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On Blaschke products associated with n -widths

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Abstract

Let E be a closed subset of the open unit disk $G = \{z : |z| < 1\}$, and let μ be a positive Borel measure with support $\text{supp } \mu = E$. Denote by A_p the restriction on E of the closed unit ball of the Hardy space $H_p(G)$, $1 \leq p \leq \infty$. In this paper we investigate orthogonality properties of the extremal functions associated with the Kolmogorov, Gelfand, and linear n -widths of A_p in $L_q(\mu, E)$, $1 \leq q < \infty$, $q \leq p$.

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1. Introduction

1.1. Overview

Let $G = \{z : |z| < 1\}$ be the open unit disk in the complex plane \mathbf{C} and let $\Gamma = \{z : |z| = 1\}$. We assume that the circle Γ is positively oriented with respect to G . Let E be a compact subset of G , and let μ be a finite positive Borel measure with support $\text{supp } \mu = E$. We further assume that E contains infinitely many points.

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Denote by $H_p(G)$, $1 \leq p \leq \infty$, the Hardy space of those analytic functions g on G such that $|g|^p$ has a harmonic majorant there. As is well known, such functions have nontangential boundary values a.e. on Γ that establish a one-to-one correspondence between $H_p(G)$ and the closed subspace of $L_p(\Gamma)$ consisting of functions whose Fourier coefficients of strictly negative index do vanish; a function in $H_p(G)$ is recovered from its boundary values through a Cauchy as well as a Poisson integral. We refer the reader to [9] for details on Hardy spaces, and we merely recall here a few facts that will be of relevance to us.

By a theorem of Szegő, we have that $\log |g| \in L_1(\Gamma)$ whenever $g \in H_p(G)$ is not the zero function. This entails that a $H_p(G)$ -function is uniquely defined by the values it assumes on a subset of Γ of positive Lebesgue measure. Conversely, whenever $\rho \in L_p(\Gamma)$ is a positive function such that $\log \rho \in L_1(\Gamma)$, the function

$$E_\rho(z) = \exp \left\{ \frac{1}{2\pi} \int_\Gamma \frac{\xi + z}{\xi - z} \log \rho(\xi) d|\xi| \right\}, \quad z \in G, \tag{1.1}$$

lies in $H_p(G)$ and has modulus ρ a.e. on Γ . This E_ρ is called the normalized *outer function* associated with ρ , the normalization being that $E_\rho(0) > 0$. More generally, a function is said to be *outer* in $H_p(G)$ if it is of the form cE_ρ with c a unimodular constant. Obviously, an outer function has no zero in G . Granted the normalization condition, the outer function E_ρ is characterized by two facts, namely:

- (i) $|E_\rho| = \rho$ a.e. on Γ ,
- (ii) among all $H_p(G)$ -functions that satisfy (i), E_ρ is largest-in-modulus pointwise on G .

A particular type of $H_\infty(G)$ -functions will also be important to us, namely *finite Blaschke products*. These are the rational functions that are analytic in G and of unit modulus on Γ ; they assume the form q/q^* , where q is an algebraic polynomial whose roots lie in G and where q^* indicates the *reciprocal polynomial* given by $q^*(z) = z^n \overline{q(1/\bar{z})}$ if n is the degree of q . The integer n is also called the degree of the Blaschke product, and the latter is called *normalized* if q is monic. For any positive integer n , we let \mathcal{B}_n denote the class of normalized Blaschke products of degree n ; upon splitting q into linear factors in the previous definition, we see that each $B \in \mathcal{B}_n$ can be uniquely written as

$$B(z) = \prod_{k=1}^n \frac{z - \xi_k}{1 - \bar{\xi}_k z}, \quad \xi_k \in G. \tag{1.2}$$

Let \mathbf{A}_p be the restriction on E of the closed unit ball of the Hardy space $H_p(G)$. Fisher and Stessin [7,8] proved that in two important cases: when $1 \leq q \leq p < \infty$, or when $1 \leq q < \infty$, $p = 2$,

$$\begin{aligned} d^n(\mathbf{A}_p, L_q(\mu, E)) &= d_n(\mathbf{A}_p, L_q(\mu, E)) = \delta_n(\mathbf{A}_p, L_q(\mu, E)) \\ &= \inf_{B \in \mathcal{B}_n} \sup_{\varphi \in \mathbf{A}_p} \|\varphi B\|_{q,\mu}, \end{aligned}$$

where d^n , d_n , and δ_n are the Kolmogorov, Gelfand and linear n -widths of \mathbf{A}_p in the space $L_q(\mu, E)$ (see, for example, [10]), and $\|\cdot\|_{q,\mu}$ is the norm in the space $L_q(\mu, E)$.

Let $1 \leq q < \infty$, $1 \leq p \leq \infty$. Set

$$m_n = m_n(p, q, \mu) = \inf_{B \in \mathcal{B}_n} \sup_{\varphi \in \mathbf{A}_p} \|\varphi B\|_{q,\mu}. \quad (1.3)$$

In this paper we investigate orthogonal properties of the extremal functions φ_n and B_n which attain the value m_n :

$$m_n = \|\varphi_n B_n\|_{q,\mu}. \quad (1.4)$$

That φ_n and B_n indeed exist follows from the fact that the “inf-sup” in (1.3) is certainly attained if the infimization is extended to Blaschke product of degree *at most* n , because the restriction of this set to E is compact in $L_\infty(E)$ and so is \mathbf{A}_p in $L_q(\mu, E)$; but the “inf” is obviously attained on \mathcal{B}_n , because each elementary factor in (1.2) has modulus strictly less than 1 on E . Necessarily φ_n is outer of $L_p(\Gamma)$ -norm exactly 1, otherwise it could not meet the “sup” in (1.3). Clearly $\varphi_n \equiv 1$ for $p = \infty$, and for $1 \leq p < \infty$ is known to satisfy the following equation:

$$m_n^q |\varphi_n(\xi)|^p = \frac{1}{2\pi} \int_E \frac{1 - |x|^2}{|\xi - x|^2} |(\varphi_n B_n)(x)|^q d\mu(x), \quad \xi \in \Gamma, \quad (1.5)$$

see [7, Proposition 1]. One consequence is that $|\varphi_n|$ extends continuously on Γ (see [6,7]). Actually, since $|\varphi_n|$ is strictly positive and C^1 -smooth on Γ by (1.5), so is $\log |\varphi_n|$ whose conjugate function $\text{Arg } \varphi_n$ is then continuous, and therefore we see upon taking the exponential that the outer function φ_n itself is continuous on Γ . If $q \leq p$ then φ_n is uniquely determined by B_n up to unimodular scalar multiples, but this may fail if $p < q$ (it is nevertheless true if E is hyperbolically small, see [7]). To avoid trivial cases of nonuniqueness, we assume throughout without loss of generality that $\varphi_n(0) > 0$.

In Sections 2 and 3 we establish orthogonality properties of the extremal functions φ_n and B_n when $q \leq p$. The authors do not know whether analogous results hold when $p < q$. In Section 4, a connection with meromorphic approximation is investigated.

1.2. Notation

Above and thereafter, $L_p(\Gamma)$, $1 \leq p \leq \infty$, stands for the Lebesgue space of functions φ measurable on Γ , with the norm

$$\|\varphi\|_p = \left(\int_\Gamma |\varphi(\xi)|^p |d\xi| \right)^{1/p} \quad \text{if } 1 \leq p < \infty \quad (1.6)$$

and

$$\|\varphi\|_\infty = \text{ess sup}_\Gamma |\varphi(\xi)| \quad \text{if } p = \infty.$$

As well, $L_q(\mu, E)$, $1 \leq q < \infty$, is the Lebesgue space of functions φ on E with the norm

$$\|\varphi\|_{q,\mu} = \left(\int_E |\varphi(x)|^q d\mu(x) \right)^{1/q} < \infty.$$

Finally, for σ a positive Borel measure with support $\text{supp } \sigma = E$, we denote by $J : H_2(G) \rightarrow L_2(\sigma, E)$ the *embedding* operator. The operator J is given by restricting an element $\varphi \in H_2(G)$ to E : $J\varphi = \varphi|_E$. Let $J^* : L_2(\sigma, E) \rightarrow H_2(G)$ be the adjoint of J . It is easily verified that for $\varphi \in H_2(G)$

$$(J^*J)(\varphi)(z) = \frac{1}{2\pi} \int_E \frac{\varphi(x)}{1 - z\bar{x}} d\sigma(x), \quad |z| < 1 \tag{1.7}$$

(see, for example, [5]).

2. Orthogonality properties of φ_n and B_n

Fix $1 \leq q < \infty$, $1 \leq p < \infty$, and a positive integer n . It is not hard to see that (1.5) is equivalent to the following relation:

$$\int_E u(x) |(\varphi_n B_n)(x)|^q d\mu(x) = m_n^q \int_\Gamma u(\xi) |\varphi_n(\xi)|^p |d\xi|, \tag{2.1}$$

where u is any function harmonic on G and continuous on \bar{G} . Equality (2.1) implies that

$$(|(\varphi_n B_n)(x)|^q d\mu(x))^*(\xi) = m_n^q |\varphi_n(\xi)|^p |d\xi|, \quad \xi \in \Gamma,$$

where $(|\varphi_n B_n|^q d\mu)^*$ is the balayage of $|\varphi_n B_n|^q d\mu$ on Γ (see, for example, [11]). It follows from (2.1) that for any $g \in H_1(G)$

$$\int_E g(x) |(\varphi_n B_n)(x)|^q d\mu(x) = m_n^q \int_\Gamma g(\xi) |\varphi_n(\xi)|^p |d\xi|. \tag{2.2}$$

We represent $B_n(x)$ in the form $B_n(x) = w_n(x)/w_n^*(x)$, where

$$w_n(x) = \prod_{k=1}^n (x - x_{k,n}), \quad w_n^*(x) = \prod_{k=1}^n (1 - \bar{x}_{k,n}x),$$

and $x_{1,n}, x_{2,n}, \dots, x_{n,n}$ are zeros of B_n , $x_{k,n} \in G$, $k = 1, \dots, n$.

We now prove that for $1 \leq q \leq p < \infty$

$$\int_E \left(\frac{x^k}{w_n^*(x)} \overline{B_n(x)} - \overline{\left(\frac{x^{n-k} w_n(x)}{(w_n^*(x))^2} \right)} B_n(x) \right) |\varphi_n(x)|^q |B_n(x)|^{q-2} d\mu(x) = 0, \tag{2.3}$$

$$k = 0, \dots, n - 1.$$

For $B \in \mathcal{B}_n$ and $\varphi \in \mathbf{A}_p$, we set

$$\Psi(B, \varphi) \triangleq \int_E |(\varphi B)(x)|^q d\mu(x)$$

and we denote by φ_B the unique function (up to unimodular scalar multiples) in \mathbf{A}_p such that

$$\Psi(B, \varphi_B) = \sup_{\varphi \in \mathbf{A}_p} \Psi(B, \varphi). \tag{2.4}$$

Necessarily φ_B is outer, so we normalize it as usual by setting $\varphi_B(0) > 0$. We write a generic $B \in \mathcal{B}_n$ as:

$$B(x) = \frac{\alpha_0 + \alpha_1 x + \dots + \alpha_{n-1} x^{n-1} + x^n}{\bar{\alpha}_0 x^n + \bar{\alpha}_1 x^{n-1} + \dots + \bar{\alpha}_{n-1} x + 1}, \quad \alpha_j \in \mathbf{C},$$

and we single out B_n to be

$$B_n(x) = \frac{a_0 + a_1 x + \dots + a_{n-1} x^{n-1} + x^n}{\bar{a}_0 x^n + \bar{a}_1 x^{n-1} + \dots + \bar{a}_{n-1} x + 1},$$

where the a_j are the coefficients of w_n .

When B ranges over \mathcal{B}_n , then $(\alpha_0, \dots, \alpha_{n-1})$ ranges over an open subset Ω of \mathbf{C}^n , and this way we coordinatize \mathcal{B}_n . Clearly, Ψ is jointly continuous with respect to $(\alpha_j)_{0 \leq j \leq n-1} \in \Omega$ and $\varphi \in \mathbf{A}_p$ when the latter is endowed with the topology induced by the sup-norm on E . Observe also that \mathbf{A}_p is compact for that topology by the Cauchy formula and the weak-compactness of balls in $H_p(G)$. Moreover, Ψ has partial derivatives with respect to the real and imaginary parts of the α_j that are likewise jointly continuous with respect to $(\alpha_j)_{0 \leq j \leq n-1}$ and φ . Since φ_B is the unique argument of the maximum in (2.4), it now follows from [4, Chapter III, Theorem 1] that $\Psi(B, \varphi_B)$ in turn has partial derivatives with respect to the real and imaginary parts of the α_j , given by

$$\frac{\partial \Psi(B, \varphi_B)}{\partial \operatorname{Re}(\alpha_j)} = \left. \frac{\partial \Psi(B, \varphi)}{\partial \operatorname{Re}(\alpha_j)} \right|_{\varphi = \varphi_B}, \quad \frac{\partial \Psi(B, \varphi_B)}{\partial \operatorname{Im}(\alpha_j)} = \left. \frac{\partial \Psi(B, \varphi)}{\partial \operatorname{Im}(\alpha_j)} \right|_{\varphi = \varphi_B}.$$

Because $B \mapsto \Psi(B, \varphi_B)$ meets a minimum on \mathcal{B}_n at $B = B_n$, or in coordinates on Ω at $\alpha_j = a_j$, $0 \leq j \leq n-1$, the above partial derivatives must vanish at this point and writing that

$$\left(\frac{\partial \Psi(B, \varphi_B)}{\partial \operatorname{Re}(\alpha_k)} - i \frac{\partial \Psi(B, \varphi_B)}{\partial \operatorname{Im}(\alpha_k)} \right) \Big|_{B=B_n} = 0,$$

while taking into account that $\varphi_{B_n} = \varphi_n$, yields (2.3) upon differentiating under the integral sign.

Let E^{-1} be the reflection of E in the unit circle and let D be the component of $\bar{\mathbf{C}} \setminus E^{-1}$ that contains the unit disk G . We have the following theorem (see [2] for $q = 2$ and $E = (-1, 1)$).

Theorem 1. *The function $\varphi_n^{p/2}$ can be extended analytically to D , and satisfies the equations*

$$m_n^q(\varphi_n^{p/2})(\xi) = \frac{1}{2\pi} \int_E \frac{|(\varphi_n B_n)(x)|^q}{(1 - \xi \bar{x}) \varphi_n^{p/2}(x)} d\mu(x), \quad \xi \in D, \quad 1 \leq q < \infty, 1 \leq p < \infty \tag{2.5}$$

and

$$m_n^q(B_n \varphi_n^{p/2})(\xi) = \frac{1}{2\pi} \int_E \frac{|(\varphi_n B_n)(x)|^q}{(1 - \xi \bar{x}) B_n(x) \varphi_n^{p/2}(x)} d\mu(x), \quad \xi \in D, 1 \leq q \leq p < \infty. \tag{2.6}$$

The following orthogonality relations are valid:

$$\int_E \frac{x^k \overline{w_n(x)}}{|w_n^*(x)|^2 \varphi_n^{p/2}(x)} |\varphi_n(x)|^q |B_n(x)|^{q-2} d\mu(x) = 0, \quad k = 0, \dots, n-1, \tag{2.7}$$

$$1 \leq q \leq p < \infty.$$

Proof. Let φ be any function in $H_2(G)$, and let $g = \varphi/\varphi_n^{p/2}$. Here and in what follows we take that branch of the $(p/2)$ th root that is positive on the positive part of the real line. By (2.2), we can write

$$\int_E \varphi(x) \overline{\varphi_n^{p/2}(x)} |\varphi_n(x)|^{q-p} |B_n(x)|^q d\mu(x) = m_n^q \int_\Gamma \varphi(\xi) \overline{\varphi_n^{p/2}(\xi)} |d\xi|.$$

Therefore,

$$J^* J(\varphi_n^{p/2}) = m_n^q \varphi_n^{p/2},$$

where $J : H_2(G) \rightarrow L_2(|\varphi_n|^{q-p} |B_n|^q d\mu, E)$ is the embedding operator. From this, on the basis of formula (1.7) where $d\sigma = |\varphi_n|^{q-p} |B_n|^q d\mu$, we obtain (2.5).

We now can rewrite (2.3) in the form

$$\int_E \frac{x^k}{w_n(x)} |(\varphi_n B_n)(x)|^q d\mu(x) = \int_E \left(\frac{x^{n-k}}{w_n^*(x)} \right) |(\varphi_n B_n)(x)|^q d\mu(x).$$

By (2.2),

$$\begin{aligned} \int_E \left(\frac{x^{n-k}}{w_n^*(x)} \right) |(\varphi_n B_n)(x)|^q d\mu(x) &= m_n^q \int_\Gamma \left(\frac{\xi^{n-k}}{w_n^*(\xi)} \right) |\varphi_n(\xi)|^p |d\xi| \\ &= m_n^q \int_\Gamma \frac{\xi^k}{w_n(\xi)} |\varphi_n(\xi)|^p |d\xi|. \end{aligned} \tag{2.8}$$

Hence, we have

$$\int_E \frac{x^k}{w_n(x)} |(\varphi_n B_n)(x)|^q d\mu(x) = m_n^q \int_\Gamma \frac{\xi^k}{w_n(\xi)} |\varphi_n(\xi)|^p |d\xi|. \tag{2.9}$$

By (2.2) and (2.9), for any $g \in H_1(G)$

$$\int_E \frac{g(x)}{w_n(x)} |(\varphi_n B_n)(x)|^q d\mu(x) = m_n^q \int_\Gamma \frac{g(\xi)}{w_n(\xi)} |\varphi_n(\xi)|^p |d\xi|. \tag{2.10}$$

Letting

$$g(x) = \frac{w_n^*(x)}{(1 - \bar{i}x)\varphi_n^{p/2}(x)}, \quad |t| < 1,$$

we get

$$\begin{aligned} & \frac{1}{2\pi} \int_E \frac{|(\varphi_n B_n)(x)|^q}{(1 - \bar{i}x)B_n(x)\varphi_n^{p/2}(x)} d\mu(x) \\ &= \frac{m_n^q}{2\pi} \int_\Gamma \frac{\overline{B_n(\xi)\varphi_n^{p/2}(\xi)} |d\xi|}{(1 - \bar{i}\xi)} = m_n^q \frac{1}{2\pi i} \int_\Gamma \frac{B_n(\xi)\varphi_n^{p/2}(\xi) d\xi}{\xi - t} \\ &= m_n^q B_n(t)\varphi_n^{p/2}(t), \end{aligned}$$

and, then, (2.6).

By (2.10), for $g = x^k / \varphi_n^{p/2}$, $k = 0, \dots, n - 1$, we obtain that

$$\begin{aligned} \int_E \frac{x^k}{w_n(x)} \frac{|(\varphi_n B_n)(x)|^q}{\varphi_n^{p/2}(x)} d\mu(x) &= m_n^q \int_\Gamma \frac{\overline{\xi^k \varphi_n^{p/2}(\xi)} |d\xi|}{w_n(\xi)} \\ &= m_n^q \int_\Gamma \frac{\overline{\xi^{n-k} \varphi_n^{p/2}(\xi)} |d\xi|}{w_n^*(\xi)} = m_n^q \int_\Gamma \frac{\overline{\xi^{n-k-1} \varphi_n^{p/2}(\xi)} d\xi}{i w_n^*(\xi)}, \end{aligned}$$

consequently,

$$\int_E \frac{x^k}{w_n(x)} \frac{|(\varphi_n B_n)(x)|^q}{\varphi_n^{p/2}(x)} d\mu(x) = 0, \quad k = 0, \dots, n - 1,$$

and (2.7) follows. \square

3. The case $p = \infty$

Let $1 \leq q < \infty$, $p = \infty$, and let n be a positive integer. Since in this case

$$\sup_{\varphi \in A_\infty} \|\varphi B\|_{q,\mu} = \|B\|_{q,\mu}, \quad B \in \mathcal{B}_n,$$

we can rewrite (1.3) and (1.4) in the form

$$m_n = \inf_{B \in \mathcal{B}_n} \|B\|_{q,\mu} = \|B_n\|_{q,\mu}. \tag{3.1}$$

Let us consider the following function:

$$g_n(\xi) = \frac{1}{2\pi} \int_E \frac{1 - |x|^2}{|\xi - x|^2} B_n(x) |d\mu(x)|, \quad \xi \in \Gamma. \tag{3.2}$$

Let u be any harmonic function on G that is continuous on the closed disk \bar{G} . By (3.2),

$$\int_E u(x)|B_n(x)|^q d\mu(x) = \int_\Gamma u(\xi)g_n(\xi)|d\xi|, \tag{3.3}$$

and consequently

$$(|B_n|^q d\mu)^*(\xi) = g_n(\xi)|d\xi|, \quad \xi \in \Gamma,$$

where $(|B_n|^q d\mu)^*$ is the balayage of $|B_n|^q d\mu$ on Γ . In particular, we have that

$$\|g_n\|_{L^1(\Gamma)} = m_n^q. \tag{3.4}$$

Consider now the function $\phi_n(z)$ defined on G by

$$\phi_n(z) = \exp\left(\frac{1}{4\pi} \int_\Gamma \frac{\xi+z}{\xi-z} \log |g_n(\xi)/m_n^q| d\xi\right).$$

Because g_n is strictly positive and continuous on Γ as is apparent from (3.2), the function ϕ_n is normalized-outer in $H_\infty(G)$ by (1.1) and, from the properties of such functions (as described in the introduction), together with (3.4), ϕ_n satisfies the following three properties:

- (a) ϕ_n is nonvanishing in G ;
- (b) $\|\phi_n\|_2 = 1$ and $\phi_n(0) > 0$;
- (c) ϕ_n satisfies on Γ the equation

$$g_n(\xi) = m_n^q |\phi_n(\xi)|^2. \tag{3.5}$$

Moreover, since g_n is C^1 -smooth and non-vanishing on Γ by (3.2), so is $|\phi_n|$ and therefore ϕ_n itself is continuous on Γ .

With the aid of ϕ_n we shall prove the following version of Theorem 1 for the case when $p = \infty$ (see [3] for $q = 2$ and $E \subset (-1, 1)$).

Theorem 2. *Let $p = \infty$ and $1 \leq q < \infty$. The function ϕ_n can be extended analytically to D , and satisfies the equations*

$$m_n^q \phi_n(\xi) = \frac{1}{2\pi} \int_E \frac{|B_n(x)|^q d\mu(x)}{(1 - \xi\bar{x})\phi_n(x)}, \quad \xi \in D \tag{3.6}$$

and

$$m_n^q (\phi_n B_n)(\xi) = \frac{1}{2\pi} \int_E \frac{|B_n(x)|^q d\mu(x)}{(1 - \xi\bar{x})\phi_n(x)B_n(x)}, \quad \xi \in D. \tag{3.7}$$

The following orthogonality relations are valid:

$$\int_E \frac{x^k \overline{w_n(x)}}{|w_n^*(x)|^2 \phi_n(x)} |B_n(x)|^{q-2} d\mu(x) = 0, \quad k = 0, \dots, n-1. \tag{3.8}$$

Proof. Let g be any function in $H_1(G)$. Using (3.2), (3.3), and (3.5), we can write

$$\int_E g(x)|B_n(x)|^q d\mu(x) = m_n^q \int_\Gamma g(\xi)|\phi_n(\xi)|^2 |d\xi|. \tag{3.9}$$

It follows from the last formula that

$$J^* J(\phi_n) = m_n^q \phi_n, \tag{3.10}$$

where $J : H_2(G) \rightarrow L_2(|B_n|^q/|\phi_n|^2 d\mu, E)$ is the embedding operator. By (3.10) and (1.7), we get (3.6).

Since for any Blaschke product $B \in \mathcal{B}_n$

$$\int_E |B(x)|^q d\mu(x) \geq \int_E |B_n(x)|^q d\mu(x), \tag{3.11}$$

it follows that for $k = 0, \dots, n - 1$,

$$\int_E \left(\frac{x^k}{w_n^*(x)} \overline{B_n(x)} - \left(\frac{x^{n-k} w_n(x)}{(w_n^*(x))^2} \right) B_n(x) \right) |B_n(x)|^{q-2} d\mu(x) = 0. \tag{3.12}$$

We can see from (3.12) and (3.9) that

$$\begin{aligned} & \int_E \frac{x^k}{w_n(x)} |B_n(x)|^q d\mu(x) \\ &= \int_E \left(\frac{x^{n-k}}{w_n^*(x)} \right) |B_n(x)|^q d\mu(x) = m_n^q \int_\Gamma \left(\frac{\xi^{n-k}}{w_n^*(\xi)} \right) |\phi_n(\xi)|^2 |d\xi| \\ &= m_n^q \int_\Gamma \frac{\xi^k}{w_n(\xi)} |\phi_n(\xi)|^2 |d\xi|. \end{aligned} \tag{3.13}$$

By (3.9) and (3.13), for any $g \in H_1(G)$

$$\int_E \frac{g(x)}{w_n(x)} |B_n(x)|^q d\mu(x) = m_n^q \int_\Gamma \frac{g(\xi)}{w_n(\xi)} |\phi_n(\xi)|^2 |d\xi|. \tag{3.14}$$

Letting

$$g(x) = \frac{w_n^*(x)}{(1-\bar{t}x)\phi_n(x)}, \quad |t| < 1,$$

we get

$$\frac{1}{2\pi} \int_E \frac{|B_n(x)|^q}{(1-\bar{t}x)B_n(x)\phi_n(x)} = \frac{m_n^q}{2\pi} \int_\Gamma \frac{\overline{B_n(\xi)\phi_n(\xi)} |d\xi|}{(1-\bar{t}\xi)},$$

$$m_n^q \frac{1}{2\pi i} \int_\Gamma \frac{\overline{B_n(\xi)\phi_n(\xi)} d\xi}{\xi - t} = m_n^q \overline{B_n(t)\phi_n(t)}$$

and, then, (3.7).

By (3.14), for

$$g = \frac{x^k}{\phi_n}, k = 0, \dots, n - 1,$$

we obtain that

$$\begin{aligned} \int_E \frac{x^k}{w_n(x)} \frac{|B_n(x)|^q}{\phi_n(x)} d\mu(x) &= m_n^q \int_{\Gamma} \frac{\xi^k \overline{\phi_n(\xi)} |d\xi|}{w_n(\xi)} \\ &= m_n^q \int_{\Gamma} \frac{\xi^{n-k} \phi_n(\xi) |d\xi|}{w_n^*(\xi)} = m_n^q \int_{\Gamma} \frac{\xi^{n-k-1} \phi_n(\xi) d\xi}{i w_n^*(\xi)}, \end{aligned}$$

consequently,

$$\int_E \frac{x^k}{w_n(x)} \frac{|B_n(x)|^q}{\phi_n(x)} d\mu(x) = 0, \quad k = 0, \dots, n - 1,$$

and (3.8) follows. \square

4. Connection with meromorphic approximation

For $q = 2$ and $E \subset (-1, 1)$ there is an important connection between the best meromorphic approximation error of the Markov function

$$f(z) = \frac{1}{2\pi i} \int_E \frac{d\mu(x)}{z - x} \tag{4.1}$$

and the extremal constant m_n .

Let $\mathcal{M}_{n,s}(G)$, $1 \leq s < \infty$, be the class of all meromorphic functions on G that can be represented in the form $h = P/Q$, where P belongs to the Hardy space $H_s(G)$ and Q is a polynomial of degree at most n , $Q \neq 0$. Denote by

$$\Delta_{n,s} = \inf_{h \in \mathcal{M}_{n,s}(G)} \|f - h\|_s$$

the error in best approximation of the Markov function f in the space $L_s(\Gamma)$ by meromorphic functions in the class $\mathcal{M}_{n,s}(G)$.

Let $1/s + 1/t = 1$. The following theorem describes a connection between $\Delta_{n,s}$ and the extremal constant $m_n(p, q, \mu)$ with $q = 2$ and $p = 2t$ (see [1,2]).

Theorem 3. (i) *We have*

$$\Delta_{n,s} = m_n^2(2t, 2, \mu);$$

(ii) *there exists a best meromorphic approximant h_n in $\mathcal{M}_{n,s}(G)$ to the Markov function f in the space $L_s(\Gamma)$ such that*

$$\Delta_{n,s} = \|f - h_n\|_s$$

and

$$h_n = P_n/B_n,$$

where $P_n \in H_s(G)$ and B_n is a solution of the extremal problem (1.3) with $q = 2$ and $p = 2t$;

(iii) the function h_n satisfies a.e. on Γ the following equations:

$$(\varphi_n^2 B_n^2)(\xi)(f - h_n)(\xi) d\xi = \Delta_{n,s} |\varphi_n(\xi)|^{2t} |d\xi| \quad \text{if } 1 < s \leq \infty,$$

and

$$(B_n^2(f - h_n))(\xi) d\xi = |(f - h_n)(\xi)| |d\xi| \quad \text{if } s = 1.$$

Proof. We shall show that this theorem follows easily from results of Sections 2 and 3. Without loss of generality we assume that $1 < s \leq \infty$. Let

$$m_n = m_n(2t, 2, \mu) = \inf_{B \in \mathcal{B}_n} \sup_{\varphi \in \mathcal{A}_{2t}} \|\varphi B\|_{2,\mu} = \|\varphi_n B_n\|_{2,\mu}. \tag{4.2}$$

Since $E \subset \mathbf{R}$ it is not hard to prove that all zeros $x_{1,n}, \dots, x_{n,n}$ of B_n belong to the smallest interval $K(E)$ containing support $\text{supp } \mu = E$ of μ (see, for example, [1]). Using (1.5) with $q = 2$ and $p = 2t$, we can write $|\varphi_n(\bar{\xi})| = |\varphi_n(\xi)|$ for $\xi \in \Gamma$. Since φ_n is outer, it follows from this that

$$\overline{\varphi_n(\bar{\xi})} = c\varphi_n(\xi), \quad \xi \in G, \quad |c| = 1. \tag{4.3}$$

As above we can assume that $\varphi_n(0) > 0$. Then (4.3) yields $\varphi_n > 0$ on $(-1, 1)$.

By (2.10), for any function $g \in H_1(G)$ we get

$$\int_E g(x) \varphi_n^2(x) B_n(x) d\mu(x) = m_n^2 \int_\Gamma g(\xi) \overline{B_n(\bar{\xi})} |\varphi_n(\xi)|^{2t} |d\xi|$$

and (see (4.1))

$$\int_\Gamma g(\xi) \varphi_n^2(\xi) B_n(\xi) f(\xi) d\xi = m_n^2 \int_\Gamma g(\xi) \overline{B_n(\bar{\xi})} |\varphi_n(\xi)|^{2t} |d\xi|. \tag{4.4}$$

Since (4.4) is valid for any $g \in H_1(G)$, it follows (see, for example, [9]) that there exists a function $p \in H_\infty(G)$ such that

$$\varphi_n^2(\xi) B_n(\xi) f(\xi) - m_n^2 \overline{B_n(\bar{\xi})} |\varphi_n(\xi)|^{2t} |d\xi|/d\xi = p(\xi)$$

a.e. on Γ . From this we obtain that

$$\varphi_n^2(\xi) B_n^2(\xi) \left(f(\xi) - \frac{p(\xi)}{\varphi_n^2(\xi) B_n(\xi)} \right) d\xi = m_n^2 |\varphi_n(\xi)|^{2t} |d\xi| \tag{4.5}$$

a.e. on Γ . Since $\|\varphi_n\|_{2t} = 1$, we can conclude from the last relation that

$$\|f - h_n\|_s = m_n^2, \tag{4.6}$$

where $h_n = P_n/B_n$ and $P_n = p/\varphi_n^2$. We remark that $h_n \in \mathcal{M}_{n,s}(G)$.

By the duality results (see [9]), (4.2) and (4.1), we get

$$\begin{aligned} \Delta_{n,s} &= \inf_{B \in \mathcal{B}_n} \sup_{\varphi \in \mathbb{A}_t} \left| \int_{\Gamma} (\varphi B f)(\zeta) d\zeta \right| \\ &= \inf_{B \in \mathcal{B}_n} \sup_{\varphi \in \mathbb{A}_t} \left| \int_E \varphi(x) B(x) d\mu(x) \right| \geq m_n^2. \end{aligned} \quad (4.7)$$

Therefore, by (4.6), (4.7), and the fact that the function $h_n \in \mathcal{M}_{n,s}(G)$, we get

$$m_n^2 = \Delta_{n,s} = \|f - h_n\|_s. \quad \square$$

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