

# Zeros of Polynomials Orthogonal over Regular $N$ -gons

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*Dedicated to Herbert Stahl on the occasion of his sixtieth birthday.*

## Abstract

We investigate the location of zeros of Bergman polynomials (orthogonal polynomials with respect to area measure) for regular  $N$ -gons in the plane. In particular, we prove two conjectures posed by M. Eiermann and H. Stahl. Furthermore, we give some consequences regarding the asymptotic behavior of such Bergman polynomials.

## 1 Introduction

Let  $G \subset \mathbb{C}$  be a bounded Jordan domain. *Bergman polynomials* for  $G$  are algebraic polynomials  $Q_n(z; G)$ ,  $\deg Q_n = n$ , in the complex variable  $z$  satisfying the orthogonality relation

$$\iint_G Q_m(z) \overline{Q_n(z)} dx dy = \delta_{m,n}, \quad z = x + iy. \quad (1.1)$$

These polynomials play an important role in different aspects of approximation theory. In particular, they have a close connection with the *interior* Riemann mapping function  $\varphi_\zeta(z)$  for  $\zeta \in G$ , that is, the conformal map of  $G$  onto the unit disk  $\{w : |w| < 1\}$  satisfying  $\varphi_\zeta(\zeta) = 0$ ,  $\varphi'_\zeta(\zeta) > 0$ . Namely,

$$\varphi'_\zeta(z) = \sqrt{\frac{\pi}{K(\zeta, \zeta)}} K(z, \zeta), \quad (1.2)$$

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where  $K(z, \zeta)$  is the *Bergman kernel*, which has the representation

$$K(z, \zeta) = \sum_{k=0}^{\infty} \overline{Q_k(\zeta)} Q_k(z). \quad (1.3)$$

We will be interested in the case when  $G = G_N$  is the regular  $N$ -gon with vertices at  $\omega_N^k$ ,  $k = 0, \dots, N-1$ , where  $\omega_N := e^{2\pi i/N}$  is the first primitive  $N$ -th root of unity. More precisely, we will investigate the properties of zeros of  $Q_n(z; G_N)$ . Note that the convexity of  $G_N$  implies that all these zeros lie in the interior of  $G_N$  (for example, see [10, Theorem 2.2]). Furthermore, from symmetry arguments, if  $n = Nl + j$ ,  $0 \leq j \leq N-1$ , we deduce that

$$Q_n(z; G_N) = z^j q_l(z^N), \quad \deg q_l = l. \quad (1.4)$$

In [3], Eiermann and Stahl presented numerical results which led them to pose the following three conjectures.

(I) For  $N = 3, 4$ , the zeros of all the  $Q_n$ 's are located exactly on the “diagonals”  $\Gamma_{k,N}$  of  $G_N$ :

$$\Gamma_{k,N} := \{z : |z| < 1, \arg z = 2\pi k/N\} \cup \{0\}, \quad k = \overline{1, N}.$$

However, for  $N \geq 5$  there are zeros of  $Q_n$ 's that are **not** on the  $\Gamma_{k,N}$ 's.

(II) For  $N = 3, 4$  and fixed  $j \in \{0, \dots, N-1\}$ , the real zeros of the  $Q_{Nl+j}$ 's interlace on  $(0, 1)$ .

(III) For  $N \geq 5$ , the only points of the boundary  $\partial G_N$  of  $G_N$  that attract zeros of the  $Q_n$ 's are its vertices, i.e., if  $Z_n$  denotes the set of zeros of  $Q_n$ , then

$$\left( \bigcap_{n=1}^{\infty} \overline{\bigcup_{m>n} Z_m} \right) \cap \partial G_N = \left\{ \omega_N^k \right\}_{k=0}^{N-1}.$$

It was shown by Andrievskii and Blatt ([1]) that (III) is false for each  $N \geq 5$  since, for such  $N$ ,  $\varphi''_{\zeta}$  blows up at the vertices of  $G_N$ . The following general result in this direction is due to Levin, Saff, and Stylianopoulos [7]:

**Theorem 1.1** *Let  $G$  be a bounded Jordan domain,  $Q_n$  the Bergman polynomials for  $G$ , and  $\nu_n$  the normalized counting measure in the zeros of  $Q_n$ . Let  $\mu_{\partial G}$  denote the equilibrium (Robin) measure for  $\partial G$  and let  $\zeta \in G$ . Then there exists a subsequence  $\{n_k\} \subseteq \mathbb{N}$  such that*

$$\nu_{n_k} \xrightarrow{*} \mu_{\partial G} \quad \text{as } n_k \rightarrow \infty \quad (1.5)$$

*if and only if the interior conformal mapping  $\varphi_{\zeta}$  cannot be analytically continued to a domain  $\tilde{G} \supset \overline{G}$ .*

The convergence in (1.5) is understood to hold in the weak-star topology.

Theorem 1.1 implies that for  $N \geq 5$ , every point of  $\partial G_N$  attracts zeros of the  $Q_n$ 's. However, if  $N = 3$  or  $4$ , Theorem 1.1 yields no information about the zeros of the  $Q_n$ 's and, in this regard, it is a main purpose of the present note to show that (I) and (II) are true statements (see Corollaries 2.5 and 2.7).

## 2 Proof of Conjectures (I) and (II)

Regarding the orthogonality relation (1.1) for  $G_N$ , we first observe the following. Let  $m = Nl + j$ ,  $n = Nr + s$ ,  $0 \leq j, s \leq N - 1$ , and suppose the polynomials  $P_m$  and  $P_n$  have the form

$$P_m(z) = z^j p_l(z^N), \quad P_n(z) = z^s p_r(z^N), \quad \text{where } \deg p_l = l, \deg p_r = r.$$

Then, clearly, for any  $A \subset \mathbb{C}$ ,

$$\iint_{\omega_N A} P_m(z) \overline{P_n(z)} dx dy = \omega_N^{j-s} \iint_A P_m(z) \overline{P_n(z)} dx dy. \quad (2.6)$$

Let  $\Delta$  denote the triangle region with vertices at  $0$ ,  $1$ , and  $\omega_N$ . Then

$$G_N = \bigcup_{k=0}^{N-1} (\omega_N^k \Delta),$$

and using (2.6) we obtain

$$\iint_{G_N} P_m(z) \overline{P_n(z)} dx dy = \sum_{k=0}^{N-1} \iint_{\omega_N^k \Delta} P_m(z) \overline{P_n(z)} dx dy = \sum_{k=0}^{N-1} \omega_N^{k(j-s)} \iint_{\Delta} P_m(z) \overline{P_n(z)} dx dy.$$

Since

$$\sum_{k=0}^{N-1} (\omega_N^{j-s})^k = \begin{cases} 0, & \text{if } j \neq s; \\ N, & \text{if } j = s, \end{cases}$$

we conclude that

$$\iint_{G_N} P_m(z) \overline{P_n(z)} dx dy = 0 \quad \text{if } m \not\equiv n \pmod{N}.$$

So, (1.1) carries useful information only for  $m \equiv n \pmod{N}$ . In this case, for  $m \neq n$ ,

$$\iint_{\Delta} Q_m(z; G_N) \overline{Q_n(z; G_N)} dx dy = \frac{1}{N} \iint_{G_N} Q_m(z; G_N) \overline{Q_n(z; G_N)} dx dy = 0. \quad (2.7)$$

Next, we show that the orthogonality relation (2.7) implies that  $Q_{Nl+j}$ ,  $0 \leq j \leq N-1$ , restricted to  $[0, 1]$ , is orthogonal to a certain system of  $l$  polynomials that depend on  $N$  and  $j$ .

For  $j = \overline{0, N-1}$  (i.e.,  $j = 0, 1, \dots, N-1$ ) and  $m = 0, 1, \dots$ , let

$$f_{N, Nm+j}(x) := \operatorname{Im} \left[ \omega_N^{j+1} (x-1 - \overline{\omega_N})^{Nm+j+1} \right]. \quad (2.8)$$

**Lemma 2.2** For  $N \geq 3$ ,  $j = \overline{0, N-1}$ , and  $l = 1, 2, \dots$ ,

$$\int_0^1 Q_{Nl+j}(x; G_N) f_{N, Nm+j}(x) dx = 0 \quad \text{for } m = \overline{0, l-1}. \quad (2.9)$$

**Proof.** In this proof we denote, for convenience,  $\omega := \omega_N$  and  $Q_n(z) := Q_n(z; G_N)$ . Let  $n > k$  and  $n \pmod N = k \pmod N = j$ . Using Green's formula (cf. [4, p. 10]) we get from (2.7) that

$$\int_{\gamma} Q_n(z) \overline{z}^{k+1} dz = 0,$$

where  $\gamma$  denotes the positively oriented boundary of the triangle  $\Delta$ . If  $\gamma_1 := [0, 1]$ ,  $\gamma_2 := [1, \omega]$ , and  $\gamma_3 := [\omega, 0]$  denote the (oriented) sides of the triangle  $\Delta$ , then we have on  $\gamma_1$ :  $\overline{z} = z$ , on  $\gamma_2$ :  $\overline{z} = -\overline{\omega}(z-1) + 1$ , on  $\gamma_3$ :  $\overline{z} = \overline{\omega}^2 z$ . Thus, using the Cauchy theorem and the fact that  $Q_n(\omega\zeta) = \omega^k Q_n(\zeta)$ , we get

$$\begin{aligned} 0 &= \int_{\gamma} Q_n(z) \overline{z}^{k+1} dz \\ &= \int_{\gamma_1} Q_n(z) z^{k+1} dz + \int_{\gamma_2} Q_n(z) (-\overline{\omega}(z-1) + 1)^{k+1} dz + \int_{\gamma_3} Q_n(z) (\overline{\omega}^2 z)^{k+1} dz \\ &= \int_{\gamma_1} Q_n(z) z^{k+1} dz + \int_{-\gamma_1 - \gamma_3} Q_n(z) (-\overline{\omega}(z-1) + 1)^{k+1} dz + \int_{\gamma_3} Q_n(z) (\overline{\omega}^2 z)^{k+1} dz \\ &= \int_{\gamma_1} Q_n(z) \left[ z^{k+1} - (-\overline{\omega}(z-1) + 1)^{k+1} \right] dz + \int_{-\gamma_3} Q_n(z) \left[ (-\overline{\omega}(z-1) + 1)^{k+1} - (\overline{\omega}^2 z)^{k+1} \right] dz \\ &= \int_{\gamma_1} Q_n(z) \left[ z^{k+1} - (-\overline{\omega}(z-1) + 1)^{k+1} \right] dz + \omega \int_{\gamma_1} Q_n(\omega\zeta) \left[ (-\overline{\omega}(\omega\zeta-1) + 1)^{k+1} - (\overline{\omega}\zeta)^{k+1} \right] d\zeta \\ &= (-1)^{k+1} \int_{\gamma_1} Q_n(z) \left[ \omega^{j+1} (z-1 - \overline{\omega})^{k+1} - \overline{\omega}^{j+1} (z-1 - \omega)^{k+1} \right] dz. \end{aligned}$$

All that remains is to note that, for real  $x$ ,

$$\omega^{j+1} (x-1 - \overline{\omega})^{k+1} - \overline{\omega}^{j+1} (x-1 - \omega)^{k+1} = 2if_{N,k}(x). \quad \blacksquare$$

**Remark 2.3** Note that, for any  $N \geq 3$ ,  $j \in \{0, 1, \dots, N-1\}$ , and  $l = 0, 1, \dots$ , the functions  $f_{N, Nl+j}(x)$  have all real zeros and exactly  $l$  of them belong to  $(0, 1)$ . Indeed, the function

$$w = g(z) := \frac{\omega_N(z - 1 - \overline{\omega_N})}{\overline{\omega_N}(z - 1 - \omega_N)}$$

maps the real axis  $\text{Im } z = 0$  onto the unit circle  $|w| = 1$ , and the image of  $(0, 1)$  is the (shorter) open subarc  $\gamma_{\omega_N}$  with endpoints 1 and  $\omega_N$ . Now, in the  $w$ -plane, the equation  $f_{N, Nl+j}(z) = 0$  is equivalent to

$$w^{Nl+j+1} = 1,$$

which has the roots  $e^{2\pi ir/(Nl+j+1)}$ ,  $r = \overline{0, Nl+j}$ . One can easily check that  $l$  of these roots belong to  $\gamma_{\omega_N}$  if and only if  $1 \leq r \leq l$ .

**Lemma 2.4** For  $N = 3$  or  $4$  and fixed  $j$ ,  $0 \leq j \leq N-1$ , the system  $\{f_{N, Nl+j}\}$ ,  $l = 0, 1, 2, \dots$ , is a Markov system on  $(0, 1)$ , i.e., any polynomial

$$p_l(x) = \sum_{r=0}^l a_r f_{N, Nr+j}(x)$$

over this system that is not identically zero has at most  $l$  zeros on  $(0, 1)$ ,  $l = 0, 1, \dots$ . Moreover, for each  $N \geq 5$  and each  $j = \overline{0, N-1}$ , the system  $\{f_{N, Nl+j}\}_{l=0}^{\infty}$  is not Markov on  $(0, 1)$ .

**Proof.** We will prove the first part of the lemma by induction on  $l$ . First, for  $l = 0$  the conclusion of the lemma holds thanks to Remark 2.3. Next, assume that, for some  $l \geq 0$ , the system  $f_{N, Nr+j}(x)$ ,  $r = 0, \dots, l$ , is a Markov system on  $(0, 1)$ , and suppose, to the contrary, that a polynomial

$$p_{l+1}(x) = \sum_{r=0}^{l+1} a_r f_{N, Nr+j}(x), \quad a_{l+1} \neq 0,$$

has  $l+2$  zeros on  $(0, 1)$ . On differentiating  $N$  times we obtain

$$p_{l+1}^{(N)}(x) = a_0 f_{N, j}^{(N)}(x) + \sum_{r=1}^{l+1} a_r f_{N, Nr+j}^{(N)}(x) = \sum_{r=1}^{l+1} a_r c_{r, N} f_{N, N(r-1)+j}(x) = \sum_{r=0}^l b_r f_{N, Nr+j}(x) =: p_l(x),$$

where  $c_{r, N} := (Nr+j+1)!/(Nr+j+1-N)!$  and  $b_r := a_{r+1} c_{r+1, N}$ . We shall show that  $p_l$  has at least  $l+1$  zeros in  $(0, 1)$ , which will yield the desired contradiction.

We remark that counting only *interior* zeros of a polynomial on  $[0, 1]$ , i.e., its zeros on  $(0, 1)$ , we can guarantee only one less zero on  $(0, 1)$  for its derivative. At the same time, each *endpoint* zero of this polynomial gives an additional zero for the derivative on  $(0, 1)$ . We claim that the polynomials

$$p_{l+1}^{(m)}(x), \quad m = \overline{0, N-1}, \tag{2.10}$$

have at least  $N - 1$  endpoint zeros in total, which would imply that  $p_l$  has at least  $l + 1$  zeros on  $(0, 1)$ .

For fixed  $j = \overline{0, N - 1}$ , let us first investigate the endpoint zeros of  $f_{N, Nr+j}^{(m)}(x)$ ,  $r = 0, 1, \dots$ . Clearly,

$$f_{N, Nr+j}^{(m)}(x) = c_{r,m} \operatorname{Im} \left[ \omega_N^{j+1} (x - 1 - \overline{\omega_N})^{Nr+j+1-m} \right], \quad c_{r,m} := (Nr + j + 1)! / (Nr + j + 1 - m)!.$$

So, after some algebra, we get

$$f_{N, Nr+j}^{(m)}(0) = (-1)^{Nr+j+1-m+r} c_{r,m} \left( 2 \cos \frac{\pi}{N} \right)^{Nr+j+1-m} \sin \left( \frac{j+1+m}{N} \pi \right) \quad (2.11)$$

and

$$f_{N, Nr+j}^{(m)}(1) = (-1)^{Nr+j+1-m} c_{r,m} \operatorname{Im}(\omega_N^m) = (-1)^{Nr+j+1-m} c_{r,m} \sin \left( \frac{2\pi m}{N} \right). \quad (2.12)$$

For  $N \geq 3$ ,  $f_{N, Nr+j}^{(m)}(0) = 0$  if and only if

$$\sin \left( \frac{j+1+m}{N} \pi \right) = 0 \quad (2.13)$$

(for  $r = 0$  and  $m > j$ , obviously  $f_{N, j}^{(m)}(x) \equiv 0$ ). But  $0 \leq j \leq N - 1$ ,  $0 \leq m \leq N - 1$  and hence  $1 \leq j + 1 + m \leq 2N - 1$ . Thus (2.13) holds only in the case  $j + m = N - 1$  regardless of  $r$ , i.e., for  $m = N - 1 - j$  and any  $r = 0, 1, \dots$ ,  $f_{N, Nr+j}^{(m)}(0) = 0$  and, therefore,

$$p_{l+1}^{(m)}(0) = 0 \quad \text{if } m = N - 1 - j.$$

Thus, to establish the claim it is enough to show that  $N - 2$  polynomials in (2.10) have a zero at  $x = 1$ , for which, according to (2.12), a sufficient condition is that

$$\sin \left( \frac{2\pi m}{N} \right) = 0 \quad (2.14)$$

for  $N - 2$  values of  $m \in \{0, \dots, N - 1\}$ . But  $0 \leq 2m/N < 2$  and so there are at most two values of  $m$  for which (2.14) is true. So, we should restrict ourselves to the case  $N \leq 4$ . For  $N = 3$ , we need just one zero at  $x = 1$ , and this happens when  $m = 0$ . For  $N = 4$ , the required two zeros occur when  $m = 0$  and  $m = 2$ . This completes the proof of the first part of the lemma.

Now we consider the case when  $N \geq 5$ . As in the proof of Corollary 2.5 below, the fact that, for some  $j \in \{0, \dots, N - 1\}$ , the system  $\{f_{N, Nl+j}\}$ ,  $l = 0, 1, \dots$ , is a Markov system on  $(0, 1)$  implies that all the zeros of the  $Q_{Nl+j}(z; G_N)$ 's,  $l = 0, 1, \dots$ , lie on the rays  $\Gamma_{k, N}$ ,  $k = \overline{1, N}$ . Since, for such  $N$ ,  $\varphi_\zeta(z)$  cannot be extended analytically to a larger region, using Theorem 1.1 we conclude that  $\{f_{N, Nl+j}\}$ ,  $l = 0, 1, \dots$ , is not Markov at least for *some*  $j \in \{0, \dots, N - 1\}$ . The fact that this

system is not Markov for *every*  $j \in \{0, \dots, N-1\}$  requires additional arguments, and we proceed as follows.

Using the representations (1.2), (1.3), and (1.4) we get, for any  $\zeta \in G_N$ ,

$$\varphi'_\zeta(z) = g_0(z^N, \zeta) + zg_1(z^N, \zeta) + \dots + z^{N-1}g_{N-1}(z^N, \zeta), \quad (2.15)$$

where

$$g_j(z^N, \zeta) := \sqrt{\frac{\pi}{K(\zeta, \zeta)}} \sum_{k=0}^{\infty} \overline{Q_{Nk+j}(\zeta)} \frac{Q_{Nk+j}(z)}{z^j}.$$

In particular, for  $\zeta = 0$ , we have  $Q_{Nk+j}(0) = 0$  for  $j = \overline{1, N-1}$ , and so

$$\varphi'_0(z) = g_0(z^N, 0) = \sqrt{\pi/K(0, 0)} \sum_{l=0}^{\infty} \overline{Q_{Nl}(0)} Q_{Nl}(z).$$

The regularity of the Lebesgue measure implies (cf. [8, Lemma 4.3]) that for the sup norm  $\|\cdot\|_{G_N}$  on  $G_N$ ,

$$\lim_{n \rightarrow \infty} \|Q_n\|_{G_N}^{1/n} = 1. \quad (2.16)$$

Since  $\varphi'_0(z)$  does not have an analytic extension to a domain  $\tilde{G} \supset G_N$ , it follows that

$$\limsup_{l \rightarrow \infty} |Q_{Nl}(0)|^{1/Nl} = 1. \quad (2.17)$$

As in the proof of Theorem 1.1 in [7] we invoke Theorem III.4.1 in [11] to conclude that, for some subsequence  $\{l_k\}_{k=1}^{\infty}$ , the normalized counting measures  $\nu_{Nl_k}$  of the zeros of  $Q_{Nl_k}(z; G_N)$  satisfy

$$\nu_{Nl_k} \xrightarrow{*} \mu_{\partial G_N} \quad \text{as} \quad l_k \rightarrow \infty. \quad (2.18)$$

Consequently,  $\{f_{N, Nl+j}\}$ ,  $l = 0, 1, \dots$ , is not Markov for  $j = 0$ .

Next we observe that, for any integer  $k$ ,

$$\varphi_0(\omega_N^k z) = \omega_N^k \varphi_0(z) \quad \text{and} \quad \varphi'_0(\omega_N^k z) = \varphi'_0(z). \quad (2.19)$$

Also note that, for any  $\zeta \in G_N$ ,

$$\varphi_\zeta(z) = \lambda \frac{\varphi_0(z) - \varphi_0(\zeta)}{1 - \overline{\varphi_0(\zeta)} \varphi_0(z)}, \quad \varphi'_\zeta(z) = \lambda \frac{\varphi'_0(z)(1 - |\varphi_0(\zeta)|^2)}{(1 - \overline{\varphi_0(\zeta)} \varphi_0(z))^2}, \quad |\lambda| = 1. \quad (2.20)$$

Setting  $\mathcal{F}_0(z, \zeta) := \varphi'_\zeta(z)$  and, for  $j = 1, \dots, N-1$ ,

$$\mathcal{F}_j(z, \zeta) := \frac{\mathcal{F}_{j-1}(z, \zeta) - g_{j-1}(z^N, \zeta)}{z},$$

and using (2.15), (2.19), and (2.20), after some algebra we get

$$\begin{aligned} Ng_j(z^N, \zeta) &= \sum_{k=0}^{N-1} \mathcal{F}_j(\omega_N^k z) \\ &= N\lambda (1 - |\varphi_0(\zeta)|^2) \varphi_0'(z) \frac{\left(\frac{\overline{\varphi_0(\zeta)}\varphi_0(z)}{z}\right)^j j + 1 + (N - j - 1) \left(\frac{\overline{\varphi_0(\zeta)}\varphi_0(z)}{z}\right)^N}{\left(1 - \left(\frac{\overline{\varphi_0(\zeta)}\varphi_0(z)}{z}\right)^N\right)^2}. \end{aligned}$$

On differentiating this equation and using the facts that  $\varphi_0''(z) \rightarrow \infty$ ,  $\varphi_0'(z)$  is bounded, and  $\varphi_0(z) \rightarrow 1$  as  $z \rightarrow 1$ ,  $z \in G_N$ , one easily concludes that  $g_j(z^N, \zeta)$  cannot be extended analytically to a larger domain for some  $\zeta \neq 0$  in  $G_N$ . Taking into account this fact, we now repeat the argument used for  $j = 0$  to conclude from (2.16) the analogs of (2.17) and (2.18); that is,

$$\limsup_{l \rightarrow \infty} |Q_{Nl+j}(\zeta)|^{1/(Nl+j)} = 1$$

and, for some subsequence  $\{l_k\}_{k=1}^\infty$  that depends on  $j$ ,

$$\nu_{Nl_k+j} \xrightarrow{*} \mu_{\partial G_N} \quad \text{as} \quad l_k \rightarrow \infty. \quad (2.21)$$

Therefore,  $\{f_{N,Nl+j}\}$ ,  $l = 0, 1, \dots$ , is not Markov for every  $j = 0, \dots, N - 1$ . ■

We remark that (2.21) provides some new information regarding the asymptotic behavior of the zeros of  $Q_n(z; G_N)$  for the cases  $N \geq 5$ .

**Corollary 2.5** *For  $N = 3$  or  $4$  and  $j = \overline{0, N-1}$ , the polynomials  $Q_{Nl+j}(x; G_N)$ ,  $l = 0, 1, \dots$ , have exactly  $l$  simple zeros on  $(0, 1)$ . Consequently, all zeros of  $Q_{Nl+j}(z; G_N)$  lie on the rays  $\Gamma_{k,N}$ ,  $k = \overline{1, N}$ .*

**Proof.** Using Lemma 2.4 and the orthogonality relation (2.9), we conclude from well-known arguments originally given by Kellog [6] (see also [9, Proposition 3.1]) that  $Q_{Nl+j}$  has at least  $l$  sign changes on  $(0, 1)$ . But it follows from the symmetry property (1.4) that  $Q_{Nl+j}$  cannot have more than  $l$  zeros on  $(0, 1)$ . ■

Next, for fixed  $j$ , we establish the interlacing property of zeros of the  $Q_{Nl+j}$ 's. This property is a consequence of the following general statement.

**Lemma 2.6** *Let  $\{g_k(t)\}_{k=0}^\infty$  be a Markov system of continuous functions on  $(a, b)$ , and suppose that polynomials  $P_n(t)$ ,  $\deg P_n \leq n$ ,  $n = 1, 2, \dots$ , are orthogonal to  $g_k(t)$ ,  $k = \overline{0, n-1}$ , on  $(a, b)$ , i.e.,*

$$\int_a^b P_n(t) g_k(t) dt = 0. \quad (2.22)$$



Then between any two consecutive zeros of  $P_n(t)$  on  $(a, b)$  there is a (unique) zero of  $P_{n-1}(t)$ .

Although similar results are known for the case when the  $P_n$ 's are in the span of the  $g_k$ 's, the authors could not find the needed form in the literature, so we provide a simple proof.

**Proof.** First of all, we note that all zeros of  $P_n(t)$ ,  $n = 1, 2, \dots$ , are simple, and lie on  $(a, b)$ . Suppose now, to the contrary, that  $\alpha$  and  $\beta$  are two consecutive zeros of  $P_{n+1}(t)$  and  $P_n(t)$  has no zeros on  $(\alpha, \beta)$ . We can assume without loss of generality that  $P_n(t) \geq 0$  and  $P_{n+1}(t) \geq 0$  on  $[\alpha, \beta]$ . Consider the polynomial

$$R_{n+1}(t) := cP_n(t) - P_{n+1}(t),$$

where the constant  $c > 0$  is chosen as follows:

- (i) if  $P_n(t) = 0$  either at  $\alpha$  or at  $\beta$ , denote this point by  $t^*$  and set

$$c := \frac{P'_{n+1}(t^*)}{P'_n(t^*)};$$

clearly,  $R_{n+1}(t)$  has a zero at  $t^*$  of multiplicity at least two.

- (ii) otherwise,  $P_n(t) > 0$  on  $[\alpha, \beta]$  and, with

$$c := \min \{C : C \geq 0, CP_n(t) - P_{n+1}(t) \geq 0 \text{ on } [\alpha, \beta]\},$$

the polynomial  $R_{n+1}(t)$  has a zero  $t^* \in (\alpha, \beta)$  of even multiplicity.

With such a choice for  $c$ , the polynomial  $R_{n+1}(t)/(t - t^*)^2$  has no more than  $n - 1$  zeros on  $(a, b)$ , and so no more than  $n - 1$  sign changes. Hence  $R_{n+1}(t)$  has no more than  $n - 1$  sign changes on  $(a, b)$ , and one can find a function

$$G_n(t) = \sum_{s=0}^{n-1} a_s g_s(t)$$

over the system  $\{g_s\}_{s=0}^{n-1}$  such that the product  $R_{n+1}(t)G_n(t)$  is nonnegative on  $(a, b)$ . On the other hand, the orthogonality relation (2.22) gives

$$\int_a^b R_{n+1}(t)G_n(t) dt = 0.$$

This implies that either  $R_{n+1}(t)$  or  $G_n(t)$  must be identically zero on  $(a, b)$ , which is impossible.  $\blacksquare$

**Corollary 2.7** For  $N = 3$  or  $4$  and fixed  $j \in \{0, \dots, N - 1\}$ , between any two consecutive zeros of  $Q_{Nl+j}(x; G_N)$ ,  $l = 2, 3, \dots$ , on  $(0, 1)$  there is a (unique) zero of  $Q_{N(l-1)+j}(x; G_N)$ .

**Proof.** We apply Lemma 2.6 to the polynomials  $P_l(t) := q_l(t)$ ,  $l = 0, 1, \dots$ , with  $q_l(t)$  defined in (1.4) and the system  $g_k(t) := t^{(j+1-N)/N} f_{N, Nk+j}(t^{1/N})$ ,  $k = 0, 1, \dots$ , with  $f_{N, Nk+j}(x)$  given by (2.8), which, by Lemma 2.4, is a Markov system on  $(0, 1)$  (since  $j$  is fixed). The orthogonality relation (2.22) follows immediately from (2.9) with the substitution  $t = x^N$ . ■

Corollaries 2.5 and 2.7 establish the truth of assertions (I) and (II).

Let  $\Phi_N(z)$  denote the *exterior* Riemann mapping function for  $G_N$ , i.e.,  $\Phi_N : \overline{\mathbb{C}} \setminus \overline{G}_N \mapsto \{|w| > 1\}$ ,  $\Phi_N(\infty) = \infty$ ,  $\Phi'_N(\infty) > 0$ . Using, for each side of  $G_N$ , the Schwarz reflection principle, we can extend  $\Phi_N$  to a function  $\tilde{\Phi}_N(z)$  that is analytic and one-to-one in  $\mathbb{C} \setminus (\cup_{k=1}^N \overline{\Gamma}_{k,N})$ .

**Corollary 2.8** For  $N = 3$  or  $4$ ,

$$\lim_{n \rightarrow \infty} Q_n(z; G_N)^{1/n} = \tilde{\Phi}_N(z)$$

locally uniformly in  $\mathbb{C} \setminus (\cup_{k=1}^N \overline{\Gamma}_{k,N})$ , where  $x^{1/n}$  denotes the branch that is positive for  $x > 0$ .

**Proof.** Indeed, the fact that all the zeros of  $Q_n(z; G_N)$ 's are located on the rays  $\Gamma_{k,N}$ ,  $k = \overline{1, N}$ , makes it possible to define single-valued analytic branches of the functions  $Q_n(z; G_N)^{1/n}$ ,  $n = 1, 2, \dots$ , in the domain  $\mathbb{C} \setminus (\cup_{k=1}^N \overline{\Gamma}_{k,N})$ . These functions form a normal family in this domain and, moreover, it is well-known ([12, Ch. 3]) that

$$\lim_{n \rightarrow \infty} Q_n(z; G_N)^{1/n} = \Phi_N(z)$$

locally uniformly in  $\mathbb{C} \setminus \overline{G}_N$ . Thus, the assertion follows from standard uniqueness theorems. ■

**Theorem 2.9** For  $N = 3$  or  $4$ , let  $\lambda_{N,j}^{(l)}$  be the normalized counting measure of the zeros of  $Q_{Nl+j}(z)$  that lie in  $(0, 1)$ , i.e.,

$$\lambda_{N,j}^{(l)} = \frac{1}{l} \sum_{\substack{x \in Z_{Nl+j} \\ x > 0}} \delta_x,$$

where  $\delta_x$  is the unit point mass at  $x$ . Then there exists a measure  $\mu_N$  such that for each  $j = \overline{0, N-1}$

$$\lambda_{N,j}^{(l)} \xrightarrow{*} \mu_N \quad \text{as} \quad l \rightarrow \infty.$$

Moreover,  $\mu_N$  is the unique measure supported on  $[0, 1]$  that satisfies the equation

$$\ln |\tilde{\Phi}_N(z)| = \frac{1}{N} \int \ln |z^N - x^N| d\mu_N(x) + \ln \frac{1}{c_N} \quad (2.23)$$

for all  $z \notin \cup_{k=1}^N \overline{\Gamma}_{k,N}$ , where  $c_N$  is the logarithmic capacity of  $G_N$ .

**Proof.** For any positive measure  $\lambda$  let  $U(z; \lambda)$  denote its logarithmic potential

$$U(z; \lambda) := \int \ln \frac{1}{|z - t|} d\lambda(t).$$

First we observe that the regularity of the Lebesgue measure over  $G_N$  implies that for each  $j = \overline{0, N-1}$

$$U(z; \nu_{Nl+j}) \rightarrow U(z; \mu_{\partial G_N}), \quad z \notin \overline{G_N}, \quad (2.24)$$

where  $\nu_{Nl+j}$  is the normalized counting measure of  $Z_{Nl+j}$ , the set of all zeros of  $Q_{Nl+j}$ . Note by symmetry, that

$$\nu_{Nl+j}(\cdot) = \frac{1}{Nl+j} \left\{ j \delta_0(\cdot) + l \sum_{k=0}^{N-1} \lambda_{N,j}^{(l)}(\omega_N^k \cdot) \right\}.$$

Hence from (2.24) it follows that if  $\lambda$  is any limit measure of  $\left\{ \lambda_{N,j}^{(l)} \right\}_{l=0}^{\infty}$ , then

$$U(z; \mu_{\partial G_N}) = U \left( z; \frac{1}{N} \sum_{k=0}^{N-1} \lambda(\omega_N^k \cdot) \right) \quad \text{for} \quad z \notin \overline{G_N}.$$

Writing

$$U(z; \mu_{\partial G_N}) = \ln \frac{1}{c_N} - \ln |\Phi_N(z)|,$$

we obtain (2.23) for  $z \notin \overline{G_N}$  and  $\mu_N = \lambda$ . Since  $\text{supp}(\lambda) \subset \cup_{k=1}^N \overline{\Gamma}_{k,N}$ , equation (2.23) holds by harmonic continuation for all  $z \in \mathbb{C} \setminus \cup_{k=1}^N \overline{\Gamma}_{k,N}$ .

Finally, we can use the unicity theorem for logarithmic potentials (cf. [11, Theorem II.2.1]) to deduce that (2.23) uniquely determines the measure  $\mu_N$  and so every limit measure  $\lambda$  must equal  $\mu_N$ . ■

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