# A Remez-Type Theorem for Homogeneous Polynomials

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**Abstract**

Remez-type inequalities provide upper bounds for the uniform norms of polynomials $p$ on given compact sets $K$, provided that $|p(x)| \leq 1$ for every $x \in K \setminus E$, where $E$ is a subset of $K$ of small measure. In this paper we prove sharp Remez-type inequalities for homogeneous polynomials on star-like surfaces in $\mathbb{R}^d$. In particular, this covers the case of spherical polynomials (when $d = 2$ we deduce a result of T. Erdélyi for univariate trigonometric polynomials).

*Key words:* Remez-type inequalities, homogeneous polynomials, star-like surfaces, logarithmic potential, equilibrium measure, Fekete polynomials

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An important question of constructive function theory is the study of the rate of change of polynomials. For instance, given a polynomial $p$ and a compact set $K$ one is interested in the size of $p$ *outside* of $K$ under the assumption that $\|p\|_K := \max_{x \in K} |p(x)| = 1$. This problem has been widely studied both for real polynomials (Chebyshev-type inequalities) and complex polynomials (Bernstein-Walsh-type inequalities). An equally interesting dual problem consists in estimating the size of the polynomials *inside* the given set under the same normalization. In other words, we are interested in *lower* bounds for $\|p\|_{K \setminus E}$ provided that $\|p\|_K = 1$ and $E$ is a subset of $K$ of small Lebesgue measure. This is the so-called Remez-type problem for polynomials. Such estimates turned out to be instrumental in proving Markov-Bernstein-type inequalities for derivatives of polynomials and Nikolskii-type inequalities comparing the size of polynomials in different norms. Hence they are considered a basic tool in approximation theory.

Let $P_n^d$ be the space of polynomials of $d$ real variables and total degree $\leq n$, $\mu_d(\cdot)$ stands for the Lebesgue measure in $\mathbb{R}^d$, $d \geq 1$, $K \subset \mathbb{R}^d$ is a compact set. Then

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the Remez problem outlined above consists in estimating the quantity

$$R_n^*(K,\delta) := \sup\left\{\frac{\|p\|_K}{\|p\|_{K\setminus E}} \,:\, p\in P_n^d,\ E\subset K,\ \mu_d(E)\le \mu_d(K)\delta^d\right\}, 0<\delta<1. \quad (1)$$

Thus $R_n^*(K,\delta)$ measures how small is $\|p\|_{K\setminus E}$ relative to $\|p\|_K$ if $E\subset K$ is a subset of measure $\le \mu_d(K)\delta^d$. The main goal is to estimate $R_n^*(K,\delta)$ in terms of $n$ and $\delta$. Clearly, $R_n^*(K,\delta)\to 1$ when $\delta\to 0$ for every fixed $n$, but finding the rate of this convergence is a nontrivial matter.

The first result related to the above problem was given by Remez [R] who showed that when $d=1$ and $K=[0,1]$ we have

$$R_n^*([0,1],\delta) = T_n\left(\frac{1+\delta}{1-\delta}\right),$$

where $T_n(x) := \frac{1}{2}\{(x+\sqrt{x^2-1})^n + (x-\sqrt{x^2-1})^n\}$ is the Chebyshev polynomial of first kind. This yields that for $0<\delta<1/2$

$$\frac{1}{n}\log R_n^*([0,1],\delta) \asymp \sqrt{\delta}. \quad (2)$$

Extensions of this result were given for trigonometric polynomials (Erdélyi [E]), complex polynomials (Erdélyi-Li-Saff [ELS]), and multivariate polynomials of total degree $\le n$ (Brudnyi-Ganzburg [BG], Kroó-Schmidt [KS], Kroó [K1]).

In this paper we shall study the multivariate Remez problem for *homogeneous* polynomials. Homogeneous polynomials arise naturally as the approximating tool in problems related to neural networks and approximation by ridge functions (see e.g. [K2], [LP]). This leads to the necessity of extending the classical polynomial inequalities to homogeneous polynomials, with the Remez inequality being one of the basic ones.

Let $H_n^d := \{\sum_{|\mathbf{k}|_1=n} a_{\mathbf{k}}\mathbf{x}^{\mathbf{k}}, a_{\mathbf{k}}\in\mathbb{R}\}$, $\mathbf{x}\in\mathbb{R}^d$, be the space of homogeneous polynomials of $d$ variables and degree $n$. (Here $|\mathbf{k}|_1$ stands for the $\ell_1$-norm of $\mathbf{k}\in\mathbb{Z}_+^d$. For $\mathbf{x}\in\mathbb{R}^n$ we denote by $|\mathbf{x}|$ the $\ell_2$-norm.) A natural domain for the study of homogeneous polynomials is a *star-like surface*. Let $r: S^{d-1}\to\mathbb{R}^+$ be a continuous even mapping of the unit sphere $S^{d-1} := \{\mathbf{x}\in\mathbb{R}^d :\ |\mathbf{x}|=1\}$ into the positive real axis $\mathbb{R}^+$. Then a star-like surface and a star-like domain corresponding to $r$ are defined, respectively, by

$$\begin{aligned}\partial K_r &:= \{\mathbf{u}\, r(\mathbf{u}) :\ \mathbf{u}\in S^{d-1}\}\\ K_r &:= \{t\mathbf{x} :\ \mathbf{x}\in\partial K_r,\ t\in[0,1]\}.\end{aligned}$$

We shall say that $r\in \mathrm{Lip}_M\alpha$, $0<\alpha\le 2$, if for every $\mathbf{x}_1,\mathbf{x}_2\in S^{d-1}$

$$\begin{aligned}&|r(\mathbf{x}_1)-r(\mathbf{x}_2)| \le M|\mathbf{x}_1-\mathbf{x}_2|^\alpha \ \text{ if }\ 0<\alpha\le 1\\ &|\nabla r(\mathbf{x}_1)-\nabla r(\mathbf{x}_2)| \le M|\mathbf{x}_1-\mathbf{x}_2|^{\alpha-1} \ \text{ if }\ 1<\alpha\le 2.\end{aligned}$$

(Naturally, if $1<\alpha\le 2$ the existence of the gradient $\nabla r$ of $r$ is assumed.)

It is shown in [K1] that for $r \in \mathrm{Lip}_M\alpha$ and $0 < \delta < 1/2$ we have

$$\frac{1}{n}\log R_n^*(K_r,\delta) = O\left(\delta^{\alpha d/(2d+2\alpha-2)}\right) \tag{3}$$

and this estimate is sharp, in general. Note that if $K_r$ is convex (so that $\alpha = 1$), then the upper bound of (3) is the same as in (2). Moreover, in case when $\alpha = 2$ ($C^2$-domain) we get from (3) that

$$\frac{1}{n}\log R_n^*(K_r,\delta) = O\left(\delta^{\frac{d}{d+1}}\right).$$

The above estimates provide sharp Remez-type results for polynomials in $P_n^d$.

Now we shall introduce a quantity similar to (1) for homogeneous polynomials on a star-like domain $K_r$. Since the norm of homogeneous polynomials is attained on the boundary $\partial K_r$ of $K_r$, the exceptional set $E$ should be a subset of $\partial K_r$, and its size will be measured by its Lebesgue surface measure $s_{d-1}(E)$ in $\mathbb{R}^d$. Now set for any $0 < \delta < 1$

$$R_n(K_r,\delta) := \sup\left\{\frac{\|h\|_{\partial K_r}}{\|h\|_{\partial K_r\setminus E}} : \ h \in H_n^d,\ E \subset \partial K_r,\ s_{d-1}(E) \le s_{d-1}(\partial K_r)\delta^{d-1}\right\},$$

$$\varphi_\alpha(\delta) := \begin{cases} \delta^\alpha, & 0 < \alpha < 1 \\ \delta \log \frac{1}{\delta}, & \alpha = 1 \\ \delta, & 1 < \alpha \le 2. \end{cases}$$

Our main result is the following.

**Main Theorem.** *Let $K_r \subset \mathbb{R}^d$ $(d \ge 2)$ be a star-like domain with $r \in \mathrm{Lip}_M\alpha$, $0 < \alpha \le 2$. Then with some $c_1 > 0$ independent of $n$ and $\delta$ we have*

$$\frac{1}{n}\log R_n(K_r,\delta) \le c_1\varphi_\alpha(\delta),\ 0 < \delta \le 1/2. \tag{4}$$

*Moreover, this estimate is sharp in the sense that a similar lower bound holds for certain $K_r$ as above.*

Clearly, whenever $K_r$ is a *convex body*, then $r \in \mathrm{Lip}_M 1$ with some $M$ depending on $K_r$. This leads to

**Corollary 1.** *For any 0-symmetric convex body $K$ in $\mathbb{R}^d$,*

$$\frac{1}{n}\log R_n(K,\delta) \le c_1\delta\log\frac{1}{\delta}, \quad 0 < \delta \le 1/2.$$

**Remark.** Note that (4) is better that (3) by roughly a square root factor. For instance, for $\alpha = 1$ (i.e. convex surface)

$$\frac{1}{n}\log R_n(K_r,\delta) = O\left(\delta\log\frac{1}{\delta}\right)$$

while for $\alpha = 2$ (smooth surface) we have

$$\frac{1}{n}\log R_n(K_r, \delta) = O(\delta).$$

This improvement of the rate of the Remez function $R_n(K_r, \delta)$ is related to the special algebraic structure of homogeneous polynomials.

Consider the space of *spherical* polynomials $P_n^d(S^{d-1})$, where $Q(K)$ denotes the restriction of functions from $Q$ to the subset $K \subset \mathbb{R}^d$. It is known that $P_n^d(S^{d-1}) = H_n^d(S^{d-1}) + H_{n-1}^d(S^{d-1})$, i.e., any $p \in P_n^d$ equals on $S^{d-1}$ the sum of 2 homogeneous polynomials of degrees $n$ and $n-1$ (see [Re], p. 43). Moreover, one of the homogeneous polynomials is even and the other one is odd. Thus, if $p \in P_n^d(S^{d-1})$ and $|p| > 1$ on a subset of $S^{d-1}$ of measure at most $\delta^{d-1}$, then it is easily seen that the moduli of the corresponding homogeneous polynomials can exceed 1 on sets of measure at most $2\delta^{d-1}$. Hence the above theorem implies the following Remez-type inequality for spherical polynomials

**Corollary 2.** *If $p \in P_n^d(S^{d-1})$ and $s_{d-1}\left\{\mathbf{x} \in S^{d-1} : |p(\mathbf{x})| > 1\right\} \le \delta^{d-1}$, then*

$$\|p\|_{S^{d-1}} \le \exp\{cn\delta\},$$

*where $c > 0$ depends only on $d$.*

Note that for $d = 2$ (univariate trigonometric polynomials) this result was obtained by T. Erdélyi [E].

The proof of the main result will be based on several lemmas. First we shall need some auxiliary geometric results which will reduce the problem to the study of 2-dimensional "diamond-shaped" domains (Lemmas 1—3). Then the problem will be transformed to a Remez-type problem for weighted univariate polynomials on $\mathbb{R}$. The study of this problem will require potential-theoretic methods (Lemmas 4—7).

**Lemma 1.** *Let $K \subset \mathbb{R}^d$ be a compact set with $\mu_d(K) = 1$, $d \ge 3$, $0 < \delta < 1$. Then for any $E \subset K$ with $\mu_d(E) \le \delta^{d-1}$ and any $\mathbf{x}^* \in K$ there exists a 2-dimensional plane $L_2^*$ passing through $\mathbf{0}$ and $\mathbf{x}^*$ such that $\mu_2(E \cap L_2^*) \le c_K\delta$, where $c_K > 0$ depends only on $K$.*

*Proof.* We may assume that $\mathbf{x}^* = (1, 0, \ldots, 0)$. Any $\mathbf{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$ can be written in cylindrical coordinates as $\mathbf{x} = (x_1, \rho, \overline{\varphi})$, where $x_1, \rho \in \mathbb{R}$, $\overline{\varphi} \in T^{d-2} := [-\pi/2, \pi/2]^{d-2}$, and $(\rho, \overline{\varphi})$ are the spherical coordinates in $\mathbb{R}^{d-1}$. Clearly, there exists an $a > 0$ such that $x_1, \rho \in [-a, a]$ whenever $\mathbf{x} \in K$. Then, using that $\mu_d(K) = 1$, we get

$$\begin{aligned} \mu_d(E) &= \int_{T^{d-2}}\int_{-a}^{a}\int_{-a}^{a} \chi_E(x_1, \rho, \overline{\varphi})|\rho|^{d-2}J(\overline{\varphi})dx_1 d\rho d\overline{\varphi} \\ &\le \delta^{d-1} = \delta^{d-1}\int_{T^{d-2}}\int_{-a}^{a}\int_{-a}^{a} \chi_K(x_1, \rho, \overline{\varphi})|\rho|^{d-2}J(\overline{\varphi})dx_1 d\rho d\overline{\varphi} \end{aligned}$$

where $\chi_E$ and $\chi_K$ are the characteristic functions of $E$ and $K$, respectively, and $\rho^{d-2}J(\overline{\varphi})$ is the Jacobian of the spherical transformation in $\mathbb{R}^{d-1}$; $J(\overline{\varphi}) \geq 0, \overline{\varphi} \in T^{d-2}$. Therefore, for some $\overline{\varphi}^* \in T^{d-2}$,

$$\begin{aligned}\int_{-a}^{a}\int_{-a}^{a} \chi_E(x_1,\rho,\overline{\varphi}^*)|\rho|^{d-2}d\rho dx_1 &\leq \delta^{d-1}\int_{-a}^{a}\int_{-a}^{a} \chi_K(x_1,\rho,\overline{\varphi}^*)|\rho|^{d-2}d\rho dx_1 \\ &\leq c'_K\delta^{d-1}. \qquad (5)\end{aligned}$$

Fixing this $\overline{\varphi}^*$ we get a 2-dimensional plane $L_2^* := \{\mathbf{x} \in \mathbb{R}^d : \mathbf{x} = (t\mathbf{x}^*, \rho\mathbf{u}^*),\ t,\rho \in \mathbb{R}\}$, where $\mathbf{u}^*$ is a point on $S^{d-2}$ with spherical coordinates $\overline{\varphi}^*$. It is clear that $\mathbf{x}^*, \mathbf{0} \in L_2^*$.

Now set $\gamma(t) := \mu_1\{\rho :\ \chi_E(t,\rho,\overline{\varphi}^*) = 1\}$. Then

$$\begin{aligned}\mu_2(E \cap L_2^*) &= \mu_2\{(t,\rho) :\ \chi_E(t,\rho,\overline{\varphi}^*) = 1\} \\ &= \int_{-a}^{a}\int_{-a}^{a} \chi_E(t,\rho,\overline{\varphi}^*)d\rho dt = \int_{-a}^{a}\gamma(t)dt. \qquad (6)\end{aligned}$$

Then by (5), (6) and Hölder's inequality, we have

$$\begin{aligned}c'_K\delta^{d-1} &\geq \int_{-a}^{a}\int_{-a}^{a} \chi_E(t,\rho,\overline{\varphi}^*)|\rho|^{d-2}d\rho dt \geq 2\int_{-a}^{a}\int_{0}^{\gamma(t)/2}\rho^{d-2}d\rho dt \\ &\geq \int_{-a}^{a}\frac{\gamma(t)^{d-1}}{2^{d-2}(d-1)}dt \geq \frac{c''_K}{2^{d-2}(d-1)}\left(\int_{-a}^{a}\gamma(t)dt\right)^{d-1}.\end{aligned}$$

Thus, using (6),

$$\mu_2(E \cap L_2^*) = \int_{-a}^{a}\gamma(t)dt \leq c_K\delta.$$

□

Let $K := K_r \subset \mathbb{R}^d$ be a star-like set and denote by

$$f_K(\mathbf{x}) := \inf\{\beta > 0 : \mathbf{x}/\beta \in K\} = \frac{|\mathbf{x}|}{r(\mathbf{x}/|\mathbf{x}|)}$$

the Minkowski functional of $K$. We shall say that $K$ is *regular* if $f_K$ is continuously differentiable on its boundary $\partial K$. Note that $f_K(\mathbf{x}) \leq 1$ if and only if $\mathbf{x} \in K$, $f_K(\mathbf{x}) = 1$ for $\mathbf{x} \in \partial K$, $f_K(t\mathbf{x}) = tf_K(\mathbf{x}),\ t > 0$, and thus $f_K(\mathbf{x}) = \langle \nabla f_K(\mathbf{x}), \mathbf{x}\rangle$ and $(\nabla f_K)(t\mathbf{x}) = \nabla f_K(\mathbf{x}),\ t > 0,\ \mathbf{x} \neq \mathbf{0}$.

Set $\mathbf{e}_j = (\delta_{ij})_{i=1}^d \in \mathbb{R}^d$, $1 \leq j \leq d$. (As usual $\delta_{ij} = 0$ if $i \neq j$, and $\delta_{ii} = 1$.) Furthermore, if $L : \mathbb{R}^d \to \mathbb{R}^d$ is a regular linear transformation, that is $L$ is a nonsingular matrix, then $\|L\|$ stands for its $\ell_2$-norm, $L(D) := \{L\mathbf{x} : \mathbf{x} \in D\},\ D \subset \mathbb{R}^d$. In addition, for a star-like set $K$, put

$$\begin{aligned}M(K) &:= \sup\{|\mathbf{x}| :\ \mathbf{x} \in K\}, \\ m(K) &:= \inf\{|\mathbf{x}| :\ \mathbf{x} \in \partial K\}, \\ M^*(K) &:= \sup\{|\nabla f_K(\mathbf{x})| :\ \mathbf{x} \in S^{d-1}\}.\end{aligned}$$

**Lemma 2.** *Let $K \subset \mathbb{R}^d$ be a star-like set such that $\nabla f_K$ exists and is bounded on $S^{d-1}$. For any $\mathbf{y} = (y_1, \dots, y_d) \in \partial K$ there exists a regular linear transformation $L : \mathbb{R}^d \to \mathbb{R}^d$ and a star-like set $D$ such that $L(D) = K$, $L\mathbf{e}_1 = \mathbf{y}$, $\nabla f_D(\mathbf{e}_1) = \mathbf{e}_1$, and $\|L\|, \|L^{-1}\| \le c_0$ with some $c_0 > 0$ depending only on $M(K)$ and $M^*(K)$.*

*Proof.* Without loss of generality we may assume (using a rotation) that $\nabla f_K(\mathbf{y}) = t\mathbf{e}_1$, $t > 0$. Note that whenever $\mathbf{x} \in \mathbb{R}^d$ we have

$$\langle \nabla f_K(\mathbf{x}), \mathbf{x} \rangle = D_{\mathbf{x}} f_K(\mathbf{x}) = f_K(\mathbf{x}). \tag{7}$$

Hence, using that $f_K(\mathbf{y}) = 1$ for $\mathbf{y} \in \partial K$,

$$1 = \langle \nabla f_K(\mathbf{y}), \mathbf{y} \rangle = t y_1, \tag{8}$$

that is, $y_1 > 0$. Now, define $L : \mathbb{R}^d \to \mathbb{R}^d$ by

$$L\mathbf{e}_1 = \mathbf{y},\ L\mathbf{e}_j = \mathbf{e}_j,\ 2 \le j \le d. \tag{9}$$

Clearly, $D := L^{-1}(K)$ is star-like and we have by (8)

$$M(K) \ge |\mathbf{y}| \ge y_1 = \frac{1}{t} = \frac{1}{|\nabla f_K(\mathbf{y})|} \ge \frac{1}{M^*(K)}.$$

It is a routine exercise to verify that $\|L\|, \|L^{-1}\| \le c_0$ with a $c_0 > 0$ depending only on $M(K)$, $M^*(K)$. Moreover, if $D := L^{-1}(K)$, i.e., $L(D) = K$, then $f_D(\mathbf{x}) = f_K(L\mathbf{x})$, and $\nabla f_D(\mathbf{x}) = L^T \nabla f_K(L\mathbf{x})$. Hence, by (9), for any $2 \le j \le d$,

$$\langle \nabla f_D(\mathbf{e}_1), \mathbf{e}_j \rangle = \left\langle L^T \nabla f_K(L\mathbf{e}_1), \mathbf{e}_j \right\rangle = \langle \nabla f_K(\mathbf{y}), L\mathbf{e}_j \rangle = \langle t\mathbf{e}_1, \mathbf{e}_j \rangle = 0.$$

Thus $\nabla f_D(\mathbf{e}_1) = \lambda \mathbf{e}_1$ where, by (7) and (9),

$$\lambda = \langle \nabla f_D(\mathbf{e}_1), \mathbf{e}_1 \rangle = f_D(\mathbf{e}_1) = f_K(L\mathbf{e}_1) = f_K(\mathbf{y}) = 1.$$

□

**Lemma 3.** *Let $K \subset \mathbb{R}^2$, $K = \{(\rho\cos\varphi, \rho\sin\varphi) : 0 \le \rho \le r(\varphi),\ 0 \le \varphi \le 2\pi\}$, where $r > 0$, $r \in \mathrm{Lip}_M \alpha$, $0 < \alpha \le 2$ on $[0, 2\pi]$, $r'(\pi/2) = 0$, when $1 < \alpha \le 2$. Assume that $r(\pi/2) = 1$. Then there exists an $a > 0$ depending only on $M$, $\alpha$, $m(K)$, $M(K)$ such that*

$$K_\alpha := \left\{(x, y) \in \mathbb{R}^2 :\ y > 0,\ y^\alpha \le 1 - a|x|^\alpha\right\} \subset K.$$

*Proof.* Let first $0 < \alpha \le 1$. Set $a := 3(M+1)r_0^{-\alpha}$, where $r_0 := m(K)$. Moreover, set

$$K_1 := \left\{(x, y) \in \mathbb{R}^2 :\ 0 \le y \le 1 - a|x|^\alpha\right\}.$$

Assume that for some $(x, y) \in \partial K \cap K_1$, $(x, y) \ne (0, 1)$. Then $x = r(\varphi)\cos\varphi$, $y = r(\varphi)\sin\varphi$, $0 \le \varphi \le \pi$, $\varphi \ne \pi/2$ and $0 \le y \le 1 - a|x|^\alpha$. Consequently,

$$\begin{aligned} a|x|^\alpha &\le |y - 1| = |r(\varphi)\sin\varphi - 1| \le |r(\varphi) - 1| + |\sin\varphi - 1| \le M\left|\frac{\pi}{2} - \varphi\right|^\alpha + \left|\frac{\pi}{2} - \varphi\right| \\ &\le \frac{\pi}{2}(M+1)\left|\frac{\pi}{2} - \varphi\right|^\alpha \le (M+1)\frac{5}{2}|\cos\varphi|^\alpha \le \frac{5}{2}\frac{(M+1)}{r_0^\alpha}|x|^\alpha, \end{aligned}$$

which is impossible by the choice of $a$. Hence we have that $K \supset K_1 \supset K_\alpha$.

In the case $1 < \alpha \le 2$ we set $a := 9(MR_0^{\alpha-1}+1)r_0^{-\alpha}$, where $R_0 := M(K)$. Then the assertion will follow from the estimates

$$\begin{aligned} a|x|^\alpha &\le |y-1| \le |y^\alpha - 1| = |r^\alpha(\varphi)\sin^\alpha\varphi - 1| \le |r^\alpha(\varphi)-1| + |\sin^\alpha\varphi - 1| \\ &\le \alpha R_0^{\alpha-1}|r'(\xi) - r'(\pi/2)| \left|\frac{\pi}{2}-\varphi\right| + \alpha|\sin^{\alpha-1}\zeta\cos\zeta| \left|\frac{\pi}{2}-\varphi\right| \\ &\le \alpha M R_0^{\alpha-1}\left|\frac{\pi}{2}-\varphi\right|^\alpha + \alpha\left|\frac{\pi}{2}-\varphi\right|^2 \le 8\frac{\left(MR_0^{\alpha-1}+1\right)}{r_0^\alpha}|x|^\alpha, \end{aligned}$$

where $\xi, \zeta$ are some constants between $\varphi$ and $\pi/2$.

□

Let $\omega$ be a nonincreasing positive continuous function on $[0,1]$, $0<\delta<1$, and let $T_n^{\omega,\delta}\left(t^2\right)$ be the normalized Chebyshev polynomial of degree $2n$ on $A_\delta := [-1,-\delta]\cup[\delta,1]$ with the weight $\omega\left(|t|\right)$, i.e.

$$\left|T_n^{\omega,\delta}\left(t^2\right)\omega(|t|)\right| \le 1,\ t\in A_\delta,\ \deg T_n^{\omega,\delta}(\cdot) = n,$$

and there exist $n+1$ points $\delta \le y_0 < \cdots < y_n \le 1$ such that

$$T_n^{\omega,\delta}\left(y_j^2\right) = \frac{(-1)^{n-j}}{\omega(|y_j|)},\ j=\overline{0,n}.$$

This polynomial exists by the Chebyshev theorem, since $\{t^k\omega(|t|)\}_{k=0}^{n-1}$ forms a T-system (cf. [KSt]) on $[0,1]$.

Denote by $\mathcal{P}_n(\delta)$ the set of real univariate algebraic polynomials of degree $\le n$ such that

$$\mu_1\left\{t\in[-1,1] : \left|\omega(|t|)p\left(t^2\right)\right| \ge 1\right\} = 2\delta.$$

Then the following lemma holds:

**Lemma 4.** *Let $\omega$, $T_n^{\omega,\delta}$, $\mathcal{P}_n(\delta)$ be defined as above. Then for any $p\in\mathcal{P}_n(\delta)$*

$$|p(0)| \le |T_n^{\omega,\delta}(0)|. \tag{10}$$

*Proof.* Denote by $\widetilde{E} := \{t\in[0,1] :\ \omega(t)|p(t^2)| \le 1\}$, $E^+ := [0,1]\setminus\widetilde{E}$. Clearly, $\mu_1(E^+) = \delta$, $\mu_1(\widetilde{E}) = 1-\delta$. Let $\psi$ be the transformation of $[0,1]$ shifting $\widetilde{E}$ to the right onto $[\delta,1]$ (or equivalently shifting $E^+$ to the left into $(0,\delta)$.) That is,

$$\psi(x) := \begin{cases} 1-\mu_1\{t\in\widetilde{E} :\ t>x\}, & x\in\widetilde{E} \\ \delta-\mu_1\{t\in E^+ :\ t>x\}, & x\in E^+. \end{cases}$$

It is easy to see that $\psi$ is a monotone increasing mapping of $\widetilde{E}$ onto $[\delta,1]$, $\psi(x)\ge x$, $x\in\widetilde{E}$, and $\psi(y)-\psi(x)\le y-x$ for $x,y\in\widetilde{E}$, $y>x$. Let $x_0<\cdots<x_n$

be points in $\widetilde{E}$ such that $y_j = \psi(x_j)$, $0 \le j \le n$. From the properties of $\psi$ we deduce that $y_j = x_j + \delta_j$ with $\delta_j \ge 0$ and $\delta_k \ge \delta_j$ whenever $k < j$.

By Lagrange interpolation formula we have

$$|p(0)| = \left|\sum_{j=0}^{n} p(x_j^2) \prod_{k=0,k\neq j}^{n} \left(\frac{-x_k^2}{x_j^2 - x_k^2}\right)\right| \le \sum_{j=0}^{n} \frac{1}{\omega(|x_j|)} \prod_{k=0,k\neq j}^{n} \left(\frac{x_k^2}{|x_j^2 - x_k^2|}\right). \tag{11}$$

Since $x_k < x_j$ and $\delta_k \ge \delta_j$ whenever $k < j$, we have for $y_j = x_j + \delta_j$ and $y_k = x_k + \delta_k$,

$$\frac{x_k^2}{x_j^2 - x_k^2} \le \frac{y_k^2}{y_j^2 - y_k^2}. \tag{12}$$

Inequality (12) implies that for $k > j$ we have

$$\frac{x_k^2}{x_k^2 - x_j^2} = 1 + \frac{x_j^2}{x_k^2 - x_j^2} \le 1 + \frac{y_j^2}{y_k^2 - y_j^2} = \frac{y_k^2}{y_k^2 - y_j^2}. \tag{13}$$

Hence combining inequalities (11), (12), and (13) we obtain

$$\begin{aligned} |p(0)| &\le \sum_{j=0}^{n} \frac{1}{\omega(|y_j|)} \prod_{k=0,k\neq j}^{n} \left(\frac{y_k^2}{|y_j^2 - y_k^2|}\right) = \sum_{j=0}^{n} \frac{(-1)^{n-j}}{\omega(|y_j|)} \prod_{k=0,k\neq j}^{n} \left(\frac{y_k^2}{y_j^2 - y_k^2}\right) \\ &= \sum_{j=0}^{n} T_n^{\omega,\delta}(y_j^2) \prod_{k=0,k\neq j}^{n} \left(\frac{y_k^2}{y_j^2 - y_k^2}\right) = |T_n^{\omega,\delta}(0)|. \end{aligned}$$

□

For the next needed lemma we shall appeal to potential theory.

**Lemma 5.** *Suppose that for some $\alpha$ and $\delta$ with $\alpha > 0$ and $0 < \delta < 1$ there holds*

$$\frac{|P_n(x)|}{(1 + |x|^\alpha)^{n/\alpha}} \le 1 \quad \text{for} \quad \delta \le |x| \le 1, \tag{14}$$

*where $P_n$ is a polynomial of degree at most $n$. Then*

$$|P_n(0)| \le e^{n g_\alpha(\delta)}, \tag{15}$$

*where*

$$g_\alpha(\delta) := \frac{1}{\alpha\pi} \int_{\delta^2}^{1} \log\left[\frac{t^{\alpha/2} + \delta^\alpha}{(t - \delta^2)^{\alpha/2}}\right] \frac{dt}{\sqrt{(1-t)(t-\delta^2)}}\,. \tag{16}$$

*Proof.* Put $p_n(x) := (x/\delta)^n P_n(\delta/x) = P_n(0)(x/\delta)^n + \cdots$. Then (14) gives

$$\omega_\alpha^n(x)|p_n(x)| \le 1 \quad \text{for} \quad \delta \le |x| \le 1, \tag{17}$$

where

$$\omega_\alpha(x) := e^{-Q_\alpha(x)}, \quad Q_\alpha(x) := \frac{1}{\alpha} \log\left(1 + \left|\frac{x}{\delta}\right|^\alpha\right). \tag{18}$$

We shall verify the following

*Claim:* There exists a probability measure $\mu_\alpha$ with support $A_\delta := [-1,-\delta]\cup[\delta,1]$ and a constant $F_\alpha$ such that

$$Q_\alpha(x) = F_\alpha - U^{\mu_\alpha}(x) \ \text{ for } \ \delta \le |x| \le 1, \tag{19}$$

where

$$U^{\mu_\alpha}(x) := \int_{A_\delta} \log \frac{1}{|x-t|} d\mu_\alpha(t)$$

is the logarithmic potential for $\mu_\alpha$.

Assuming the validity of (19), we can rewrite (17) as

$$\frac{1}{n}\log|p_n(x)| + U^{\mu_\alpha}(x) \le F_\alpha \ \text{ for } \ \delta \le |x| \le 1. \tag{20}$$

But the left-hand side of (20) is subharmonic in $\overline{\mathbb{C}} \setminus \{x : \ \delta \le |x| \le 1\}$, where $\overline{\mathbb{C}}$ denotes the extended complex plane. Hence (20) holds for all $x \in \overline{\mathbb{C}}$. Letting $x \to \infty$, we get

$$\frac{1}{n}\log|P_n(0)| \le F_\alpha + \log\delta,$$

and so

$$|P_n(0)| \le e^{n(F_\alpha + \log\delta)}. \tag{21}$$

We remark that $\mu_\alpha$ is the *weighted equilibrium measure* (cf. [ST, Chapter I]) for the weight $\omega_\alpha$ on the set $A_\delta$. To obtain a formula for $F_\alpha$, it is convenient to make the change of variables $t = x^2$. By [ST, Theorem IV.1.10(f)], we have

$$d\mu_\alpha(x) = \frac{1}{2} d\widetilde{\mu}_\alpha(t) \ \text{ and } \ F_\alpha = \frac{1}{2}\widetilde{F}_\alpha, \tag{22}$$

where $\widetilde{\mu}_\alpha$ is the weighted equilibrium measure for the weight $\widetilde{\omega}_\alpha(t) = \left[\omega_\alpha\left(\sqrt{t}\right)\right]^2$ on $[\delta^2, 1]$, and $\widetilde{F}_\alpha$ is the corresponding weighted Robin constant such that

$$U^{\widetilde{\mu}_\alpha}(x) + \widetilde{Q}_\alpha(t) = \widetilde{F}_\alpha \ \text{ for } \ t \in \operatorname{supp}(\widetilde{\mu}_\alpha), \tag{23}$$

where

$$\widetilde{Q}_\alpha(t) := \log\frac{1}{\widetilde{\omega}_\alpha(t)} = \frac{2}{\alpha}\log\left[1 + \left(\frac{\sqrt{t}}{\delta}\right)^\alpha\right]. \tag{24}$$

From our claim we have $\operatorname{supp}(\widetilde{\mu}_\alpha) = [\delta^2, 1]$, and hence $\widetilde{F}_\alpha = -F([\delta^2,1])$, where $F(K)$ is the "F-functional" of Mhaskar and Saff (cf. [ST, Theorem IV.1.5(b)]). This gives

$$\widetilde{F}_\alpha = \frac{1}{\pi}\int_{\delta^2}^1 \widetilde{Q}_\alpha(t)\frac{dt}{\sqrt{(1-t)(t-\delta^2)}} - \log\left[\frac{1-\delta^2}{4}\right], \tag{25}$$

where we have used the facts that $\operatorname{cap}([\delta^2,1]) = (1-\delta^2)/4$, and the unweighted equilibrium measure (Robin measure) for the interval $[\delta^2, 1]$ is

$$d\lambda_\delta(t) := \frac{1}{\pi}\frac{dt}{\sqrt{(1-t)(t-\delta^2)}}, \ \delta^2 \le t \le 1. \tag{26}$$

Since (26) is a unit measure and, as is well-known, $U^{\lambda_\delta}(s) = -\log[(1-\delta^2)/4]$ for all $s \in [\delta^2, 1]$, we obtain from (22), (24) and (25) that

$$\begin{aligned} F_\alpha + \log\delta &= \frac{1}{2}\left[\widetilde{F}_\alpha + \log\left(\delta^2\right)\right] \\ &= \frac{1}{2}\int_{\delta^2}^{1}\left[\widetilde{Q}_\alpha(t) + \log\left(\delta^2\right) + \log\frac{1}{|t-s|}\right] d\lambda_\delta(t) \\ &= \frac{1}{\alpha\pi}\int_{\delta_2}^{1}\log\left[\frac{t^{\alpha/2}+\delta^\alpha}{|t-s|^{\alpha/2}}\right]\frac{dt}{\sqrt{(1-t)(t-\delta^2)}} \end{aligned}$$

for every $s \in [\delta^2, 1]$. Taking $s = \delta^2$ we see from (21) that the estimate (15) holds.

It remains to verify the claim concerning (19). For this purpose it is equivalent and more convenient to establish that $\mathrm{supp}(\widetilde{\mu}_\alpha) = [\delta^2, 1]$, where as above, $\widetilde{\mu}_\alpha$ is the weighted equilibrium measure for $\exp(-\widetilde{Q}_\alpha(t))$ on $[\delta^2, 1]$. It is readily verified that

$$(t-\delta^2)\widetilde{Q}'_\alpha(t) = \frac{t^{\alpha/2} - \delta^2 t^{\alpha/2-1}}{\delta^\alpha + t^{\alpha/2}}$$

is increasing on $[\delta^2, 1]$. Thus the support of $\widetilde{\mu}_\alpha$ is an interval (cf. [ST, Theorem IV.1.10(c)], [B, Theorem 9]). Also, since $\widetilde{Q}'_\alpha(t) > 0$, it follows from [ST, Theorem IV.1.11(ii)], [B, Theorem 10(ii)] that $\mathrm{supp}(\widetilde{\mu}_\alpha) = [\delta^2, b]$ for some $\delta^2 < b \le 1$. To show that $b = 1$, assume the contrary. Then

$$F\left([\delta^2, \beta]\right) = \log\left(\mathrm{cap}[\delta^2, \beta]\right) - \frac{1}{\pi}\int_{\delta^2}^{\beta}\widetilde{Q}_\alpha(t)\frac{dt}{\sqrt{(\beta-t)(t-\delta^2)}}$$

must attain its maximum for $\beta = b$. Consequently,

$$\frac{d}{d\beta}F([\delta^2, \beta])\bigg|_{\beta=b} = 0,$$

which after setting $y = -1 + 2(t-\delta^2)/(\beta-\delta^2)$ gives

$$1 = \frac{1}{\pi}\int_{-1}^{1}(\beta-\delta^2)\left(\frac{t^{\alpha/2-1}}{\delta^\alpha + t^{\alpha/2}}\right)\left(\frac{y+1}{2}\right)\frac{dy}{\sqrt{1-y^2}} \tag{27}$$

when $\beta = b$. But (27) is impossible since

$$(\beta-\delta^2)\left(\frac{t^{\alpha/2-1}}{\delta^\alpha + t^{\alpha/2}}\right)\left(\frac{y+1}{2}\right) = (t-\delta^2)\widetilde{Q}'_\alpha(t) < 1$$

for $|y| < 1$. Thus $b = 1$ and so $\mathrm{supp}(\widetilde{\mu}_\alpha) = [\delta^2, 1]$, which completes the proof of (15).

□

Concerning sharpness of Lemma 5 we establish

**Lemma 6.** *For each* $\alpha > 0$ *and* $0 < \delta < 1$ *there exists a sequence of polynomials* $\{P_n^{\delta,\alpha}\}$, $\deg P_n^{\delta,\alpha} = n$ *satisfying*

$$\frac{|P_n(x)|}{(1+|x|^\alpha)^{n/\alpha}} \le 1 \quad \text{for} \quad |x| \ge \delta, \tag{28}$$

*such that*

$$\lim_{n\to\infty} |P_n^{\delta,\alpha}(0)|^{1/n} = e^{g_\alpha^*(\delta)}, \tag{29}$$

*where*

$$g_\alpha^*(\delta) := \frac{1}{\alpha\pi}\int_0^1 \log\left(1+\frac{\delta^\alpha}{t^{\alpha/2}}\right)\frac{dt}{\sqrt{t(1-t)}}\ . \tag{30}$$

*Proof.* Set

$$P_n^{\delta,\alpha}(x) := \frac{x^n\Phi_n(\delta/x)}{\|\omega_\alpha^n\Phi_n\|_{[-1,1]}},$$

where $\Phi_n(x)$ is the *Fekete polynomials associated with weight* $\omega_\alpha$ (cf. [ST, Section III.1]) and $\omega_\alpha$ is defined by (18). Then

$$p_n^{\delta,\alpha}(x) := \left(\frac{x}{\delta}\right)^n P_n^{\delta,\alpha}\left(\frac{\delta}{x}\right) = \frac{\Phi_n(x)}{\|\omega_\alpha^n\Phi_n\|_{[-1,1]}}.$$

Thus, by construction, $P_n^{\delta,\alpha}$ satisfies (28). It follows from [ST, Corollary III.1.10] that

$$\lim_{n\to\infty}\frac{1}{n}\log\left(\frac{|\Phi_n(z)|}{\|\omega_\alpha^n\Phi_n\|_{[-1,1]}}\right) = F_\alpha^* - U^{\mu_\alpha^*}(z),\ \forall z\in\mathbb{C}\setminus[-1,1],$$

where $\mu_\alpha^*$ is the weighted equilibrium measure for the weight $\omega_\alpha$ on the interval $[-1,1]$, and $F_\alpha^*$ is the corresponding modified Robin constant. Existence of $\mu_\alpha^*$ and $F_\alpha^*$ can be shown exactly in the same way as it was done in Lemma 5.

Now, since for each $n$, the function $\frac{1}{n}\log|p_n^{\delta,\alpha}(z)|+U^{\mu_\alpha^*}(z)$ is harmonic at $\infty$, we deduce that

$$\lim_{n\to\infty}\left(\frac{1}{n}\log|p_n^{\delta,\alpha}(z)|+U^{\mu_\alpha^*}(z)\right)\bigg|_{z=\infty} = F_\alpha^*,$$

or, equivalently

$$\lim_{n\to\infty}\left(\frac{1}{n}\log|P_n^{\delta,\alpha}(0)|-\log\delta\right) = F_\alpha^*.$$

Then the assertion of the lemma can be derived in a similar fashion as in Lemma 5.

□

Concerning the behavior of the quantities $g_\alpha(\delta)$ and $g_\alpha^*(\delta)$ we prove the following.

**Lemma 7.** *Let $\alpha$ and $\delta$ be defined as in Lemmas 5 and 6 and let $g_\alpha(\delta)$ be defined by (16), $g^*_\alpha(\delta)$ be defined by (30). Then for $0 < \delta \le 1/2$*

$$g_\alpha(\delta) \asymp \varphi_\alpha(\delta) \asymp g^*_\alpha(\delta), \tag{31}$$

*where $f(\delta) \asymp g(\delta)$ means that $c_2 g(\delta) \le f(\delta) \le c_1 g(\delta)$, $0 < \delta \le 1/2$ with positive constants $c_1, c_2$ depending only on $\alpha$.*

*Proof.* Fix $0 < \alpha \le 2$. By (16) we have that

$$g_\alpha(\delta) := \frac{1}{\alpha\pi}\int_{\delta^2}^{1} \log\left[\frac{t^{\alpha/2}+\delta^\alpha}{(t-\delta^2)^{\alpha/2}}\right]\frac{dt}{\sqrt{(1-t)(t-\delta^2)}} = \frac{1}{\alpha\pi}\left(\int_{\delta^2}^{1/2} + \int_{1/2}^{1}\right).$$

We shall estimate these last two integrals separately.

$$\begin{aligned}
I_1 &:= \int_{\delta^2}^{1/2} \log\left[\frac{t^{\alpha/2}+\delta^\alpha}{(t-\delta^2)^{\alpha/2}}\right]\frac{dt}{\sqrt{(1-t)(t-\delta^2)}} \\
&= \delta\int_{1}^{1/2\delta^2} \log\left[\frac{1+u^{\alpha/2}}{(u-1)^{\alpha/2}}\right]\frac{du}{\sqrt{(1-\delta^2 u)(u-1)}} \\
&\asymp \delta\int_{1}^{1/2\delta^2} \log\left[\frac{1+u^{\alpha/2}}{(u-1)^{\alpha/2}}\right] d\left(\sqrt{u-1}\right) \\
&\asymp \delta\sqrt{u-1}\,\log\left[\frac{1+u^{\alpha/2}}{(u-1)^{\alpha/2}}\right]\Bigg|_1^{1/2\delta^2} + \delta\int_{1}^{1/2\delta^2}\frac{u^{-1+\alpha/2}+1}{(u^{\alpha/2}+1)\sqrt{u-1}}\,du \\
&\asymp \delta^\alpha + \varphi_\alpha(\delta) \asymp \varphi_\alpha(\delta).
\end{aligned}$$

It is easy to see that for $t \in [1/2, 1]$

$$\log\left[\frac{t^{\alpha/2}+\delta^\alpha}{(t-\delta^2)^{\alpha/2}}\right] \asymp \delta^\alpha.$$

Hence

$$I_2 := \int_{1/2}^{1} \log\left[\frac{t^{\alpha/2}+\delta^\alpha}{(t-\delta^2)^{\alpha/2}}\right]\frac{dt}{\sqrt{(1-t)(t-\delta^2)}} \asymp \delta^\alpha.$$

Combining estimates for $I_1$ and $I_2$ we obtain the first part of (31).

The same type of arguments can be applied to derive that $g^*_\alpha(\delta) \asymp \varphi_\alpha(\delta)$.

□

We are now ready to give the

*Proof of the Main Theorem.* Let $h \in H^d_n$, $E \subset \partial K_r$ with $s_{d-1}(E) \le s_{d-1}(\partial K_r)\delta^{d-1}$ and $0 < \delta \le 1/2$. We may assume that $\|h\|_{\partial K_r \setminus E} = 1$ and $s_{d-1}(\partial K_r) = 1$. Now we need a proper upper bound for $\|h\|_{\partial K_r}$. Set

$$E_h := \{t\mathbf{x} :\ t \in [0,1],\ \mathbf{x} \in \partial K_r,\ |h(\mathbf{x})| > 1\}.$$

Clearly,

$$\mu_d(E_h) \le cs_{d-1}(E) \le c\delta^{d-1} \tag{32}$$

with some $c > 0$ depending only on $K_r$. Now the proof can be completed in several steps.

*Step 1.* First using Lemma 1 we can reduce our problem to the 2-dimensional case. Indeed, if $\|h\|_{\partial K_r} = |h(\mathbf{x}^*)|$ with some $\mathbf{x}^* \in \partial K_r$ then by Lemma 1 and (32) there exists a 2-dimensional plane $L_2^*$ passing through $\mathbf{0}$ and $\mathbf{x}^*$ such that $\mu_2(E_h \cap L_2^*) \le c\delta$ with a $c > 0$ depending only on $K_r$. Moreover, $h\big|_{L_2^*} \in H_n^2$ , and $\widetilde{K} := \partial K_r \cap L_2^*$ is a star-like surface in $\mathbb{R}^2$ containing $\mathbf{x}^*$, which satisfies the $\mathrm{Lip}_M\alpha$ property. Moreover, $m(\widetilde{K}) \ge m(K_r)$, $M(\widetilde{K}) \le M(K)$, $M^*(\widetilde{K}) \le M^*(K)$, i.e. Lemmas 2 and 3 can be applied to $\widetilde{K}$ with the corresponding constants being independent of $\mathbf{x}^*$. Hence we may assume that $d = 2$.

*Step 2.* Now we shall use Lemmas 2 and 3 to reduce the problem to "diamond-shaped" domains. For $K_r \subset \mathbb{R}^2$ we have in polar coordinates

$$K_r = \{(\rho, \varphi) :\ 0 \le \rho \le r^*(\varphi),\ 0 \le \varphi \le 2\pi\}$$

where $r^*(\varphi) := r(\cos\varphi, \sin\varphi)$, and $r^* \in \mathrm{Lip}_{\widetilde{M}}\alpha$ on $[0, 2\pi]$ with some $\widetilde{M} > 0$ depending only on $K_r$. We may assume that $\mathbf{x}^* = (1, 0)$. In addition, in view of Lemma 2 we may also assume without loss of generality that $\partial K_r$ possesses a vertical tangent line at $\mathbf{x}^*$ if $1 < \alpha \le 2$. Otherwise, by Lemma 2 there exists a regular linear transformation $L : \mathbb{R}^2 \to \mathbb{R}^2$ such that $L\mathbf{x}^* = \mathbf{x}^*$, $D = L^{-1}(K_r)$, $\mathbf{x}^* \in \partial D$, and $\nabla f_D(\mathbf{x}^*) = \mathbf{x}^*$. Hence $h$ can be replaced by $h^*(\mathbf{x}) = h(L\mathbf{x}) \in H_n^2$, and $K_r$ by the star-like domain $D$ corresponding to some $r^* \in \mathrm{Lip}_{M_2}\alpha$, $M_2 > 0$ depending only on $K_r$. Then by Lemma 3 there exists an $a > 0$ such that

$$K_\alpha := \{(x, y) \in \mathbb{R}^2 :\ |x|^\alpha + a|y|^\alpha \le 1\} \subset K_r. \tag{33}$$

Since $a$ depends only on $K_r$ we may set $a = 1$. Recalling that $\mu_2(E_h) \le c\delta$ we obtain from (33) that

$$\begin{aligned}&\mu_1 \{\varphi \in [0, 2\pi] : |h\left(r_\alpha^*(\varphi)\cos\varphi, r_\alpha^*(\varphi)\sin\varphi\right)| > 1\} \\ &\le \mu_1 \{\varphi \in [0, 2\pi] : |h\left(r^*(\varphi)\cos\varphi, r^*(\varphi)\sin\varphi\right)| > 1\} \le \frac{2\mu_2(E_h)}{m^2(K_r)} \le c_1\delta, \end{aligned} \tag{34}$$

where $r_\alpha^*(\varphi) := (|\cos\varphi|^\alpha + |\sin\varphi|^\alpha)^{-1/\alpha}$, $m(K_r) := \inf\{|\mathbf{x}| : \mathbf{x} \in \partial K_r\}$.

*Step 3.* Finally, we transform the problem to weighted univariate polynomials. We may assume that $n$ is even since otherwise $h(x, y)$ can be multiplied by $x$. So let $n = 2m$. Clearly, given that $h(x, y) = \sum_{j=0}^{2m} a_j x^j y^{2m-j}$ we have

$$\begin{aligned} h\left(r_\alpha^*(\varphi)\cos\varphi, r_\alpha^*(\varphi)\sin\varphi\right) &= (r_\alpha^*(\varphi))^{2m} \sum_{j=0}^{2m} a_j \cos^j\varphi \sin^{2m-j}\varphi \\ &= (1 + |\tan\varphi|^\alpha)^{-2m/\alpha} \sum_{j=0}^{2m} a_j \tan^{2m-j}\varphi = \frac{p_{2m}(t)}{(1 + |t|^\alpha)^{2m/\alpha}} \end{aligned} \tag{35}$$

where $t := \tan\varphi$, $p_{2m}(t) := \sum_{j=0}^{2m} a_j t^{2m-j}$. By (34)

$$\mu_1 \left\{ t \in [-1,1] : \frac{|p_{2m}(t)|}{(1+|t|^\alpha)^{2m/\alpha}} > 1 \right\} \le 2c_1\delta. \tag{36}$$

In addition, $|p_{2m}(0)| = |h(1,0)| = |h(\mathbf{x}^*)|$. Thus we arrive at the extremal problem of finding the maximal value of $|p_{2m}(0)|$ under condition (36). Evidently, it can be assumed that $p_{2m}$ is even, i.e., $p_{2m}(t) = q_m(t^2)$. Finally, using Lemmas 4, 5 and 7 we obtain

$$|h(\mathbf{x}^*)| = |p_{2m}(0)| \le e^{cm\varphi_\alpha(\delta)},$$

which gives the upper bound of the theorem.

In order to verify the sharpness of the above upper bound we proceed as follows. By Lemmas 6 and 7 there exists a sequence of univariate polynomials $P_n^{\delta,\alpha}$ of degree $n$ satisfying (28) such that

$$\lim_{n\to\infty} \frac{1}{n} \log|P_n^{\delta,\alpha}(0)| \ge c\varphi_\alpha(\delta).$$

Reversing transformation (35) we obtain homogeneous polynomials $h_n^{\delta,\alpha} \in H_n^2$ such that by (28)

$$\mu_1\{(x,y) \in \mathbb{R}^2 : |x|^\alpha + |y|^\alpha = 1, \ |h_n^{\delta,\alpha}(x,y)| > 1\} \le c\delta,$$

and

$$\frac{1}{n} \log|h_n^{\delta,\alpha}(1,0)| \ge c_1\varphi_\alpha(\delta).$$

Then the lower bound of the theorem holds for the star-like surface

$$K := \{(x_1,\ldots,x_d) \in \mathbb{R}^d : |x_1|^\alpha + |x_2|^\alpha = 1, \ |x_j| \le 1, \ 3 \le j \le d\}.$$

□

**Remark.** While Lemma 6 yields the sharpness of Lemma 5 and hence the main theorem, it does not provide an explicit expression for the extremal polynomials. Nevertheless, in the special case when $\alpha \ne 1$ extremal polynomials can be given explicitly; namely, in the case when $0 < \alpha < 1$ we can take

$$P_{2n}^{\delta,\alpha}(x) := (1 + \delta^\alpha - x^2)^n,$$

and, for $1 < \alpha \le 2$,

$$P_{2n}^{\delta,\alpha}(x) := T_n\left(\frac{2x^2 - 9 - \delta^2}{9 - \delta^2}\right),$$

where $T_n(x) := \cos n \arccos x$ is the Chebyshev polynomial.

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