Potential Theoretic Tools in Polynomial and Rational Approximation

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Logarithmic potential theory is an elegant blend of real and complex analysis that has had a profound effect on many recent developments in approximation theory. Since logarithmic potentials have a direct connection with polynomial and rational functions, the tools provided by classical potential theory and its extensions to cases when an external field (or weight) is present, have resolved some long-standing problems concerning orthogonal polynomials, rates of polynomial and rational approximation, convergence behavior of Padé approximants (both classical and multi-point), to name but a few.

In this article we provide an introduction to the tools of classical and "weighted" potential theory, along with a taste of various applications. We begin by introducing three "different" quantities associated with a compact (closed and bounded) set in the plane.

1 Classical Logarithmic Potential Theory

Potential theory has its origin in the following

Electrostatics Problem. Let E be a compact set in the complex plane $\mathbb{C}$. Place a unit positive charge on E so that equilibrium is reached in the sense that the energy is minimized.

To create a mathematical framework for this problem, we let $\mathcal{M}(E)$ denote the collection of all positive unit measures μ supported on E (so that $\mathcal{M}(E)$ contains all possible distributions of charges placed on E). The *logarithmic potential* associated with μ is

$$U^{\mu}(z) := \int \log \frac{1}{|z-t|} d\mu(t),$$

which is harmonic outside the support $S(\mu)$ of μ and is *superharmonic* in $\mathbb{C}$. The latter means that the value of the potential at any point z is not less than its average over any circle centered at z. Notice that, since μ is a unit measure,

$$\lim_{z\to\infty} (U^{\mu}(z) + \log|z|) = 0. \tag{1.1}$$

The *energy* of such a potential is defined by

$$I(\mu) := \int U^{\mu} d\mu = \int\int \log \frac{1}{|z-t|} d\mu(t) d\mu(z).$$

Thus, the electrostatics problem involves the determination of

$$V_E := \inf\{I(\mu) : \ \mu \in \mathcal{M}(E)\},$$

which is called the *Robin constant* for E. Note that since E is bounded, we have

$$\operatorname{diam} E := \sup_{z,t \in E} |z - t| < \infty,$$

which implies that

$$-\infty < V_E \le +\infty.$$

The *logarithmic capacity* of E , denoted by $\operatorname{cap}(E)$, is defined by

$$\operatorname{cap}(E) := e^{-V_E}.$$

If $V_E = +\infty$, we set $\operatorname{cap}(E) = 0$. Such sets are called *polar* and they are very "thin". In particular, the "area" (= planar Lebesgue measure) and the "length" (= one-dimensional Hausdorff measure) of any polar set, are both equal to zero. For example, any countable set has capacity zero. (However, the classical Cantor set has positive capacity.)

A fundamental theorem of Frostman asserts that if $\operatorname{cap}(E) > 0$, there exists a unique measure $\mu_E \in \mathcal{M}(E)$ such that $I(\mu_E) = V_E$. This extremal measure is called the *equilibrium measure* (or *Robin measure*) for E.

We do not dwell on the proof of the Frostman result, but only mention that it utilizes three important properties:

(i) $\mathcal{M}(E)$ is compact with respect to weak-star convergence of measures;

(ii) $I(\mu)$ is a lower semi-continuous function on $\mathcal{M}(E)$;

(iii) $I(\mu)$ is a strictly convex function on $\mathcal{M}(E)$.

The existence of μ_E follows from (i), (ii), while (iii) guarantees the uniqueness. The weak-star convergence (denoted weak*) is defined as follows: we say that a sequence $\{\mu_n\}$ converges weak* to μ (write $\mu_n \xrightarrow{*} \mu$), if

$$\int f d\mu_n \to \int f d\mu \quad \text{as} \quad n \to \infty$$

for any function f continuous in $\mathbb{C}$.

The potential U^{μ_E} associated with μ_E is called the *equilibrium potential* (or *conductor potential*) for E. Some basic facts about $\operatorname{cap}(E)$ and U^{μ_E} are:

(a) Let $\partial_\infty E$ denote the *outer boundary* of E (that is, the boundary of the unbounded component of $\mathbb{C} \setminus E$; see Fig. 1). Then μ_E is supported on $\partial_\infty E$:

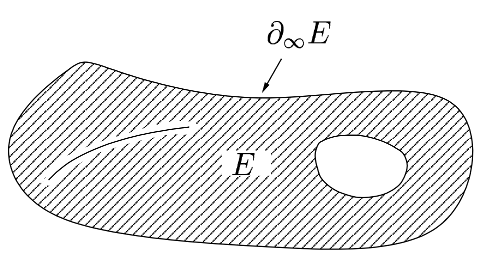

Figure 1: Outer boundary of E

$$S(\mu_E) \subseteq \partial_\infty E.$$

Moreover, if strict inclusion takes place, then the set $\partial_\infty E \setminus S(\mu_E)$ has capacity zero. It follows from the above inclusion that, being unique, the equilibrium measures for E and for $\partial_\infty E$ coincide. Therefore

$$\operatorname{cap}(E) = \operatorname{cap}(\partial_\infty E).$$

(b) For all $z \in \mathbb{C}$,

$$U^{\mu_E}(z) \leq V_E$$

with equality holding *quasi-everywhere* on E; that is, except possibly for a set of capacity zero. We write this as

$$U^{\mu_E}(z) = V_E = \log \frac{1}{\operatorname{cap}(E)} \quad \text{q.e. on } E. \tag{1.2}$$

Moreover, such equality *characterizes* μ_E:
If the potential of some $\mu \in \mathcal{M}(E)$ is constant q.e. on E and $I(\mu) < \infty$, then $\mu = \mu_E$.

(c) A point $z \in E$ is called *regular* if (1.2) holds at z. If the interior[1] Int$E$ of E is not empty, it follows from (a) that the conductor potential is harmonic there. Then (b) guarantees that (1.2) holds at every point of Int$E$. The following fact is deeper: if $\partial_\infty E$ is *connected*, then every point of $\partial_\infty E$ is regular. Furthermore, at every regular point the conductor potential is *continuous*.

It is helpful to keep in mind the following two simple examples.

Example 1.1. Let E be the closed disk of radius R, centered at 0. Then $d\mu_E = ds/2\pi R$, where ds is the arclength on the circle $|z| = R$. One way to derive this is to observe that E is invariant under rotations. The equilibrium measure, being unique and supported on $|z| = R$, must enjoy the same property, and therefore must be of the above form. Calculating the potential, we obtain

$$U^{\mu_E}(z) = \log \frac{1}{|z|}, \quad |z| > R \quad \text{and} \quad U^{\mu_E}(z) = \log \frac{1}{R}, \quad |z| \leq R.$$

Therefore (see (1.2)), $\operatorname{cap}(E) = R$.

Example 1.2. Let $E = [a, b]$ be a segment on the real line. Then $\operatorname{cap}(E) = (b-a)/4$ and $d\mu_E$ is the arcsine measure; i.e.

$$d\mu_E = \frac{1}{\pi} \frac{dx}{\sqrt{(x-a)(b-x)}}, \quad x \in [a, b].$$

If $a = -1$, $b = 1$, the conductor potential is given by

$$U^{\mu_E}(z) = \log 2 - \log |z + \sqrt{z^2 - 1}|$$

(for arbitrary a, b the expression is a bit more complicated). These results can be obtained from Example 1.1 by applying the Joukowski conformal map of $\mathbb{C} \setminus [-1, 1]$ onto $|w| > 1$.

There is an important relation between the equilibrium potential and the notion of *Green function*. Assume, for simplicity, that $\partial_\infty E$ is connected and let Ω denote the unbounded

[1] A point $z_0 \in$Int E if and only if there is some open disk with center at z_0 that lies entirely in E.

component of $\mathbb{C} \setminus E$ (so that $\partial\Omega = \partial_\infty E$ and $\Omega \cup \{\infty\}$ is a simply connected domain in the extended complex plane).

Let $w = \Phi(z)$ denote the conformal map of Ω onto $|w| > 1$, normalized by $\Phi(\infty) = \infty$, $\Phi'(\infty) > 0$. That is, for some constant $c > 0$,

$$\Phi(z) = \frac{1}{c} z + \text{ lower order terms,} \quad \text{as} \quad z \to \infty.$$

By the Riemann Mapping Theorem, such a Φ exists and is unique. Moreover, its absolute value $|\Phi|$ becomes a continuous function in the whole plane if we set

$$|\Phi(z)| = 1, \quad z \in E.$$

Let us examine some properties of the function $g = \log|\Phi|$.

First, g is the real part of $\log \Phi(z)$ which is analytic in Ω. Therefore

(i) *g is harmonic in Ω;*

Second, our normalization implies that

(ii) $\lim_{z\to\infty} (g(z) - \log|z|)$ *exists and is finite;*

Finally,

(iii) *g is continuous in the closed domain $\overline{\Omega}$ and equals zero on its boundary.*

There is a unique function that enjoys these three properties. It is called the *Green function for Ω with pole at infinity* and is denoted by $g_\Omega(\cdot, \infty)$. So we have just shown that

$$\log|\Phi(z)| = g_\Omega(z, \infty)$$

(and that the limit in (ii) is equal to $\log(1/c)$). It is now easy to see that

$$U^{\mu_E}(z) = \log \frac{1}{\operatorname{cap}(E)} - g_\Omega(z, \infty). \tag{1.3}$$

Indeed, let h denote the difference of the two sides of (1.3). Then h is harmonic in the domain Ω, and is equal to zero on its boundary. Moreover, h has a finite limit at infinity, namely $\log(1/c) - \log(1/\operatorname{cap}(E))$, recall (1.1). By the maximum principle, h is identically zero and we are done. We also obtain that the constant c is just $\operatorname{cap}(E)$.

In the case when $\partial_\infty E$ is a smooth closed Jordan curve, there is a simple representation for μ_E. The equilibrium measure of any arc γ on $\partial_\infty E$ is given by

$$\mu_E(\gamma) = \frac{1}{2\pi} \int_\gamma \frac{\partial g_\Omega}{\partial n} ds = \frac{1}{2\pi} \int_\gamma |\Phi'| ds,$$

where the derivative in the first integral is taken in the direction of the *outer* normal on $\partial_\infty E$. Alternatively, $\mu_E(\gamma)$ is given by the normalized angular measure of the image $\Phi(\gamma)$:

$$\mu_E(\gamma) = \frac{1}{2\pi} \int_{\Phi(\gamma)} d\theta \tag{1.4}$$

(for this representation, the smoothness of $\partial_\infty E$ is not needed).

The reader is invited to carry out the above calculations, for the special case of a disk, considered in Example 1.1.

We now introduce another quantity associated with E. It arises in the following

Geometric Problem. Place n points on E so that they are "as far apart" as possible in the sense of the geometric mean of the distances between the points. Since the number of different pairs of n points is $n(n-1)/2$, we consider the quantity

$$\delta_n(E) := \max_{z_1,\dots,z_n \in E} \left(\prod_{1 \le i < j \le n} |z_i - z_j| \right)^{2/n(n-1)} .$$

Any system of points $\mathcal{F}_n = \left\{ z_1^{(n)}, \dots, z_n^{(n)} \right\}$ for which the maximum is attained, is called an *n-point Fekete set* for E; the points $z_i^{(n)}$ in $\mathcal{F}_n$ are called *Fekete points.*

For example, if $n = 2$, then $\mathcal{F}_2 = \left\{ z_1^{(2)}, z_2^{(2)} \right\}$, where $\left| z_1^{(2)} - z_2^{(2)} \right| = \operatorname{diam} E$. Obviously, these 2 points lie on the outer boundary of E. In general, it follows from the maximum modulus principle for analytic functions, that for all n, the Fekete sets lie on the outer boundary of E.

It turns out (cf. [R], [SaTo]), that the sequence δ_n decreases, so we may define

$$\tau(E) := \lim_{n \to \infty} \delta_n(E).$$

The quantity $\tau(E)$ is called the *transfinite diameter* of E.

Example 1.3. Let E be the closed unit disk. Then one can show that the set of n-th roots of unity is an n-point Fekete set for E (and so is any of its rotations). Furthermore, $\tau(E) = 1$.

Example 1.4. Let $E = [-1, 1]$. Then (cf. [Sz]) the set $\mathcal{F}_n$ turns out to be unique and it coincides with the zeros of $(1 - x^2) P_{n-2}^{(1,1)}(x)$, where $P_{n-2}^{(1,1)}$ is the Jacobi polynomial with parameters $(1, 1)$ of degree $n - 2$. Also, $\tau(E) = 1/2$.

Finally, we introduce a third quantity — the *Chebyshev constant*, $\operatorname{cheb}(E)$ — which arises in a mini-max problem.

Polynomial Extremal Problem: Determine the minimal sup-norm on E for monic polynomials of degree n. That is, determine

$$t_n(E) := \min_{p \in \mathcal{P}_{n-1}} \|z^n + p(z)\|_E,$$

where $\mathcal{P}_{n-1}$ denotes the collection of all polynomials of degree $\le n - 1$ and $\|\cdot\|_E$ is defined by

$$\|f\|_E := \max_{z \in E} |f(z)|.$$

We assume that E contains infinitely many points (which is always the case if $\operatorname{cap}(E) > 0$). Then for every n there is a unique monic polynomial $T_n(z) = z^n + \cdots$ such that $\|T_n\|_E = t_n(E)$. It is called the n-th *Chebyshev polynomial* for E.

In view of the simple inequality

$$t_{m+n}(E) = \|T_{m+n}\|_E \le \|T_m T_n\|_E \le \|T_m\|_E \|T_n\|_E = t_m(E) t_n(E),$$

one can show (cf. [R], [SaTo]) that the sequence $t_n(E)^{1/n}$ converges, so we may define

$$\mathrm{cheb}(E) := \lim_{n\to\infty} t_n(E)^{1/n}.$$

Example 1.5. Let E be the closed disk of radius R, centered at 0. For any $p \in \mathcal{P}_{n-1}$, the ratio $(z^n + p(z))/z^n$ represents an analytic function in $|z| \ge 1$ that takes the value 1 at ∞. By the maximum principle,

$$\|z^n + p(z)\|_E = \max_{|z|=R} |z^n + p(z)| = R^n \max_{|z|=R} \left| \frac{z^n + p(z)}{z^n} \right| \ge R^n,$$

and strict inequality takes place if $p(z)$ is not identically zero. It follows that $T_n(z) = z^n$. Therefore $t_n(E) = R^n$ and $\mathrm{cheb}(E) = R$.

Example 1.6. Let $E = [-1, 1]$. Then T_n is the classical monic Chebyshev polynomial

$$T_n(x) = 2^{1-n} \cos(n \arccos x), \quad x \in [-1, 1], \quad n \ge 1.$$

Also, $t_n(E) = 2^{1-n}$ from which it follows that $\mathrm{cheb}(E) = 1/2$.

Closely related to Chebyshev polynomials are *Fekete polynomials*. An n-th Fekete polynomial $F_n(z)$ is a monic polynomial having all its zeros at the n points of a Fekete set $\mathcal{F}_n$.

Example 1.7. If E is the closed unit disk centered at 0, then one can take $F_n(z) = z^n - 1$, so that $\|F_n\|_E = 2$. Comparing this with Example 1.5 we see that the F_n's are asymptotically optimal for the Chebyshev problem:

$$\lim_{n\to\infty} \|F_n\|_E^{1/n} = \lim_{n\to\infty} \|T_n\|_E^{1/n} = 1 = \mathrm{cheb}(E).$$

Moreover, uniformly on compact subsets of $|z| > 1$, we have

$$\lim_{n\to\infty} |F_n(z)|^{1/n} = \lim_{n\to\infty} |T_n(z)|^{1/n} = |z| = \exp\{-U^{\mu_E}(z)\},$$

(the last equality follows from Example 1.1). Finally, it is easy to see that the zeros of F_n are asymptotically uniformly distributed on $|z| = 1$. By that we mean that for any arc γ on this circle,

$$\frac{1}{n} \times \{\text{number of zeros of } F_n \text{ in } \gamma\} \to \frac{1}{2\pi} \times \{\text{length of } \gamma\}, \quad n \to \infty.$$

Note that the second ratio coincides with $\mu_E(\gamma)$ (cf. Example 1.1).

The examples of this section illustrate the following fundamental theorem, various parts of which are due to Fekete, Frostman, and Szegő.

Fundamental Theorem of Classical Potential Theory. *For any compact set* $E \subset \mathbb{C}$,
(a) $\operatorname{cap}(E) = \tau(E) = \operatorname{cheb}(E)$;
(b) *Fekete polynomials are asymptotically optimal for the Chebyshev problem:*

$$\lim_{n\to\infty} \|F_n\|_E^{1/n} = \operatorname{cheb}(E) = \operatorname{cap}(E).$$

If $\operatorname{cap}(E) > 0$ *(so that* μ_E *is defined), then we also have:*
(c) *Uniformly on compact subsets of the unbounded component of* $\mathbb{C} \setminus E$,

$$\lim_{n\to\infty} |F_n(z)|^{1/n} = \exp\{-U^{\mu_E}(z)\};$$

(d) *Fekete points (the zeros of* F_n*) have asymptotic distribution* μ_E.

The last statement is illustrated in Example 1.7, but let us make it more precise. Let

$$P_n(z) = \prod_{k=1}^{n}(z - z_k)$$

and let δ_{z_k} denote the unit mass placed at z_k. Then $U^{\delta_{z_k}}(z) = \log \dfrac{1}{|z - z_k|}$, and we see that

$$|P_n(z)|^{1/n} = e^{-U^{\nu}(z)},$$

where ν is the unit measure (*normalized zero counting measure for* P_n) given by

$$\nu = \nu_{P_n} := \frac{1}{n}\sum_{k=1}^{n} \delta_{z_k}.$$

Notice that for any set K,

$$\nu(K) = \frac{1}{n} \times \{\text{number of zeros of } P_n \text{ in } K\}.$$

We can now rigorously formulate part (d) of the Fundamental Theorem:
The normalized zero counting measures for Fekete polynomials converge weak to* μ_E.

In applications, the following result is also useful:
Let $\{P_n\}$ *be any sequence of monic polynomials having all their zeros in* E *and such that* $\nu_{P_n} \xrightarrow{*} \mu_E$. *If* $\partial_\infty E$ *is regular (e.g., if it is connected), then the assertions* (b) *and* (c) *of the Fundamental Theorem hold for the* P_n*'s.*

Such sequences can be constructed by various "discretization" techniques. One of the simplest discretizations was employed by J.L. Walsh in his work on polynomial and rational approximation; see Remark (a) at the end of the next section.

2 Polynomial Approximation of Analytic Functions

Let f be a continuous function on a compact set E (symbolically, $f \in C(E)$) and let

$$e_n(f;E) = e_n(f) := \min_{p \in \mathcal{P}_n} \|f - p\|_E \tag{2.1}$$

be the error in best uniform approximation of f by polynomials of degree at most n. We denote by p_n^* the polynomial of best approximation: $\|f - p_n^*\|_E = e_n(f)$.

If $e_n(f) \to 0$ as $n \to \infty$, the series

$$p_1^* + \sum_{n=1}^{\infty} (p_{n+1}^* - p_n^*)$$

converges to f uniformly on E, so that the continuous function f must be analytic at every interior point of E. (The collection of all functions that are continuous on E and analytic in Int$E$ is denoted by $\mathcal{A}(E)$.) Furthermore, it follows from the maximum principle, that the above series automatically converges on every bounded component of $\mathbb{C} \setminus E$, so that its sum represents an analytic continuation of f to these components (e.g., if E is the unit circle $|z| = 1$, then the convergence holds in the unit disk $|z| \leq 1$). Such a continuation, however, may be impossible. Therefore, in order to ensure that $e_n(f) \to 0$ for *every function* f in $\mathcal{A}(E)$, it is necessary to assume that the only component of $\mathbb{C} \setminus E$ is the unbounded one; that is, $\mathbb{C} \setminus E$ is connected (so that E does not separate the plane).

A celebrated theorem of S.N. Mergelyan (cf. [Ga]) asserts that this assumption is also sufficient. Here we prove this result in a special case when E is connected and f is analytic in some neighborhood of E. The proof will also give the *rate of approximation.*

So let $\mathbb{C} \setminus E$ and E both be connected. Then the complement of E with respect to the extended complex plane is a simply-connected domain. Let Φ be the conformal map considered in Section 1 and recall that

$$\log|\Phi(z)| = \log \frac{1}{\operatorname{cap}(E)} - U^{\mu_E}(z) = g_{\mathbb{C} \setminus E}(z, \infty), \quad z \in \mathbb{C} \setminus E. \tag{2.2}$$

For any $R > 1$, let Γ_R denote the level curve $\{z : |\Phi(z)| = R\}$, see Fig. 2 (we call such a curve a *level curve with index* R).

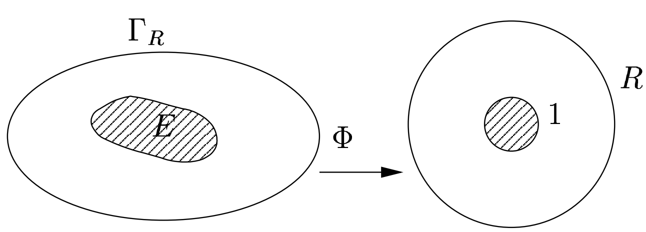

Figure 2: Level curve of Φ

Note that Γ_R is also a level curve for the potential:

$$U^{\mu_E}(z) = \log \frac{1}{R \operatorname{cap}(E)}, \quad z \in \Gamma_R. \tag{2.3}$$

Let F_{n+1} be the $(n+1)$-st Fekete polynomial for E and let P_n be the polynomial of degree $\leq n$ that interpolates f at the zeros of F_{n+1}. We are given that f is analytic in a neighborhood of E; hence there exists $R > 1$ such that f is analytic on and inside Γ_R. For any such R, the *Hermite interpolation formula* yields

$$f(z) - P_n(z) = \frac{1}{2\pi i} \int_{\Gamma_R} \frac{F_{n+1}(z)}{F_{n+1}(t)} \frac{f(t)dt}{t-z}, \quad z \text{ inside } \Gamma_R. \tag{2.4}$$

(The validity of the Hermite formula follows by first observing that the right-hand side vanishes at the zeros of $F_{n+1}(z)$, and then by replacing $f(z)$ by its Cauchy integral representation to deduce that the difference between f and the right-hand side is indeed a polynomial of degree at most n).

Formula (2.4) leads to a simple estimate:

$$e_n(f) \leq \|f - P_n\|_E \leq K \frac{\|F_{n+1}\|_E}{\min_{\Gamma_R} |F_{n+1}(t)|},$$

where K is some constant independent of n. Applying parts (b), (c) of the Fundamental Theorem we obtain, with the aid of (2.3), that

$$\limsup_{n\to\infty} e_n(f)^{1/n} \leq \frac{\operatorname{cap}(E)}{R\operatorname{cap}(E)} = \frac{1}{R} < 1. \tag{2.5}$$

We have proved that indeed $e_n(f) \to 0$ and that the convergence is geometrically fast. Since $R > 1$ was arbitrary (but such that f is analytic on and inside Γ_R), we have actually proved that (2.5) holds with R replaced by $R(f)$, where

$$R(f) := \sup\{R : \ f \text{ admits analytic continuation to the interior of } \Gamma_R\}.$$

Can we improve on this? The answer is — no! In order to show this, we need the following very useful result.

Bernstein-Walsh Lemma. *Assume that both E and $\mathbb{C} \setminus E$ are connected. If a polynomial p of degree n satisfies $|p(z)| \leq M$ for $z \in E$, then $|p(z)| \leq Mr^n$ for $z \in \Gamma_r$, $r > 1$.*

The proof uses essentially the same argument as in Example 1.5. The function $p(z)/\Phi^n(z)$ is analytic outside E, even at ∞. Since $|\Phi| = 1$ on ∂E, we know that $|p(z)/\Phi^n(z)| \leq M$ for $z \in \partial E$. Hence the maximum principle yields

$$\left|\frac{p(z)}{\Phi^n(z)}\right| \leq M, \quad z \in \mathbb{C} \setminus E$$

and the result follows by the definition of Γ_r.

Assume now that (2.5) holds for some $R > R(f)$ and let $R(f) < \rho < R$. Then for some constant $c > 1$,

$$e_n(f) \leq \frac{c}{\rho^n}, \quad n \geq 1.$$

Since, from the triangle inequality,

$$\|p^*_{n+1} - p^*_n\|_E = \|p^*_{n+1} - f + f - p^*_n\|_E \leq e_{n+1}(f) + e_n(f) \leq 2c\rho^{-n},$$

we obtain from the Bernstein-Walsh Lemma that for any $r > 1$,

$$\|p_{n+1}^* - p_n^*\|_{\Gamma_R} \leq 2c \left(\frac{r}{\rho}\right)^n, \quad n \geq 1.$$

If we choose $R(f) < r < \rho$, we obtain that the series $p_1^* + \sum_{n=1}^{\infty}(p_{n+1}^* - p_n^*)$ converges uniformly inside Γ_r. Hence it gives an analytic continuation of f to the interior of Γ_r, which contradicts the definition of $R(f)$.

Let us summarize what we have proved.

Theorem 2.1. (Walsh [W, Ch. VII]). *Let the compact set E be connected and have a connected complement. Then for any $f \in \mathcal{A}(E)$,*

$$\limsup_{n\to\infty} e_n(f)^{1/n} = \frac{1}{R(f)}.$$

Remarks.

(a) The proof of this theorem shows that on interpolating f at Fekete points we obtain a sequence of polynomials that gives, asymptotically, the best possible rate of approximation. It may be not easy, however, to find these points and it is desirable to have other methods at hand. Assume, for example, that E is bounded by a smooth Jordan curve Γ. With Φ as above, let the points $w_1, \ldots, w_n$ be equally-spaced on $|w| = 1$ and let $z_i = \Phi^{-1}(w_i) \subset \partial E$ be their preimages. These points (called the *Fejér points*) divide Γ into n subarcs, each having μ_E-measure $1/n$ (the latter can be derived from the formula (1.4)). Therefore, the Fejér points have asymptotic distribution μ_E. Let P_n be the monic polynomial with zeros at $z_1, \ldots, z_n$. According to the statement in the end of Section 1, the sequence $\{P_n\}$ enjoys the same properties (b), (c) as $\{F_n\}$ does, and the proof of Theorem 2.1 shows that

$$\limsup_{n\to\infty} \|f - P_n\|_E^{1/n} = \frac{1}{R(f)}.$$

(b) $R(f)$ is the first value of R for which the level curve Γ_R contains a singularity of f. It may well be possible that f is analytic at some other points of $\Gamma_{R(f)}$, but the geometric rate of best polynomial approximation "does not feel this" — whether every point of $\Gamma_{R(f)}$ is a singularity or merely one point is a singularity, the rate of approximation remains the same as if f was analytic only inside of $\Gamma_{R(f)}$! To take advantage of any extra analyticity, different approximation tools are needed; e.g., rational functions. We demonstrate this in Section 6.

(c) It follows from (2.2) that

$$\Gamma_R = \{z \in \mathbb{C} \setminus E : \ g_{\mathbb{C}\setminus E}(z, \infty) = \log R\}. \tag{2.6}$$

Assume now that $\mathbb{C} \setminus E$ is connected but is E *not.* Then one can still define the Green function $g_{\mathbb{C}\setminus E}$ via the formula

$$g_{\mathbb{C}\setminus E} = \log \frac{1}{\text{cap}(E)} - U^{\mu_E},$$

from which it follows that properties (i)–(iii) described in Section 1 will hold, provided E is regular. Then, with Γ_R defined by (2.6), it is easy to modify the above proof to show that Walsh's Theorem 2.1 holds in this case as well.

Example 2.1. Let $E = [-1, -\alpha] \cup [\alpha, 1]$, $0 < \alpha < 1$, and let $f = 0$ on $[-1, -\alpha]$ and $f = 1$ on $[\alpha, 1]$. Some level curves Γ_R of $g_{\mathbb{C}\setminus E}$ are depicted on Fig. 3. For R small, Γ_R consists of two pieces, while for R large, Γ_R is a single curve. There is a "critical value" $R_0 = g_{\mathbb{C}\setminus E}(0, \infty)$ for which Γ_{R_0} represents a self-intersecting lemniscate-like curve (the bold curve in Fig. 3). Clearly, f can be extended as an analytic function to the interior of Γ_{R_0} (define $f = 0$ inside the left lobe and $f = 1$ inside the right lobe). For $R > R_0$, the interior of Γ_R is a (connected) domain; hence there is no function analytic inside of Γ_R that is equal to 0 on $[-1, -\alpha]$ and to 1 on $[\alpha, 1]$. Therefore

$$R(f) = R_0 = \exp\left\{g_{\mathbb{C}\setminus E}(0, \infty)\right\},$$

and by the (extension of) Walsh's theorem:

$$\limsup_{n\to\infty} e_n(f)^{1/n} = \exp\left\{-g_{\mathbb{C}\setminus E}(0, \infty)\right\}.$$

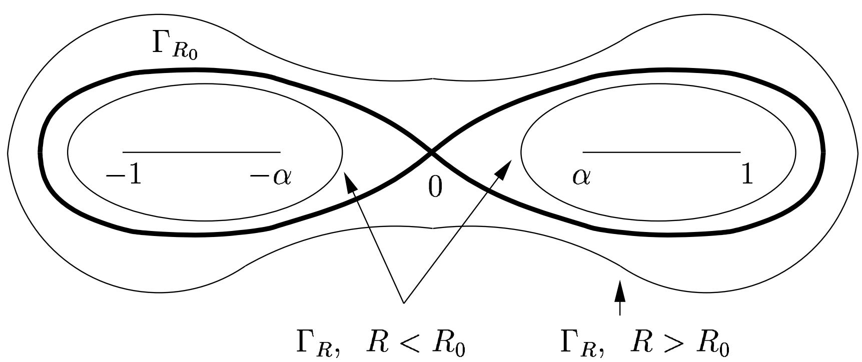

Figure 3: Level curves of $g_{\mathbb{C}\setminus E}$

3 Approximation with Varying Weights — a background

We start with two problems that have triggered much of the recent potential theoretic research on polynomial and rational approximation and on orthogonal polynomials.

Let $0 < \theta < 1$. A polynomial $P(x) = \sum_{k=0}^{n} a_k x^k$ is said to be *incomplete of type* θ ($P \in I_\theta$), if $a_k = 0$ for $k < n\theta$. The study of such polynomials was introduced in [L] by Lorentz who proved the following.

Theorem 3.1 (G.G. Lorentz, 1976). *If* $P_n \in I_\theta$, $\deg P_n \to \infty$ *as* $n \to \infty$ *and*

$$\|P_n\|_{[0,1]} = \max_{[0,1]} |P_n(x)| \le M, \quad \textit{all } n,$$

then

$$P_n(x) \to 0 \quad \textit{for} \quad x \in [0, \theta^2).$$

Concerning the sharpness of this result, we state

Problem 1: Is $[0, \theta^2)$ the largest interval where the convergence to zero is guaranteed?

Another problem, dealing with the asymptotic behavior of recurrence coefficients for orthogonal polynomials, was posed by G. Freud, also in 1976 (cf. [F]). Let

$$w_\alpha(x) := e^{-|x|^\alpha}, \quad \alpha > 0 \tag{3.1}$$

be a weight on the real line and let $\{p_n\}$ be *orthonormal polynomials* with respect to this weight:

$$\int_{-\infty}^{\infty} p_m(x) p_n(x) e^{-|x|^\alpha} dx = \delta_{mn}$$

(for $\alpha = 2$ these are the classical Hermite polynomials). Since the weight is even, the polynomials p_n satisfy the following 3-term recurrence relation

$$x p_n(x) = a_{n+1} p_{n+1}(x) + a_n p_{n-1}(x),$$

where $\{a_n\}$ is some sequence of real numbers (cf. [Sz]). For the weights (3.1), G. Freud conjectured that

$$\lim_{n\to\infty} n^{1/\alpha} a_n \quad \text{exists.}$$

Problem 2: Resolve this conjecture.

Seemingly very different, these two problems are connected by a common thread — both can be formulated in terms of *weighted polynomials* of the form

$$w^n(x) P_n(x), \quad \deg P_n \le n.$$

For the Lorentz Problem, one simply observes that any $P \in I_\theta$ of degree $n/(1-\theta)$ (which for simplicity we assume to be an integer) can be written in the form

$$P(x) = x^{n\theta/(1-\theta)} P_n(x),$$

where P_n is a polynomial of degree $\le n$. Therefore, this problem deals with sequences of weighted polynomials that satisfy

$$\|w^n P_n\|_{[0,1]} \le M, \quad w(x) = x^{\theta/(1-\theta)}, \quad \deg P_n \le n.$$

Regarding Problem 2, we observe that from the normalization

$$\int_{-\infty}^{\infty} p_n^2(x) e^{-|x|^\alpha} dx = 1$$

the substitution

$$x \to n^{1/\alpha} x, \quad p_n(x) \to P_n(x) := n^{1/2\alpha} p_n(n^{1/\alpha} x)$$

leads again to a sequence of weighted polynomials for which

$$\|w^n P_n\|_{L_2(\mathbb{R})} = 1, \quad w(x) = e^{-|x|^\alpha/2}, \quad \deg P_n \le n,$$

where $\|\cdot\|_{L_2(\mathbb{R})}$ is defined by

$$\|f\|_{L_2(\mathbb{R})} := \left(\int_{\mathbb{R}} |f(x)|^2 dx\right)^{1/2}.$$

In this framework, the following question is of fundamental importance:

Problem 3(Generalized Weierstrass Approximation Problem)
For $E \subset \mathbb{R}$ closed, $w : E \to [0, \infty)$, characterize those functions f continuous on E that are uniform limits on E of some sequence of weighted polynomials $\{w^n P_n\}$, $\deg P_n \le n$.

It turns out that Problems 1, 2, and 3 can be resolved with the aid of potential theory, when an *external field* is introduced.

4 Logarithmic Potentials with External Fields

Let E be a closed (not necessarily compact) subset of $\mathbb{C}$ and let $w(z)$ be a nonnegative weight on E. We define a new "distance function" on E, replacing $|z-t|$ by $|z-t|w(z)w(t)$. This gives rise to weighted versions of logarithmic capacity, transfinite diameter and Chebyshev constant.

Weighted capacity: $\text{cap}(w, E)$.

As before, let $\mathcal{M}(E)$ denote the collection of all unit measures supported on E. We set

$$Q := \log \frac{1}{w}$$

and call it the *external field*. Consider the modified energy integral for $\mu \in \mathcal{M}(E)$:

$$\begin{aligned} I_w(\mu) &:= \int\int \log \frac{1}{|z-t|w(z)w(t)} d\mu(z) d\mu(t) \\ &= \int\int \log \frac{1}{|z-t|} d\mu(z) d\mu(t) + 2 \int Q(z) d\mu(z) \end{aligned} \tag{4.1}$$

and let

$$V_w := \inf_{\mu \in \mathcal{M}(E)} I_w(\mu).$$

The *weighted capacity* is defined by

$$\text{cap}(w, E) := e^{-V_w}.$$

In the sequel, we assume that w satisfies the following conditions:

(i) $w > 0$ on a subset of positive logarithmic capacity;

(ii) w is continuous (or, more generally, upper semi-continuous);

(iii) If E is unbounded, then $|z|w(z) \to 0$ as $|z| \to \infty$, $z \in E$.

Under these restrictions on w, there exists a unique measure $\mu_w \in \mathcal{M}(E)$, called the *weighted equilibrium measure*, such that

$$I(\mu_w) = V_w.$$

The above integral (4.1) can be interpreted as the total energy of the unit charge μ, in the presence of the external field Q (in this electrostatics interpretation, the field is actually $2Q$). Since this field has a strong repelling effect near points where $w = 0$ (i.e. $Q = \infty$), assumption (iii) physically means that, for the equilibrium distribution, no charge occurs near ∞. In other words, the support $S(\mu_w)$ of μ_w is necessarily *compact*. However, unlike the unweighted case, the support need not lie entirely on $\partial_\infty E$ and, in fact, it can be quite an arbitrary closed subset of E. Determining this set is one of the most important aspects of weighted potential theory.

Weighted transfinite diameter: $\tau(w, E)$.

Let

$$\delta_n(w) := \max_{z_1,\dots,z_n \in E} \left(\prod_{1 \le i < j \le n} |z_i - z_j| w(z_i) w(z_j) \right)^{2/n(n-1)}.$$

Points $z_1^{(n)}, \dots, z_n^{(n)}$ at which the maximum is attained are called *weighted Fekete points*. The corresponding *Fekete polynomial* is the monic polynomial with all its zeros at these points.

As in the unweighted case, the sequence $\delta_n(w)$ is decreasing, so one can define

$$\tau(w, E) := \lim_{n\to\infty} \delta_n(w),$$

which we call the *weighted transfinite diameter* of E.

Weighted Chebyshev constant: $\mathrm{cheb}(w, E)$.

Let

$$t_n(w) := \min_{p \in \mathcal{P}_{n-1}} \|w^n(z)(z^n - p(z))\|_E.$$

Then the *weighted Chebyshev constant* is defined by

$$\mathrm{cheb}(w, E) := \lim_{n\to\infty} t_n(w)^{1/n}.$$

The following theorem (due to Mhaskar and Saff) generalizes the classical results of Section 1.

Generalized Fundamental Theorem. *Let E be a closed set of positive capacity. Assume that w satisfies the conditions* (i)–(iii) *and let $Q = \log(1/w)$. Then*

$$\mathrm{cap}(w, E) = \tau(w, E) = \mathrm{cheb}(w, E) \exp\left\{ - \int Q \, d\mu_w \right\}.$$

Moreover, weighted Fekete points have asymptotic distribution μ_w as $n \to \infty$, and weighted Fekete polynomials are asymptotically optimal for the weighted Chebyshev problem.

How can one find μ_w?

In most applications, the weight w is continuous and the set E is regular. Recall that the latter means that the classical (unweighted) equilibrium potential for E is equal to V_E

everywhere on E, not just quasi-everywhere. Under these assumptions, the equilibrium measure $\mu = \mu_w$ is characterized by the conditions that $\mu \in \mathcal{M}(E)$, $I(\mu) < \infty$ and, for some constant c_w, the following *variational conditions* hold:

$$\begin{cases} U^\mu + Q = c_w & \text{on } S(\mu) \\ U^\mu + Q \geq c_w & \text{on } E. \end{cases} \tag{4.2}$$

On integrating (against $\mu = \mu_w$) the first condition, we obtain that the constant is given by

$$c_w = I_w(\mu_w) + \int Q d\mu_w = V_w - \int Q d\mu_w.$$

When trying to find μ_w, an essential step (and a nontrivial problem in its own right!) is to determine the support $S(\mu_w)$. There are several methods by which $S(\mu_w)$ can be numerically approximated, but they are complicated from the computational point of view. Therefore, knowing properties of the support can be useful and we list some of them.

Properties of the support $S(\mu_w)$

(a) The sup-norm of weighted polynomials "lives" on $S(\mu_w)$. That is, for any n and for any polynomial P_n of degree at most n, there holds

$$\|w^n P_n\|_E = \|w^n P_n\|_{S(\mu_w)}.$$

(b) Let K be a compact subset of E of positive capacity, and define

$$F(K) := \log \operatorname{cap}(K) - \int_K Q d\mu_K,$$

where μ_K is the classical (unweighted) equilibrium measure for K. This so-called **F-functional** of Mhaskar and Saff is often a helpful tool in finding $S(\mu_w)$. Since $\operatorname{cap}(K)$ and μ_K remain the same if we replace K by $\partial_\infty K$, we obtain that $F(K) = F(\partial_\infty K)$. It turns out that the outer boundary of $S(\mu_w)$ maximizes the F-functional:

$$\max_K F(K) = F(\partial_\infty S(\mu_K)).$$

This result is especially useful when E is a real interval and Q is *convex*. It is then easy to derive from (4.2) that $S(\mu_w)$ is an *interval*. Thus, to find the support, one merely needs to maximize $F(K)$ only over intervals $K \subset E$, which amounts to a standard calculus problem for the determination of the endpoints of $S(\mu_w)$.

Example 4.1. Incomplete polynomials

Here $E = [0, 1]$ and $Q(x) = \log(1/w(x)) = -\dfrac{\theta}{1-\theta} \log x$ is convex. Maximizing the F-functional one gets $S(\mu_w) = [\theta^2, 1]$. (For details, see [SaTo, Sec. IV.1].)

Example 4.2. Freud Weights

Here $E = \mathbb{R}$ and $w(x) = \exp(-|x|^\alpha)$. Hence $Q(x) = |x|^\alpha$ is convex provided that $\alpha > 1$, and we obtain $S_w = [-a_\alpha, a_\alpha]$, where a_α can be given explicitly in terms of the Gamma function. (Actually, this result also holds for all $\alpha > 0$; see [SaTo, Sec. IV.1].) For example, when $\alpha = 2$, we get $S_w = [-1, 1]$.

5 Generalized Weierstrass Approximation Problem

We address here Problems 1-3 of Section 3. Let E be a regular closed subset of $\mathbb{R}$ and $w(x)$ be continuous on E. Then we have the following weighted analogue of the Bernstein-Walsh lemma:

$$|w^n(x)P_n(x)| \leq \|w^n P_n\|_{S(\mu_w)} \exp\{-n(U^{\mu_n}(x) + Q(x) - c_w)\}, \quad x \in E \setminus S(\mu_w).$$

With the aid of (4.2) and a variant of the Stone-Weierstrass theorem (cf. [SaTo]), one can show that if a sequence $\{w^n(x)P_n(x)\}$, $\deg P_n \leq n$, converges uniformly on E, then it tends to 0 for every $x \in E \setminus S(\mu_w)$.

Thus, if some $f \in C(E)$ is a uniform limit on E of such a sequence, it must vanish on $E \setminus S(\mu_w)$. The converse is not true, in general, but it is true in many important cases.

Incomplete polynomials.

For the weight $w = x^{\theta/(1-\theta)}$, we have mentioned that $S(\mu_w) = [\theta^2, 1]$. It was proved by Saff and Varga and, independently, by M. v. Golitschek (cf. [SaVa], [Go]), that any $f \in C[0,1]$ that vanishes on $[0, \theta^2]$ is a uniform limit on $[0,1]$ of incomplete polynomials of type θ.

In particular, choosing $f(x) = 0$ for $x \in [0, \theta^2]$, and $f(x) = x - \theta^2$ for $x > \theta^2$, the sequence of type θ polynomials converging uniformly to f on $[0,1]$ is uniformly bounded on $[0,1]$, but does not tend to zero for $x > \theta^2$. Thus the answer to Problem 1 of Section 3 is — yes, Lorentz's Theorem 3.1 is indeed sharp!

Freud Conjecture.

For $\alpha > 1$, let $[-a_\alpha, a_\alpha]$ be the support of the equilibrium measure for the weight $e^{-|x|^\alpha}$. Lubinsky and Saff showed in [LuSa], that any $f \in C(\mathbb{R})$ that vanishes outside this support is a uniform limit of a sequence of the form $\exp\{-n|x|^\alpha\}P_n(x)$, $n \geq 1$. This result was the major ingredient in the argument given by Mhaskar, Lubinsky, and Saff [MLS], that resolved the Freud Conjecture in the affirmative.

Concerning more general weights, Saff made the following conjecture:

Conjecture. *Let E be a real interval, and assume that $Q = \log(1/w)$ is convex on E. Then any function $f \in C(E)$ that vanishes on $E \setminus S(\mu_E)$ is the uniform limit on E of some sequence of weighted polynomials $\{w^n P_n\}$, $\deg P_n \leq n$.*

This conjecture was proved by V. Totik [To] utilizing a careful analysis of the smoothness of the density of the weighted equilibrium measure. We remark that for more general Q and E, the conjecture is false, and additional requirements on f are needed.

6 Rational Approximation

For a rational function $R(z) = P_1(z)/P_2(z)$, where P_1 and P_2 are monic polynomials of degree n, one can write

$$-\frac{1}{n}\log|R(z)| = U^{\nu_1}(z) - U^{\nu_2}(z),$$

where ν_1, ν_2 are the normalized zero counting measures for P_1, P_2, respectively. The right-hand side represents the logarithmic potential of the *signed measure* $\mu = \nu_1 - \nu_2$:

$$U^{\nu_1}(z) - U^{\nu_2}(z) = U^{\mu}(z) = \int \log \frac{1}{|z-t|} d\mu(t).$$

The theory of such potentials can be developed along the same lines as in Section 1. We present below only the very basic notions of this theory that are needed to formulate the approximation results. A more in-depth treatment can be found in the works of Bagby [B], Gonchar [Gon], as well as [SaTo].

The analogy with electrostatics problems suggests considering the following energy problem. Let $E_1, E_2 \subset \mathbb{C}$ be two closed sets that are a positive distance apart. The pair (E_1, E_2) is called a *condenser* and the sets E_1, E_2 are called the *plates.* Let μ_1 and μ_2 be positive unit measures supported on E_1 and E_2, respectively. Consider the energy integral of the signed measure $\mu = \mu_1 - \mu_2$:

$$I(\mu) = \int\int \log \frac{1}{|z-t|} d\mu(z) d\mu(t).$$

Since $\mu(\mathbb{C}) = 0$, the integral is well-defined, even if one of the sets is unbounded. While not obvious, it turns out that such $I(\mu)$ is always *positive.* We assume that E_1 and E_2 have positive logarithmic capacity. Then the minimal energy (over all signed measures of the above form)

$$V(E_1, E_2) := \inf_{\mu} I(\mu)$$

is finite and positive. We then define the *condenser capacity* $\mathrm{cap}(E_1, E_2)$ by

$$\mathrm{cap}(E_1, E_2) := 1/V(E_1, E_2).$$

One can show, as with the Frostman theorem, that there exists a unique signed measure $\mu^* = \mu_1^* - \mu_2^*$ (the *equilibrium measure* for the condenser) for which $I(\mu^*) = V(E_1, E_2)$. Furthermore, the corresponding potential (called the *condenser potential*) is constant on each plate:

$$U^{\mu^*} = c_1 \text{ on } E_1, \quad U^{\mu^*} = -c_2 \text{ on } E_2, \tag{6.1}$$

(we assume throughout that E_1, E_2 are regular — otherwise the above equalities hold only quasi-everywhere). On integrating against μ^*, we deduce from (6.1) that

$$c_1 + c_2 = V(E_1, E_2) = 1/\mathrm{cap}(E_1, E_2). \tag{6.2}$$

We mention that (similar to the case of the conductor potential) the relations of type (6.1) *characterize* μ^*. Moreover, one can deduce from (6.1) that the measure μ_i^* is supported on the boundary (not necessarily the outer one) of E_i, $i = 1, 2$. Therefore, on replacing each E_i by its boundary, we do not change the condenser capacity or the condenser potential.

Example 6.1. Let E_1, E_2 be, respectively, the circles $|z| = r_1$, $|z| = r_2$, $r_1 < r_2$.
These sets are invariant under rotations. Being unique, the measure μ^* is therefore also invariant under rotations and we obtain that

$$\mu_1^* = \frac{1}{2\pi r_1} ds, \quad d\mu_2^* = \frac{1}{2\pi r_2} ds,$$

where ds denotes the arclength over the respective circles E_1, E_2. Applying the result of Example 1.1, we find that

$$U^{\mu^*}(z) = \begin{cases} 0, & |z| > r_2 \\ \log(r_2/|z|), & r_1 \le |z| \le r_2 \\ \log(r_2/r_1), & |z| < r_1. \end{cases}$$

Therefore (recall (6.2))

$$\operatorname{cap}(E_1, E_2) = 1/\log \frac{r_2}{r_1}. \tag{6.3}$$

Assume now that each plate of a condenser is a single Jordan arc or curve (without self-intersections), and let G be the doubly-connected domain that is bounded by E_1 and E_2, see Fig. 4. We call such a G a *ring domain.*

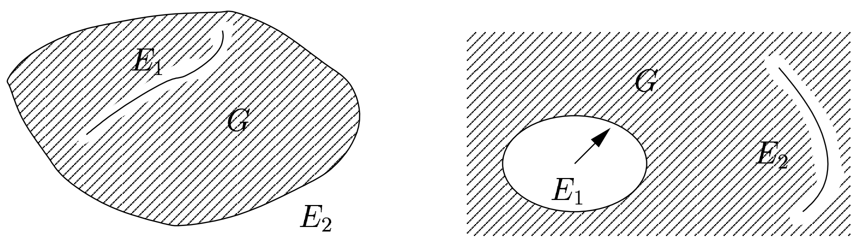

Figure 4: Ring domains

For ring domains one can give an alternative definition of condenser capacity. Let

$$u(z) := \int \log(z-t)d\mu^*(t) + c_1.$$

This function is locally analytic but not single-valued in G (notice that there is no modulus sign in the integral). Moreover, if we fix t and let z move along a simple closed counterclockwise oriented curve in G that encircles E_1, say, then the imaginary part of $\log(z-t)$ increases by 2π, for $t \in E_1$, while for $t \in E_2$ it returns to the original value. Since μ_1^* and μ_2^* are unit measures, it follows that the function $\phi : z \to w = \exp(u(z))$ is analytic and single-valued. Moreover, it can be shown to be one-to-one in G. By its definition, ϕ satisfies

$$\log|\phi| = -U^{\mu^*} + c_1 = 0 \text{ on } E_1; \quad \log|\phi| = -U^{\mu^*} + c_1 = c_1 + c_2 \text{ on } E_2.$$

Therefore ϕ maps G conformally onto the annulus $1 < |w| < e^{c_1+c_2}$.

It is known from the theory of conformal mapping, that, for a ring domain G, there exists unique $R > 1$, called the *modulus of* G (we denote it by $\operatorname{mod}(G)$), such that G can be mapped conformally onto the annulus $1 < |w| < R$. We have thus shown that

$$\operatorname{cap}(E_1, E_2) = 1/\log(\operatorname{mod}(G)). \tag{6.4}$$

We remark that if $G_1 \supset G_2$ are two ring domains, then $\operatorname{mod}(G_1) \geq \operatorname{mod}(G_2)$.

Example 6.2. Let E_1, E_2 be as above, and assume that E_2 is the R-th level curve for E_1. That is, $|\Phi(z)| = R$ for $z \in E_2$, where Φ maps conformally the unbounded component of $\mathbb{C} \setminus E_1$ onto $|w| > 1$. In particular, Φ maps the corresponding ring domain G onto the annulus $1 < |w| < R$, and we conclude that $\text{mod}(G) = R$ (so that $\text{cap}(E_1, E_2) = 1/\log R$). Applying this to the configuration of Example 6.1, we see that $\Phi(z) = z/r_1$, so that $R = r_2/r_1$, and we obtain again (6.3).

We now turn to rational approximation. Let $E \subset \mathbb{C}$ be compact. We denote by $\mathcal{R}_n$ the collection of all rational functions of the form $R = P/Q$, where P, Q are polynomials of degree at most n, and Q has no zeros in E. For $f \in \mathcal{A}(E)$, let

$$r_n(f;E) = r_n(f) := \inf_{r \in \mathcal{R}_n} \|f - r\|_E$$

be the error in best approximation of f by rational functions from $\mathcal{R}_n$. Clearly, since polynomials are rational functions, we have (cf. (2.1)) $r_n(f) \le e_n(f)$. A basic theorem regarding the rate of rational approximation was proved by Walsh [W, Ch.IX]. Following is a special case of this theorem.

Theorem 6.3. (Walsh) *Let E be a single Jordan arc or curve and let f be analytic on a simply connected domain $D \supset E$. Then*

$$\limsup_{n\to\infty} r_n(f)^{1/n} \le \exp\{-1/\text{cap}(E, \partial D)\}. \tag{6.5}$$

The proof of (6.5) follows the same ideas as the proof of inequality (2.5). Let Γ be a contour in $D \setminus E$ that is arbitrarily close to ∂D. Let $\mu^* = \mu_1^* - \mu_2^*$ be the equilibrium measure for the condenser (E, Γ). For any n, let $\alpha_1^{(n)}, \ldots, \alpha_n^{(n)}$ be equally spaced on E (with respect to μ_1^*) and let $\beta_1^{(n)}, \ldots, \beta_n^{(n)}$ be equally spaced on Γ (with respect to μ_2^*). Then one can show that the rational functions $r_n(z)$ with zeros at the $\alpha_i^{(n)}$'s and poles at the $\beta_i^{(n)}$'s satisfy

$$\left(\frac{\max_E |r_n|}{\min_\Gamma |r_n|} \right)^{1/n} \to e^{-1/\text{cap}(E,\Gamma)}. \tag{6.6}$$

Let $R_n = p_{n-1}/q_n$ be the rational function with poles at the $\beta_i^{(n)}$'s that interpolates f at the points $\alpha_i^{(n)}$'s. Then the Hermite formula (cf. (2.4)) takes the following form:

$$f(z) - R_n(z) = \frac{1}{2\pi i} \int_\Gamma \frac{r_n(z)}{r_n(t)} \frac{f(t)}{t - z} dt, \quad z \text{ inside } \Gamma,$$

and it follows from (6.6) that

$$\limsup_{n\to\infty} r_n(f)^{1/n} \le \limsup_{n\to\infty} \|f - R_n\|_E^{1/n} \le e^{-1/\text{cap}(E,\Gamma)}.$$

Letting Γ approach ∂D, we get the result.

Remarks.
(a) Unlike in the polynomial approximation, no rate of convergence of $r_n(f)$ to 0 can ensure that a function $f \in C(E)$ is analytic somewhere beyond E.
(b) One can construct a function for which equality holds in (6.5), so that this bound is sharp.

Such a function necessarily has a singularity at every point of ∂D; otherwise f would be analytic in a larger domain, so that the corresponding condenser capacity will become smaller. In view of Theorem 6.3, this would violate the assumed equality in (6.5).

Although sharp, the bound (6.5) is unsatisfactory, in the following sense. Assume, for example, that E is connected and has a connected complement, and let Γ_R, $R > 1$, be a level curve for E. Let f be a function that is analytic in the domain D bounded by Γ_R and such that the equality holds in (6.5). According to Example 6.2 we then obtain that

$$\limsup_{n\to\infty} r_n(f)^{1/n} = \frac{1}{R}. \tag{6.7}$$

By Remark (b) above, such f must have singularities on Γ_R. Hence (recall Remark (b) following Theorem 2.1) the relation (6.7) holds with $r_n(f)$ replaced by $e_n(f)$. But the family $\mathcal{R}_n$ contains $\mathcal{P}_n$ and it is much more rich than $\mathcal{P}_n$ — it depends on $2n+1$ parameters while $\mathcal{P}_n$ depends only on $n+1$ parameters. One would expect, therefore, that at least for a subsequence of n's, $r_n(f)$ behaves asymptotically like $e_{2n}(f)$. This was a motivation for the following conjecture.

Conjecture. (A.A. Gonchar) *Let E be a compact set and f be analytic in an open set D containing E. Then*

$$\liminf_{n\to\infty} r_n(f;E)^{1/n} \le \exp\{-2/\mathrm{cap}(E,\partial D)\}. \tag{6.8}$$

This conjecture was proved by O. Parfenov [Pa] for the case when E is a continuum with connected complement and in the general case by V. Prokhorov [P]; they used a very different method — the so-called "AAK Theory" (cf. [Y]). However this method is not constructive, and it remains a challenging problem to find such a method. Yet, potential theory can be used to obtain bounds like (6.8) in the stronger form

$$\lim_{n\to\infty} r_n(f;E)^{1/n} = \exp\{-2/\mathrm{cap}(E,\partial D)\}$$

for some important *subclasses* of analytic functions, such as Markov functions (cf. [Gon]) and functions with a finite number of algebraic branch-points (cf. [St]).

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