Riesz spherical potentials with external fields and minimal energy points separation

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Abstract

In this paper we consider the minimal energy problem on the sphere S^d for Riesz potentials with external fields. Fundamental existence, uniqueness, and characterization results are derived about the associated equilibrium measure. The discrete problem and the corresponding weighted Fekete points are investigated. As an application we obtain the separation of the minimal s-energy points for $d-2 < s < d$. The explicit form of the separation constant is new even for the classical case of $s = d-1$.

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1 Introduction and main results

In this article we shall further develop and apply the theory of minimal s-energy problems for Riesz spherical potentials with external field, where the potential varies inversely with respect to the s-power of the Euclidean distance between points. The restriction to spherical potentials is mainly motivated by the applications to minimal energy points on the sphere, but the analysis may be carried out on more general manifolds, as well as with other kernels. This we intend to address in a subsequent work. For more on the general theory of equilibrium potentials with external fields we refer to the recent works of Zorii [22], [23], and [24].

As the main application of our results we derive optimal order separation of the minimal s-energy points on the sphere $S^d \subset \mathbf{R}^{d+1}$ for the range of the parameter $d-2 < s < d$. The explicit form of our separation constant is new even for the classical case $s = d-1$ considered by Dahlberg [2] in 1978, and improves upon the (mainly) implicit constants obtained in [15] by Kuijlaars, Saff, and Sun for the cases $d-1 < s < d$. In addition, for the important particular case of S^2, our results with what was previously known settle the question of well-separation of minimal s-energy points for all $s \geq 0$, except for the critical value $s = 2$.

1.1 *Energy problems on the sphere with external fields.*

Let $S^d := \{x \in \mathbf{R}^{d+1} \ : \ |x| = 1\}$ be the unit sphere in $\mathbf{R}^{d+1}$, where $|\cdot|$ denotes the Euclidean norm. Given a compact set $E \subseteq S^d$, consider the class $\mathcal{M}(E)$ of unit positive Borel measures supported on E. For $0 < s < d$ the *Riesz s-potential* and *Riesz s-energy* of a measure $\mu \in \mathcal{M}(E)$ are given respectively by

$$U_s^\mu(x) := \int k_s(x,y)\, d\mu(y), \quad I_s(\mu) := \int\int k_s(x,y)\, d\mu(x) d\mu(y),$$

where $k_s(x, y) := |x - y|^{-s}$ is the so-called *Riesz kernel*. For the case $s = 0$ we use the logarithmic kernel $k_0(x, y) := \log(1/|x - y|)$ instead. The *s-capacity* of E is then defined as $C_s(E) := 1/W_s(E)$ for $s > 0$, where $W_s(E) := \inf\{I_s(\mu) : \mu \in \mathcal{M}(E)\}$. A property is said to hold *quasi-everywhere* (q.e.), if the exceptional set has s-capacity zero. When $C_s(E) > 0$, there exists a unique minimizer $\mu_E = \mu_{s,E}$, called *the s-equilibrium measure for E*, such that $I_s(\mu_E) = W_s(E)$. For more details see [16, Chapter II].

We shall refer to a non-negative lower semi-continuous function $Q : S^d \to [0, \infty]$, such that $Q(x) < \infty$ on a set of positive Lebesgue surface measure, as an *external field*. The weighted energy associated with $Q(x)$ is then given by

$$I_Q(\mu) := I_s(\mu) + 2 \int Q(x)\, d\mu(x). \tag{1.1}$$

Definition 1.1 *The energy problem on the sphere in the presence of the external field $Q(x)$ refers to the minimal quantity*

$$V_Q := \inf \left\{ I_Q(\mu) \ : \ \mu \in \mathcal{M}(S^d) \right\}. \tag{1.2}$$

*A measure $\mu_Q = \mu_{Q,s} \in \mathcal{M}(S^d)$ such that $I_Q(\mu_Q) = V_Q$ is called an s-**equilibrium measure associated with** $Q(x)$.*

We first state a Frostman-type theorem which deals with the existence and uniqueness of the measure μ_Q, as well as a criterion that characterizes μ_Q in terms of its potential. The proof of this theorem follows closely the proof of [21, Theorem I.1.3]. It could also be derived as a particular case from the more general results in [22] (see in particular Theorem 1 and Proposition 1).

Theorem 1.2 *Let $0 \leq s < d$. For the minimal energy problem on S^d with external field $Q(x)$ the following properties hold:*

(a) V_Q *is finite.*

(b) There exists a unique s-equilibrium measure $\mu_Q = \mu_{Q,s} \in \mathcal{M}(S^d)$ *associated with* $Q(x)$*. Moreover, the support* S_Q *of this measure is contained in the compact set* $E_M := \{x \in S^d : Q(x) \leq M\}$ *for some* $M > 0$*.*

(c) The measure μ_Q *satisfies the variational inequalities*

$$U_s^{\mu_Q}(x) + Q(x) \geq F_Q \quad \text{q.e. on} \quad S^d, \tag{1.3}$$

$$U_s^{\mu_Q}(x) + Q(x) \leq F_Q \quad \text{for all} \quad x \in S_Q, \tag{1.4}$$

where

$$F_Q := V_Q - \int Q(x)\, d\mu_Q(x). \tag{1.5}$$

(d) Inequalities (1.3) and (1.4) completely characterize the equilibrium measure μ_Q *in the sense that if* $\nu \in \mathcal{M}(S^d)$ *is a measure with finite s-energy such that for some constant* C *we have*

$$U_s^{\nu}(x) + Q(x) \geq C \quad \text{q.e. on} \quad S^d, \tag{1.6}$$

$$U_s^{\nu}(x) + Q(x) \leq C \quad \text{for all} \quad x \in \operatorname{supp}(\nu), \tag{1.7}$$

then $\nu = \mu_Q$ *and* $C = F_Q$*.*

Remark For a given compact subset $E \subset S^d$ with positive capacity, we may consider a problem similar to (1.2) but with $\mu \in \mathcal{M}(E)$ instead. For the external field Q on E the same theorem holds with E instead of S^d (one can set $Q \equiv \infty$ on $S^d \setminus E$ and use the theorem above).

In the case when $d - 1 \leq s < d$ we are able to analyze further the characterization property from Theorem 1.2 (d). More precisely, the following result holds.

Theorem 1.3 *Let $d-1 \le s < d$, Q be an external field on S^d, and F_Q be defined as in (1.5). For any measure $\lambda \in \mathcal{M}(S^d)$ we have*

$$\text{“}\inf_{x\in S_Q}\text{”}(U_s^\lambda(x) + Q(x)) \le F_Q, \tag{1.8}$$

and

$$\sup_{x\in \text{supp}(\lambda)} (U_s^\lambda(x) + Q(x)) \ge F_Q, \tag{1.9}$$

where “inf” *means that the infimum is taken quasi-everywhere. If equality holds in both inequalities, then $\lambda = \mu_Q$.*

Remark The proof of this theorem utilizes the Principle of Domination for Riesz potentials, which in general holds for the parameter range $d-1 \le s < d+1$ and measures supported on any subsets of $\mathbf{R}^{d+1}$. Since the measures in our case are supported on the d-dimensional manifold S^d, a restricted version of the Principle of Domination holds for $d-2 < s < d$ (see Lemma 5.1). Nonetheless, it remains an open problem whether Theorem 1.3 holds for this larger range. However, for the particular case of *Riesz external fields* $Q_a(x) = Q_{a,s}(x) := k_s(x,a)$, where $a \in S^d$ is fixed, we are able to establish this assertion directly. Because of its importance for the applications to the separation of minimum energy points, we formulate the result as a separate theorem.

Theorem 1.4 *For any positive multiple of a Riesz external field, that is $Q(x) := c\,Q_{a,s}(x)$ with $c > 0$, the conclusions of Theorem 1.3 hold for $d-2 < s < d$.*

Observe that if δ_a is the unit Dirac-delta measure supported at a, then the Riesz external field is also the Riesz potential of δ_a, i.e. $Q_{a,s}(x) = U_s^{\delta_a}(x)$. Henceforth, we will assume that the point a is the North Pole of S^d.

1.2 *Minimal s-energy points on the sphere and their separation*

The initial motivation for considering the minimal energy problem with external field, as well as the main application in this paper, is related to minimal s-energy points and their separation. Given a configuration of N points on the unit sphere $\omega_N = \{x_1, x_2, \ldots, x_N\} \subset S^d$ we define its *discrete Riesz s-energy* as

$$\mathcal{E}_s(\omega_N) := \sum_{1 \le i \ne j \le N} k_s(x_i, x_j), \quad s \ge 0, \tag{1.10}$$

where $k_s(x, y)$ is the Riesz kernel for $s > 0$ and the logarithmic kernel for $s = 0$ (see Section 1.1). A configuration $\omega_N^* = \omega_{N,s}^* = \{x_1^*, x_2^*, \ldots, x_N^*\}$ that minimizes (1.10) is called an *N-point minimal s-energy arrangement*. Such arrangements serve as "well-distributed" point sets on the sphere and have been a source of intensive investigations with a wide variety of applications in chemistry, physics, crystallography, morphology, etc. (see [1], [9], [13], [18], [19]). The celebrated Thomson problem [18] is the special case $d = 2$, $s = 1$. Here we are particularly interested in the separation properties of such minimal arrangements. We say that a sequence of configurations $\omega_N = \{x_1^{(N)}, x_2^{(N)}, \ldots, x_N^{(N)}\} \subset S^d$, $N = 2, 3, \ldots$ is *well separated* if there is a positive constant C independent of N such that

$$\Delta_N = \Delta(\omega_N) := \min_{i \ne j} |x_i^{(N)} - x_j^{(N)}| \ge \frac{C}{N^{1/d}}. \tag{1.11}$$

Among the most studied extremal arrangements in the literature are the *best-packing* points, in which case the smallest distance between all the various pairs of N points is maximized. Such points, for fixed N, correspond to limiting minimal s-energy arrangements as $s \to \infty$. When $d = 2$, Habicht and van der Waerden [11] have not only established the separation for these points, they have actually found $\lim \sqrt{N}\Delta_N$. Another distinguished sequence of configurations is the *Fekete points* ($s = d - 1$), and their well separated property was shown by Dahlberg [2] for any d. In

[14], Kuijlaars and Saff among other results handled the case $s \geq d$, but for $s = d$ they obtained the weaker inequality $\Delta_N \geq C/(N \log N)^{1/d}$. It is still an open problem whether one can omit the $\log N$ term from this estimate. For the minimal s-energy arrangements when $d = 2$ and $s = 0$, called *logarithmic points*, Rakhmanov, Saff, and Zhou in [20] showed that $\Delta_N \geq (3/5)/\sqrt{N}$, which was subsequently improved by Dubickas [6] to $\Delta_N \geq (7/4)/\sqrt{N}$, and Dragnev [5], who obtained the estimate $\Delta_N \geq 2/\sqrt{N-1}$.

In this paper we shall prove that N-point minimal s-energy arrangements are well separated for $d - 2 < s < d$. More precisely, if $\omega_N^* = \{x_1^*, x_2^*, \dots, x_N^*\}$ is an N-point minimal s-energy arrangement and

$$\Delta_{N,s,d} = \Delta(\omega_N^*) := \min_{i \neq j} |x_i^* - x_j^*| \tag{1.12}$$

we have the following result.

Theorem 1.5 *For $d-2 < s < d$, any sequence $\{\omega_N^*\}_{N=2}^\infty$ of minimal s-energy point configurations is well-separated. More precisely, for any $N(> 2)$-point minimal s-energy configuration ω_N^* for S^d we have*

$$\Delta_{N,s,d} \geq \frac{K_{s,d}}{N^{1/d}}, \quad K_{s,d} := \left(\frac{2\mathcal{B}(d/2, 1/2)}{\mathcal{B}(d/2, (d-s)/2)} \right)^{1/d} = \frac{2^{1-s/d}}{I_s(\sigma/\|\sigma\|)^{1/d}}, \tag{1.13}$$

where $\mathcal{B}(x,y) := \int_0^1 u^{x-1}(1-u)^{y-1}\, du$ is the Beta function. In particular, when $s = d-1$

$$\Delta_{N,d-1,d} \geq \frac{2^{1/d}}{N^{1/d}}, \tag{1.14}$$

and, when $d = 2$

$$\Delta_{N,s,2} \geq \frac{2\sqrt{1-s/2}}{\sqrt{N}}. \tag{1.15}$$

Remark For the case $s = d - 2 > 0$ we can deduce from the above theorem that there is at least one sequence of minimal $(d-2)$-energy configurations which satisfies (1.13). Indeed, let us consider a sequence $s_1, s_2, \ldots, s_n, \cdots \searrow d-2$ and let $\{\omega^*_{N,i}\}_{N=2}^{\infty}$ be a minimal s_i-energy point configuration sequence which satisfies (1.13). For every fixed N from the collection of configurations $\{\omega^*_{N,i}\}_{i=1}^{\infty}$, we select a subsequence, so that $\omega^*_{N,i} \to \omega^*_N$ as $i \to \infty$. From the continuity of the discrete energy functional it is easy to derive that ω^*_N is a minimal $(d-2)$-energy configuration, which satisfies (1.13) with $K_{d-2,d}$. Then the sequence $\{\omega^*_N\}_{N=2}^{\infty}$ has the desired properties.

We note that in a concurrent independent work [15] the separation property was established for $d-1 < s < d$, but no explicit forms of the separation constants were given there. The explicit forms of $K_{s,d}$ in (1.13)-(1.15) here are new, even for the classical case $s = d-1$. The separation property in the range $d-2 < s < d-1$ is also established for the first time here. Thus, except for $s = 2$, all sequences of minimal s-energy configurations are well separated for the classical case of S^2. When $d = s = 2$ the estimate from [14] is $d_{N,2} \geq K_d/\sqrt{N \log N}$, and it is open whether the $\log N$ term can be dropped. Unfortunately, our result doesn't help in this direction, because in (1.15) $K_{s,d} \to 0$ as $s \to 2$. Finally, we note that when $d = 2$ and $s = 0$ the estimate (1.15) is the same as the one in [5].

Remark We are grateful to J. Brauchart for pointing out that for $d = 2$, a careful analysis of the proof of Theorem 1.5 shows that N can be replaced by $N-1$ in (1.13)-(1.15).

We recall the approach from [5]. Let us fix N and choose one of the points of a minimal energy arrangement ω^*_N to be at the North Pole a. Next we apply stereographic projection with center a of the unit sphere onto the extended complex plane. The image of a is the point at infinity and the images $z_1, z_2, \ldots, z_{N-1}$ of the rest of the points become weighted Fekete points in the complex plane with an external field $Q_N(z) = (N-1)\log(1+|z|^2)/(2(N-2))$ (see [21, Chapter 3]). But the weighted Fekete points, no matter how many points we consider, all lie in the support of the equilibrium measure of the corresponding weighted energy problem, which turns out to be the disk $\{z : |z| \leq \sqrt{N-2}\}$. Therefore, $z_1, z_2, \ldots, z_{N-1}$ belong to this disk, and thus, stay away from the point at infinity, which in turn implies the separation of a from $x_1^*, x_2^*, \ldots, x_{N-1}^*$.

In this paper our approach is somewhat similar. The difference is that instead of projecting we solve the discrete energy problem directly on the sphere. Let N be fixed and let $\omega_N^* = \{x_1^*, x_2^*, \ldots, x_N^*\}$ be a minimal s-energy configuration, $s > 0$. Without loss of generality we may place x_N^* at the North Pole a and let

$$Q_N(x) = Q_{N,a,s}(x) := k_s(x,a)/(N-2), \quad N > 2. \tag{1.16}$$

Since this field is a multiple of the Riesz external field, we will be able to utilize Theorem 1.4.

Definition 1.6 *Let $Q(x)$ be an external field and $n \geq 2$ be an arbitrary natural number. A set of n points $\mathcal{R}_n := \{y_1, y_2, \ldots, y_n\} \subset S^d$ that solves the discretized minimization problem*

$$\text{minimize} \left\{ \sum_{1 \leq i \neq j \leq n} [k_s(x_i, x_j) + Q(x_i) + Q(x_j)] \ : \ x_i \in S^d, i = 1, 2, \ldots, n \right\}. \tag{1.17}$$

*is called an n-**point minimal Q-weighted Riesz set**.*

We note that the problem (1.17) is a discretized version of (1.2). The existence of Riesz sets is an easy consequence of the lower semi-continuity of the energy functional and the compactness of the unit sphere. Observe, that in the particular case when $Q(x) = Q_N(x)$ and $n = N-1$ the points $\{x_1^*, x_2^*, \ldots, x_{N-1}^*\}$ form an $(N-1)$-point Q_N-weighted Riesz set.

The normalized counting measure of an n-point set $E_n = \{x_1, x_2, \ldots, x_n\}$ is defined as

$$\mu_{E_n} := \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i},$$

where δ_x is the Dirac-delta measure with unit mass at x. In addition, denote with $h_{E_n}(x)$ the weighted potential of μ_{E_n}, i.e.

$$h_{E_n}(x) := U_s^{\mu_{E_n}}(x) + Q(x) = \frac{1}{n}\sum_{i=1}^{n} \frac{1}{|x - x_i|^s} + Q(x). \tag{1.18}$$

As an application of Theorems 1.3 we deduce the following theorem.

Theorem 1.7 *Let $d-1 \leq s < d$. Let $E_n \subset S^d$ be a set of n distinct points, and suppose that the associated weighted potential satisfies the inequality*

$$h_{E_n}(x) \geq M \quad \text{q.e. on} \quad S_Q, \tag{1.19}$$

for some constant M. Then for all $x \in \mathbf{R}^{d+1}$ we have

$$U_s^{\mu_{E_n}}(x) \geq M + U_s^{\mu_Q}(x) - F_Q. \tag{1.20}$$

Furthermore,

$$h_{E_n}(x) \geq M \quad \text{q.e. on} \quad S^d. \tag{1.21}$$

If $Q(x) = c\,Q_a(x)$, $c > 0$, then Theorem 1.4 allows us to extend the range of the parameter s.

Theorem 1.8 *Let $d-2 < s < d$. Let $Q(x) = c\,Q_a(x)$, E_n be a set of n points, and suppose that the associated weighted potential satisfies the inequality*

$$h_{E_n}(x) \geq M \quad \text{q.e. on} \quad S_Q, \tag{1.22}$$

for some constant M. Then for all $x \in S^d$ we have

$$U_s^{\mu_{E_n}}(x) \geq M + U_s^{\mu_Q}(x) - F_Q. \tag{1.23}$$

Theorems 1.7 and 1.8 can be applied to show that the minimal Q-weighted Riesz energy sets from Definition 1.6 are contained in the *essential support* of μ_Q

$$S_Q^* := \{x \in S^d \ : \ U_s^{\mu_Q}(x) + Q(x) \leq F_Q\}. \tag{1.24}$$

Corollary 1.9 *For any $d-1 \leq s < d$ and any positive integer n, the weighted Riesz sets $\mathcal{R}_n$ are contained in the essential support S_Q^*. Moreover, when $Q(x) = c\, Q_a(x)$, this is true for $d-2 < s < d$.*

1.3 *The equilibrium problem for* $Q_N(x)$

In the paragraph after Definition 1.6, we noted that if $\omega_N^* = \{x_1^*, x_2^*, \dots, x_N^*\}$ is a minimal s-energy configuration and x_N^* is fixed at the North Pole a, then $\{x_1^*, x_2^*, \dots, x_{N-1}^*\}$ forms an $(N-1)$-point Q_N-weighted Riesz set. In light of Corollary 1.9 we are able to conclude that these points belong to the essential support $S_{Q_N}^*$ (see (1.24)). In this case the essential support coincides with the equilibrium support S_{Q_N} (see Theorem 1.10). Therefore, since x_N^* was arbitrary we get that

$$\Delta_{N,s,d} \geq \mathrm{dist}(a, S_{Q_N}).$$

This latter observation motivates us to consider the equilibrium problem for the Riesz external field $Q_N(x)$ defined in (1.17).

Throughout the paper we shall denote by $\sigma(x) = \sigma_d(x)$ the Lebesgue surface measure on S^d. For future reference, we recall that the total mass of $\sigma(x)$ is given by

$$W_{d+1} := \|\sigma\| = \int_{S^d} d\sigma(x) = \frac{2\pi^{(d+1)/2}}{\Gamma((d+1)/2)}. \tag{1.25}$$

For a fixed $r > 0$ let C_r be the polar cap $\{x \in S^d \ : \ |x-a| < r\}$ and Σ_r be its complement $S^d \setminus C_r$. For $y \in C_r$ denote by

$$\epsilon_y = \epsilon_{y,r} := \mathrm{Bal}(\delta_y, \Sigma_r)$$

the balayage measure of the unit Dirac-delta measure δ_y onto Σ_r. When $y = a$ we simply write

$$\epsilon_r := \epsilon_a = \epsilon_{a,r}.$$

In Section 3 we show that for $d-2 < s < d$ the balayage measure ϵ_y is well-defined and absolutely continuous with respect to the Lebesgue surface measure σ restricted to Σ_r. Let $\epsilon'_y(x)$ be its density, i.e.

$$d\epsilon_y(x) = \epsilon'_y(x) d\sigma|_{\Sigma_r}.$$

We note that

$$U_s^{\epsilon_y}(z) = U_s^{\delta_y}(z) = 1/|z-y|^s, \quad z \in \Sigma_r \tag{1.26}$$

and on the rest of the sphere S^d we have $U_s^{\epsilon_y}(z) < U_s^{\delta_y}(z)$ (see [16, p. 401]).

Next, we define the measure ν_r supported on Σ_r as

$$d\nu_r(x) := \left\{ \int_{C_r} \epsilon'_y(x)\, d\sigma(y) + 1 \right\} d\sigma(x)|_{\Sigma_r}. \tag{1.27}$$

It is easy to verify that the potential of ν_r is constant on Σ_r. Indeed, using (1.26) for any $z \in \Sigma_r$ we obtain

$$\begin{aligned} U_s^{\nu_r}(z) &= \int_{\Sigma_r} \frac{1}{|z-x|^s} \left(\int_{C_r} \epsilon'_y(x)\, d\sigma(y) \right) d\sigma(x) + \int_{\Sigma_r} \frac{1}{|z-x|^s}\, d\sigma(x) \\ &= \int_{C_r} U_s^{\epsilon_y}(z) d\sigma(y) + \int_{\Sigma_r} \frac{1}{|z-x|^s}\, d\sigma(x) \\ &= \int_{S^d} \frac{1}{|z-y|^s}\, d\sigma(y) = \frac{2^{d-s}\pi^{d/2}\Gamma((d-s)/2)}{\Gamma(d-s/2)} = W_{d+1} I_s(\sigma/\|\sigma\|). \end{aligned} \tag{1.28}$$

Therefore, the measure from (1.27) is a multiple of the equilibrium measure on Σ_r.

Consider now the class of signed measures, depending on r

$$\eta_r = \eta_{r,s} := \frac{1 + \frac{\|\epsilon_r\|}{N-2}}{\|\nu_r\|} \nu_r - \frac{1}{N-2} \epsilon_r. \tag{1.29}$$

For

$$r_0 = r_{0,s} := \min\{r \ : \ \eta_r \geq 0\} \tag{1.30}$$

we obtain the following result which is used to prove Theorem 1.5.

Theorem 1.10 *Let $d-2 < s < d$. Then $\mu_{Q_N} = \eta_{r_0}$ and $S_{Q_N} = \Sigma_{r_0}$. Moreover, $\Delta_{N,s,d} \geq r_0$.*

The paper is organized as follows. In Section 2 we prove Theorems 1.3 and 1.7, postponing the proofs of Theorems 1.4, 1.5, 1.8, and Corollary 1.9 until Section 5. Section 3 investigates balayage and equilibrium measures of spherical caps. The proof of Theorem 1.10 is in Section 4.

2 Energy problems on the sphere with external fields - Proofs

We begin with the proof of Theorem 1.3. For this we need the Principle of Domination for Riesz potentials. The following theorem of Landkof can be found in [16, Theorem 1.29].

Theorem L *Let $p-2 \leq s < p$. Suppose μ is a positive measure in $\mathbf{R}^p$ whose potential U_s^μ is finite μ-almost everywhere, and that $f(x)$ is a $(p-s)$-superharmonic function. Then if the inequality $U_s^\mu(x) \leq f(x)$ holds μ-almost everywhere, it holds everywhere in $\mathbf{R}^p$.*

Proof of Theorem 1.3. Suppose to the contrary that there is a measure $\lambda \in \mathcal{M}(S^d)$ such that (1.8) fails, i.e. there is a constant $L_1 > F_Q$ such that

$$U_s^\lambda(x) + Q(x) \geq L_1, \quad \text{q.e. on} \ \ S_Q.$$

Applying (1.4) we obtain then that

$$U_s^\lambda(x) \geq U_s^{\mu_Q}(x) - F_Q + L_1 \quad \text{q.e. on} \ \ S_Q.$$

From Theorem 1.2 we have that μ_Q has finite s-energy, therefore its support S_Q will have positive s-capacity. Hence, the s-equilibrium measure μ_{S_Q} of S_Q is well-defined. Let $\nu := (L_1 - F_Q)C_s(S_Q)\mu_{S_Q}$. Then $U_s^{\nu}(x) = L_1 - F_Q$ q.e. on S_Q. Thus,

$$U_s^{\lambda}(x) \geq U_s^{\mu_Q+\nu}(x) \quad \text{q.e. on} \quad S_Q. \tag{2.1}$$

Observe that both ν and μ_Q have finite s-energy, which implies that the inequality will be true $(\mu_Q + \nu)$-a.e. Since for this range of s, potentials are $(d+1-s)$-superharmonic, we can apply Theorem L with $p = d+1$ to derive the inequality (2.1) for all $x \in \mathbf{R}^{d+1}$. Multiplying by $|x|^s$ and letting $|x| \to \infty$ we obtain that $1 \geq 1 + (L_1 - F_Q)C_s(S_Q)$, which is a contradiction.

We derive (1.9) similarly, utilizing (1.3) instead. Let there be a measure $\lambda \in \mathcal{M}(S^d)$ and a constant $L_2 < F_Q$, such that

$$U_s^{\lambda}(x) + Q(x) \leq L_2 \ , \qquad x \in \text{supp}(\lambda).$$

Integration with respect to λ yields that λ has finite s-energy (recall that F_Q is finite and $Q(x) \geq 0$). This implies that $C_s(\text{supp}(\lambda)) > 0$, and that the measure $\mu_{\text{supp}(\lambda)}$ is well-defined. From (1.3) we get

$$U_s^{\mu_Q}(x) \geq U_s^{\lambda}(x) + F_Q - L_2 \quad \text{q.e. on} \quad \text{supp}(\lambda).$$

With $\nu := (F_Q - L_2)C_s(\text{supp}(\lambda))\mu_{\text{supp}(\lambda)}$ we can write

$$U_s^{\mu_Q}(x) \geq U_s^{\lambda+\nu}(x) \quad \text{q.e. on} \quad \text{supp}(\lambda).$$

Applying again the Principle of Domination we obtain a similar contradiction.

Finally, suppose equality holds in both (1.8) and (1.9). Using (1.4) we can extend (1.8) to

$$U_s^{\lambda}(x) \geq F_Q - Q(x) \geq U_s^{\mu_Q}(x) \ \ \text{q.e. on} \ \ S_Q,$$

which from the Principle of Domination can be extended to

$$U_s^{\lambda}(x) \geq U_s^{\mu_Q}(x) \quad \text{on} \quad \mathbf{R}^{d+1}.$$

In particular, from (1.3) we have $U_s^{\lambda}(x) + Q(x) \geq U_s^{\mu_Q}(x) + Q(x) \geq F_Q$ q.e. on S^d. Using (1.9) and Theorem 1.2 (d) we conclude that $\lambda = \mu_Q$. □

As a consequence of Theorem 1.3 we can deduce Theorem 1.7.

Proof of Theorem 1.7. From Theorem 1.3 we conclude that $M \leq F_Q$. Using (1.4), the inequality (1.19) yields that

$$U_s^{\mu_{E_n}}(x) + F_Q - M \geq U_s^{\mu_Q}(x) \quad \text{q.e. on} \quad S_Q.$$

Now let λ be a multiple of the Lebesgue surface measure on S^d, so that $U_s^{\lambda}(x) = F_Q - M$ for all $x \in S^d$. Then

$$U_s^{\mu_{E_n}+\lambda}(x) \geq U_s^{\mu_Q}(x) \quad \text{q.e. on} \quad S_Q.$$

Since μ_Q has a finite s-energy, this inequality holds μ_Q almost everywhere. The aforementioned principle of domination implies (1.20).

Finally, we note that (1.21) is an immediate consequence of (1.20) and (1.3). □

3 Balayage and equilibrium measures for spherical caps

In this section we lay the potential-theoretical groundwork for the proofs of Theorems 1.5 and 1.10. We have to find the balayage measures ϵ_y and ϵ_r, as well as the equilibrium measure ν_r (see (1.27)). We shall follow the approach to balayage measures for Riesz kernels as given by Landkof in [16, Chapter IV, §5].

Let a_* be the South Pole of S^d. With the notation of the previous section, we fix $y \in C_r$ and consider the Kelvin transform $\mathcal{K}_y : S^d \to \mathbf{R}^{d+1}$, called also stereographic projection or inversion with pole y and radius $\sqrt{2}$, i.e. if $x^* := \mathcal{K}_y(x)$ is the image of x under this transform, we have that x^* lies on the ray determined by y and x, and

$$|y - x| \cdot |y - x^*| = 2. \tag{3.1}$$

The transformation of the distance is given by the formula

$$|x^* - z^*| = \frac{2|x - z|}{|x - y||z - y|}. \tag{3.2}$$

The image of S^d is a hyperspace orthogonal to the radius-vector y, which we can identify with $\mathbf{R}^d$ in a natural way. Let x_1 and x_2 be points that minimize, respectively maximize, the distance from y to the hypersphere (hypercircle) $\partial\Sigma_r$. Clearly, if $y \neq a, a_*$ the points are unique and lie on the two-dimensional plane determined by a, a_*, and y. Then $\mathcal{K}_y(\Sigma_r)$ is a hyperball D_y with diameter $\overline{x_1^* x_2^*}$. Let b^* be the center and R_y be the radius of this hyperball. Since $|x_1^* - b^*| = |x_2^* - b^*|$ we obtain using (3.2) that

$$\frac{|y - x_1|}{|y - x_2|} = \frac{|b - x_1|}{|b - x_2|}. \tag{3.3}$$

Let $\overline{yw}$ be the bisector of $\angle y$ in $\triangle x_1 x_2 y$. Then (3.3) implies that $\overline{bw}$ will be the bisector of $\angle b$ in $\triangle x_1 x_2 b$. Thus, we can construct b from y as follows. First, we construct w as the intersection of the segments $\overline{y\, a_*}$ and $\overline{x_1 x_2}$, and then b as the intersection of the ray $aw^{\rightarrow}$ with the sphere S^d. If x_0 is the midpoint of $\overline{x_1 x_2}$ we see that the quadrilateral $x_0\, w\, y\, a$ can be inscribed in a circle with diameter $\overline{aw}$. Therefore, $\angle x_0 a w = \angle x_0 y w$. If y' is the intersection of the ray $y x_0^{\rightarrow}$ with the sphere, then y' is symmetrical to b with respect to $\overline{a a_*}$.

Next, we recall the definition of the Kelvin transform of measures. Given a measure ν with no point mass at y, its Kelvin transformation $\nu^* = \mathcal{K}_y(\nu)$ is a measure defined by

$$d\nu^*(x^*) := \frac{2^{s/2}}{|x-y|^s} d\nu(x). \tag{3.4}$$

Clearly, (3.1) and (3.4) imply the duality $(\nu^*)^* = \nu$.

We now focus on determining the balayage ϵ_y of the Dirac-delta measure δ_y onto Σ_r. In general, the existence of this balayage measure onto a compact set is guaranteed only for $d-1 \leq s < d$, but we will show that in our particular case when the set is the complement of a spherical cap (which is a spherical cap itself), it exists for $d-2 < s < d$.

Let λ_y be the equilibrium measure of the hyperball D_y, normalized so that its potential is one. From [16, Appendix 1] (with $p = d$, $0 < \alpha = d - s < 2$), we have that

$$d\lambda_y(x^*) = A_{s,d} \frac{dx^*}{(R_y^2 - |x^* - b^*|^2)^{\frac{d-s}{2}}}, \quad |x^* - b^*| \leq R_y, \tag{3.5}$$

where dx^* is the regular Lebesgue measure in the hyperplane (restricted to the ball $|x^* - b^*| \leq R_y$) and

$$A_{s,d} := \frac{\Gamma(d/2)\sin(\pi(d-s)/2)}{\pi^{d/2+1}}. \tag{3.6}$$

The balayage measure in question is then

$$\epsilon_y := 2^{-s/2}(\lambda_y)^*. \tag{3.7}$$

Indeed, for any $z \in \Sigma_r$ its potential satisfies

$$\begin{aligned} U_s^{\epsilon_y}(z) &= \int_{\Sigma_r} \frac{1}{|x-z|^s}\, d\epsilon_y(x) \\ &= \int_{D_y} \frac{|x^*-y|^s|z^*-y|^s}{2^s|x^*-z^*|^s} \frac{1}{|x^*-y|^s}\, d\lambda_y(x^*) \\ &= \frac{|z^*-y|^s}{2^s} U_s^{\lambda_y}(z^*) = \frac{1}{|z-y|^s} = U_s^{\delta_y}(z). \end{aligned} \tag{3.8}$$

Observe, that on the rest of the sphere S^d we have $U_s^{\epsilon_y}(z) < U_s^{\delta_y}(z)$ (see [16, p. 401, (A.1) and (A.2)]).

In the five steps below we determine the quantities involved in the definition of η_r from (1.29).

Step 1. *The balayage measure* ϵ_y.

To compute ϵ_y explicitly, we find from (3.5) and the definition of ϵ_y (3.7) that

$$d\epsilon_y(x) = A_{s,d} \frac{dx^*}{|y - x^*|^s (|x_1^* - b^*|^2 - |x^* - b^*|^2)^{\frac{d-s}{2}}}, \quad |x^* - b^*| \leq R_y, \tag{3.9}$$

where $A_{s,d}$ is the constant defined in (3.6). The relation between the hyperplanar Lebesgue measure dx^* and the surface Lebesgue measure $d\sigma(x)$ on the unit sphere S^d is given by

$$\frac{dx^*}{|y - x^*|^d} = \frac{d\sigma(x)}{|y - x|^d}. \tag{3.10}$$

Since $|x_1^* - b^*| = |x_1^* - x_2^*|/2$, using (3.1) and (3.2), from (3.9) and (3.10) we obtain

$$d\epsilon_y(x) = A_{s,d} \left(\frac{|y - x|^2 \, |x_0 - x_1|^2}{|y - x_1|^2 \, |y - x_2|^2} - \frac{|x - b|^2}{|y - b|^2} \right)^{\frac{s-d}{2}} \frac{d\sigma(x)}{|x - y|^d}, \quad x \in \Sigma_r. \tag{3.11}$$

Because the points x_0, x_1, x_2, y and b all lie in one two-dimensional plane, in cylindrical coordinates we have:

$$x = (\sqrt{1 - u^2}\, \overline{x}, u), \ \ y = (\sqrt{1 - t^2}\, \overline{y}, t), \ \ b = (\sqrt{1 - q^2}\, \overline{y}, q), \ \ x_{1,2} := (\pm\sqrt{1 - t_0^2}\, \overline{y}, t_0), \ \ x_0 = (\overline{0}, t_0),$$

where $\overline{x}, \overline{y} \in S^{d-1}$. Note that in these coordinates $C_r = \{y \ : \ t > t_0\}$ and $\Sigma_r = \{x \ : \ u \leq t_0\}$. Recall that y' is symmetrical to b about the aa_*-axis. From $|y' - x_0||y - x_0| = |x_1 - x_0||x_2 - x_0|$ we can find q in terms of t and t_0, namely

$$q = \frac{2t_0 - t(1 + t_0^2)}{1 - 2tt_0 + t_0^2}.$$

We evaluate that

$$\sqrt{1 - q^2} = \frac{\sqrt{1 - t^2}(1 - t_0^2)}{1 - 2tt_0 + t_0^2},$$

and

$$|y-x_1|^2|y-x_2|^2 = 4(t-t_0)^2.$$

The quantity $|y-b|^2$ is found to be

$$|y-b|^2 = \frac{4(t-t_0)^2}{1-2tt_0+t_0^2}.$$

The density of ϵ_y from (3.11) now becomes

$$\begin{aligned}
d\epsilon_y(x) &= A_{s,d}\left(\frac{|y-x|^2(1-t_0^2)}{4(t-t_0)^2} - \frac{|x-b|^2(1-2tt_0+t_0^2)}{4(t-t_0)^2}\right)^{\frac{s-d}{2}} \frac{d\sigma(x)}{|x-y|^d} \\
&= A_{s,d}\left(\frac{2(1-x\cdot y)(1-t_0^2)-2(1-x\cdot b)(1-2tt_0+t_0^2)}{4(t-t_0)^2}\right)^{\frac{s-d}{2}} \frac{d\sigma(x)}{|x-y|^d} \\
&= A_{s,d}\left(\frac{4t_0(t-t_0)-2x\cdot(y(1-t_0^2)-b(1-2tt_0+t_0^2))}{4(t-t_0)^2}\right)^{\frac{s-d}{2}} \frac{d\sigma(x)}{|x-y|^d} \\
&= A_{s,d}\left(\frac{4t_0(t-t_0)-4u(t-t_0)}{4(t-t_0)^2}\right)^{\frac{s-d}{2}} \frac{d\sigma(x)}{|x-y|^d} \\
&= A_{s,d}\left(\frac{t_0-u}{t-t_0}\right)^{\frac{s-d}{2}} \frac{d\sigma(x)}{|x-y|^d} =: \epsilon_y'(x)d\sigma(x), \quad x\in\Sigma_r. \qquad (3.12)
\end{aligned}$$

The latter representation shows that ϵ_y is absolutely continuous with respect to the restriction of the Lebesgue surface measure σ to Σ_r.

Step 2. *The measure ϵ_r.*

To find the measure ϵ_r, we observe that in this case $t=1$ and $|y-x|^2 = 2(1-u)$. In addition, recall that in polar coordinates we have the relation

$$d\sigma(x) = d\sigma_d(x) = (1-u^2)^{\frac{d-2}{2}}du\, d\sigma_{d-1}(\overline{x}),$$

where σ_d and σ_{d-1} are the Lebesgue surface measures on S^d and S^{d-1} respectively. So, the measure in (3.12) simplifies to

$$d\epsilon_r(x) = A_{s,d}\left(\frac{t_0-u}{1-t_0}\right)^{\frac{s-d}{2}} \frac{(1+u)^{\frac{d-2}{2}}du\, d\sigma_{d-1}(\overline{x})}{2^{d/2}(1-u)}, \quad -1\le u<t_0,\ \overline{x}\in S^{d-1}. \qquad (3.13)$$

Step 3. *The norm* $\|\epsilon_r\|$.

Integrating (3.13) over Σ_r and using (1.25) we evaluate the norm as follows:

$$\begin{aligned} \|\epsilon_r\| &= A_{s,d}\int_{S^{d-1}}\left(\int_{-1}^{t_0}\left(\frac{t_{-0}-u}{1-t_0}\right)^{\frac{s-d}{2}}\left(\frac{1+u}{2}\right)^{\frac{d-2}{2}}\frac{du}{2(1-u)}\right)d\sigma_{d-1}(\overline{x}) \\ &= W_dA_{s,d}\int_{-1}^{t_0}\left(\frac{t_{-0}-u}{1-t_0}\right)^{\frac{s-d}{2}}\left(\frac{1+u}{2}\right)^{\frac{d-2}{2}}\frac{du}{2(1-u)} \\ &= W_dA_{s,d}\frac{(1-t_0)^{\frac{d-s}{2}}(1+t_0)^{\frac{s}{2}}}{2^{\frac{d}{2}}}\int_0^1\frac{v^{\frac{d}{2}-1}(1-v)^{\frac{s-d}{2}}\,dv}{2-(1+t_0)v}. \end{aligned} \tag{3.14}$$

Step 4. *The equilibrium measure* ν_r.

Recall that the measure ν_r, supported on Σ_r, is defined in (1.27), and is a multiple of the equilibrium measure on Σ_r. To determine ν_r we have to evaluate the integral (see also (3.12))

$$J_r(x) := A_{s,d}\int_{C_r}\left(\frac{t_0-u}{t-t_0}\right)^{\frac{s-d}{2}}\frac{d\sigma(y)}{|x-y|^d}\ . \tag{3.15}$$

By substituting $d\sigma(y) = (1-t^2)^{(d-2)/2}dt\,d\sigma_{d-1}(\overline{y})$ and $|x-y|^2 = 2(1-ut-\sqrt{1-u^2}\sqrt{1-t^2}\,\overline{x}\cdot\overline{y})$ in (3.15) we get that

$$J_r(x) = A_{s,d}\int_{t_0}^1\left(\frac{t_0-u}{t-t_0}\right)^{\frac{s-d}{2}}\left(\int_{S^{d-1}}\frac{(1-t^2)^{\frac{d-2}{2}}\,d\sigma_{d-1}(\overline{y})}{2^{\frac{d}{2}}(1-ut-\sqrt{1-u^2}\sqrt{1-t^2}\,\overline{x}\cdot\overline{y})^{\frac{d}{2}}}\right)dt.$$

Using the Funk-Hecke formula [17, p. 20] we evaluate the inner integral (save for a constant) as follows

$$\begin{aligned} I_1 &:= \int_{S^{d-1}}\frac{d\sigma_{d-1}(\overline{y})}{(1-ut-\sqrt{1-u^2}\sqrt{1-t^2}\,\overline{x}\cdot\overline{y})^{\frac{d}{2}}} \\ &= W_{d-1}\int_{-1}^1\frac{(1-v^2)^{\frac{d-3}{2}}\,dv}{(1-ut-\sqrt{1-u^2}\sqrt{1-t^2}\,v)^{\frac{d}{2}}} \\ &= W_{d-1}\int_0^\pi\frac{\sin^{d-2}\theta\,d\theta}{(1-ut-\sqrt{1-u^2}\sqrt{1-t^2}\cos\theta)^{\frac{d}{2}}} \\ &= W_{d-1}\frac{2^{d/2}}{(1+u)^{d/2}(1-t)^{d/2}}\int_0^\pi\frac{\sin^{d-2}\theta\,d\theta}{(\rho^2-2\rho\cos\theta+1)^{\frac{d}{2}}}, \end{aligned} \tag{3.16}$$

where $\rho := \sqrt{(1-u)(1+t)}/\sqrt{(1+u)(1-t)}$. From [16, p. 400] we have that

$$\int_0^\pi \frac{\sin^{d-2}\theta\, d\theta}{(\rho^2 - 2\rho\cos\theta + 1)^{\frac{d}{2}}} = \frac{1}{\rho^{d-2}(\rho^2-1)} \int_0^\pi \sin^{d-2}\alpha\, d\alpha,$$

which when substituted in (3.16) yields

$$I_1 = W_{d-1} \frac{2^{\frac{d-2}{2}}}{(1-u)^{\frac{d-2}{2}}(1+t)^{\frac{d-2}{2}}(t-u)} \int_0^\pi \sin^{d-2}\alpha\, d\alpha.$$

It is not difficult to show that

$$W_{d-1}\int_0^\pi \sin^{d-2}\alpha\, d\alpha = W_d,$$

thus reducing the integral in (3.16) to

$$J_r(x) = W_d A_{s,d} \frac{(t_0-u)^{\frac{s-d}{2}}}{2(1-u)^{\frac{d-2}{2}}} \int_{t_0}^1 (t-t_0)^{\frac{d-s}{2}}(1-t)^{\frac{d-2}{2}} \frac{dt}{t-u}. \tag{3.17}$$

Since $d\sigma(x) = d\sigma_d(x) = (1-u^2)^{\frac{d-2}{2}}\, du d\sigma_{d-1}(\overline{x})$, substituting $v = (1-t)/(1-t_0)$ in (3.17) we get

$$d\nu_r(x) = \left((1-u^2)^{\frac{d-2}{2}} + W_d A_{s,d} \frac{(1+u)^{\frac{d-2}{2}}(1-t_0)^{d-\frac{s}{2}}}{(t_0-u)^{\frac{d-s}{2}}} \int_0^1 \frac{v^{\frac{d-2}{2}}(1-v)^{\frac{d-s}{2}}\, dv}{2(1-u-(1-t_0)v)} \right) du d\sigma_{d-1}(\overline{x}). \tag{3.18}$$

Step 5. *The norm $\|\nu_r\|$.*

We evaluate the norm as

$$\begin{aligned} \|\nu_r\| &= W_d \left(\int_{-1}^{t_0} (1-u^2)^{\frac{d-2}{2}} du \right. \\ &+ \left. W_d A_{s,d}(1-t_0)^{d-\frac{s}{2}} \int_{-1}^{t_0} (t_0-u)^{\frac{s-d}{2}}(1+u)^{\frac{d-2}{2}} \int_0^1 \frac{v^{\frac{d-2}{2}}(1-v)^{\frac{d-s}{2}}\, dv}{2(1-u-(1-t_0)v)} du \right). \end{aligned} \tag{3.19}$$

4 The equilibrium problem for $Q_N(x)$

In this section we present the proof of Theorem 1.10. First, we need a lemma concerning the densities of the signed measures η_r defined in (1.29). Since both ϵ_r and ν_r have densities with rotational symmetry about the polar axis, η_r can be written in polar coordinates as $d\eta_r = \eta_r'(u)du\, d\sigma_{d-1}(\overline{x})$.

Lemma 4.1 *Let $d-2 < s < d$. If $\lim_{u\to t_0^-} \eta_r'(u) \geq 0$, then $\eta_r'(u) \geq 0$ for all $-1 \leq u \leq t_0$.*

Proof From the definition (1.29) and the formulas (3.13) and (3.18) we get that

$$\begin{aligned}\eta_r'(u) &= \frac{1+\frac{\|\epsilon_r\|}{N-2}}{\|\nu_r\|}\left((1-u^2)^{\frac{d-2}{2}} + W_d A_{s,d}\frac{(1+u)^{\frac{d-2}{2}}(1-t_0)^{d-\frac{s}{2}}}{(t_0-u)^{\frac{d-s}{2}}}\int_0^1 \frac{v^{\frac{d-2}{2}}(1-v)^{\frac{d-s}{2}}\,dv}{2(1-u-(1-t_0)v)}\right)\\ &\quad -\frac{A_{s,d}}{N-2}\left(\frac{1-t_0}{t_0-u}\right)^{\frac{d-s}{2}}\frac{(1+u)^{\frac{d-2}{2}}}{2^{d/2}(1-u)},\end{aligned} \tag{4.1}$$

which can also be written as

$$\begin{aligned}\eta_r'(u) &= \frac{(N-2+\|\epsilon_r\|)(1+u)^{\frac{d-2}{2}}(1-t_0)^{d-\frac{s}{2}}}{(N-2)\|\nu_r\|(t_0-u)^{\frac{d-s}{2}}(1-u)}\times\\ &\quad \left[\left(\frac{1-u}{1-t_0}\right)^{\frac{d}{2}}\left(\frac{t_0-u}{1-t_0}\right)^{\frac{d-s}{2}} + \frac{W_d A_{s,d}}{2}\int_0^1 \frac{v^{\frac{d-2}{2}}(1-v)^{\frac{d-s}{2}}(1-u)}{t_0-u+(1-t_0)(1-v)}\,dv\right.\\ &\quad \left. -\frac{\|\nu_r\|A_{s,d}}{(N-2+\|\epsilon_r\|)2^{d/2}(1-t_0)^{d/2}}\right].\end{aligned}$$

Observe that the first multiple is always positive when $-1 \leq u < t_0$. Therefore, we focus on the expression in the brackets. The first two terms depend on u and the third is a constant, so let us denote them respectively $f(u)$, $g(u)$, and C. Since $f(t_0) = 0$, the assumption of the lemma implies that

$$\lim_{u\to t_0^-}\left[\; f(u) + g(u) - C\;\right] = g(t_0) - C \geq 0. \tag{4.2}$$

For future reference we note that (4.2) is equivalent to

$$\frac{W_d A_{s,d}\,\mathcal{B}(\frac{d}{2},\frac{d-s}{2})}{2} \geq \frac{\|\nu_r\|A_{s,d}}{(N-2+\|\epsilon_r\|)2^{d/2}(1-t_0)^{d/2}}, \tag{4.3}$$

where the Beta function has also the representation

$$\mathcal{B}(x,y) = \int_0^\infty \frac{u^{x-1}}{(1+u)^{x+y}}\,du.$$

To finish the proof of the lemma it is enough to establish the inequality

$$f(u) \geq g(t_0) - g(u) \qquad \text{for all} \quad u \in [-1, t_0), \tag{4.4}$$

i.e. we have to show that

$$\begin{aligned}
\left(\frac{1-u}{1-t_0}\right)^{\frac{d}{2}} \left(\frac{t_0-u}{1-t_0}\right)^{\frac{d-s}{2}} &\geq \frac{W_d A_{s,d}}{2} \int_0^1 v^{\frac{d-2}{2}} (1-v)^{\frac{d-s-2}{2}} \left(1 - \frac{(1-u)(1-v)}{t_0 - u + (1-t_0)(1-v)}\right) dv \\
&= \frac{W_d A_{s,d}}{2} \left(\frac{t_0-u}{1-t_0}\right) \int_0^1 \frac{v^{\frac{d}{2}} (1-v)^{\frac{d-s-2}{2}}}{\frac{t_0-u}{1-t_0} + 1 - v}\, dv.
\end{aligned} \tag{4.5}$$

The formulas (1.25) and (3.6) yield that

$$\frac{W_d A_{s,d}}{2} = \frac{\sin \pi(d-s)/2}{\pi} = \frac{1}{\mathcal{B}\left(\frac{d-s}{2}, 1 - \frac{d-s}{2}\right)}, \tag{4.6}$$

which together with the substitution $x = \dfrac{t_0 - u}{1 - t_0}$ transforms (4.5) to

$$(1+x)^{d/2} x^{(d-s-2)/2} \mathcal{B}\left(\frac{d-s}{2}, 1 - \frac{d-s}{2}\right) \geq \int_0^1 \frac{v^{\frac{d-s-2}{2}} (1-v)^{\frac{d}{2}}}{x+v}\, dv. \tag{4.7}$$

Next, we divide (4.7) by $(1+x)^{d/2} x^{(d-s-2)/2}$ and obtain that (4.4) is equivalent to

$$\mathcal{B}\left(\frac{d-s}{2}, 1 - \frac{d-s}{2}\right) \geq \int_0^1 \frac{\left(\frac{v}{x}\right)^{\frac{d-s-2}{2}} \left(\frac{1-v}{1+x}\right)^{\frac{d}{2}}}{1 + v/x}\, d(v/x) = \int_0^{1/x} \frac{v^{\frac{d-s-2}{2}} \left(\frac{1-vx}{1+x}\right)^{\frac{d}{2}}}{1+v}\, dv. \tag{4.8}$$

Since $0 \leq \dfrac{1-vx}{1+x} \leq \dfrac{1}{1+x} < 1$ and $0 \leq x \leq \dfrac{1+t_0}{1-t_0}$ we can estimate the right-hand side as follows:

$$\int_0^{1/x} \frac{v^{\frac{d-s-2}{2}} \left(\frac{1-vx}{1+x}\right)^{\frac{d}{2}}}{1+v}\, dv \leq \int_0^{1/x} \frac{v^{\frac{d-s-2}{2}}}{1+v}\, dv \leq \int_0^\infty \frac{v^{\frac{d-s-2}{2}}}{1+v}\, dv = \mathcal{B}\left(\frac{d-s}{2}, 1 - \frac{d-s}{2}\right),$$

which proves (4.8). Therefore, (4.4) holds and the proof is complete. □

We now present the proof of Theorem 1.10.

Proof of Theorem 1.10. From the uniqueness of the equilibrium measure μ_{Q_N} and the rotational symmetry of the external field about the polar axis, we conclude that the equilibrium support S_{Q_N} shares the same rotational symmetry. Let r_1 be defined as

$$r_1 := \max\{r \ : \ S_{Q_N} \subseteq \Sigma_r\}, \tag{4.9}$$

and let $t_1 = 1 - r_1^2/2$ (observe that the set Σ_{r_1} is given in polar coordinates as $u \leq t_1$). From Theorem 1.2 (b) we have that $r_1 > 0$. Let λ_1 be the equilibrium measure of Σ_{r_1}. It is clear that $\lambda_1 = \nu_{r_1}/\|\nu_{r_1}\|$. Thus, $U_s^{\lambda_1}(x) = 1/\text{cap}(\Sigma_{r_1})$ for all $x \in \Sigma_{r_1}$. In addition, from (3.8) we have that $U_s^{\frac{\epsilon_{r_1}}{N-2}}(x) = Q_N(x)$ when $x \in \Sigma_{r_1}$. Hence, the inequalities (1.3) and (1.4) from Theorem 1.2 yield that

$$U_s^{\mu_{Q_N}}(x) + Q_N(x) = U_s^{\mu_{Q_N} + \frac{\epsilon_{r_1}}{N-2}}(x) \geq F_{Q_N} = F_{Q_N}\text{cap}(\Sigma_{r_1})U_s^{\lambda_1}(x) \quad \text{q.e. on } \ \Sigma_{r_1}, \tag{4.10}$$

$$U_s^{\mu_{Q_N}}(x) + Q_N(x) = U_s^{\mu_{Q_N} + \frac{\epsilon_{r_1}}{N-2}}(x) \leq F_{Q_N} = F_{Q_N}\text{cap}(\Sigma_{r_1})U_s^{\lambda_1}(x) \quad \text{for all } \ x \in S_{Q_N}. \tag{4.11}$$

But the measure λ_1 has clearly finite energy, so inequality (4.10) also holds λ_1-a.e., and from Lemma 5.1 we deduce that it holds everywhere on S^d, and consequently equality holds in (4.11) everywhere on S_{Q_N}. Using (5.1) from the proof of the same lemma we could extend the inequalities (4.10) and (4.11) to the Kelvin transforms of the two measures. Applying an extension of the de La Vallée Poussin theorem (see [12, Theorem 2.5] or [8, Section 3]) we obtain,

$$(F_{Q_N}\text{cap}(\Sigma_{r_1})\lambda_1)^* |_{S^*_{Q_N}} \geq \left(\mu_{Q_N} + \frac{\epsilon_{r_1}}{N-2}\right)^* |_{S^*_{Q_N}},$$

where $S^*_{Q_N}$ denotes the stereographic projection image of S_{Q_N}. This inequality can be transferred back to the original measures (see (3.4)) to get

$$(F_{Q_N}\mathrm{cap}(\Sigma_{r_1})\lambda_1)\,|_{S_{Q_N}} \geq \left(\mu_{Q_N} + \frac{\epsilon_{r_1}}{N-2}\right)|_{S_{Q_N}}.$$

By integrating the inequality (4.10) over λ_1 we obtain that

$$\frac{1}{\mathrm{cap}(\Sigma_{r_1})}\left(1 + \frac{\|\epsilon_{r_1}\|}{N-2}\right) \geq F_{Q_N},$$

which can be used to derive that

$$\left(1 + \frac{\|\epsilon_{r_1}\|}{N-2}\right)\frac{\nu_{r_1}}{\|\nu_{r_1}\|} \geq F_{Q_N}\mathrm{cap}(\Sigma_{r_1})\lambda_1 \geq \mu_{Q_N} + \frac{\epsilon_{r_1}}{N-2} \quad \text{on} \quad S_{Q_N}.$$

This inequality together with the definition (1.29) implies that

$$\eta_{r_1}|_{S_{Q_N}} \geq \mu_{Q_N}|_{S_{Q_N}}. \tag{4.12}$$

From the definition of r_1 and (4.12) we easily obtain that $\lim_{u\to r_1^-}\eta'_{r_1}(u) \geq 0$. Now Lemma 4.1 allows us to conclude that η_{r_1} is a positive measure, which coupled with (1.30) and (4.9) shows that $r_1 \geq r_0$. But then μ_{Q_N} is also an equilibrium measure for the external field $Q_N(x)$ when restricted to the set Σ_{r_0} (see the remark after theorem 1.2). But η_{r_0} is a positive measure on Σ_{r_0}, whose weighted potential is constant on the entire Σ_{r_0}, and hence, by uniqueness $\mu_{Q_N} = \eta_{r_0}$ and $S_{Q_N} = \Sigma_{r_0}$.

To finish the proof we have to show that $U_s^{\eta_{r_0}}(z) + Q_N(z) > F_{Q_N}$ for all $z \in S^d \setminus \Sigma_{r_0}$. With $x = (\sqrt{1-u^2}\,\overline{x}, u)$ and $z = (\sqrt{1-\xi^2}\,\overline{z}, \xi)$, $u \leq t_0 < \xi$, $\overline{x}, \overline{z} \in S^{d-1}$, we write

$$\begin{aligned} U_s^{\eta_{r_0}}(z) + Q_N(z) &= \int_{-1}^{t_0}\eta_{r_0}(u)\left(\int_{S^{d-1}}\frac{d\sigma_{d-1}(\overline{x})}{|z-x|^s}\right)du + \frac{1}{(N-2)|a-z|^s} \\ &=: \int_{-1}^{t_0}\eta_{r_0}(u)\kappa(u,\xi)\,du + \frac{1}{(N-2)2^{s/2}(1-\xi)^{s/2}}, \end{aligned} \tag{4.13}$$

where the density $\eta_{r_0}(u)$ is given in (4.1). Clearly, the weighted potential has a rotational symmetry about the polar axis, and thus it is a function of the variable ξ only. We will show that this function is strictly convex on $(t_0, 1]$, and since it is constant on $[-1, t_0]$, and is greater or equal to that constant on $[t_0, 1]$, it has to be strictly greater than that constant on $(t_0, 1]$. Since the second term in (4.13) is strictly convex in $(t_0, 1]$, it is enough to focus on the first term only. More precisely, we shall establish the convexity of the kernel $\kappa(u, \xi)$ as a function of $\xi \in (t_0, 1]$ for any $u \in [-1, t_0]$, which implies the convexity of the entire integral.

To evaluate the kernel, we utilize again the Funk-Hecke formula [17, p. 20] to deduce

$$\begin{aligned}\kappa(u,\xi) = \int_{S^{d-1}} \frac{d\sigma_{d-1}(\overline{x})}{|z-x|^s} &= \int_{S^{d-1}} \frac{d\sigma_{d-1}(\overline{x})}{(2-2u\xi-2\sqrt{1-u^2}\sqrt{1-\xi^2}\,\overline{x}\cdot\overline{z})^{s/2}} \\ &= W_{d-1}\int_{-1}^{1} \frac{(1-v^2)^{\frac{d-3}{2}}\,dv}{(2-2u\xi-2\sqrt{1-u^2}\sqrt{1-\xi^2}\,v)^{s/2}}.\end{aligned}$$

Making the substitution $v = \cos\theta$ and denoting $\rho := \sqrt{\dfrac{(1+u)(1-\xi)}{(1-u)(1+\xi)}}$ we get that

$$\begin{aligned}\kappa(u,\xi) &= \frac{W_{d-1}}{(1-u)^{s/2}(1+\xi)^{s/2}} \int_0^{\pi} \frac{\sin^{d-2}\theta\, d\theta}{(\rho^2-2\rho\cos\theta+1)^{s/2}} \\ &= \frac{W_d}{(1-u)^{s/2}(1+\xi)^{s/2}}\ {}_2F_1\left(\frac{s}{2}, \frac{s+2-d}{2}; \frac{d}{2}; \frac{(1+u)(1-\xi)}{(1-u)(1+\xi)}\right),\end{aligned} \tag{4.14}$$

where we used [10, Formula 3.665 (2)] and the fact that $W_{d-1}\mathcal{B}\left(\frac{d-1}{2}, \frac{1}{2}\right) = W_d$. Here ${}_2F_1$ is the hypergeometric function

$${}_2F_1(\alpha, \beta; \gamma; x) = \sum_{k=0}^{\infty} \frac{(\alpha)_k(\beta)_k}{(\gamma)_k k!} x^k. \tag{4.15}$$

Observe, that for our choice of u and ξ, we have $0 < \rho < 1$, so the hypergeometric function is well defined. Thus, (4.14) becomes

$$\kappa(u,\xi) = \frac{W_d}{(1-u)^{s/2}} \sum_{k=0}^{\infty} \frac{(\frac{s}{2})_k(\frac{s+2-d}{2})_k(1+u)^k}{(\frac{d}{2})_k\ k!\ (1-u)^k} \frac{(1-\xi)^k}{(1+\xi)^{k+s/2}}. \tag{4.16}$$

We shall establish that all the functions

$$g_k(\xi) := \frac{(1-\xi)^k}{(1+\xi)^{k+s/2}} \ , \quad k = 0, 1, 2, \ldots$$

are convex. Since all coefficients in the sum (4.16) are positive, and the series is uniformly convergent for $\xi \in [t_0 + \epsilon, 1]$, we deduce the convexity of the right-hand side in (4.16) by differentiation.

For $k = 0$ the convexity of $g_k(\xi)$ is obvious. For $k \geq 1$, we shall establish the strict convexity on $(-1, 1)$ of the more general functions

$$h(\xi) = \frac{(1-\xi)^\alpha}{(1+\xi)^\beta} \ , \quad \alpha \geq 1, \ \beta > 0.$$

Indeed, the second derivative of $h(\xi)$

$$\frac{d^2h}{d\xi^2}(\xi) = \frac{(1-\xi)^\alpha}{(1+\xi)^\beta}\left[\frac{\alpha(\alpha-1)}{(1-\xi)^2} + \frac{2\alpha\beta}{(1-\xi)(1+\xi)} + \frac{\beta(\beta+1)}{(1+\xi)^2}\right],$$

is found to be positive for $\xi \in (-1, 1)$ whenever $\alpha \geq 1, \ \beta > 0$.

This establishes that for any fixed $u \leq t_0$, the kernel $\kappa(u, \xi)$ in (4.14) is convex as a function of ξ in the interval $[t_0, 1]$, and therefore we deduce that the integral term in (4.13), and hence the weighted potential, is also strictly convex for $\xi \in [t_0, 1]$. This shows that $U_s^{\eta_{r_0}}(z) + Q_N(z) > F_{Q_N}$ on $S^d \setminus \Sigma_{r_0}$, which implies that $S_{Q_N} = S^*_{Q_N}$. Therefore, $\Delta_{N,s,d} \geq r_0$. □

5 Minimal s-energy points on the sphere and their separation - Proofs

Before we proceed with the proof of Theorem 1.4, we need the following restricted version of the Principle of Domination to measures supported on S^d.

Lemma 5.1 *Let $d-2 < s < d$ and let μ and ν be two measures supported on $S^d \setminus \{a\}$ (recall that a is the North Pole). Suppose further that U_s^μ is finite μ-a.e. and that the inequality $U_s^\mu(x) \leq U_s^\nu(x)$ holds μ-a.e. Then it holds everywhere on S^d.*

Proof Under a stereographic projection with center a and radius $\sqrt{2}$ we transform the sphere S^d into the hyperplane $\{x^* = (x_1, x_2, \ldots, x_d, 0)\}$. If x^* is the image of x under the stereographic projection, i.e. x and x^* lie on one ray stemming from a and $|x-a||x^*-a| = 2$, then we have the distance conversion formula $|x^* - y^*| = 2|x-y|/(|x-a||y-a|)$. For the Kelvin transform λ^* of the measure λ (see (3.4)) we have

$$U_s^{\lambda^*}(x^*) = \int_{S_{\lambda^*}} \frac{d\lambda^*(y^*)}{|x^*-y^*|^s} = \int_{S_\lambda} \frac{|x-a|^s}{2^{s/2}|x-y|^s}\, d\lambda(y) = \frac{|x-a|^s}{2^{s/2}} U_s^\lambda(x). \tag{5.1}$$

Denote with E the exceptional set from S_μ where the inequality $U_s^\mu(x) \leq U_s^\nu(x)$ does not hold, and let E^* be its image under the stereographic projection. We claim that $\mu^*(E^*) = 0$. Indeed,

$$\mu^*(E^*) = \int_{E^*} d\mu^*(x^*) = \int_E \frac{2^{s/2}}{|x-a|^s}\, d\mu(x) \leq \frac{2^{s/2}}{[\mathrm{dist}(a, S_\mu)]^s} \int_E d\mu(x) = 0.$$

Using (5.1) and the assumption in the lemma we obtain that the inequality $U_s^{\mu^*}(x^*) \leq U_s^{\nu^*}(x^*)$ holds μ^*-almost everywhere. But μ^* and ν^* both are supported in the hyperplane, which can be identified with $\mathbf{R}^d$. Therefore, we may use Theorem L with $p = d$ to extend the inequality everywhere in the hyperplane. Again using (5.1) we then have the inequality holding true on $S^d \setminus \{a\}$. Since both potentials are continuous at a, we can extend the inequality there too. □

We are now ready to prove Theorem 1.4.

Proof of Theorem 1.4. We have that $Q(x) = c\, Q_a(x)$. From the proof of Theorem 1.10 with c instead of $1/(N-2)$, we know that $\mu_Q = (1 + c\|\epsilon_r\|)\nu_r/\|\nu_r\| - c\epsilon_r$ (see (1.29)) and $S_Q = \Sigma_r$, where

r is such that equality holds in (4.3). Suppose to the contrary, that there is $s \in (d-2, d)$, τ and C_τ, such that

$$U_s^\tau(x) + Q(x) \geq C_\tau > F_Q \quad \text{q.e. on } \Sigma_r. \tag{5.2}$$

Let $\hat{\tau} = \text{Bal}(\tau, \Sigma_r)$ be the balayage measure of τ onto $S_Q = \Sigma_r$ (see [16, Chapter IV, § 5, p. 260]), which is well defined for $s \in (d-2, d)$. We have that $\|\hat{\tau}\| \leq \|\tau\| = 1$ (when we form the balayage of measures supported on the sphere, the mass may decrease). Also $U_s^{\hat{\tau}}(x) = U_s^\tau(x)$ q.e. on Σ_r and $U_s^\tau(x) \geq U_s^{\hat{\tau}}(x)$ on S^d. So, from (5.2) we have consecutively

$$U_s^{\hat{\tau}}(x) + Q(x) = \ U^\tau(x)_s + Q(x) \geq C_\tau = \ U_s^{\mu_Q}(x) + Q(x) + C_\tau - F_Q \ \text{ q.e. on } \Sigma_r.$$

Recall that ν_r is a multiple of the equilibrium measure of Σ_r, so if $\eta := (C_\tau - F_Q)\text{cap}(\Sigma_r)\nu_r/\|\nu_r\|$, then $U_s^\eta(x) = C_\tau - F_Q$ for all $x \in \Sigma_r$. Therefore,

$$U_s^{\hat{\tau}}(x) \geq U_s^{\mu_Q+\eta}(x) \ \text{ q.e. on } \Sigma_r$$

We now can apply Lemma 5.1 to extend this inequality on the entire sphere, in particular

$$U_s^{\hat{\tau}}(a) \geq U_s^{\mu_Q+\eta}(a) > U_s^{\mu_Q}(a). \tag{5.3}$$

On the other hand, the inequality $U_s^\tau(x) + Q(x) \geq C_Q$ holds q.e. on Σ_Q, and thus μ_Q-a.e., so after integration with respect to μ_Q we obtain

$$\int U_s^\tau(x)\, d\mu_Q(x) + c\, U_s^{\mu_Q}(a) \geq C_\tau. \tag{5.4}$$

Similarly, integrating with respect to $\hat{\tau}$ the equality $U_s^{\mu_Q}(x) + Q(x) = F_Q$, $x \in \Sigma_r$, we get

$$\int U_s^{\hat{\tau}}(x)\, d\mu_Q(x) + c\, U_s^{\hat{\tau}}(a) = F_Q\|\hat{\tau}\| \leq F_Q < C_\tau. \tag{5.5}$$

Recall that $U_s^{\hat{\tau}}(x) = U_s^\tau(x)$ q.e. on Σ_r; hence

$$\int U_s^{\hat{\tau}}(x)\, d\mu_Q(x) = \int U_s^\tau(x)\, d\mu_Q(x),$$

which together with (5.4) and (5.5) implies that $U_s^{\hat{\tau}}(a) < U_s^{\mu_Q}(a)$. This contradicts (5.3), which proves (1.8) for the extended range of s.

To derive (1.9) for this extended range, assume to the contrary, that there is a measure τ and a constant C_τ such that

$$U_s^\tau(x) + Q(x) \leq C_\tau < F_Q \quad \text{for every} \quad x \in \operatorname{supp}(\tau).$$

Using (1.3), which holds everywhere for this choice of Q, we can extend this inequality and get $U_s^\sigma(x) \leq U_s^{\mu_Q}(x)$ for all $x \in \operatorname{supp}(\tau)$. By Lemma 5.1 this holds everywhere on S^d, and in particular

$$U_s^\tau(a) \leq U_s^{\mu_Q}(a). \tag{5.6}$$

Integrating the inequality $U_s^\tau(x) + Q(x) \leq C_\tau$ with respect to τ we obtain

$$I_s(\tau) + c\, U_s^\tau(a) \leq C_\tau. \tag{5.7}$$

On the other hand we can integrate $U_s^{\mu_Q}(x) + Q(x) = F_Q$ with respect to μ_Q, which yields

$$I_s(\mu_Q) + c\, U_s^{\mu_Q}(a) = F_Q. \tag{5.8}$$

Combining (1.1), (5.6)-(5.8), we derive

$$I_Q(\mu_Q) = I_s(\mu_Q) + 2c\, U_s^{\mu_Q}(a) = F_Q + c\, U_s^{\mu_Q}(a) \geq F_Q + c\, U_s^\tau(a) > C_\tau + c\, U_s^\tau(a) \geq I_Q(\tau)$$

which contradicts the minimization property of μ_Q. □

When $Q(x) = c\, Q_a(x)$, $c > 0$, we invoke Theorem 1.4 to extend the range of the parameter s in Theorem 1.8.

Proof of Theorem 1.8. As in the proof of Theorem 1.7 we can utilize Theorem 1.4 to derive that $M \leq F_Q$, and then apply (1.4) to (1.18) to get

$$U_s^{\mu_{E_n}}(x) + F_Q - M \geq U_s^{\mu_Q}(x) \quad \text{q.e. on} \quad S_Q.$$

Now let τ be a multiple of the Lebesgue surface measure σ on S^d, so that $U_s^\sigma(x) = F_Q - M$ for all $x \in S^d$. Then

$$U_s^{\mu_{E_n}+\tau}(x) \geq U_s^{\mu_Q}(x) \quad \text{q.e. on} \quad S_Q.$$

Using Lemma 5.1 we extend this inequality to all of S^d, which establishes (1.23). $\square$

The proof of Corollary 1.9 proceeds as follows.

Proof of Corollary 1.9. Let $\mathcal{R}_n = \{y_1, y_2, \ldots, y_n\}$ be a weighted Riesz set. Let $h_E(x)$ be the function defined in (1.18), associated with the set $E := \{y_1, y_2, \ldots, y_{n-1}\}$. From Definition 1.6 it is clear that the point y_n is a global minimum of h_E over S^d. Then (1.19) holds with $M = h_E(y_n) = U_s^{\mu_E}(y_n) + Q(y_n)$. Thus, (1.20) implies that

$$U_s^{\mu_E}(x) \geq U_s^{\mu_E}(y_n) + Q(y_n) + U_s^{\mu_Q}(x) - F_Q.$$

In particular, when $x = y_n$ we obtain $U_s^{\mu_Q}(y_n) + Q(y_n) \leq F_Q$, which yields that $y_n \in S_Q^*$.

In the particular case when $Q = c\, Q_a$ in the argument above we use (1.22) and (1.23) instead.

$\square$

We now are ready to show the separation result in Theorem 1.5.

Proof of Theorem 1.5. The starting point in our proof is that the weighted equilibrium measure $\mu_{Q_N} = \eta_{r_0}$ from Theorem 1.10 satisfies the condition (4.2), which implies that r_0 satisfies (4.3). With the relation $r_0^2 = 2(1 - t_0)$ in mind we write

$$(N - 2 + \|\epsilon_{r_0}\|)(2(1 - t_0))^{d/2} \geq \frac{2}{\mathcal{B}(\frac{d}{2}, \frac{d-s}{2})} \cdot \frac{\|\nu_{r_0}\|}{W_d}. \tag{5.9}$$

Using (3.19) and the substitutions $1+u=(1-t_0)\tilde{w}$ and $1+u=(1+t_0)w$ we have

$$
\begin{aligned}
\frac{\|\nu_{r_0}\|}{W_d} &= \mathcal{B}\left(\frac{d}{2},\frac{1}{2}\right) - \int_{t_0}^{1}(1-u^2)^{\frac{d-2}{2}}\,du \\
&\quad + W_d A_{s,d}(1-t_0)^{d-\frac{s}{2}}\int_{-1}^{t_0}(t_0-u)^{\frac{s-d}{2}}(1+u)^{\frac{d-2}{2}}\int_0^1 \frac{v^{\frac{d-2}{2}}(1-v)^{\frac{d-s}{2}}\,dv}{2(1-u-(1-t_0)v)}du \\
&= \mathcal{B}\left(\frac{d}{2},\frac{1}{2}\right) - \frac{(2(1-t_0))^{d/2}}{2}\int_0^1 \tilde{w}^{\frac{d-2}{2}}(1-(1-t_0)\tilde{w}/2)^{\frac{d-2}{2}}\,d\tilde{w} \qquad (5.10)\\
&\quad + \frac{W_d A_{s,d}}{2}(1-t_0)^{d-\frac{s}{2}}(1+t_0)^{\frac{s}{2}}\int_0^1 w^{\frac{d-2}{2}}(1-w)^{\frac{s-d}{2}}\left(\int_0^1 \frac{v^{\frac{d-2}{2}}(1-v)^{\frac{d-s}{2}}\,dv}{2-(1+t_0)w-(1-t_0)v}\right)dw
\end{aligned}
$$

The first integral can be estimated as

$$
\int_0^1 \tilde{w}^{\frac{d-2}{2}}(1-(1-t_0)\tilde{w}/2)^{\frac{d-2}{2}}\,d\tilde{w} \le \int_0^1 \frac{\tilde{w}^{\frac{d-2}{2}}}{(1-\tilde{w})^{\frac{s+2-d}{2}}}\,d\tilde{w} = \mathcal{B}\left(\frac{d}{2},\frac{d-s}{2}\right). \qquad (5.11)
$$

Interchanging the order of integration in the double integrable in (5.11) yields

$$
\begin{aligned}
&\int_0^1 w^{\frac{d-2}{2}}(1-w)^{\frac{s-d}{2}}\left(\int_0^1 \frac{v^{\frac{d-2}{2}}(1-v)^{\frac{d-s}{2}}\,dv}{2-(1+t_0)w-(1-t_0)v}\right)dw \\
&\qquad = \int_0^1 v^{\frac{d-2}{2}}(1-v)^{\frac{d-s}{2}}\left(\int_0^1 \frac{w^{\frac{d-2}{2}}(1-w)^{\frac{s-d}{2}}\,dw}{2-(1+t_0)w-(1-t_0)v}\right)dv \\
&\qquad \ge \int_0^1 v^{\frac{d-2}{2}}(1-v)^{\frac{d-s}{2}}\left(\int_0^1 \frac{w^{\frac{d-2}{2}}(1-w)^{\frac{s-d}{2}}\,dw}{2-(1+t_0)w}\right)dv \\
&\qquad = \mathcal{B}\left(\frac{d}{2},\frac{d-s+2}{2}\right)\int_0^1 \frac{w^{\frac{d-2}{2}}(1-w)^{\frac{s-d}{2}}}{2-(1+t_0)w}\,dw \\
&\qquad = \frac{d-s}{2d-s}\,\mathcal{B}\left(\frac{d}{2},\frac{d-s}{2}\right)\int_0^1 \frac{w^{\frac{d-2}{2}}(1-w)^{\frac{s-d}{2}}}{2-(1+t_0)w}\,dw. \qquad (5.12)
\end{aligned}
$$

Substituting (5.11) and (5.12) back in (5.10), and using formula (3.14), we derive the estimate

$$
\frac{\|\nu_{r_0}\|}{W_d} \ge \mathcal{B}\left(\frac{d}{2},\frac{1}{2}\right) - \frac{(2(1-t_0))^{d/2}}{2}\mathcal{B}\left(\frac{d}{2},\frac{d-s}{2}\right) + \frac{(2(1-t_0))^{d/2}}{2}\frac{d-s}{2d-s}\,\mathcal{B}\left(\frac{d}{2},\frac{d-s}{2}\right)\|\epsilon_{r_0}\|. \qquad (5.13)
$$

In light of (5.13) and (5.9) we obtain

$$
\frac{2\mathcal{B}(\frac{d}{2},\frac{1}{2})}{\mathcal{B}(\frac{d}{2},\frac{d-s}{2})} \le \left(N-1+\|\epsilon_{r_0}\|\frac{d}{2d-s}\right)r_0^d \le N r_0^d,
$$

where we used the fact that $\|\epsilon_{r_0}\| \le 1$. This proves the theorem. □

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