APPROXIMATION BY RATIONAL AND MEROMORPHIC FUNCTIONS HAVING A BOUNDED NUMBER OF FREE POLES

BY

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1. **Introduction.** If a function $g(z)$ defined on a smooth Jordan curve Γ of the z-plane is the uniform limit on Γ of a sequence of polynomials $p_n(z)$ of respective degrees n, say

$$|g(z)-p_n(z)| \leqq \varepsilon_n(\to 0), \qquad z \text{ on } \Gamma,$$

then one can deduce certain properties of the function $g(z)$ and the sequence $p_n(z)$. As a consequence of the Maximum Principle $g(z)$ is the set of boundary values on Γ of a function $f(z)$ which is analytic in the interior D of Γ and continuous on the closed region $D+\Gamma$. It is also clear that the sequence $p_n(z)$ converges to $f(z)$ at each point of D. In addition, some continuity properties of $g(z)$ on Γ may be deduced if an estimate is known on the rapidity of convergence of the sequence ε_n. Indeed, the inequality $\varepsilon_n \leqq A/n^{k+\alpha}$, where k is a nonnegative integer and $0<\alpha<1$, implies that the kth derivative of $g(z)$ exists on Γ (in the one-dimensional sense) and satisfies there a Lipschitz condition of order α.

In this paper we make the weaker assumption that the function $g(z)$ is the uniform limit on Γ of a sequence of rational functions each having at most ν free poles, and we establish analogues of the above mentioned conclusions. Specifically we shall deal with rational functions of type (n, ν), i.e., rational functions of the form

$$r_{n\nu}(z) = \frac{a_0 z^n + a_1 z^{n-1} + \cdots + a_n}{b_0 z^\nu + b_1 z^{\nu-1} + \cdots + b_\nu}, \qquad \sum_0^\nu |b_k| \neq 0,$$

for fixed ν.

In §2 we show that the condition

$$|g(z)-r_{n\nu}(z)| \leqq \varepsilon_n(\to 0), \qquad z \text{ on } \Gamma,$$

implies the existence of a function $f(z)$ which is meromorphic with at most ν poles in D, is continuous on $D+\Gamma$, and is equal to $g(z)$ for z on Γ. If the function $f(z)$ is known to have precisely ν poles in D, it is further shown that the rational functions

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$r_{n\nu}(z)$ must converge to $f(z)$ at each point of D. The last result is similar to one obtained by J. L. Walsh [1, p. 3].

In §3 and §4 we establish theorems on the Lipschitz continuity and analyticity of $g(z)$ on Γ as a consequence of certain hypotheses on the degree of convergence of the $r_{n\nu}(z)$ and on the location of the limit points of their poles.

2. **Uniform convergence of meromorphic functions.** An easy extension of a theorem on polynomial approximation is

THEOREM 1. *Let E be a closed bounded point set whose complement is connected and whose interior is nonempty. Suppose $f(z)$ is meromorphic in the interior of E with precisely ν poles there and is otherwise finite and continuous on E. Then there exists a sequence of rational functions $r_{n\nu}(z)$ of respective types (n, ν) which converges uniformly to $f(z)$ on the boundary of E.*

Proof. Let $q(z)=z^{\nu}+a_1z^{\nu-1}+\cdots+a_{\nu}$ be the polynomial of the form indicated having as its zeros the ν poles of $f(z)$ in the interior of E. By a well-known theorem of Mergelyan [2, §A1] the analytic function $q(z)f(z)$ can be uniformly approximated on E as closely as desired by a polynomial, and hence [2, p. 89] there exists a sequence of polynomials $p_n(z)$ of respective degrees n which converges uniformly on the boundary of E to $q(z)f(z)$. Theorem 1 now follows by taking $r_{n\nu}(z)=p_n(z)/q(z)$.

If a function $g(z)$ defined merely on the boundary ∂E of E is the uniform limit of polynomials, then as mentioned in §1 there exists a function $f(z)$ analytic in the interior of E and continuous on E such that $f(z)\equiv g(z)$ for z on ∂E. Hence the converse to Theorem 1 is valid for $\nu=0$. To establish a converse result for $\nu>0$ we appeal to the following special case of a result due to S. Warschawski [3]:

THEOREM 2. *Let $h(z)$ be analytic in a Jordan region D_0 and continuous on $D_0+\partial D_0$. For fixed α on ∂D_0 let*

$$|h(z)-h(\alpha)| \leqq L|z-\alpha| \tag{1}$$

hold for all z on ∂D_0. Then (1) *holds for all z on $D_0+\partial D_0$.*

We may now prove

THEOREM 3. *Let D be a Jordan region and $g(z)$ a function defined on ∂D. Suppose $f_n(z)$ is a sequence of functions each meromorphic with at most ν poles in D and otherwise finite and continuous on $D+\partial D$. If $\lim_{n\to\infty} f_n(z)=g(z)$ uniformly for z on ∂D, then there exists a function $f(z)$ which is meromorphic with at most ν poles in D and is otherwise finite and continuous on $D+\partial D$ such that $f(z)\equiv g(z)$ for z on ∂D.*

Proof. Theorem 3 holds for $\nu=0$, so assume that it holds for $\nu=k-1$ and suppose that each of the functions $f_n(z)$ has at most k poles in D. Clearly we may assume that each $f_n(z)$ has at least one pole in D, say at a point α_n. Let α be a limit point of the α_n and let α_{n_i} be a subsequence which converges to α. Then $\{(z-\alpha_{n_i})f_{n_i}(z)\}$ is a sequence of functions each meromorphic with at most $k-1$ poles in D which converges to the function $(z-\alpha)g(z)$ uniformly for z on ∂D. Thus by the induction hypothesis there exists a function $h(z)$ which is meromorphic in D with at most $k-1$ poles there and continuous on $D+\partial D$ such that $h(z)=(z-\alpha)g(z)$ for z on ∂D. Set

$$\begin{aligned} f(z) &\equiv h(z)/(z-\alpha), && z \text{ in } D, \\ &\equiv g(z), && z \text{ on } \partial D. \end{aligned}$$

If $\alpha \in D$, then clearly $f(z)$ is the desired function. If $\alpha \in \partial D$, it remains to show that $f(z)$ is continuous at α.

Let Γ_1 be a closed subarc of ∂D which contains the point α and terminates at the distinct points β_1 and β_2, where $\beta_1 \neq \alpha$, $\beta_2 \neq \alpha$. Join the points β_1, β_2 by an open Jordan arc Γ_2 which lies in D, contains no pole of $h(z)$, and is such that $h(z)$ and hence $f(z)$ is analytic in the Jordan region D_0 bounded by $\Gamma_1+\Gamma_2$. Since $g(z)$ is continuous on Γ_1 it is bounded there, say by a constant M, and so

$$|h(z)| \leqq M|z-\alpha| \tag{2}$$

for z on Γ_1. For M large enough inequality (2) also holds for z on Γ_2 since $h(z)/(z-\alpha)$ is finite and continuous on the closure of Γ_2. Theorem 2 thus implies that for an appropriate choice of the constant M inequality (2) is valid for z in D_0, and hence $f(z)$ is bounded in D_0. Finally, note that $f(z)$ is continuous on $D_0+\Gamma_1+\Gamma_2-\{\alpha\}$, and that

$$\lim_{z\to\alpha;\, z\in\Gamma_1} f(z) = f(\alpha),$$

and hence the continuity of $f(z)$ at α follows from a theorem of Lindelöf [4, p. 460].

Theorem 3 equivalently states that the family $D(\nu)$ composed of all those functions which are meromorphic with at most ν poles in D and which are otherwise finite and continuous on $D+\partial D$ is complete with respect to the Tchebycheff (uniform) norm taken over ∂D.

COROLLARY 1. *Let D be a Jordan region and let the function $F(z)$ be analytic in D except for a finite number of isolated singularities at the points $z_1, \ldots, z_k$ in D, where at least one z_i is an essential singularity of $F(z)$. Suppose $F(z)$ is contiuous on $D+\partial D-\{z_1, \ldots, z_k\}$, and suppose $F_n(z)$ is a sequence of functions each meromorphic in D and continuous on $D+\partial D$ which converges uniformly to $F(z)$ on ∂D. Then if $p(n)$ denotes the number of poles of $F_n(z)$ in D, we have $p(n)\to\infty$ as $n\to\infty$.*

Proof. The contrary assumption implies that there exists an integer ν and a subsequence n_i such that $p(n_i)\leqq\nu$ for $i=1, 2, \ldots$. Since the subsequence $F_{n_i}(z)$ converges uniformly to $F(z)$ on ∂D, Theorem 3 asserts the existence of a function $f(z) \in D(\nu)$ such that $f(z)=F(z)$ for z on ∂D. But then $f(z)\equiv F(z)$ for z in D, which is absurd.

We remark that although Corollary 1 is presented here as a consequence of Theorem 3, it may be proved directly from the theory of normal families.

Theorem 1 and Theorem 3 yield

THEOREM 4. *Let Γ be a Jordan curve with interior D and let $g(z)$ be a function defined (finite) on Γ. A necessary and sufficient condition that $g(z)$ be the set of boundary values on Γ of a function which is meromorphic with at most ν poles in D and which is otherwise finite and continuous on $D+\Gamma$ is that there exist a sequence of rational functions $r_{n\nu}(z)$ of respective types (n, ν) such that $\lim_{n\to\infty} r_{n\nu}(z)=g(z)$ uniformly for z on Γ.*

If the function $g(z)$ is the set of boundary values of a meromorphic function $f(z)$ known to have *precisely* ν poles in D, then any sequence of rational functions of respective types (n, ν) which converges uniformly to $g(z)$ on Γ necessarily converges to $f(z)$ in D. More generally we prove

THEOREM 5. *Suppose $f_0(z)$ is meromorphic with precisely ν poles in a bounded domain D and is otherwise finite and continuous on $D+\partial D$. If $f_n(z)$ is a sequence of functions each meromorphic with at most ν poles in D and continuous on $D+\partial D$ which satisfies*

$$(3) \qquad |f_0(z)-f_n(z)| \leqq \varepsilon_n(\to 0), \qquad z \text{ on } \partial D,$$

then:

(i) *For n sufficiently large each $f_n(z)$ has precisely ν poles in D, and these poles approach respectively the ν poles of $f_0(z)$ in D.*

(ii) *The $f_n(z)$ converge to $f_0(z)$ in the domain D' obtained from D by deleting the ν poles of $f_0(z)$.*

(iii) *For each closed set $S\subset D'$ we have for n large*

$$[\max |f_0(z)-f_n(z)|;\ z \text{ on } S] \leqq M(S)\varepsilon_n,$$

where $M(S)$ is a constant dependent only on S, D, and on the sequence $f_n(z)$.

Proof. For $n=0, 1, 2, \ldots$, let $q_n(z)=z^{\mu_n}+\cdots+a_n$ denote the polynomial of the form indicated having as its zeros the poles of $f_n(z)$ in D. Since D is bounded and since $\mu_n\leqq\nu$ for each n, the sequence $q_n(z)$ is uniformly bounded in D. A well-known application of Lagrange's Interpolation Formula thus implies that the $q_n(z)$ form a normal family in the finite plane and that each limit function of the family is a polynomial of the form $z^\mu+\cdots+a$, $0\leqq\mu\leqq\nu$.

Let $q(z)$ be any such limit function and $q_{n_i}(z)$ a subsequence which converges uniformly to $q(z)$ on compact sets of the plane. From (3) we have

$$\lim_{i\to\infty} q_{n_i}(z)f_{n_i}(z) = q(z)f_0(z),$$

uniformly for z on ∂D, and so the analyticity of the functions $q_{n_i}(z)f_{n_i}(z)$ implies that $q(z)f_0(z)$ is analytic in D. Hence the polynomial $q_0(z)$ must be a factor of

$q(z)$; and since $q(z)$ is monic and has at most ν zeros, it follows that $q(z) \equiv q_0(z)$. Thus the only limit function of the $q_n(z)$ is $q_0(z)$, and hence the sequence $q_n(z)$ converges to $q_0(z)$ uniformly on compact sets of the plane. Conclusion (i) now follows from Hurwitz's Theorem.

Now let $S \subset D'$ be closed. Since the $q_n(z)$ are uniformly bounded on ∂D we obtain from (3)

$$(4) \qquad |q_0(z)q_n(z)f_0(z) - q_0(z)q_n(z)f_n(z)| \leqq A\varepsilon_n, \qquad z \text{ on } \partial D.$$

The function whose absolute value appears in (4) is analytic in D, and so (4) holds for z on S. By conclusion (i) the set S contains no limit points of the poles of the $f_n(z)$ and hence for n large enough we have $|q_n(z)q_0(z)| \geqq m > 0$ for z on S. There follows

$$|f_0(z) - f_n(z)| \leqq A\varepsilon_n/m, \qquad z \text{ on } S,$$

which completes the proof of Theorem 5.

From conclusion (i) we deduce

COROLLARY 2. *If $f_0(z)$ and D are as in Theorem 5 and $f_n(z)$ is a sequence of functions each meromorphic with at most μ $(<\nu)$ poles in D and continuous on $D+\partial D$, then the $f_n(z)$ do not converge uniformly to $f_0(z)$ on ∂D.*

The assumption that the number of poles of the functions $f_n(z)$ not exceed the number of poles of the limit function $f_0(z)$ cannot be weakened in Theorem 5. Indeed the sequence $f_n(z) \equiv (z-1+1/n)/z(z-1)$ converges uniformly to $1/z$ on $|z|=2$, but does not converge to $1/z$ for $z=1$. The method of proof of Theorem 5 does however yield

COROLLARY 3. *Suppose $f_0(z)$ and D are as in Theorem 5 and $f_n(z)$ is a sequence of functions each meromorphic with at most η poles in D and continuous on $D+\partial D$. If $\lim_{n\to\infty} f_n(z) = f_0(z)$ uniformly for z on ∂D, then each pole of $f_0(z)$ in D is a limit point of poles of the $f_n(z)$, and $\lim_{n\to\infty} f_n(z) = f_0(z)$ uniformly on each closed subset of D which contains no limit points of the poles of the $f_n(z)$.*

An easy generalization of Hurwitz's Theorem is

COROLLARY 4. *Suppose, in addition to the hypotheses of Theorem 5, that the function $f_0(z)$ does not vanish on ∂D. Then for n sufficiently large $f_n(z)$ and $f_0(z)$ have the same number of zeros in D.*

The proof of Corollary 4 is left to the reader.

If the function $f_0(z)$ of Theorem 5 has a pole at a point α in D, then we can apply Corollary 4 to obtain an estimate on the degree of divergence of the sequence $f_n(\alpha)$. We choose a constant δ (>0) so small that $f_0(z)$ is analytic and nonzero in $0 < |z-\alpha| \leqq \delta$. From conclusion (iii) of Theorem 5 there follows

$$(5) \qquad |1/f_0(z) - 1/f_n(z)| \leqq M\varepsilon_n, \qquad |z-\alpha| = \delta.$$

Since for n sufficiently large $f_n(z)$ and $f_0(z)$ have the same number of poles in $|z-\alpha| \leqq \delta$, Corollary 4 implies that the $f_n(z)$ do not vanish there for n large. Hence inequality (5) holds for $z=\alpha$, and so $|f_n(\alpha)| \geqq 1/M\varepsilon_n$.

Theorem 3 and Theorem 5 yield the following dual theorems which are obtained by interchanging the poles and zeros of the functions $f_n(z)$:

THEOREM 6. *Let D be a Jordan region and let $G(z)$ be defined and different from zero on ∂D. Suppose $F_n(z)$ is a sequence of functions each meromorphic with at most ν zeros in D, continuous on $D+\partial D$, and finite on ∂D. If* $\lim_{n\to\infty} F_n(z)=G(z)$ *uniformly for z on ∂D, then $G(z)$ is the set of boundary values on ∂D of a function which is meromorphic with at most ν zeros in D and continuous on $D+\partial D$.*

THEOREM 7. *Suppose the function $F_0(z)$ is meromorphic with precisely ν zeros in a bounded domain D, and is continuous on $D+\partial D$ and finite and different from zero on ∂D. If $F_n(z)$ is a sequence of functions each meromorphic with at most ν zeros in D and continuous on $D+\partial D$ which satisfies*

$$|F_0(z)-F_n(z)| \leqq \varepsilon_n \ (\to 0), \qquad z \text{ on } \partial D,$$

then:

(i) *For n sufficiently large each $F_n(z)$ has precisely ν zeros in D, and these zeros approach respectively the ν zeros of $F_0(z)$ in D.*

(ii) *Each pole of $F_0(z)$ in D is a limit point of poles of the $F_n(z)$, multiplicity included, and the $F_n(z)$ have no other limit point of poles in D.*

(iii) $\operatorname{Lim}_{n\to\infty} F_n(z)=F_0(z)$ *uniformly on each closed set $S\subset D$ which contains no poles of $F_0(z)$, and for n large*

$$[\max |F_0(z)-F_n(z)|;\ z \text{ on } S] \leqq M(S)\varepsilon_n,$$

where $M(S)$ is a constant dependent only on S, D, and on the sequence $F_n(z)$.

The proofs of Theorem 6 and Theorem 7, which follow from methods used by J. L. Walsh [5], are left to the reader.

3. **Lipschitz continuity.** We now apply the results of §2 to obtain theorems which relate the boundary continuity of a meromorphic function $f(z)$ to the degree of approximation of $f(z)$ by rational functions.

Let Γ be an analytic Jordan curve and D its interior. We say that a function $f(z)$ *belongs to class $L_\nu(k, \alpha)$ on Γ*, where ν and k are nonnegative integers and $0<\alpha<1$, if $f(z)$ is meromorphic with at most ν poles in D and is otherwise finite and continuous on $D+\Gamma$, and if $f^{(k)}(z)$ exists on Γ in the one-dimensional sense and satisfies there a Lipschitz condition of order α, i.e.,

$$(6) \qquad |f^{(k)}(z_1)-f^{(k)}(z_2)| \leqq L|z_1-z_2|^\alpha, \qquad z_1, z_2 \text{ on } \Gamma,$$

where L is a constant independent of z_1 and z_2.

It is of importance to mention here that the property of a function that it has a kth derivative satisfying condition (6) is invariant under conformal mapping. This fact is well-illustrated by the following theorem [6, p. 24]:

THEOREM 8. *Let the function $g(z)$ be defined on an analytic Jordan curve Γ. A necessary and sufficient condition that $g(z)$ possess a kth derivative on Γ which satisfies a Lipschitz condition of order α $(0<\alpha<1)$ on Γ is that there exist a region D_1 containing Γ and a sequence of functions $f_n(z)$ analytic in D_1 and satisfying*

$$|f_n(z)| \leqq AR^n, \qquad z \text{ in } D_1,$$
$$|g(z)-f_n(z)| \leqq A_1/n^{k+\alpha}, \qquad z \text{ on } \Gamma.$$

The fundamental theorem relating the degree of best polynomial approximation on Γ to the existence of functions of class $L_0(k, \alpha)$ on Γ was established by J. H. Curtiss, W. E. Sewell, and J. L. Walsh [6, p. 27] and is stated as

THEOREM 9. *Let Γ be an analytic Jordan curve and $f(z)$ a function defined on Γ. Then the following statements are equivalent:*

(i) *$f(z)$ is the set of boundary values on Γ of a function of class $L_0(k, \alpha)$ on Γ.*

(ii) *There exists a sequence of polynomials $p_n(z)$ of respective degrees n such that*

$$|f(z)-p_n(z)| \leqq A/n^{k+\alpha}, \qquad z \text{ on } \Gamma.$$

(iii) *There exists a domain D_0 containing Γ and its interior and a sequence of functions $f_n(z)$ analytic in D_0 satisfying the inequalities*

$$|f_n(z)| \leqq A_1R^n, \qquad z \text{ in } D_0,$$
$$|f(z)-f_n(z)| \leqq A_2/n^{k+\alpha}, \qquad z \text{ on } \Gamma.$$

An extension of Theorem 9 to the case $\nu>0$ is given in

THEOREM 10. *Let Γ be an analytic Jordan curve and D its interior. If $f(z)$ is meromorphic in D with precisely ν poles there and is otherwise finite and continuous on $D+\Gamma$, then the following statements are equivalent:*

(i) *$f(z)$ belongs to class $L_\nu(k, \alpha)$ on Γ.*

(ii) *There exists a sequence of rational functions $r_{n\nu}(z)$ of respective types (n, ν) such that*

$$|f(z)-r_{n\nu}(z)| \leqq A/n^{k+\alpha}, \qquad z \text{ on } \Gamma.$$

(iii) *There exists a domain D_0 containing $D+\Gamma$ and a sequence of meromorphic functions $f_n(z)$ of the form*

$$f_n(z) = f_{n1}(z)/f_{n2}(z), \tag{7}$$

where $f_{n1}(z)$ is analytic in D_0 and $f_{n2}(z)$ is a polynomial of the form $z^\lambda+a_1z^{\lambda-1}+\cdots+a_\lambda$, $0\leqq\lambda\leqq\nu$, such that the following inequalities hold:

$$|f_{n1}(z)| \leqq A_1R^n, \qquad z \text{ in } D_0,$$
$$|f(z)-f_n(z)| \leqq A_2/n^{k+\alpha}, \qquad z \text{ on } \Gamma.$$

Proof. Suppose $f(z) \in L_\nu(k, \alpha)$ on Γ, and let $q(z)$ be the monic polynomial of degree ν whose zeros are the poles of $f(z)$ in D. It is easy to see that $q(z)f(z) \in L_0(k, \alpha)$ on Γ, and hence there exists a sequence of polynomials $p_n(z)$ of respective degrees n such that

$$|q(z)f(z)-p_n(z)| \leqq B/n^{k+\alpha}, \qquad z \text{ on } \Gamma.$$

Whence

$$|f(z)-p_n(z)/q(z)| \leqq A/n^{k+\alpha}, \qquad z \text{ on } \Gamma,$$

and so (i) implies (ii).

Now assume (ii) holds and write $r_{n\nu}(z)=P_n(z)/Q_n(z)$, where $P_n(z)$ is a polynomial of degree n and $Q_n(z)$ is the polynomial of the form $z^\mu+\cdots+a$, $0 \leqq \mu \leqq \nu$, whose zeros are the finite poles of $r_{n\nu}(z)$. Theorem 5 implies that the $Q_n(z)$ are uniformly bounded on Γ, and since the $r_{n\nu}(z)$ are also uniformly bounded there, the same must be true of the $P_n(z)$. Thus by the Generalized Bernstein Lemma [2, p. 77] we deduce for z on any bounded domain D_0, $|P_n(z)| \leqq A_1 R^n$.

It remains to show that (iii) implies (i). The uniform convergence of the $f_n(z)$ on Γ implies, by Theorem 5, that the zeros of the $f_{n2}(z)$ approach the poles of $f(z)$ in D. Hence there exists an annular region D_1 containing Γ such that for n sufficiently large each $f_n(z)$ is analytic in D_1 and satisfies there the inequality $|f_n(z)| \leqq A_3 R^n$. Statement (i) now follows from Theorem 8.

Theorem 10, in contrast with Theorem 9, assumes not merely that $f(z)$ be defined on Γ, but that $f(z)$ be the boundary values on Γ of a meromorphic function known to have precisely ν poles interior to Γ. This hypothesis can be weakened by assuming, instead, that all the finite poles of the rational functions $r_{n\nu}(z)$ lie in D. In the proof of such a result it is convenient to have for reference

LEMMA 1. *Suppose $f(z)$ is meromorphic in $U: |z|<1$ with precisely μ $(\geqq 0)$ poles there, and is otherwise finite and continuous on $|z| \leqq 1$. If $\alpha_1, \alpha_2, \ldots, \alpha_\mu$ are the poles of $f(z)$ in U and if $r(z)$ is a rational function of type (n, ν), $n \geqq \nu$, having all its finite poles in U, then there exists a rational function $R(z)$ of the form*

$$R(z) = \frac{c_0 z^{n+\mu}+c_1 z^{n+\mu-1}+\cdots+c_{n+\mu}}{(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_\mu)},$$

such that

$$[\max |f(z)-R(z)|;\ |z| = 1] \leqq 2^\nu[\max |f(z)-r(z)|;\ |z| = 1].$$

Proof. If $r(z)$ has no finite pole, we simply take $R(z) \equiv r(z)$. Otherwise let $\beta_1, \beta_2, \ldots, \beta_\lambda$ be the finite poles of $r(z)$ and let

$$T(z) \equiv \prod_{i=1}^{\mu} (z-\alpha_i)/(1-\bar{\alpha}_i z), \qquad B(z) \equiv T(z) \prod_{i=1}^{\lambda} (z-\beta_i)/(1-\bar{\beta}_i z).$$

We note that $S_0(z) \equiv B(z)r(z)$ is a rational function of the form

$$S_0(z) = q_0(z)/(1-\bar{\alpha}_1 z)\cdots(1-\bar{\alpha}_\mu z)(1-\bar{\beta}_1 z)\cdots(1-\bar{\beta}_\lambda z),$$

where $q_0(z)$ is a polynomial of degree $n+\mu$. Setting $M\equiv[\max|f(z)-r(z)|;\ |z|=1]$ we obtain from the Maximum Principle

$$|B(z)f(z)-S_0(z)| \leqq M, \qquad |z| \leqq 1. \tag{8}$$

Since $B(\beta_\lambda)f(\beta_\lambda)=0$, the triangle inequality yields

$$|B(z)f(z)-(S_0(z)-S_0(\beta_\lambda))| \leqq 2M, \qquad |z| = 1,$$

and hence

$$|[(1-\bar{\beta}_\lambda z)/(z-\beta_\lambda)]B(z)f(z)-S_1(z)| \leqq 2M, \qquad |z| = 1, \tag{9}$$

where $S_1(z)\equiv(S_0(z)-S_0(\beta_\lambda))(1-\bar{\beta}_\lambda z)/(z-\beta_\lambda)$. Note that $S_1(z)$ is a rational function of the form

$$S_1(z) = q_1(z)/(1-\bar{\alpha}_1 z)\cdots(1-\bar{\alpha}_\mu z)(1-\bar{\beta}_1 z)\cdots(1-\bar{\beta}_{\lambda-1}z),$$

where $q_1(z)$ is a polynomial of degree $n+\mu$.

Since inequality (9) holds for $|z|\leqq 1$, the same reasoning used to deduce (9) as a consequence of (8) yields

$$\left|\left[\frac{1-\bar{\beta}_{\lambda-1}z}{z-\beta_{\lambda-1}}\right]\left[\frac{1-\bar{\beta}_\lambda z}{z-\beta_\lambda}\right]B(z)f(z)-S_2(z)\right| \leqq 4M, \qquad |z| = 1,$$

where $S_2(z)$ is a rational function of the form

$$S_2(z) = q_2(z)/(1-\bar{\alpha}_1 z)\cdots(1-\bar{\alpha}_\mu z)(1-\bar{\beta}_1 z)\cdots(1-\bar{\beta}_{\lambda-2}z),$$

and $q_2(z)$ is a polynomial of degree $n+\mu$.

After λ steps we obtain a polynomial $q_\lambda(z)$ of degree $n+\mu$ which satisfies

$$\left|T(z)f(z)-\frac{q_\lambda(z)}{(1-\bar{\alpha}_1 z)\cdots(1-\bar{\alpha}_\mu z)}\right| \leqq 2^\lambda M \leqq 2^\nu M, \qquad |z| = 1,$$

and so Lemma 1 follows by taking $R(z)\equiv q_\lambda(z)/(z-\alpha_1)\cdots(z-\alpha_\mu)$.

We may now prove

THEOREM 11. *Let $g(z)$ be a function defined (finite) on an analytic Jordan curve Γ with interior D. Then the following statements are equivalent:*

(i) *$g(z)$ is the set of boundary values on Γ of a function which belongs to class $L_\nu(k,\alpha)$ on Γ.*

(ii) *There exists a sequence of rational functions $r_{n\nu}(z)$ of respective types (n,ν) having all their finite poles in D and satisfying*

$$|g(z)-r_{n\nu}(z)| \leqq A/n^{k+\alpha}, \qquad z \text{ on } \Gamma.$$

(iii) *There exists a domain D_0 containing $D+\Gamma$ and a sequence of meromorphic functions $f_n(z)$ of the form (7), where all the zeros of $f_{n2}(z)$ lie in D, such that*

$$|f_{n1}(z)| \leqq A_1 R^n, \qquad z \text{ in } D_0, \tag{10}$$

$$|g(z)-f_n(z)| \leqq A_2/n^{k+\alpha}, \qquad z \text{ on } \Gamma. \tag{11}$$

Proof. That (i) implies (ii) and (iii) is immediate from Theorem 10. Therefore since (ii) clearly implies (iii), we need only show (iii) implies (i).

Assuming statement (iii) holds, Theorem 3 asserts the existence of a function $f(z)$ which is meromorphic with precisely μ $(\leqq \nu)$ poles in D, is continuous on $D+\Gamma$, and equal to $g(z)$ for z on Γ. Let $z=\psi(w)$ map $U:|w|<1$ conformally onto D, and set $F(w)\equiv f(\psi(w))$. We shall prove $F(w)\in L_\nu(k,\alpha)$ on $C:|w|=1$, which is equivalent to statement (i).

Since Γ is analytic, there exists a constant ρ (>1) such that $\psi(w)$ is analytic on $|w|\leqq\rho$ and such that the image of $|w|\leqq\rho$ under $z=\psi(w)$ lies in D_0. From (10) and the fact that all the zeros of $f_{n2}(z)$ lie in D we have

$$|f_n(\psi(w))| \leqq MR^n, \qquad w \text{ on } C_\rho:|w| = \rho. \tag{12}$$

The function $s_n(w)\equiv f_n(\psi(w))$ is analytic on $|w|\leqq\rho$ except for a finite number of poles, say at the points $\beta_{n1}, \beta_{n2}, \ldots, \beta_{n,\lambda(n)}$. Since $\psi(w)$ is schlicht and since all the poles of $f_n(z)$ lie in D, it follows that $\lambda(n)\leqq\nu$, and that $|\beta_{nj}|<1$ for $j=1, 2, \ldots, \lambda(n)$.

Now let $P_{n,N}(w)$ be the polynomial in w of degree $N+\lambda(n)-1$ which interpolates to the analytic function

$$S_n(w) \equiv s_n(w)(w-\beta_{n1})(w-\beta_{n2})\cdots(w-\beta_{n,\lambda(n)})$$

in the points $\beta_{n1}, \beta_{n2}, \ldots, \beta_{n,\lambda(n)}$ and in the origin taken of multiplicity N. The Hermite Interpolation Formula asserts

$$S_n(w)-P_{n,N}(w) = \frac{1}{2\pi i}\int_{C_\rho} \frac{(w-\beta_{n1})\cdots(w-\beta_{n,\lambda(n)})w^N S_n(t)}{(t-\beta_{n1})\cdots(t-\beta_{n,\lambda(n)})t^N(t-w)}\,dt$$

for w on C, and so

$$s_n(w)-\frac{P_{n,N}(w)}{(w-\beta_{n1})\cdots(w-\beta_{n,\lambda(n)})} = \frac{1}{2\pi i}\int_{C_\rho}\frac{w^N s_n(t)}{t^N(t-w)}\,dt,$$

for w on C. Thus from (12) we deduce

$$|s_n(w)-T_{n,N}(w)| \leqq M_1R^n/\rho^N, \qquad w \text{ on } C,$$

where $T_{n,N}(w)\equiv P_{n,N}(w)/(w-\beta_{n1})\cdots(w-\beta_{n,\lambda(n)})$.

Now choose a positive integer τ so large that $\gamma\equiv R/\rho^\tau<1$. Then

$$|s_n(w)-T_{n,\tau n}(w)| \leqq M_1\gamma^n, \qquad w \text{ on } C,$$

and hence from (11) and the triangle inequality there follows

$$|F(w)-T_{n,\tau n}(w)| \leqq A_2/n^{k+\alpha}+M_1\gamma^n \leqq A_3/n^{k+\alpha},$$

for w on C. We note that $T_{n,\tau n}(w)$ is a rational function of type $(\tau n+\nu-1, \nu)$ having all its finite poles in U. Thus since $F(w)$ is meromorphic with precisely μ poles in U and is continuous on $|w|\leqq 1$, Lemma 1 implies that there exists a sequence of rational functions $R_n(w)$ of respective types $(\tau n+\nu-1+\mu, \mu)$ satisfying

$$|F(w)-R_n(w)| \leqq A_4/n^{k+\alpha}, \qquad w \text{ on } C,$$

and having all their finite poles on a closed set interior to C. It is easy to see that for $\delta(>0)$ sufficiently small we have

$$|R_n(w)| \leqq M_2(\rho^\tau)^n, \qquad 1-\delta < |w| < \rho,$$

and so Theorem 8 implies $F(w) \in L_\nu(k, \alpha)$ on C, which completes the proof.

Theorem 11 and [2, §9.7, Lemma I] yield

COROLLARY 5. *Suppose $g(z)$ is defined on an analytic Jordan curve Γ and $r_{n\nu}(z)$ is a sequence of rational functions of respective types (n, ν) satisfying*

$$|g(z)-r_{n\nu}(z)| \leqq A/n^{k+\alpha}, \qquad z \text{ on } \Gamma,$$

where k is a nonnegative integer and $0<\alpha<1$. If no point of Γ is a limit point of those poles of the $r_{n\nu}(z)$ which lie exterior to Γ, then the kth derivative of $g(z)$ exists on Γ and satisfies a Lipschitz condition of order α there.

In the theorems of this section the case $\alpha=1$ is excluded. However, Theorem 9 holds [7] if the Lipschitz condition of order unity on Γ is replaced by the Zygmund condition

$$|f(x+h)+f(x-h)-2f(x)| \leqq L|h|,$$

with respect to arc length on Γ. The extensions of Theorem 10 and Theorem 11 to this exceptional case are immediate.

4. **Overconvergence.** The theorems of §2 and §3 dealt with approximation to a function meromorphic interior to a closed curve Γ and continuous on Γ. We turn now to the questions of analyticity on Γ and its relationship to the overconvergence of sequences of rational functions. The term *overconvergence* is here meant to describe the phenomenon that certain sequences which converge sufficiently rapidly on Γ necessarily converge on a point set containing Γ in its interior.

Of fundamental importance in the study of overconvergence of sequences of rational functions of type (n, ν) is a lemma [8] due to J. L. Walsh. We state this result in the following slightly more general form:

LEMMA 2. *Let E, with boundary Γ, be a closed bounded point set whose complement (with respect to the extended plane) K is connected and regular in the sense that K possesses a Green's function $G(z)$ with pole at infinity. Let $\Gamma_\sigma(\sigma>1)$ denote generically the locus $G(z)=\log\sigma$, and suppose that rational functions $r_{n\nu}(z)$ of respective types (n, ν) satisfy the inequality*

$$\limsup_{n\to\infty}\,[\max|r_{n\nu}(z)|;\ z \text{ on } \Gamma]^{1/n} \leqq 1/\rho, \qquad 1 < \rho \leqq \infty. \tag{13}$$

Let S be a closed set in the interior of Γ_σ, $1<\sigma<\rho$, and containing no limit point of the poles of the $r_{n\nu}(z)$. Then the sequence $r_{n\nu}(z)$ converges uniformly to zero on S, and we have

$$\limsup_{n\to\infty}\,[\max|r_{n\nu}(z)|;\ z \text{ on } S]^{1/n} \leqq \sigma/\rho. \tag{14}$$

The $r_{n\nu}(z)$ need not be defined for every n.

Because of the lemma's frequent use we submit a new and brief

Proof. Let $\sigma<\mu<\infty$ and let $q_n(z)$ be the polynomial of the form $q_n(z)=z^{\lambda_n}+\cdots+a_{\lambda_n}$, $0\leqq\lambda_n\leqq\nu$, whose zeros are those poles of $r_{n\nu}(z)$ which lie interior to Γ_μ. We note that the function $s_n(z)\equiv q_n(z)r_{n\nu}(z)$ is a rational function of type (n, n) whose poles lie on or exterior to Γ_μ. From (13) and the uniform boundedness of the $q_n(z)$ on Γ we have

$$\limsup_{n\to\infty}\,[\max|s_n(z)|;\ z \text{ on } E]^{1/n} \leqq 1/\rho,$$

and so from [2, §9.7, Lemma I] there follows

$$\limsup_{n\to\infty}\,[\max|s_n(z)|;\ z \text{ on } S]^{1/n} \leqq (\mu\sigma-1)/(\mu-\sigma)\rho.$$

Since S contains no limit point of the poles of the $r_{n\nu}(z)$, the functions $q_n(z)$ are for n sufficiently large uniformly bounded below in modulus by a positive constant on S and hence

$$\limsup_{n\to\infty}\,[\max|r_{n\nu}(z)|;\ z \text{ on } S]^{1/n} \leqq (\mu\sigma-1)/(\mu-\sigma)\rho.$$

Letting $\mu\to\infty$ we obtain (14).

An easy application of Lemma 2 to analytic continuation is

THEOREM 12. *Let E, Γ, and Γ_σ be as in Lemma 2 and let E_σ denote the interior of Γ_σ. Suppose that $g(z)$ is a function defined (finite) on Γ and $r_{n\nu}(z)$ is a sequence of rational functions of respective types (n, ν) which satisfy the inequality*

$$\limsup_{n\to\infty}\,[\sup|g(z)-r_{n\nu}(z)|;\ z \text{ on } \Gamma]^{1/n} \leqq 1/\rho < 1. \tag{15}$$

If no point of Γ_ρ is a limit point of those poles of the $r_{n\nu}(z)$ which lie in E_ρ, then there exists a function $f(z)$ which is meromorphic with at most ν poles in E_ρ such that $f(z)\equiv g(z)$ for z on Γ.

Proof. Set $t_n(z)\equiv r_{n\nu}(z)-r_{n-1,\nu}(z)$, and note that the $t_n(z)$ form a sequence of rational functions of respective types $(n+\nu, 2\nu)$ which satisfies

$$\limsup_{n\to\infty}\,[\max|t_n(z)|;\ z \text{ on } \Gamma]^{1/n} \leqq 1/\rho.$$

For $\varepsilon>0$ sufficiently small, none of the poles of the $r_{n\nu}(z)$ and hence of the $t_n(z)$ lie on $\Gamma_{\rho-\varepsilon}$. Thus from Lemma 2 we deduce

$$\limsup_{n\to\infty}\,[\max|t_n(z)|;\ z \text{ on } \Gamma_{\rho-\varepsilon}]^{1/n} \leqq (\rho-\varepsilon)/\rho,$$

which implies that the $r_{n\nu}(z)$ are uniformly bounded on $\Gamma_{\rho-\varepsilon}$.

Now write $r_{n\nu}(z)=h_n(z)/q_n(z)$, where $h_n(z)$ is analytic in $E_{\rho-\varepsilon}$ and $q_n(z)$ is the monic polynomial whose zeros are those poles of $r_{n\nu}(z)$ which lie in $E_{\rho-\varepsilon}$. The uniform boundedness of the $r_{n\nu}(z)$ and the $q_n(z)$ on $\Gamma_{\rho-\varepsilon}$ implies that the $h_n(z)$ form a normal family in $E_{\rho-\varepsilon}$. Thus there exists a subsequence $s_k(z)$ of the $r_{n\nu}(z)$ and a function $f(z)$ meromorphic with at most ν poles in $E_{\rho-\varepsilon}$ such that $\lim_{k\to\infty} s_k(z) = f(z)$ uniformly on each closed subset of an open set D obtained from $E_{\rho-\varepsilon}$ by the omission of at most ν points. Clearly the identity $f(z)=g(z)$ holds with at most ν exceptions for z on Γ, and hence by the continuity of $f(z)$ and $g(z)$ on Γ the identity holds everywhere on Γ. Theorem 12 now follows from the arbitrariness of ε.

COROLLARY 6. *With the geometric conditions of Lemma 2 suppose that the function $F(z)$ is meromorphic with precisely ν poles in the interior T of E and is otherwise finite and continuous on E. If there exists a sequence of rational functions $r_{n\nu}(z)$ of respective types (n, ν) satisfying*

$$\limsup_{n\to\infty} [\max |F(z)-r_{n\nu}(z)|;\ z \text{ on } \Gamma]^{1/n} \leqq 1/\rho < 1,$$

then $F(z)$ can be extended so as to be analytic on $E_\rho - T$.

Proof. By Theorem 5 the sequence $r_{n\nu}(z)$ converges to $F(z)$ on E and the finite poles of the $r_{n\nu}(z)$ approach the ν poles of $F(z)$ in T. It then follows from the proof of Theorem 12 that there exists a function $f(z)$ meromorphic with at most ν poles in E_ρ such that $f(z)\equiv F(z)$ for z on E. Since $f(z)$ must be analytic on $E_\rho - T$ it is the desired continuation.

We conclude with an extension of [1, Theorem 3]:

THEOREM 13. *With the geometric conditions of Lemma 2 suppose the function $f(z)$ is analytic on Γ and is meromorphic with precisely ν poles in E_ρ $(\rho > 1)$. Suppose $r_{n\nu}(z)$ is a sequence of rational functions of respective types (n, ν) which satisfy*

$$\limsup_{n\to\infty} [\max |f(z)-r_{n\nu}(z)|;\ z \text{ on } \Gamma]^{1/n} \leqq 1/\rho.$$

Then for n sufficiently large each $r_{n\nu}(z)$ has precisely ν finite poles, which approach respectively the ν poles of $f(z)$ in E_ρ; and the $r_{n\nu}(z)$ converge uniformly to $f(z)$ on each compact subset of E_ρ which contains no pole of $f(z)$.

Theorem 13 generalizes [1, Theorem 3] since it does not assume that $f(z)$ is analytic on E. The proof of Theorem 13, which is left to the reader, follows from Theorem 5, Lemma 2, and the methods used in [1].

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