

ZERO DISTRIBUTION OF MÜNTZ EXTREMAL POLYNOMIALS IN $L_p[0, 1]$

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ABSTRACT. Let $\{\lambda_j\}_{j=0}^\infty$ be a sequence of distinct positive numbers. Let $1 \leq p \leq \infty$ and $T_{n,p} = T_{n,p}\{\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_n\}(x)$ denote the L_p extremal Müntz polynomial in $[0, 1]$ with exponents $\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_n$. We investigate the zero distribution of $\{T_{n,p}\}_{n=1}^\infty$. In particular, we show that if

$$\lim_{n \rightarrow \infty} \frac{\lambda_n}{n} = \alpha > 0,$$

then the normalized zero counting measure of $T_{n,p}$ converges weakly as $n \rightarrow \infty$ to

$$\frac{\alpha}{\pi} \frac{t^{\alpha-1}}{\sqrt{t^\alpha(1-t^\alpha)}} dt,$$

while if $\alpha = 0$ or ∞ , the limiting measure is a Dirac delta at 0 or 1 respectively.

1. INTRODUCTION AND RESULTS

Let $\lambda_1, \lambda_2, \dots$ be a sequence of distinct positive numbers. An expression of the form

$$(1.1) \quad \sum_{j=0}^n c_j x^{\lambda_j}$$

is called a Müntz polynomial. The name refers, of course, to the famous theorem of Müntz that if $\inf_j \lambda_j > 0$, these polynomials are dense in L_p spaces iff

$$\sum_{j=0}^{\infty} \frac{1}{\lambda_j} = \infty.$$

Müntz polynomials share many of the properties of ordinary algebraic polynomials. The most fundamental is that a polynomial of the form (1.1) has at most n distinct zeros in $(0, \infty)$, or is identically zero.

Müntz extremal polynomials are generalizations of classical orthogonal and Chebyshev polynomials. They have been investigated by amongst others, Borwein and Erdelyi [2], Milovanovic and his coworkers [3]. Let $1 \leq p \leq \infty$. We denote by

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$T_{n,p}(x) = T_{n,p}\{\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_n\}(x)$ the linear combination of $\{x^{\lambda_j}\}_{j=0}^n$ with coefficient of x^{λ_n} equal to 1, satisfying

$$(1.2) \quad \|T_{n,p}\{\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_n\}\|_{L_p[0,1]} = \min_{c_0, \dots, c_{n-1}} \|x^{\lambda_n} - \sum_{j=0}^{n-1} c_j x^{\lambda_j}\|_{L_p[0,1]}.$$

It is known that $T_{n,p}$ exists and is unique, has exactly n distinct (and simple) zeros in $(0, 1)$, and the zeros of $T_{n,p}$ and $T_{n+1,p}$ interlace. Moreover, if we swap λ_n with some λ_j , the extremal polynomial changes only by a non-zero multiplicative constant. Thus when dealing with a fixed n , and studying zeros of extremal polynomials, we may assume that $\{\lambda_j\}_{j=0}^n$ are in increasing order. However, we shall not need to assume that $\{\lambda_j\}_{j=0}^\infty$ is increasing. Concerning the zeros as $n \rightarrow \infty$, an important result of Borwein [2, Thm. 4.1.1, p. 155] asserts that the corresponding Müntz polynomials are dense iff the maximum spacing between successive zeros of $T_{n,p}$ has limit 0 as $n \rightarrow \infty$. Saff and Varga [6] studied the related zero distribution of lacunary incomplete polynomials.

In this paper, we study the asymptotic zero distribution of $\{T_{n,p}\}_{n=1}^\infty$. Let ν_n denote the normalized zero counting measure of $T_{n,p}$, so that

$$\nu_n([a, b]) = \frac{1}{n} \times \text{Number of zeros of } T_{n,p} \text{ in } [a, b].$$

In the case of polynomials, where $\lambda_j = j$, $j \geq 0$, it is a classical result [5, p. 170], [7, Thm. 3.4.1, p. 84 and Thm. 3.6.1, p. 98] that for $0 \leq a < b \leq 1$,

$$\lim_{n \rightarrow \infty} \nu_n([a, b]) = \int_a^b \frac{dx}{\pi \sqrt{x(1-x)}}.$$

Equivalently we write

$$d\nu_n \xrightarrow{*} \frac{dx}{\pi \sqrt{x(1-x)}}, \quad n \rightarrow \infty$$

and say that $d\nu_n$ converges weakly to the arcsine distribution on $[0, 1]$. This type of result has been studied in detail for the case $p = 2$ of orthogonal polynomials, and when there is a weight w in the norm in (1.2). The monograph of Stahl and Totik [7] gives a comprehensive account, while the monograph of Andrievskii and Blatt [1] considers discrepancy, or rate of convergence, to the limiting distribution.

In a loose sense, our conclusion is that when $\lim_{n \rightarrow \infty} \lambda_n/n$ exists, all the possible zero distributions are those provided by

$$\lambda_j = \alpha j, \quad j \geq 0$$

for some $\alpha \in [0, \infty]$. Extremal polynomials for these exponents are essentially L_p extremal polynomials with the substitution of variable $x \rightarrow x^\alpha$. Accordingly, we define for $0 < \alpha < \infty$, a probability measure on $(0, 1)$,

$$(1.3) \quad d\mu_\alpha(t) = \frac{\alpha}{\pi} \frac{t^{\alpha-1}}{\sqrt{t^\alpha(1-t^\alpha)}} dt.$$

For $\alpha = 0$, we set

$$(1.4) \quad d\mu_0 = d\delta_0,$$

a unit mass at 0, and for $\alpha = \infty$, we set

$$(1.5) \quad d\mu_\infty = d\delta_1,$$

a unit mass at 1. We prove:

Theorem 1 *Let $1 \leq p \leq \infty$, $0 \leq \alpha \leq \infty$, and $\{\lambda_j\}_{j=0}^\infty$ denote a sequence of distinct positive numbers with*

$$(1.6) \quad \lim_{j \rightarrow \infty} \frac{\lambda_j}{j} = \alpha.$$

Then if $0 \leq a \leq b \leq 1$,

$$(1.7) \quad \lim_{n \rightarrow \infty} \nu_n([a, b]) = \mu_\alpha([a, b]),$$

that is,

$$d\nu_n \xrightarrow{*} d\mu_\alpha, \quad n \rightarrow \infty.$$

Remarks

(a) An interesting feature of the theorem is that asymptotic zero distribution has no relation to the density of Müntz polynomials - in stark contrast to the Borwein-Erdelyi result on spacing. Thus if $\lambda_n = n \log n, n \geq 2$, then the corresponding Müntz polynomials are dense, while the asymptotic zero distribution is a Dirac delta at 1. If $\lambda_n = n^2, n \geq 0$, then the limiting zero distribution is still a Dirac delta at 1, but the corresponding Müntz polynomials are not dense.

(b) We can somewhat weaken the hypothesis (1.6): roughly speaking we can ignore $o(n)$ of the exponents in $\{\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_n\}$. To make this more precise, assume $\alpha < \infty$. We write

$$(1.8) \quad \lim_{j \rightarrow \infty \text{ a.e.}} \frac{\lambda_j}{j} = \alpha$$

if for each $\varepsilon \in (0, 1)$, there exists for large enough n , a set

$$(1.9) \quad S_{n,\varepsilon} \subset \{0, 1, 2, \dots, n\}$$

with at most εn elements such that

$$(1.10) \quad j \in \{0, 1, 2, \dots, n\} \setminus S_{n,\varepsilon} \Rightarrow \left| \frac{\lambda_j}{j} - \alpha \right| < \varepsilon.$$

In the case $\alpha = \infty$, we replace this by for each $K > 0$, there exists for large enough n , a set $S_{n,\varepsilon} \subset \{0, 1, 2, \dots, n\}$ with at most εn elements such that

$$j \in \{0, 1, 2, \dots, n\} \setminus S_{n,\varepsilon} \Rightarrow \frac{\lambda_j}{j} > K.$$

Theorem 2 *Let $1 \leq p \leq \infty$, $0 \leq \alpha \leq \infty$, and $\{\lambda_j\}_{j=0}^\infty$ denote a sequence of distinct positive numbers with*

$$(1.11) \quad \lim_{j \rightarrow \infty \text{ a.e.}} \frac{\lambda_j}{j} = \alpha.$$

Then the conclusion (1.7) of Theorem 1 persists.

We shall also show that one cannot ignore more than $o(n)$ exponents in $\{\lambda_j\}_{j=0}^n$ without affecting the zero distribution:

Theorem 3 *Let $1 \leq p \leq \infty$ and $\varepsilon \in (0, 1)$. Let $\{\lambda_j\}_{j=0}^\infty$, $\{\gamma_j\}_{j=0}^\infty$, $\{\rho_j\}_{j=0}^\infty$ denote sequences of distinct positive numbers with*

$$(1.12) \quad \lim_{j \rightarrow \infty} \frac{\gamma_j}{j} = 0; \quad \lim_{j \rightarrow \infty} \frac{\rho_j}{j} = \infty.$$

Assume also that for large enough n , there is the disjoint union

$$(1.13) \quad \{\lambda_j\}_{j=0}^n := \{\gamma_j\}_{j=0}^{k(n)} \cup \{\rho_j\}_{j=0}^{\ell(n)},$$

where

$$\lim_{n \rightarrow \infty} \frac{k(n)}{n} = \varepsilon.$$

Then

$$(1.14) \quad d\nu_n \xrightarrow{*} \varepsilon d\mu_0 + (1 - \varepsilon) d\mu_\infty, \quad n \rightarrow \infty.$$

We are not sure if this result generalizes to the case where 0 and ∞ are replaced in (1.12) by other limits. What is clear is that for a general choice of $\{\lambda_j\}_{j=0}^\infty$, the asymptotic zero distribution can be quite complicated, and there need not be a weak limit. For example, by adjoining sufficiently large blocks of exponents $\{\alpha_j\}_{j=n_1}^{n_2}$, one may construct $\{\lambda_n\}_{n=0}^\infty$, such that every μ_α , $\alpha \in [0, \infty]$, is a weak limit of some subsequence of $\{\nu_n\}$. We prove the results in the next section.

2. Proofs

We begin with some notation. We abbreviate $T_{n,p} \{\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_n\}$ as $T_{n,p} \{\lambda_0 \dots \lambda_n\}$. Let $Z_p(\lambda_0 \dots \lambda_n)[a, b]$ denote the total number of zeros of $T_{n,p} \{\lambda_0 \dots \lambda_n\}(x)$ in $[a, b]$. We say that $\{\gamma_0, \gamma_1, \gamma_2, \dots, \gamma_m\}$ is a *refinement* of $\{\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_n\}$ if

$$\{\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_n\} \subset \{\gamma_0, \gamma_1, \gamma_2, \dots, \gamma_m\}.$$

The main tools of proof are interlacing properties of successive Chebyshev polynomials, monotonicity properties with respect to the exponents, and zero distribution for the specific choice $\{\alpha_j\}_{j=0}^\infty$.

Lemma 2.1 *Let $\{\gamma_j\}_{j=0}^m$ be distinct positive numbers and $\{\lambda_j\}_{j=0}^n$ be distinct positive numbers.*

(a) *Suppose that $\{\gamma_0, \gamma_1, \gamma_2, \dots, \gamma_m\}$ is a refinement of $\{\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_n\}$. Then for $[a, b] \subset [0, 1]$,*

$$(2.1) \quad |Z_p(\lambda_0 \dots \lambda_n)[a, b] - Z_p(\gamma_0 \dots \gamma_m)[a, b]| \leq 2(m - n).$$

(b) *Suppose that $\{\gamma_0, \gamma_1, \gamma_2, \dots, \gamma_k\}$ and $\{\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_n\}$ have ℓ exponents in common. Then for $[a, b] \subset [0, 1]$,*

$$(2.2) \quad |Z_p(\lambda_0 \dots \lambda_n)[a, b] - Z_p(\gamma_0 \dots \gamma_k)[a, b]| \leq 2(n + k + 2 - 2\ell).$$

Proof

(a) We may rewrite $\{\gamma_0, \gamma_1, \gamma_2, \dots, \gamma_m\}$ as $\{\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_m\}$. Since any subset of $\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_m}\}$ is a Chebyshev system on $[\varepsilon, 1]$ for any $0 < \varepsilon < 1$, the zeros of $T_{n,p} \{\lambda_0 \dots \lambda_j\}(x)$ and $T_{n,p} \{\lambda_0 \dots \lambda_{j+1}\}(x)$ interlace [4, Theorem 1.1, p. 1]. It then follows that for every interval $[a, b]$,

$$|Z_p(\lambda_0 \dots \lambda_j)[a, b] - Z_p(\lambda_0 \dots \lambda_{j+1})[a, b]| \leq 2.$$

Applying this for $j = n, n + 1, \dots, m$ gives (2.1).

(b) We may find a refinement of both $\{\gamma_0, \gamma_1, \gamma_2, \dots, \gamma_k\}$ and $\{\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_n\}$ consisting of $n + k + 2 - \ell$ elements. Applying (a) to the refinement and each of the sets $\{\gamma_0, \gamma_1, \gamma_2, \dots, \gamma_k\}$ and $\{\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_n\}$, and then combining the two inequalities gives the result. ■

Apart from interlacing, we shall also use the lexicographic property:

Lemma 2.2 *Let $\{\lambda_j\}_{j=0}^n$ be a sequence of distinct positive numbers and $\{\gamma_j\}_{j=0}^n$ be a sequence of distinct positive numbers with*

$$(2.3) \quad \lambda_j \leq \gamma_j, \quad 0 \leq j \leq n.$$

Then for $0 \leq a \leq 1$,

$$(2.4) \quad Z_p(\lambda_0 \dots \lambda_n)[a, 1] \leq Z_p(\gamma_0 \dots \gamma_n)[a, 1].$$

Proof

We may assume that the two sets have n exponents in common. For then, one can apply the result for this special case n times, using monotonicity each time. Let $0 < \varepsilon < 1$. Then in $[\varepsilon, 1]$, the combined set of powers $\{x^{\lambda_j}\}_{j=0}^n \cup \{x^{\gamma_j}\}_{j=0}^n$ (with duplicates deleted, and exponents placed in increasing order) is a Descartes system. If $T_{n,p}^\varepsilon\{\lambda_0 \dots \lambda_n\}(x)$ and $T_{n,p}^\varepsilon\{\gamma_0 \dots \gamma_n\}(x)$ denote the corresponding Müntz extremal polynomials on $[\varepsilon, 1]$, it is known that the zeros of $T_{n,p}^\varepsilon\{\lambda_0 \dots \lambda_n\}(x)$ lie to the left of those of $T_{n,p}^\varepsilon\{\gamma_0 \dots \gamma_n\}(x)$, in the sense that the j th smallest zero of the former Müntz polynomial is \leq the j th smallest zero of the latter Müntz polynomial. For $p = \infty$, a proof of this is given in the book of Borwein and Erdelyi [2, Thm. 3.3.4, pp. 116-117]. For $1 < p \leq \infty$, a proof is given in Pinkus and Ziegler [4, Thm. 5.1, p. 13], while when $p = 1$, we can apply the remarks there (or a continuity argument involving $p \rightarrow 1+$). As $\varepsilon \rightarrow 0+$, $T_{n,p}^\varepsilon\{\gamma_0 \dots \gamma_n\}(x)$ must converge uniformly to $T_{n,p}\{\gamma_0 \dots \gamma_n\}(x)$ because of uniqueness of $T_{n,p}\{\gamma_0 \dots \gamma_n\}(x)$, and the fact that the extremal error increases as $[\varepsilon, 1]$ grows to $[0, 1]$. Hence the zeros of $T_{n,p}\{\lambda_0 \dots \lambda_n\}(x)$ lie to the left of those of $T_{n,p}\{\gamma_0 \dots \gamma_n\}(x)$ and (2.4) follows. ■

The next result asserts essentially that if for "most" indices j , we have $\lambda_j \leq \gamma_j$, then the asymptotic proportion of zeros in $[a, 1]$ of extremal polynomials with exponents $\{\lambda_j\}$ does not exceed that for $\{\gamma_j\}$.

Lemma 2.3 *Let $\{\lambda_j\}_{j=0}^\infty$ and $\{\gamma_j\}_{j=0}^\infty$ be sequences of distinct positive numbers with the following property: for each $\varepsilon > 0$, there exists for large enough n , a set*

$$(2.5) \quad S_{n,\varepsilon} \subset \{0, 1, 2, \dots, n\}$$

with at most εn elements such that

$$(2.6) \quad j \in \{0, 1, 2, \dots, n\} \setminus S_{n,\varepsilon} \Rightarrow \lambda_j \leq \gamma_j.$$

Then for $0 \leq a \leq 1$,

$$(2.7) \quad \limsup_{n \rightarrow \infty} \frac{1}{n} Z_p(\lambda_0 \dots \lambda_n)[a, 1] \leq \limsup_{n \rightarrow \infty} \frac{1}{n} Z_p(\gamma_0 \dots \gamma_n)[a, 1]$$

and

$$(2.8) \quad \liminf_{n \rightarrow \infty} \frac{1}{n} Z_p(\lambda_0 \dots \lambda_n)[a, 1] \leq \liminf_{n \rightarrow \infty} \frac{1}{n} Z_p(\gamma_0 \dots \gamma_n)[a, 1].$$

Proof

Let us fix $\varepsilon > 0$, n large, and $S_{n,\varepsilon}$ be as in the statement. We define for the given n , a modified set of exponents $\{\lambda_j^*\}_{j=0}^n$ by

$$\lambda_j^* = \begin{cases} \lambda_j, & j \in \{0, 1, 2, \dots, n\} \setminus S_{n,\varepsilon} \\ \gamma_j, & j \in S_{n,\varepsilon} \end{cases}.$$

Then

$$\lambda_j^* \leq \gamma_j, \quad 0 \leq j \leq n.$$

By the previous lemma, for $0 \leq a \leq 1$,

$$Z_p(\lambda_0^* \dots \lambda_n^*)[a, 1] \leq Z_p(\gamma_0 \dots \gamma_n)[a, 1].$$

Also $\{\lambda_j^*\}_{j=0}^n$ and $\{\lambda_j\}_{j=0}^n$ have at least $1 + n(1 - \varepsilon)$ elements in common, so by Lemma 2.1(b),

$$|Z_p(\lambda_0^* \dots \lambda_n^*)[a, 1] - Z_p(\lambda_0 \dots \lambda_n)[a, 1]| \leq 4\varepsilon n + 4.$$

Combining these inequalities gives

$$Z_p(\lambda_0 \dots \lambda_n)[a, 1] \leq Z_p(\gamma_0 \dots \gamma_n)[a, 1] + 4\varepsilon n + 4.$$

Dividing by n and letting $n \rightarrow \infty$ gives

$$\limsup_{n \rightarrow \infty} \frac{1}{n} Z_p(\lambda_0 \dots \lambda_n)[a, 1] \leq \limsup_{n \rightarrow \infty} \frac{1}{n} Z_p(\gamma_0 \dots \gamma_n)[a, 1] + 4\varepsilon.$$

As $\varepsilon > 0$ is arbitrary, (2.7) follows. Similarly, (2.8) follows. ■

Next, we study the zero distribution for the comparison sequence $\{\alpha j\}_{j=0}^{\infty}$:

Lemma 2.4 *Let $\alpha \in (0, \infty)$ and*

$$(2.9) \quad \gamma_j = \alpha j, \quad j \geq 0.$$

Then for $0 \leq a < b \leq 1$,

$$(2.10) \quad \lim_{n \rightarrow \infty} \frac{1}{n} Z_p(\gamma_0 \dots \gamma_n)[a, b] = \mu_\alpha([a, b]).$$

Proof

Suppose first $p < \infty$. Let $T_{n,p}^*$ denote the monic (ordinary) polynomial of degree n satisfying

$$\int_0^1 |T_{n,p}^*(x)|^p \frac{1}{\alpha} x^{1/\alpha-1} dx = \min_{\deg(P) \leq n-1} \int_0^1 |x^n - P(x)|^p \frac{1}{\alpha} x^{1/\alpha-1} dx.$$

The substitution $x = t^\alpha$ gives

$$\int_0^1 |T_{n,p}^*(t^\alpha)|^p dt = \min_{\deg(P) \leq n-1} \int_0^1 |t^{\alpha n} - P(t^\alpha)|^p dt.$$

It follows from uniqueness that

$$(2.11) \quad T_{n,p}^*(t^\alpha) = T_{n,p}\{\gamma_0 \dots \gamma_n\}(t).$$

We see then that the total multiplicity of zeros of $T_{n,p}\{\gamma_0 \dots \gamma_n\}$ in $[a, b]$ is the total multiplicity of zeros of $T_{n,p}^*$ in $[a^\alpha, b^\alpha]$. Since the weight $\frac{1}{\alpha} x^{1/\alpha-1}$ is positive a.e.

in $[0, 1]$, classical results assert that the limiting zero distribution of $\{T_{n,p}^*\}_{n=0}^\infty$ is the arcsine distribution [1, Cor. 5.7, p. 261]. Hence as $n \rightarrow \infty$,

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{1}{n} \times \text{Number of zeros of } T_{n,p} \text{ in } [a, b] \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \times \text{Number of zeros of } T_{n,p}^* \text{ in } [a^\alpha, b^\alpha] \\ &= \int_{a^\alpha}^{b^\alpha} \frac{dx}{\pi \sqrt{x(1-x)}} = \frac{\alpha}{\pi} \int_a^b \frac{t^{\alpha-1}}{\sqrt{t^\alpha(1-t^\alpha)}} dt = \int_a^b d\mu_\alpha(t). \blacksquare \end{aligned}$$

Proof of Theorem 2

Our hypothesis is

$$\lim_{j \rightarrow \infty \text{ a.e.}} \frac{\lambda_j}{j} = \alpha.$$

Assume first that $0 < \alpha < \infty$. Let $\varepsilon \in (0, \alpha)$. We then obtain for large enough n , from (1.10),

$$j \in \{0, 1, 2, \dots, n\} \setminus S_{n,\varepsilon} \Rightarrow (\alpha - \varepsilon)j \leq \frac{\lambda_j}{j} \leq (\alpha + \varepsilon)j.$$

Applying Lemma 2.3, with $\gamma_j = (\alpha + \varepsilon)j$, $j \geq 0$, we deduce that

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \frac{1}{n} Z_p(\lambda_0 \dots \lambda_n) [a, 1] \\ & \leq \limsup_{n \rightarrow \infty} \frac{1}{n} Z_p(0, (\alpha + \varepsilon), 2(\alpha + \varepsilon), \dots, n(\alpha + \varepsilon)) [a, 1] \end{aligned}$$

and similarly applying Lemma 2.3 to $(\alpha - \varepsilon)j$, $j \geq 0$, and λ_j , $j \geq 0$ (with roles swapped),

$$\begin{aligned} & \liminf_{n \rightarrow \infty} \frac{1}{n} Z_p(0, (\alpha - \varepsilon), 2(\alpha - \varepsilon), \dots, n(\alpha - \varepsilon)) [a, 1] \\ & \leq \liminf_{n \rightarrow \infty} \frac{1}{n} Z_p(\lambda_0 \dots \lambda_n) [a, 1]. \end{aligned}$$

Applying Lemma 2.4 with $\gamma_j = (\alpha \pm \varepsilon)j$, $j \geq 0$, gives

$$\begin{aligned} \int_a^1 d\mu_{\alpha-\varepsilon}(t) & \leq \liminf_{n \rightarrow \infty} \frac{1}{n} Z_p(\lambda_0 \dots \lambda_n) [a, 1] \\ & \leq \limsup_{n \rightarrow \infty} \frac{1}{n} Z_p(\lambda_0 \dots \lambda_n) [a, 1] \leq \int_a^1 d\mu_{\alpha+\varepsilon}(t). \end{aligned}$$

Letting $\varepsilon \rightarrow 0+$, and using dominated convergence gives

$$\lim_{n \rightarrow \infty} \frac{1}{n} Z_p(\lambda_0 \dots \lambda_n) [a, 1] = \int_a^1 d\mu_\alpha(t).$$

This gives the result when $[a, b] = [a, 1]$. For general $[a, b]$, we use

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{1}{n} Z_p(\lambda_0 \dots \lambda_n) [a, b] \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} Z_p(\lambda_0 \dots \lambda_n) [a, 1] - \lim_{n \rightarrow \infty} \frac{1}{n} Z_p(\lambda_0 \dots \lambda_n) (b, 1] \\ &= \int_a^1 d\mu_\alpha(t) - \int_b^1 d\mu_\alpha(t). \end{aligned}$$

Note that because μ_α is absolutely continuous, the number of zeros in a neighborhood of the point b is negligible in the sense of asymptotic distribution. Finally, if $\alpha = 0$, the arguments above give for $0 < a \leq 1$,

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \frac{1}{n} Z_p(\lambda_0 \dots \lambda_n) [a, 1] \\ & \leq \limsup_{n \rightarrow \infty} \frac{1}{n} Z_p(0, \varepsilon, 2\varepsilon, \dots, n\varepsilon) [a, 1] = \int_a^1 d\mu_\varepsilon(t). \end{aligned}$$

Letting $\varepsilon \rightarrow 0+$ (and using some straightforward estimates) gives

$$\lim_{n \rightarrow \infty} \frac{1}{n} Z_p(\lambda_0 \dots \lambda_n) [a, 1] = 0 = \int_a^1 d\mu_0(t).$$

Since $\frac{1}{n} Z_p(\lambda_0 \dots \lambda_n) [0, 1] = 1$, we obtain

$$\lim_{n \rightarrow \infty} \frac{1}{n} Z_p(\lambda_0 \dots \lambda_n) [0, 1] = 1 = \int_0^1 d\mu_0(t).$$

The case $\alpha = \infty$ is similar. ■

Proof of Theorem 1

This is a special case of Theorem 2. ■

Proof of Theorem 3

Let $0 < a < b < 1$. Because of (1.13) and interlacing properties, to the left of each zero of $T_{n,p} \{ \gamma_0 \dots \gamma_{k(n)} \} (x)$ in $[0, a]$, there is a zero of $T_{n,p} \{ \lambda_0 \dots \lambda_n \} (x)$. Moreover,

$$Z_p(\lambda_0 \dots \lambda_n) [0, a] \geq Z_p(\gamma_0 \dots \gamma_{k(n)}) [0, a]$$

so applying Theorem 1 to $\{ \gamma_j \}_{j=0}^\infty$,

$$\begin{aligned} \liminf_{n \rightarrow \infty} \frac{1}{n} Z_p(\lambda_0 \dots \lambda_n) [0, a] & \geq \liminf_{n \rightarrow \infty} \frac{k(n)}{n} \frac{1}{k(n)} Z_p(\gamma_0 \dots \gamma_{k(n)}) [0, a] \\ (2.12) \qquad \qquad \qquad & = \varepsilon \int_0^a d\mu_0 = \varepsilon. \end{aligned}$$

Similarly,

$$\begin{aligned} \liminf_{n \rightarrow \infty} \frac{1}{n} Z_p(\lambda_0 \dots \lambda_n) [b, 1] & \geq \liminf_{n \rightarrow \infty} \frac{\ell(n)}{n} \frac{1}{\ell(n)} Z_p(\rho_0 \dots \rho_{\ell(n)}) [b, 1] \\ (2.13) \qquad \qquad \qquad & = (1 - \varepsilon) \int_b^1 d\mu_\infty = 1 - \varepsilon. \end{aligned}$$

Then it follows that

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \frac{1}{n} Z_p(\lambda_0 \dots \lambda_n) (a, b) \\ & \leq 1 - \liminf_{n \rightarrow \infty} \frac{1}{n} Z_p(\lambda_0 \dots \lambda_n) [0, a] - \liminf_{n \rightarrow \infty} \frac{1}{n} Z_p(\lambda_0 \dots \lambda_n) [b, 1] \leq 0. \end{aligned}$$

So for $0 < a < b < 1$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} Z_p(\lambda_0 \dots \lambda_n) (a, b) = 0.$$

Next, by (2.12) and (2.13),

$$\begin{aligned} \varepsilon &\leq \liminf_{n \rightarrow \infty} \frac{1}{n} Z_p(\lambda_0 \dots \lambda_n) [0, a] \\ &\leq \limsup_{n \rightarrow \infty} \frac{1}{n} Z_p(\lambda_0 \dots \lambda_n) [0, a] \\ &\leq 1 - \liminf_{n \rightarrow \infty} \frac{1}{n} Z_p(\lambda_0 \dots \lambda_n) (a, 1] \leq \varepsilon, \end{aligned}$$

so

$$\lim_{n \rightarrow \infty} \frac{1}{n} Z_p(\lambda_0 \dots \lambda_n) [0, a] = \varepsilon.$$

Similarly,

$$\lim_{n \rightarrow \infty} \frac{1}{n} Z_p(\lambda_0 \dots \lambda_n) [b, 1] = 1 - \varepsilon.$$

It follows that as $n \rightarrow \infty$,

$$d\nu_n \xrightarrow{*} \varepsilon d\delta_0 + (1 - \varepsilon) d\delta_1.$$

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