Jackson-Type Theorems on Some Transcendental Curves in $\mathbb{R}^d$

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Abstract

"Efficient curves" in the sense of the best rate of multivariate polynomial approximation to contractive functions on these curves, were first introduced by D. J. Newman and L. Raymon in 1969. They proved that algebraic curves are efficient, but claim that the exponential curve $\gamma := \{(t, e^t) : 0 \le t \le 1\}$ is not. We prove *to the contrary* that this exponential curve and its generalization to higher dimensions are indeed efficient. We also investigate helical curves in $\mathbb{R}^d$ and show that they too are efficient. Transcendental curves of the form $\{(t, t^\lambda) : \delta \le t \le 1\}$ are shown to be efficient for $\delta > 0$, contradicting another claim of Newman and Raymon.

Let γ be a continuous curve of finite length in $\mathbb{R}^d$, $d \ge 2$, and denote by C_γ the space of continuous real-valued functions f on γ endowed with the usual supremum norm $\|f\|_\gamma$. Consider the set $K_\gamma \subset C_\gamma$ of all contractions on γ given by

$$K_\gamma := \{\, f \in C_\gamma \,:\, |f(x) - f(y)| \le |x - y|\,,\; x,\, y \in \gamma \}\,.$$

In this paper we shall study the rate of approximation of functions in K_γ by elements of P_n^d, the space of real algebraic polynomials of d variables and degree $\le n$ in each variable. Denote by $P_n^d\big|_\gamma$ the restriction of P_n^d to

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γ, and $r_n(\gamma) := \dim P_n^d\big|_\gamma$. Then our goal is to estimate the error in best approximation of contractions given by

$$E_n(\gamma) := \sup_{f\in K_\gamma} \inf_{p\in P_n^d} \|f-p\|_\gamma \,.$$

According to a general result of Lorentz [6] the optimal rate of approximation of contractions that can possibly be attained is $1/r_n(\gamma)$, i.e., $E_n(\gamma) \geq c/r_n(\gamma)$ for some $c>0$ independent of n. (Lorentz [6] gives the lower bound for $E_n(\gamma)$ in terms of the so called "massivity" of γ, but in case of a continuous curve γ of finite length his estimate is equivalent to the one given above.) Therefore we shall say that the curve γ is "efficient" if

$$E_n(\gamma) = O(1/r_n(\gamma)) \quad \text{as} \quad n\to\infty \,.$$

Thus efficiency of a curve means that optimal rate of polynomial approximation is achieved on the curve.

Efficient curves were first studied by Newman and Raymon [7], [8]. In particular, they show in [7] that algebraic curves in $\mathbb{R}^2$ are efficient. Note that $r_n(\gamma) \asymp n$ when $d=2$ and γ is algebraic. (In what follows we write $a_n \asymp b_n$ if there exist positive constants $c_1,\ c_2$ so that $c_2 b_n \leq a_n \leq c_1 b_n$ for all n large.) In [8] an example of a non-algebraic efficient curve in $\mathbb{R}^2$ is given. However, this curve is "almost" algebraic in the sense that it is given by a parametric representation $\left\{\left(t, \sum_{k=0}^\infty a_k t^k\right) \,:\, 0\leq t\leq 1\right\}$ with $a_k \to 0$ extremely rapidly.

In addition, Newman and Raymon [7] studied the exponential curve $\gamma^* := \{(t,e^t) : 0\leq t\leq 1\}$ in $\mathbb{R}^2$. For the corresponding quantities in the L_2 setting, it is claimed in [7] and [10] that $E_n(\gamma^*)_{L_2} \asymp n^{-3/2}$. Note that $r_n(\gamma^*) = (n+1)^2$, i.e., this estimate seems to indicate that the exponential curve is inefficient. However, the proof of the lower bound for $E_n(\gamma^*)_{L_2}$ given in [7] is not correct (see Remark 2 at the end of this paper). We shall show in this paper that any general exponential curve in $\mathbb{R}^d$ is, in fact, efficient. Newman and Raymon [7] also make a similar mistake while considering the curve $\{(t,t^\alpha) : \delta\leq t\leq 1\}$ in $\mathbb{R}^2$ ($\delta>0$, α irrational), claiming that it, too, is inefficient. It turns out that curves of this form are efficient on $[\delta,1]$, $\delta>0$, and inefficient on $[0,1]$.

Now we introduce the basic curves we shall investigate in this paper. Let $\boldsymbol{\lambda} = (\lambda_1,\ldots,\lambda_{d-1}) \in \mathbb{R}^{d-1}$, and denote by $S_{\boldsymbol{\lambda}}$ the maximal number of algebraically independent λ_j's, $1\leq j\leq d-1$. (Recall that algebraic independence means linear independence over the rationals.) Consider the general exponential curve in $\mathbb{R}^d$ given by

$$e_{\boldsymbol{\lambda}} := \left\{ \left(t, e^{\lambda_1 t}, \ldots, e^{\lambda_{d-1} t}\right) \,:\, 0 \le t \le 1 \right\} . \tag{1}$$

Similarly, a general trigonometric curve in $\mathbb{R}^{2d-1}$ is given by

$$h_{\boldsymbol{\lambda}} := \{(t,\, \sin \lambda_1 t,\, \cos \lambda_1 t,\, \ldots,\, \sin \lambda_{d-1} t,\, \cos \lambda_{d-1} t) : 0 \le t \le 1\} . \tag{2}$$

(In $\mathbb{R}^3$ this curve is usually called a "helix".)

We shall prove:

Theorem 1. *For any $\boldsymbol{\lambda} \in \mathbb{R}^{d-1}$ the curves $e_{\boldsymbol{\lambda}}$ and $h_{\boldsymbol{\lambda}}$ are efficient. Moreover*

$$r_n(e_{\boldsymbol{\lambda}}) \asymp n^{S_{\boldsymbol{\lambda}}+1}, \quad r_n(h_{\boldsymbol{\lambda}}) \asymp n^{S_{\boldsymbol{\lambda}}+1},$$

and

$$E_n(e_{\boldsymbol{\lambda}}) \asymp n^{-(S_{\boldsymbol{\lambda}}+1)}, \quad E_n(h_{\boldsymbol{\lambda}}) \asymp n^{-(S_{\boldsymbol{\lambda}}+1)}. \tag{3}$$

In particular, for $\gamma^* = \{(t, e^t) : 0 \le t \le 1\}$, we have $E_n(\gamma^*) \asymp n^{-2}$ ($S_{\boldsymbol{\lambda}} = 1$). Also, for the helix $h^* := \{(t, \sin t, \cos t) : 0 \le t \le 1\}$ in $\mathbb{R}^3$ we obtain $E_n(h^*) \asymp n^{-2}$.

Consider now the curve

$$q_{\boldsymbol{\lambda},\delta} := \left\{ \left(t^{\lambda_1}, \ldots, t^{\lambda_d}\right) \,:\, \delta \le t \le 1 \right\} , \tag{4}$$

where $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_d) \in \mathbb{R}^d$ and $\delta \ge 0$.

Theorem 2. *Let $\boldsymbol{\lambda} = (1, \lambda_2, \ldots, \lambda_d) \in \mathbb{R}^d$ and $\delta > 0$. Then the curve $q_{\boldsymbol{\lambda},\delta} \subset \mathbb{R}^d$ is efficient, and*

$$E_n(q_{\boldsymbol{\lambda},\delta}) \asymp n^{-S_{\boldsymbol{\lambda}}} \asymp 1/r_n(q_{\boldsymbol{\lambda},\delta}) . \tag{5}$$

Moreover, if $\boldsymbol{\lambda} \in \mathbb{R}^d_+$, then

$$E_n(q_{\boldsymbol{\lambda},0}) \asymp n^{-(S_{\boldsymbol{\lambda}}+1)/2}, \quad r_n(q_{\boldsymbol{\lambda},0}) \asymp n^{S_{\boldsymbol{\lambda}}} . \tag{6}$$

Hence the curve $q_{\boldsymbol{\lambda},0}$ is inefficient if $S_{\boldsymbol{\lambda}} > 1$.

In the special case $d = S_{\boldsymbol{\lambda}} = 2$, estimate (6) for the L^2-norm appears in Passow and Raymon [9]. Note that (5) yields in this case $E_n(q_{\boldsymbol{\lambda},\delta}) \asymp n^{-2}$, contrary to the erroneous claim $E_n(q_{\boldsymbol{\lambda},\delta}) \asymp n^{-3/2}$ given in [7] and [10].

Observe that all the above examples of efficient curves (algebraic and "almost" algebraic curves, exponential and trigonometric curves, "polynomial" curves (4) with $\delta > 0$) are locally analytic, while the only inefficient curve $q_{\boldsymbol{\lambda},0}$ is not analytic. This leads to the following open problem: is every locally analytic curve $\gamma(t) \subset \mathbb{R}^d$ $(a \le t \le b)$ efficient?

In order to verify our results we shall need some Markov and Nikolskii-type inequalities for the exponential polynomials of the form

$$p(z) = \sum_{j=0}^{m-1} \sum_{l=0}^{n-1} C_{j,l} z^j e^{\lambda_l z}\,, \tag{7}$$

where $C_{j,l} \in \mathbb{C}$, $\boldsymbol{\lambda} = (\lambda_0, \dots, \lambda_{n-1}) \in \mathbb{C}^n$. Markov-type inequalities for these functions are studied in [2] and [5]. The main tool in verifying Markov-type inequalities for polynomials of the form (7) is a result of Tijdeman (see [1], proof of Lemma 1, p. 120) stating that for any $p(z)$ of the form (7) we have

$$\max_{|z| \le R} |p(z)| \le C^{mn+\Lambda_n} \max_{|z| \le r} |p(z)|\,, \tag{8}$$

where $0 < r < R$, $\Lambda_n = \max_{0 \le l \le n-1} |\lambda_l|$, and $C > 1$ is independent of m, n and $\boldsymbol{\lambda}$. In particular, using (8) it is shown in [5] that for any exponential polynomial p of the form (7) we have

$$\max_{0 \le x \le 1} |p'(x)| \le C_1 (mn + \Lambda_n)^2 \max_{0 \le x \le 1} |p(x)| \tag{9}$$

for some absolute constant $C_1 > 1$. Even though estimate (9) is formulated in [5] for real λ_l's it extends to the case of complex λ_l's as well, since Tijdeman's inequality (8) holds for complex exponents. A standard application of Markov-type inequalities leads to the estimate

$$\|p\|_{C[0,1]} \le C_0 (mn + \Lambda_n) \|p\|_{L_2[0,1]}\,, \tag{10}$$

where p is of the form (7), and $C_0 > 0$ is an absolute constant (see, e.g. [3], p. 281 for details). We shall use estimate (10) in order to obtain upper bounds in Theorems 1 and 2. We begin with the following key lemma.

Lemma 1. *Let a, n, $m \in \mathbb{N}$, $b > 0$, $\{\mu_j\}$ be $an+1$ distinct complex numbers with $|\mu_j| \leq bmn$ for $1 \leq j \leq an$ and $|\mu_{an+1}| \leq mn$. Then there exist univariate complex-valued polynomials p_j of deg $\leq m-1$ for $1 \leq j \leq an$ such that*

$$\left\| e^{\mu_{an+1}x} - \sum_{j=1}^{an} p_j(x) e^{\mu_j x} \right\|_{C[0,1]} \leq C^* e^{(2b+4-a)mn},$$

where $C^* = C_0(a+b+2)$ and C_0 is the constant in (10).

Proof. Set $r = \frac{1}{2}(e+1)(b+1)$, $N := anm + 1$,

$$\begin{aligned} \Lambda &= \{\lambda_1, \ldots, \lambda_N\} \\ &:= \Big\{ \underbrace{rmn + \mu_j, \ldots, rmn + \mu_j}_{m}, \, 1 \leq j \leq an \, , \, rmn + \mu_{an+1} \Big\} \qquad (11) \end{aligned}$$

i.e. each number $rnm + \mu_j$ $(1 \leq j \leq an)$ appears in Λ with multiplicity m, and $\lambda_N = rmn + \mu_{an+1}$. Consider now the so-called Müntz-Legendre polynomials for Λ (see [3], p. 126):

$$L_\Lambda(t) := \frac{1}{2\pi i} \int_\Gamma \prod_{j=1}^{N-1} \frac{z + \overline{\lambda}_j + 1}{z - \lambda_j} \frac{t^z}{z - \lambda_N} \, dz \, , \quad t > 0,$$

where Γ is a positively oriented simple closed contour surrounding all poles of the integrand. Then, by (11) and the Residue Theorem,

$$\begin{aligned} L_\Lambda(t) &= \frac{1}{2\pi i} \int_\Gamma \left(\prod_{j=1}^{an} \frac{z + \overline{\mu}_j + rmn + 1}{z - \mu_j - rmn} \right)^m \frac{t^z}{z - \mu_{an+1} - rmn} \, dz \\ &= A t^{\mu_{an+1} + rmn} + \sum_{j=1}^{an} \widetilde{p_j}(\log t) t^{\mu_j + rmn} \, , \qquad (12) \end{aligned}$$

where $\widetilde{p_j}$ are univariate complex polynomials of deg $\leq m-1$, and

$$A := \left(\prod_{j=1}^{an} \frac{\mu_{an+1} + \overline{\mu}_j + 2rmn + 1}{\mu_{an+1} - \mu_j} \right)^m . \qquad (13)$$

It is shown in [3], Theorem 3.4.3 on p. 127, that

$$\begin{aligned}\|L_\Lambda\|_{L_2[0,1]} &= (1+2\mathfrak{Re}\,(rmn+\mu_{an+1}))^{-1/2} \le (1+2rmn-2mn)^{-1/2} \\ &\le (2r-1)^{-1/2} \le \frac{1}{\sqrt{e}}\,. \qquad (14)\end{aligned}$$

Furthermore, by (13) using that $|\mu_j| \le bmn$ $(1 \le j \le an)$ and $|\mu_{an+1}| \le mn$, we obtain

$$\begin{aligned}|A| &= \prod_{j=1}^{an} \left| \frac{\mu_{an+1}+\overline{\mu}_j+2rmn+1}{\mu_{an+1}-\mu_j} \right|^m \ge \prod_{j=1}^{an} \left(\frac{2mnr+1-mn-bmn}{mn+bmn} \right)^m \\ &\ge \left(\frac{2r}{b+1} - 1 \right)^{amn} = e^{amn}. \qquad (15)\end{aligned}$$

Now set $t = e^{x-1}$ in (12), and define

$$Q(x) := \frac{1}{A}\, e^{\mu_{an+1}-rmn(x-1)} L_\Lambda(e^{x-1}) = e^{\mu_{an+1}} + \sum_{j=1}^{an} p_j(x) e^{\mu_j x}\,, \qquad (16)$$

where $p_j(x) := \frac{1}{A}\, e^{\mu_{an+1}-\mu_j} \widetilde{p}_j(x)$, $(1 \le j \le an)$. Then by (14), (15), and (16) we have

$$1 \ge \sqrt{e}\,\|L_\Lambda\|_{L_2[0,1]} \ge \sqrt{e}\,\|L_\Lambda\|_{L_2[e^{-1},1]} \ge \|L_\Lambda(e^{x-1})\|_{L_2[0,1]}$$

$$= \left| A e^{-rmn-\mu_{an+1}} \right| \|e^{rmnx} Q(x)\|_{L_2[0,1]} \ge e^{mn(a-r-1)} \|Q\|_{L_2[0,1]}\,.$$

This and the Nikolskii-type inequality (10) applied to Q of the form (16) yield

$$\begin{aligned}\|Q\|_{C[0,1]} &\le C_0\,((an+1)m + mn + bmn)\, \|Q\|_{L_2[0,1]} \\ &\le C_0 \left(a + \frac{1}{n} + 1 + b \right) mn e^{(r+1-a)mn} \le C_0 (a+b+2) e^{(2b+4-a)mn}\,.\end{aligned}$$

□

Lemma 2. *Let $b > 0$, $N, m, a(b) \in \mathbb{N}$ and $n := \lfloor N/a(b) \rfloor$, where $a(b)$ will be specified below, $f \in C[0,1]$, and let $0 = \mu_0, \ldots, \mu_N \in \mathbb{C}$ satisfy $|\mu_j| \le bN$ for*

$j=\overline{0,N}$, with μ_j's distinct. Then there exist univariate complex polynomials p_j^* of degree $\le m-1$ for $m \ge a(b)$ such that

$$\left\| f(x) - \sum_{j=0}^{N} p_j^*(x) e^{\mu_j x} \right\|_{C[0,1]} \le C^*(b)\, \omega\left(f, \frac{1}{mN}\right) . \qquad (17)$$

In addition, if $|\mu_j| \le bn$ for $j=\overline{0,N}$, then there exist $b_j \in \mathbb{R}$ for $j=\overline{0,N}$, such that

$$\left\| f(x) - \sum_{j=0}^{N} b_j e^{\mu_j x} \right\|_{C[0,1]} \le C^{**}(b)\, \omega\left(f, \frac{1}{N}\right) . \qquad (18)$$

Proof. Since $\mu_0 = 0$ we may assume that $\|f(x)\|_{C[0,1]} \le \omega(f,1)$. A simple transformation of Jackson's theorem yields that there exist $a_k \in \mathbb{R}$, $k=\overline{0,mn}$, such that with $q(x) := \sum_{k=0}^{mn} a_k e^{kx}$ we have for some absolute constant $C > 0$

$$\|f-q\|_{C[0,1]} \le C\omega\left(f, \frac{1}{mn}\right) . \qquad (19)$$

Clearly, $\|q\|_{C[0,1]} \le C_1 \omega(f,1)$. Therefore, by a well known estimate for the coefficients a_k of q, we have

$$|a_k| \le \omega(f,1)\, e^{Amn} , \quad k=\overline{0,mn} , \qquad (20)$$

where $A > 0$ is an absolute constant. Set $a(b) := \lfloor 2(A+2b+4) \rfloor + 1$. Applying Lemma 1 (with $a(b)$ replacing a and assuming that $m \ge a(b)$) we obtain that for every $0 \le k \le mn$ there exist univariate polynomials $p_{j,k}$ of degree $\le m-1$ so that

$$\left\| e^{kx} - \sum_{j=0}^{N} p_{j,k}(x) e^{\mu_j x} \right\|_{C[0,1]} \le C^* \exp\left\{ \left(\frac{2b+4}{a(b)} - 1 \right) mN \right\} . \qquad (21)$$

Hence setting $p_j^*(x) := \sum_{k=0}^{mn} a_k p_{j,k}(x)$, $0 \le j \le N$, we have by (19)-(21)

$$
\begin{aligned}
\left\| f(x) - \sum_{j=0}^{N} p_j^* e^{\mu_j x} \right\|_{C[0,1]} &\le C\omega\left(f, \frac{1}{mn}\right) \\
&\quad + \left\| \sum_{k=0}^{mn} a_k \left(e^{kx} - \sum_{j=0}^{N} p_{j,k}(x) e^{\mu_j x} \right) \right\|_{C[0,1]} \\
&\le C\omega\left(f, \frac{1}{mn}\right) + C^* \sum_{k=0}^{mn} \omega(f,1) e^{Amn} e^{\left(\frac{2b+4}{a(b)} - 1\right) mN} \\
&\le C_1(b)\, \omega\left(f, \frac{1}{mn}\right) + C^* \omega(f,1)(mN) e^{\left(\frac{A+2b+4}{a(b)} - 1\right) mN} \\
&\le C_1(b)\, \omega\left(f, \frac{1}{mn}\right) + C^*(b)\, \omega\left(f, \frac{1}{mn}\right) (mN)^2 e^{-\frac{mN}{2}} \\
&\le C^*(b)\, \omega\left(f, \frac{1}{mn}\right) .
\end{aligned}
$$

For the second case one should use all previous arguments with $m = 1$, taking to the account that the condition $m \ge a(b)$ is not necessary any more, since $|\mu_j| \le bn$. So, we have

$$
\left\| f(x) - \sum_{j=0}^{N} b_j e^{\mu_j x} \right\|_{C[0,1]} \le C^{**}(b) \omega\left(f, \frac{1}{N}\right) ,
$$

for some real numbers b_j, $j = \overline{0, N}$. □

Proof of Theorem 1. We first need to calculate the quantities $r_n(e_{\boldsymbol{\lambda}})$ and $r_n(h_{\boldsymbol{\lambda}})$, where the curves $e_{\boldsymbol{\lambda}}$ and $h_{\boldsymbol{\lambda}}$ are given by (1) and (2), respectively. Any $p \in P_n^d$ restricted to the exponential curve $e_{\boldsymbol{\lambda}}$ appears in the form

$$
p(t) = \sum_{j=1}^{N_{\boldsymbol{\lambda}}} p_j(t) e^{\xi_j t} ,
$$

where the p_j's are real univariate polynomials of deg $\le n$,

$$
\Lambda := \{\xi_1, \ldots, \xi_{N_\lambda}\} := \left\{ \sum_{j=1}^{d-1} \lambda_j k_j \,:\, k_j \in \mathbb{Z},\ 0 \le k_j \le n \right\} ,
$$

and $N_{\boldsymbol\lambda} := \#\,\Lambda$, the cardinality of Λ.

Without loss of generality we may assume that λ_j, $1 \le j \le S_{\boldsymbol\lambda}$, are algebraically independent, and any λ_j with $S_{\boldsymbol\lambda}+1 \le j \le d-1$ is a rational linear combination of λ_j, $1 \le j \le S_{\boldsymbol\lambda}$. Then it is easy to see that there exists $q \in \mathbb{N}$ such that

$$\xi_j = \frac{1}{q}\sum_{k=1}^{S_{\boldsymbol\lambda}} p_{k,j}\lambda_k\,,$$

where $p_{k,j} \in \mathbb{N}$ satisfy $p_{k,j} \le Mn$, $1 \le k, j \le S_{\boldsymbol\lambda}$, for some $M \in \mathbb{N}$ depending only on $\lambda_1, \ldots, \lambda_{d-1}$. Hence

$$(n+1)^{S_{\boldsymbol\lambda}} \le N_{\boldsymbol\lambda} = \#\Lambda \le (Mn+1)^{S_{\boldsymbol\lambda}}\,,$$

i.e.,

$$r_n(e_{\boldsymbol\lambda}) = (n+1)N_{\boldsymbol\lambda} \sim n^{S_{\boldsymbol\lambda}+1}\,.$$

The lower estimate for $E_n(e_{\boldsymbol\lambda})$ now follows from Lorentz's result quoted in the introduction. Furthermore, set

$$\Lambda^* := \left\{\sum_{j=1}^{S_{\boldsymbol\lambda}} \lambda_j k_j \;:\; 0 \le k_j \le n\right\} = \{\mu_0, \ldots, \mu_{N-1}\}\,,$$

where $N := \#\Lambda^* = (n+1)^{S_{\boldsymbol\lambda}}$. Consider the space of exponential polynomials

$$P_{\boldsymbol\lambda,n} := \left\{\sum_{j=0}^{N-1} p_j(t)e^{t\mu_j} \;:\; p_j \in P_n^1\right\}.$$

Clearly, $P_n^d\big|_{e_{\boldsymbol\lambda}} \supset P_{\boldsymbol\lambda,n}$. Moreover, since the curve $e_{\boldsymbol\lambda}$ is differentiable it follows that for any $f \in K_\gamma$ the function $\widetilde{f}(t) := f(t, e^{\lambda_1 t}, \ldots, e^{\lambda_{d-1}t})$ is $\mathrm{Lip}\,1$ on $[0,1]$ and the Lipschitz constant is independent of f. (The same remark holds with respect to the curve $h_{\boldsymbol\lambda}$ below.) Hence by Lemma 2 (with $N-1=n$ and $m=n$) we have

$$E_n(e_{\boldsymbol\lambda}) \le C/nN \le C_1/n^{S_{\boldsymbol\lambda}+1}\,.$$

Thus $E_n(e_{\boldsymbol\lambda}) \asymp n^{-(S_{\boldsymbol\lambda}+1)}$.

Next we consider the curve $h_{\boldsymbol\lambda} \subset \mathbb{R}^{2d-1}$. Any $p \in P_n^{2d-1}$ on $h_{\boldsymbol\lambda}$ can be written as

$$p(t) = \sum_{j=0}^{N_{\boldsymbol\lambda}} p_j(t)e^{it\mu_j}\,,$$

where

$$\mu_j \in \Lambda_1 := \left\{ \sum_{j=1}^{d-1} \lambda_j k_j \ : \ -n \le k_j \le n, \ 1 \le j \le d-1 \right\},$$

$N_{\boldsymbol{\lambda}} := \#\Lambda_1$, and p_j's are univariate complex polynomials of deg $\le n$. As above,

$$(2n+1)^{S_{\boldsymbol{\lambda}}} \le N_{\boldsymbol{\lambda}} \le C n^{S_{\boldsymbol{\lambda}}}$$

for some $C > 0$ depending only on $\boldsymbol{\lambda}$. Hence

$$r_n(h_{\boldsymbol{\lambda}}) = (n+1) N_{\boldsymbol{\lambda}} \sim n^{S_{\boldsymbol{\lambda}}+1}. \tag{22}$$

Clearly, for any $\lambda^* \in \mathbb{R}$

$$\text{span}\{(\sin \lambda^* t)^r (\cos \lambda^* t)^k : 0 \le r, k \le n\} \supset \text{span}\{\sin \lambda^* kt, \cos \lambda^* kt : 0 \le k \le n\}.$$

Therefore, for any $0 \le k, e_1, \ldots, e_{d-1} \le n$,

$$\mathfrak{Re}\, t^k e^{it(e_1\lambda_1 + \cdots + e_{d-1}\lambda_{d-1})} \in P_n^{2d-1}\big|_{h_{\boldsymbol{\lambda}}}. \tag{23}$$

Assuming again that λ_j, $1 \le j \le S_{\boldsymbol{\lambda}}$, are algebraically independent, set

$$\Lambda_2 := \left\{ \sum_{j=1}^{S_{\boldsymbol{\lambda}}} \lambda_j k_j \ : \ 0 \le k_j \le n, \ 1 \le j \le S_{\boldsymbol{\lambda}} \right\} = \{\eta_1, \ldots, \eta_N\},$$
$$N := \#\Lambda_2 = (n+1)^{S_{\boldsymbol{\lambda}}}.$$

Then by (23)

$$\mathfrak{Re}\, t^k e^{it\eta_j} \in P_n^{2d-1}\big|_{h_{\boldsymbol{\lambda}}}, \quad 0 \le k \le n, \ 1 \le j \le N.$$

This and Lemma 2 yield that

$$E_n(h_{\boldsymbol{\lambda}}) \le \frac{C}{nN} \le C_1 n^{-S_{\boldsymbol{\lambda}}-1}.$$

A similar lower estimate follows from (22) and Lorentz's result. This completes the proof of Theorem 1.

□

Proof of Theorem 2. Consider first the curve $q_{\boldsymbol{\lambda},\delta} \subset \mathbb{R}^d$ given by (4) for $1 > \delta > 0$. Note that this curve is differentiable, i.e., any $f \in K_\gamma$ composed with $q_{\boldsymbol{\lambda},\delta}$ is Lip 1. Clearly

$$P_n^d\big|_{q_{\boldsymbol{\lambda},\delta}} = \operatorname{span}\{t^{\mu_j}, 1 \le j \le N\}, \qquad t \in [\delta, 1],$$

where $\{\mu_1, \ldots, \mu_N\} = \{\lambda_1 e_1 + \ldots + \lambda_N e_N : 0 \le j \le N\}$. As in the proof of Theorem 1 we can show that $N \asymp n^{S_{\boldsymbol{\lambda}}}$, where $S_{\boldsymbol{\lambda}}$ is the number of algebraically independent λ_j's, $1 \le j \le d$. Substituting $t = e^{(1-x)\log\delta}$ yields with $\mu_j' := -\mu_j \log\delta$

$$P_n^d\big|_{q_{\boldsymbol{\lambda},\delta}} = \operatorname{span}\left\{e^{\mu_j' x}, 1 \le j \le N\right\}, \qquad x \in [0,1].$$

Hence by Lemma 2 (with $m = 1$) in the case when $S_{\boldsymbol{\lambda}} > 1$

$$E_n(q_{\boldsymbol{\lambda},\delta}) \le C/N \le C_1 n^{-S_{\boldsymbol{\lambda}}}.$$

For $S_{\boldsymbol{\lambda}} = 1$ the above inequality follows from Jackson's theorem. So, since $r_n(q_{\boldsymbol{\lambda},\delta}) = N \asymp n^{S_{\boldsymbol{\lambda}}}$ we again obtain the corresponding lower bound by Lorentz's result.

Let us now verify the estimates (6) in Theorem 2. Let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_d) \in \mathbb{R}_+^d$ and $\lambda_1 = 1$. Without loss of generality we may assume that $\lambda_j \ge 1$, $2 \le j \le d$ (this can always be achieved by a proper parametrization), yielding that $q_{\boldsymbol{\lambda},0}$ is differentiable. As above, it follows that $r_n(q_{\boldsymbol{\lambda},0}) \asymp n^{S_{\boldsymbol{\lambda}}}$. Furthermore, let us assume again that λ_j, $1 \le j \le S_{\boldsymbol{\lambda}}$, are algebraically independent, and set

$$\Omega_0 := \{\eta_1, \ldots, \eta_N\} := \left\{\sum_{j=1}^{S_{\boldsymbol{\lambda}}} \lambda_j k_j : 0 \le k_j \le n,\ 1 \le j \le S_{\boldsymbol{\lambda}}\right\}.$$

Clearly, $N = (n+1)^{S_{\boldsymbol{\lambda}}}$. Then

$$P_n^d\big|_{q_{\boldsymbol{\lambda},0}} \supset \operatorname{span}\{t^{\eta_j},\ 1 \le j \le N\}.$$

Set now $M := \max_{1 \le j \le d} \lambda_j,\ n_1 := \lfloor n/MS_{\boldsymbol{\lambda}} \rfloor$,

$$\Omega_1 := \{\eta_1^*, \ldots, \eta_{N^*}^*\} := \left\{\sum_{j=1}^{S_{\boldsymbol{\lambda}}} \lambda_j k_j : 0 \le k_j \le n_1,\ 1 \le j \le S_{\boldsymbol{\lambda}}\right\}.$$

Clearly, $N^* = (n_1+1)^{S_{\boldsymbol{\lambda}}}$, and

$$\eta_k^* \le n_1 S_{\boldsymbol{\lambda}} M \le n\,, \quad 1 \le k \le N^*\,. \tag{24}$$

Consider now the set $\Omega_2 := \Omega_0 \cap [0,n]$. By (24) $\Omega_1 \subset \Omega_2$. Moreover, recalling that $\lambda_1 = 1$ we have $k \in \Omega_2$, $0 \le k \le n$. This implies that for any two consecutive elements $\mu_1 < \mu_2 \in \Omega_2$ (i.e., $(\mu_1,\mu_2)\cap\Omega_2 = \emptyset$) we have $\mu_2 - \mu_1 \le 1$. Then by a result from [4] (see Theorem 4.1 on p. 360) for any contraction f on $[0,1]$

$$\min_{a_\beta\in\mathbb{R}} \left\| f(t) - \sum_{\beta\in\Omega_2} a_\beta t^\beta \right\|_{C[0,1]} \le C/\sqrt{S}\,, \tag{25}$$

where $S = \sum_{\beta\in\Omega_2}\beta$. Since $\Omega_1 \subset \Omega_2$ we obtain that

$$S = \sum_{\beta\in\Omega_2}\beta \ge \sum_{\beta\in\Omega_1}\beta \ge C_1 n^{S_{\boldsymbol{\lambda}}+1}\,. \tag{26}$$

Moreover, using that $\Omega_2 \subset \Omega_0$, i.e.,

$$P_n^d\big|_{q_{\boldsymbol{\lambda},0}} \supset \operatorname{span}\left\{t^\beta \,:\, \beta\in\Omega_0\right\} \supset \operatorname{span}\left\{t^\beta \,:\, \beta\in\Omega_2\right\}$$

we obtain from (25) and (26) that

$$E_n(q_{\boldsymbol{\lambda},0}) \le C_2 n^{-(S_{\boldsymbol{\lambda}}+1)/2}\,.$$

In addition, by [4], Theorems 3.1 and 4.1, estimate (25) is sharp in the class of all contractions in $C[0,1]$, i.e., for some contractions f the inequality (25) can be reversed (with another absolute constant). Since

$$S = \sum_{\beta\in\Omega_2}\beta \le \sum_{\beta\in\Omega_0}\beta \le C_3 n^{S_{\boldsymbol{\lambda}}+1}\,,$$

the desired lower bound for $E_n(q_{\boldsymbol{\lambda},0})$ also follows.

□

Remark 1. A weaker version of the estimate of Lemma 1 appears in the proof of the lower bound of Theorem 5 in [5]. However, the proof of it given in [5] has a gap. Lemma 1 above gives a correct version of the estimate needed to complete the lower bound of Theorem 5 in [5].

Remark 2. The mistakes by Newman and Raymon in Theorems 2 and 5 of [7] and repeated by Raymon in Theorem 2 of [10] can be traced to the lower bound estimate on page 254 of [7]. There a choice for $G(z)$ is made which is not possible, since G must be entire and so cannot have any poles.

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