where $\lfloor t \rfloor$ is the floor function of t, and

$$r_{i,N} := \min_{j:j \neq i} |x_{i,N} - x_{j,N}|.$$

Pick a point $x_{i_1,N} \in \omega_N^*$ with $r_{i_1,N} \leq \delta_N(A)$. In $\omega_N^* \setminus \{x_{i_1,N}\}$ pick a point $x_{i_2,N}$ so that $r_{i_2,N} \leq \delta_{N-1}(A)$. Continue this process until we pick a point $x_{i_{\lfloor \epsilon N \rfloor + 1},N} \in \omega_N^* \setminus \{x_{i_1,N}, \ldots, x_{i_{\lfloor \epsilon N \rfloor},N}\}$ such that $r_{i_{\lfloor \epsilon N \rfloor + 1},N} \leq \delta_{N-\lfloor \epsilon N \rfloor}(A)$. Then

$$\mathcal{E}_s(A, N) = E_s(\omega_N^*) \geq \sum_{k=1}^{\lfloor \epsilon N \rfloor + 1} \frac{1}{(r_{i_k,N})^s} \geq \sum_{k=1}^{\lfloor \epsilon N \rfloor + 1} \frac{1}{(\delta_{N-k+1}(A))^s} \geq \frac{\epsilon N}{(\delta_{N_\epsilon}(A))^s}.$$

Hence,
(3.2)

$$\overline{g}_{s,\alpha}(A) \geq \limsup_{N \to \infty} \frac{\epsilon}{(\delta_{N_\epsilon}(A))^s N^{s/\alpha}} = \frac{\epsilon(1 - \epsilon)^{s/\alpha}}{\left(\liminf\limits_{N \to \infty} \delta_{N_\epsilon}(A) \cdot N_\epsilon^{1/\alpha}\right)^s} = \frac{\epsilon(1 - \epsilon)^{s/\alpha}}{\left(\underline{g}_{\infty,\alpha}(A)\right)^s},$$

since N_ϵ passes through all natural numbers. Similarly,

$$(3.3) \qquad \underline{g}_{s,\alpha}(A) \geq \frac{\epsilon(1-\epsilon)^{s/\alpha}}{\left(\overline{g}_{\infty,\alpha}(A)\right)^s}.$$

Then, letting first $s \to \infty$ and then $\epsilon \to 0$, we get

$$(3.4) \quad \liminf_{s\to\infty} \left(\overline{g}_{s,\alpha}(A)\right)^{1/s} \geq \frac{1}{\underline{g}_{\infty,\alpha}(A)} \quad \text{and} \quad \liminf_{s\to\infty} \left(\underline{g}_{s,\alpha}(A)\right)^{1/s} \geq \frac{1}{\overline{g}_{\infty,\alpha}(A)}.$$

Upper estimates. Let, for every $N(\geq 2)$ fixed, $X_N = \{x, x_1, \ldots, x_{N-1}\} \subset \mathbf{R}^{d'}$ be such that $a := \delta(X_N) > 0$ and for every $k \in \mathbf{N}$ let M_k be the set of points from X_N contained in $B(x, a(k+1))$ but not in $B(x, ak)$, where $B(x, r)$ is the open ball in $\mathbf{R}^{d'}$ centered at x with radius r. Then, from a volume argument,

$$\#M_k \cdot \mathcal{L}_{d'}\left[B\left(0, a/2\right)\right] \leq \mathcal{L}_{d'}\left[B\left(x, a(k+3/2)\right) \setminus B\left(x, a(k-1/2)\right)\right],$$

and so $\#M_k \leq (2k+3)^{d'} - (2k-1)^{d'} \leq 4d'(2k+3)^{d'-1}$. Hence,

$$P_s(x, X_N) := \sum_{i=1}^{N-1} \frac{1}{|x - x_i|^s} = \sum_{k=1}^{\infty} \sum_{x_i \in M_k} \frac{1}{|x - x_i|^s}$$

$$\leq \sum_{k=1}^{\infty} \frac{\#M_k}{a^s k^s} \leq \frac{4d'}{a^s} \sum_{k=1}^{\infty} \frac{(2k+3)^{d'-1}}{k^s} \leq \frac{\eta_s}{a^s}, \quad s > d',$$

where $\eta_s := \theta_{d'}\zeta(s - d' + 1)$ and $\theta_{d'}$ is a constant depending only on d'.

Now let $\overline{\omega}_N := \{\overline{x}_{1,N}, \ldots, \overline{x}_{N,N}\}$ be a best-packing N-point configuration on A; that is, $\delta(\overline{\omega}_N) = \delta_N(A)$. Then, using the above estimate, for $s > d'$ we get

$$\mathcal{E}_s(A, N) \leq E_s(\overline{\omega}_N) = \sum_{i=1}^{N} P_s(\overline{x}_{i,N}, \overline{\omega}_N) \leq \frac{\eta_s N}{(\delta_N(A))^s}.$$

Hence, for $s > d'$ we have
(3.5)
$$\overline{g}_{s,\alpha}(A) \leq \limsup_{N\to\infty} \frac{\eta_s}{\left(\delta_N(A) \cdot N^{1/\alpha}\right)^s} = \frac{\eta_s}{\left(\underline{g}_{\infty,\alpha}(A)\right)^s}, \quad \underline{g}_{s,\alpha}(A) \leq \frac{\eta_s}{\left(\overline{g}_{\infty,\alpha}(A)\right)^s}.$$

Then, since $\eta_s^{1/s} \to 1$ as $s \to \infty$, we have

$$(3.6) \quad \limsup_{s\to\infty} \left(\overline{g}_{s,\alpha}(A)\right)^{1/s} \leq \frac{1}{\underline{g}_{\infty,\alpha}(A)}, \qquad \limsup_{s\to\infty} \left(\underline{g}_{s,\alpha}(A)\right)^{1/s} \leq \frac{1}{\overline{g}_{\infty,\alpha}(A)}.$$

Inequalities (3.4) and (3.6) yield relations (3.1). Propositions 3.1—3.3 are proved.

Proof of Theorem 2.3. Using (2.1) and Proposition 3.2 we get

$$\lim_{s\to\infty} C_{s,d}^{1/s} = \lim_{s\to\infty} g_{s,d}([0,1]^d)^{1/s} = \frac{1}{g_{\infty,d}([0,1]^d)} = \frac{1}{C_{\infty,d}}.$$

Proof of Theorem 2.2. Taking into account Theorem 2.1, Proposition 3.2 and Theorem 2.3, we get equation (2.3):

$$g_{\infty,d}(A) = \left(\lim_{s\to\infty} (g_{s,d}(A))^{1/s}\right)^{-1} = \lim_{s\to\infty} \frac{\mathcal{H}_d(A)^{1/d}}{C_{s,d}^{1/s}} = C_{\infty,d}\mathcal{H}_d(A)^{1/d}.$$

Now suppose that A is d-rectifiable with $\mathcal{H}_d(A) > 0$, and $\{\overline{\omega}_N\}_{N=2}^{\infty}$ is a sequence of best-packing configurations on A such that $\#\overline{\omega}_N = N$. To show that $\{\overline{\omega}_N\}_{N=2}^{\infty}$

is asymptotically uniformly distributed on A, choose any subset $B \subset A$ whose boundary relative to A has $\mathcal{H}_d$-measure zero. Let $\overline{B}$ be the closure of the set B.

Set $p_N := \#(\overline{\omega}_N \cap B)$ and let $\mathcal{N} \subset \mathbf{N}$ be an infinite subset such that the limit

$$p(\mathcal{N}) := \lim_{\mathcal{N} \ni N \to \infty} \frac{p_N}{N}$$

exists. If $p(\mathcal{N}) > 0$, then for sufficiently large $N \in \mathcal{N}$ we get

$$\delta_N(A) = \delta(\overline{\omega}_N) \leq \delta(\overline{\omega}_N \cap B) \leq \delta_{p_N}(B) \leq \delta_{p_N}(\overline{B}).$$

Since $\overline{B}$ is a closed d-rectifiable set and $\mathcal{H}_d(\overline{B}) = \mathcal{H}_d(B)$, using (2.3), we have

$$(3.7) \qquad p(\mathcal{N}) \leq \lim_{\mathcal{N} \ni N \to \infty} \frac{\delta_{p_N}(\overline{B})^d \cdot p_N}{\delta_N(A)^d \cdot N} = \left(\frac{g_{\infty,d}(\overline{B})}{g_{\infty,d}(A)} \right)^d = \frac{\mathcal{H}_d(B)}{\mathcal{H}_d(A)}.$$

If $p(\mathcal{N}) = 0$, then the inequality $p(\mathcal{N}) \leq \mathcal{H}_d(B)/\mathcal{H}_d(A)$ is trivial. Thus,

$$\limsup_{N \to \infty} \frac{p_N}{N} \leq \mathcal{H}_d(B)/\mathcal{H}_d(A).$$

Next, let $q_N := \#(\overline{\omega}_N \cap (A \setminus B))$. Since the boundary of $A \setminus B$ relative to A also has $\mathcal{H}_d$-measure zero, using the same argument we can write

$$\limsup_{N \to \infty} \frac{q_N}{N} \leq \frac{\mathcal{H}_d(A \setminus B)}{\mathcal{H}_d(A)},$$

which implies that $\liminf_{N \to \infty} p_N/N \geq \mathcal{H}_d(B)/\mathcal{H}_d(A)$. This shows that (1.10) holds.

To prove (2.4) we will need the following lemma. Denote $\mu_{d'} := \mathcal{L}_{d'}(B(0,2))$.

Lemma 3.4. *Let $0 < \alpha \leq d'$, G and F be two sets in $\mathbf{R}^{d'}$ and assume that for some positive numbers c, γ and $\rho < (\gamma/\mu_{d'})^{1/\alpha}$ there holds $\mathcal{L}_{d'}[G(\rho) \setminus F((c+1)\rho)] > \gamma \rho^{d'-\alpha}$. Then for $N = \lfloor \gamma/(\mu_{d'}\rho^\alpha) \rfloor + 1$ we have $\delta_N(G \setminus F(c\rho)) \geq \rho$.*

Proof. Let $k \in \mathbf{N} \cup \{0\}$ be the largest number of pairwise disjoint balls of radius $\rho/2$ centered at points of $G \setminus F(c\rho)$. We just need to show that $k > \gamma/(\mu_{d'}\rho^\alpha)$. Assume the contrary. Choose points $x_1, \ldots, x_k \in G \setminus F(c\rho)$ such that $|x_i - x_j| \geq \rho$, $1 \leq i \neq j \leq k$. Then

$$\mathcal{L}_{d'}\left(\bigcup_{i=1}^{k} B(x_i, 2\rho) \right) \leq k\mu_{d'}\rho^{d'} \leq \gamma\rho^{d'-\alpha} < \mathcal{L}_{d'}[G(\rho) \setminus F((c+1)\rho)].$$

This means that there is a point $y \in G(\rho) \setminus F((c+1)\rho)$ such that $|y - x_i| \geq 2\rho$, $i = 1, \ldots, k$. Also, there exists a point $x_{k+1} \in G$ such that $|y - x_{k+1}| < \rho$. Hence, $\text{dist}(x_{k+1}, F) \geq c\rho$. Thus, $x_{k+1} \in G \setminus F(c\rho)$ and $|x_{k+1} - x_i| > \rho$, $i = 1, \ldots, k$, and so we have $k+1$ pairwise disjoint balls of radius $\rho/2$ centered at points of $G \setminus F(c\rho)$ which contradicts to the maximality of k. Lemma 3.4 is proved.

Another fact needed to show (2.4) is the left inequality in (2.8). We can assume that $\overline{\mathcal{M}}_\alpha(A) > 0$. Choose any $0 < M < \overline{\mathcal{M}}_\alpha(A)$. Then there is a sequence $\{r_m\}_{m=1}^{\infty}$, $r_m \searrow 0$, $m \to \infty$, such that $\mathcal{L}_{d'}(A(r_m)) > M\beta_{d'-\alpha}r_m^{d'-\alpha}$, $m \in \mathbf{N}$. By Lemma 3.4 (with $F = \emptyset$) for the sequence $N_m := \lfloor M\beta_{d'-\alpha}/(\mu_{d'}r_m^\alpha) \rfloor + 1$, $m \in \mathbf{N}$, we have $\delta_{N_m}(A) \geq r_m \geq (M\beta_{d'-\alpha}/(\mu_{d'}N_m))^{1/\alpha}$ for sufficiently large m. Hence, $\overline{g}_{\infty,\alpha}(A) \geq (M\beta_{d'-\alpha}/\mu_{d'})^{1/\alpha}$. Letting $M \to \overline{\mathcal{M}}_\alpha(A)$, gives the lower estimate in (2.8).

Proof of inequality (2.4). In the case $\mathcal{H}_d(A) = 0$ we have $\overline{\mathcal{M}}_d(A) > 0$ and by the left inequality in (2.8) there holds $\overline{g}_{\infty,d}(A) > 0 = C_{\infty,d}\mathcal{H}_d(A)^{1/d}$. Assume that

$\mathcal{H}_d(A) > 0$ and set $d'' = d' - d$. Let $c_0 \in (0,1)$ be such that $(c_0 + 1)^{d''} \mathcal{H}_d(A) < \overline{\mathcal{M}}_d(A)$ and $M_1, M_2 > 0$ be such numbers that

$$(c_0 + 1)^{d''} \mathcal{H}_d(A) < (c_0 + 1)^{d''} M_1 < M_2 < \overline{\mathcal{M}}_d(A).$$

Choose any $\epsilon \in (0,1)$ and a d-rectifiable compact subset $K_\epsilon \subset A$ such that $\mathcal{H}_d(K_\epsilon) > \mathcal{H}_d(A)(1-\epsilon)$. By definition (1.8) there is a sequence of positive numbers $\{r_m\}_{m=1}^{\infty}$, $r_m \searrow 0$, $m \to \infty$, such that $\mathcal{L}_{d'}(A(r_m)) > M_2 \beta_{d''} \cdot r_m^{d''}$, $m \in \mathbf{N}$. By (1.9) we have $\mathcal{M}_d(K_\epsilon) = \mathcal{H}_d(K_\epsilon) < M_1$. Then, for sufficiently large m

$$\mathcal{L}_{d'} [K_\epsilon((c_0 + 1)r_m)] < M_1 \beta_{d''} \cdot (c_0 + 1)^{d''} r_m^{d''}$$

and hence,

$$\mathcal{L}_{d'} [A(r_m) \setminus K_\epsilon((c_0 + 1)r_m)] > \left(M_2 - (c_0 + 1)^{d''} M_1 \right) \beta_{d''} \cdot r_m^{d''}.$$

By Lemma 3.4 with $\alpha = d$, there is a constant $\nu_1 > 0$ independent of m and ϵ, such that for $k_m = \lfloor \nu_1/r_m^d \rfloor + 1$ and m sufficiently large we have $\delta_{k_m}(A \setminus K_\epsilon(c_0 r_m)) \geq r_m$. Let $X_m \subset A \setminus K_\epsilon(c_0 r_m)$ be a best-packing collection of k_m points.

Set $\nu := C_{\infty,d} \mathcal{H}_d(A)^{1/d}$. By (2.3) and the choice of K_ϵ, for sufficiently large N, we have $\delta_N(K_\epsilon) > \nu(1 - \epsilon)^{1/d} N^{-1/d}$. Choose N_m to be the largest integer such that $\nu(1 - \epsilon)^{1/d} N_m^{-1/d} \geq c_0 r_m$ and denote by Y_m the best-packing collection of N_m points on K_ϵ. Since $\text{dist}(X_m, K_\epsilon) \geq c_0 r_m$, we have that $\delta(X_m \cup Y_m) \geq c_0 r_m$ for m sufficiently large. Hence,

$$\overline{g}_{\infty,d}(A) \geq \limsup_{m \to \infty} \delta_{k_m + N_m}(A)(k_m + N_m)^{1/d} \geq$$

$$\geq \limsup_{m \to \infty} c_0 r_m \left(\frac{\nu_1}{r_m^d} + \frac{\nu^d(1 - \epsilon)}{c_0^d r_m^d} - 1 \right)^{1/d} = \left(c_0^d \nu_1 + \nu^d(1 - \epsilon) \right)^{1/d}.$$

Letting $\epsilon \to 0$, we get

$$\overline{g}_{\infty,d}(A) \geq \left(c_0^d \nu_1 + \nu^d \right)^{1/d} > \nu = C_{\infty,d} \mathcal{H}_d(A)^{1/d}.$$

This completes the proof of Theorem 2.2.

Proof of Proposition 2.4. Using Proposition 3.1, Theorem 2.3, and inequality (2.4) we have

$$\lim_{s \to \infty} \left(\frac{\underline{g}_{s,d}(A)}{C_{s,d} \mathcal{H}_d(A)^{-s/d}} \right)^{1/s} = \frac{C_{\infty,d} \mathcal{H}_d(A)^{1/d}}{\overline{g}_{\infty,d}(A)} < 1,$$

and the required inequality follows for sufficiently large s.

Proof of the Proposition 2.5. We only need to prove (2.7) and (2.8) since the upper estimates in (2.5) and (2.6) will follow from (3.5) and the lower estimates in (2.7) and (2.8). Analogously, the lower estimates in (2.5) and (2.6) are obtained from the upper estimates in (2.7) and (2.8), using (3.2) or (3.3) with ϵ equal, say $1/2$. We remark that (3.2), (3.3) and (3.5) hold for any infinite set A.

Since we do not look for sharp constants, redefine

$$\underline{\mathcal{M}}_\alpha(A) := \liminf_{r \to 0^+} \mathcal{L}_{d'}(A(r))/r^{d'-\alpha} \quad \text{and} \quad \overline{\mathcal{M}}_\alpha(A) := \limsup_{r \to 0^+} \mathcal{L}_{d'}(A(r))/r^{d'-\alpha}.$$

To show the lower estimate in (2.7), assume that $\underline{\mathcal{M}}_\alpha(A) > 0$ (otherwise it is trivial). Pick any $0 < M < \underline{\mathcal{M}}_\alpha(A)$ and set $r_N := (M/(\mu_{d'} N))^{1/\alpha}$. Then, for N sufficiently large $\mathcal{L}_{d'}(A(r_N)) > M r_N^{d'-\alpha}$. By Lemma 3.4 (with $F = \emptyset$), for $k_N = \lfloor M/(\mu_{d'} r_N^\alpha) \rfloor + 1$ (k_N will be greater that N) we have $\delta_N(A) \geq \delta_{k_N}(A) \geq r_N = (M/(\mu_{d'} N))^{1/\alpha}$ for

sufficiently large N. Hence, $\underline{g}_{\infty,\alpha}(A) \geq \mu_{d'}^{-1/\alpha} M^{1/\alpha}$. Letting $M \to \underline{\mathcal{M}}_\alpha(A)$, we get the lower estimate in (2.7). We need the following lemma for the upper estimate.

Lemma 3.5. *Let $0 < \alpha \leq d'$, $A \neq \emptyset$ be a set in $\mathbf{R}^{d'}$ and, for some positive numbers γ and $\rho < (\gamma/\beta_{d'})^{1/\alpha}$, assume that there holds $\mathcal{L}_{d'}(A(\rho)) < \gamma\rho^{d'-\alpha}$. Then for any $N > \gamma/(\beta_{d'}\rho^\alpha)$ we have $\delta_N(A) \leq 2\rho$.*

Proof. Suppose $k \geq 2$ is an integer such that $\delta_k(A) > 2\rho$, and let $x_1, \ldots, x_k \in A$ be a collection of distinct points with separation at least 2ρ. Then

$$\mathcal{L}_{d'}\left(\bigcup_{i=1}^k B(x_i, \rho)\right) = k\beta_{d'}\rho^{d'} \leq \mathcal{L}_{d'}(A(\rho)) < \gamma\rho^{d'-\alpha}.$$

Hence, $k \leq \gamma/(\beta_{d'}\rho^\alpha)$, and so for any $N > \gamma/(\beta_{d'}\rho^\alpha)$ we have $\delta_N(A) \leq 2\rho$, which proves the lemma.

To get the upper estimate in (2.7), we can assume that $\underline{\mathcal{M}}_\alpha(A) < \infty$. Choose any $M > \underline{\mathcal{M}}_\alpha(A)$. There is a sequence of positive numbers $\{r_m\}_{m=1}^\infty$, $r_m \searrow 0$, $m \to \infty$, such that $\mathcal{L}_{d'}(A(r_m)) < Mr_m^{d'-\alpha}$, $m \in \mathbf{N}$. Set $N_m := \lfloor M/(\beta_{d'}r_m^\alpha) \rfloor + 1$. By Lemma 3.5 we have $\delta_{N_m}(A) \leq 2r_m$ for sufficiently large m. Consequently,

$$\underline{g}_{\infty,\alpha}(A) \leq \liminf_{m\to\infty} \delta_{N_m}(A)N_m^{1/\alpha} \leq \liminf_{m\to\infty} 2r_m N_m^{1/\alpha} = 2\beta_{d'}^{-1/\alpha}M^{1/\alpha}.$$

Letting $M \to \underline{\mathcal{M}}_\alpha(A)$ completes the proof of (2.7).

The left inequality in (2.8) was shown before the proof of inequality (2.4). Thus, it remains to prove the right inequality in (2.8) for the case $\overline{\mathcal{M}}_\alpha(A) < \infty$. Pick any $M > \overline{\mathcal{M}}_\alpha(A)$ and let $r_N := (M/(\beta_{d'}(N-1)))^{1/\alpha}$, $N \geq 2$. Then $\mathcal{L}_{d'}(A(r_N)) < Mr_N^{d'-\alpha}$ for N sufficiently large. Since, $N > M/(\beta_{d'}r_N^\alpha)$, by Lemma 3.5 we get $\delta_N(A) \leq 2r_N = 2(M/(\beta_{d'}(N-1)))^{1/\alpha}$. Hence, $\overline{g}_{\infty,\alpha}(A) \leq 2\beta_{d'}^{-1/\alpha}M^{1/\alpha}$. Letting $M \to \overline{\mathcal{M}}_\alpha(A)$ completes the proof of (2.8) and Proposition 2.5.

Proof of Proposition 2.6. It was shown in [13] (see also [17, Theorem 4.14]) that for any set $K \in \mathcal{K}$ there are constants $c_1, c_2 > 0$ such that

(3.8) $c_1 r^\lambda \leq \mathcal{H}_\lambda(K \cap B(x, r)) \leq c_2 r^\lambda$, $x \in K$, $0 < r < 1$.

Using an argument analogous to the proof of Lemma 3.5, one can show that $\overline{g}_{\infty,\lambda}(K) < \infty$. Since for any set $K \in \mathcal{K}$ we have $\underline{\mathcal{M}}_\lambda(K) \geq C\mathcal{H}_\lambda(K) > 0$ with $C > 0$ independent of K (cf. e.g. [17, p. 79]), by (2.7) we have $\underline{g}_{\infty,\lambda}(K) > 0$.

Assume that $g_{\infty,\lambda}(K)$ exists. Let $S_1, \ldots, S_p : \mathbf{R}^{d'} \to \mathbf{R}^{d'}$ be the similitudes with the same contraction coefficient $\sigma \in (0,1)$ such that relations (2.9) hold. Set $h := \min_{i \neq j} \text{dist}(S_i(K), S_j(K))$ and choose $k \in \mathbf{N}$ so that $\delta_k(K) < h$.

Let $m \in \mathbf{N}$ and for $\mathbf{i} = (i_1, \ldots, i_m) \in \{1, \ldots, p\}^m =: Z_p^m$ put $F_\mathbf{i} := S_{i_1} \circ \cdots \circ S_{i_m}$. Then

(3.9) $\text{dist}(F_\mathbf{i}(K), F_\mathbf{j}(K)) \geq h\sigma^{m-1} > \sigma^{m-1}\delta_k(K)$, $\mathbf{i} \neq \mathbf{j}$,

and $\bigcup_{\mathbf{i} \in Z_p^m} F_\mathbf{i}(K) = K$. Let $\overline{\omega}_k \subset K$ be a collection of k points such that $\delta(\overline{\omega}_k) = \delta_k(K)$, and $\omega_m := \cup_{\mathbf{i} \in Z_p^m} F_\mathbf{i}(\overline{\omega}_k)$. In view of (3.9), it is not difficult to see that $\delta_{kp^m}(K) \geq \delta(\omega_m) = \sigma^m\delta_k(K)$. On the other hand, from any collection of $c_m := (k-1)p^m + 1$ points on K at least k must belong to the same $F_\mathbf{i}(K)$, and hence, $\delta_{c_m}(K) \leq \sigma^m\delta_k(K)$. Since $c_m \leq kp^m$, we have $\delta_{kp^m}(K) = \delta_{c_m}(K)$ and

$$g_{\infty,\lambda}(K) = \lim_{m\to\infty} \delta_{kp^m}(K)(kp^m)^{1/\lambda} = \lim_{m\to\infty} \delta_{c_m}(K)(kp^m)^{1/\lambda} =$$

$$= g_{\infty,\lambda}(K) \lim_{m\to\infty} (kp^m/c_m)^{1/\lambda} = g_{\infty,\lambda}(K)\,(k/(k-1))^{1/\lambda},$$

which contradicts the finiteness and positiveness of $g_{\infty,\lambda}(K)$. Hence, $0 < \underline{g}_{\infty,\lambda}(K) < \overline{g}_{\infty,\lambda}(K) < \infty$. Taking into account Propositions 3.1 and 3.3, we get Proposition 2.6.

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Department of Mathematics, Vanderbilt University, Nashville, TN, 37240
E-mail address: sergiy.v.borodachov@vanderbilt.edu

Department of Mathematics, Vanderbilt University, Nashville, TN, 37240
E-mail address: doug.hardin@vanderbilt.edu

Department of Mathematics, Vanderbilt University, Nashville, TN, 37240
E-mail address: Edward.B.Saff@Vanderbilt.edu