Zero Distribution of
Bergman Orthogonal Polynomials for
Certain Planar Domains

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Abstract

Let $G$ be a simply-connected domain in the complex plane bounded by a closed Jordan curve $L$, and let $P_n$, $n \geq 0$, be polynomials of respective degrees $n = 0, 1, \ldots$ that are orthonormal in $G$ with respect to the area measure (the so-called Bergman polynomials). Let $\varphi$ be a conformal map of $G$ onto the unit disk. We characterize in terms of the asymptotic behavior of the zeros of $P_n$'s the case when $\varphi$ has a singularity on $L$. To investigate the opposite case we consider a special class of lens-shaped domains $G$ that are bounded by two orthogonal circular arcs. Utilizing the theory of logarithmic potentials with external fields, we show that the limiting distribution of the zeros of the $P_n$'s for such lens domains is supported on a Jordan arc joining the two vertices of $G$. We determine this arc along with the distribution function.

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1 Introduction

Let $G$ be a bounded simply-connected domain in the complex plane $\mathbb{C}$, whose boundary $L := \partial G$ is a Jordan curve and let $\{P_n\}_{n=0}^{\infty}$ denote the sequence of Bergman polynomials of $G$. This is defined as the sequence

$$P_n(z) = \gamma_n z^n + \cdots, \quad \gamma_n > 0, \quad n = 0, 1, 2, \ldots,$$

of polynomials that are orthonormal with respect to the inner product

$$(f, g) := \int_G f(z) \overline{g(z)} \, dm(z),$$

where $dm$ stands for the 2-dimensional Lebesgue measure.

Bergman polynomials form a complete orthonormal system for the corresponding Hilbert space $L_2(G)$ and they provide a convenient representation for its reproducing kernel function. The latter is intimately related to the derivative of a conformal map of $G$ onto the unit disk $\mathbb{D} := \{w : |w| < 1\}$. Consequently, Bergman polynomials are useful for the construction of approximations to the conformal map (via the so-called Bieberbach polynomials); see e.g. [4, pp. 34–35].

Let $\Omega := \mathbb{C} \setminus G$ and $\Delta := \overline{\mathbb{C}} \setminus \mathbb{D}$ denote, respectively, the exterior (in $\mathbb{C}$) of $G$ and $\mathbb{D}$. Then the exterior conformal map $\Phi$ associated with $G$ is the conformal map

$$\Phi : \Omega \to \Delta,$$  \hspace{2cm} (1.1)

normalized so that

$$\Phi(z) = cz + O(1), \quad z \to \infty, \quad c > 0.$$  \hspace{2cm} (1.2)

We note that the (logarithmic) capacity $\text{cap} L$ of $L$ (or $\overline{G}$) is given by

$$\text{cap} L = 1/c.$$  \hspace{2cm} (1.3)
It is well-known that (cf. [9, Lemma 4.3])
\[
\lim_{n \to \infty} \| P_n \|_{\mathcal{C}}^{1/n} = 1. \tag{1.4}
\]
(Here and in the sequel \( \| \cdot \| \) means the uniform norm on the subscripted set.)
Furthermore, (1.4) implies that
\[
\lim_{n \to \infty} \gamma_n^{1/n} = \frac{1}{\text{cap} L} \tag{1.5}
\]
and
\[
\lim_{n \to \infty} |P_n(z)|^{1/n} = |\Phi(z)|, \tag{1.6}
\]
locally uniformly in \( \overline{\mathbb{C}} \setminus \text{Co}(\overline{G}) \), where \( \text{Co}(\overline{G}) \) denotes the convex hull of \( \overline{G} \); see [13, Theorem 3.1.1].

If \( G \) is convex, the limit (1.6) holds for all \( z \in \overline{\mathbb{C}} \setminus \overline{G} \). Moreover, in this case all zeros of \( P_n \) lie in \( G \) (cf. [10, p. 366]) and, as we shall show in Corollary 2.3 below, there are positive constants \( c_1 \) and \( c_2 \) such that for all \( z \in \overline{\mathbb{C}} \setminus G \) and all \( n \),
\[
|P_n(z)||\Phi(z)|^{-n} \geq c_1 n^{-c_2}. \tag{1.7}
\]
A matching upper bound of the form \( c_3 n^{c_4} \) follows from [1, p. 360], and these estimates show that a stronger form of (1.6) holds for all \( z \in \overline{\mathbb{C}} \setminus G \), at least for convex \( G \).

The behavior of the \( P_n \)'s inside \( G \) is much more delicate and depends, in particular, on the analytic properties of an interior conformal mapping \( \varphi \) of \( G \) onto \( \mathbb{D} \). We shall consider the following two cases separately:

**Case (i)** \( \varphi \) has a singularity on \( L \) (\( = \partial G \));

**Case (ii)** \( \varphi \) has no singularities on \( L \) (that is, \( \varphi \) can be extended analytically to a domain \( G_0 \supset \overline{G} \)).

Note that the above splitting is independent of the choice of \( \varphi \), since any two conformal mappings of \( G \) onto \( \mathbb{D} \) are related by a Möbius transformation. As we show in Theorem 2.1, Case (i) can be characterized in terms of the
asymptotic behavior of the zeros of Bergman polynomials. It follows, in particular, that in this case every point of $L$ is a limit point of zeros of the $P_n$'s. Under additional assumptions on $G$, Case (i) has been illuminated by the work of Andrievskii and Blatt [1], who have established Erdős-Turán type theorems (Theorem 5 in [1]) for the distribution of zeros of the $P_n$'s.

Case (ii) is more complicated, and different situations may arise. For example, if $L$ is an analytic curve, then the exterior map $\Phi$ can also be extended to a larger domain $\Omega_0 \supset (\mathbb{C} \setminus G)$, and it turns out (cf. [4, pp. 12–13]) that the zeros of the $P_n$'s have no limit points in $\Omega_0$ (in particular, on $L$). On the other hand, if $G$ is a square or an equilateral triangle, then all the zeros of the $P_n$'s must lie on the segments that join the vertices of $G$ with its center, and every point of these segments attracts the zeros of the $P_n$'s. This fact was conjectured by Eiermann and Stahl [3], and has recently been proved in [8].

Regarding the Case (ii), one goal of this paper is to consider a special class of domains for which the location of zeros of Bergman polynomials is much less transparent. Let $G$ be a lens-shaped domain bounded by two circular arcs that are orthogonal to each other. We show (in Theorem 3.3) that there is a "critical arc" $\Gamma$ in $G$, which joins the two vertices of $L$, such that the limiting distribution of the zeros of the $P_n$'s is given by a certain measure supported on $\Gamma$. This is in contrast to the behavior of the zeros of Faber polynomials for such regions (see [7]), which is somewhat surprising since many results concerning Bergman polynomials are obtained via comparison with Faber polynomials.

The paper is organized as follows: In the next section we address Case (i) and we also establish there the estimate (1.7). Our main results concerning lens-shaped domains are formulated in Section 3 and their proofs appear in Sections 4, 5. A strengthened version of Theorem 3.3 (the discrepancy of zeros) is discussed in Section 6.
2 Some general results

We begin with some definitions that are needed for the statements of our results. Let $Q$ be a polynomial of degree $n$ with zeros $z_1, z_2, \ldots, z_n$. The \textit{normalized counting measure of the zeros of} $Q$ is defined by

$$\nu_Q := \frac{1}{n} \sum_{k=1}^{n} \delta_{z_k}, \quad (2.1)$$

where $\delta_z$ denotes the unit point mass at the point $z$. In other words, for any subset $A$ of $\mathbb{C}$,

$$\nu_Q(A) = \frac{\text{number of zeros of } Q \text{ in } A}{n}.$$  \hspace{1cm} (2.1)

Next, given a sequence $\{\sigma_n\}$ of Borel measures, we say that $\{\sigma_n\}$ \textit{converges in the weak* sense} to a measure $\sigma$, symbolically $\sigma_n \rightharpoonup \sigma$, if

$$\int f \, d\sigma_n \to \int f \, d\sigma, \quad n \to \infty,$$

for every function $f$ continuous on $\overline{\mathbb{C}}$.

The exterior mapping function defined by (1.1) and (1.2) can be naturally extended to a homeomorphism between the corresponding closed domains, so that

$$\Phi(L) = \mathbb{T} := \{ w : |w| = 1 \}.$$  \hspace{1cm} (2.1)

Then the normalized angular measure $d\theta/2\pi$ on $\mathbb{T}$ gives rise, in a natural way, to a unit measure $\mu_L$ on $L$. Namely, for any Borel set $A \subset L$, \hspace{1cm} (2.1)

$$\mu_L(A) := \frac{1}{2\pi} \int_{\Phi(A)} d\theta.$$  \hspace{1cm} (2.1)

This measure is called the \textit{equilibrium measure for} $L$ (or for $\overline{\mathbb{C}}$).

We can now formulate our first result whose proof is given later in this section.
Theorem 2.1 Let $G \subset \mathbb{C}$ be a simply connected domain bounded by a closed Jordan curve $L$, and let $\varphi$ be a conformal map of $G$ onto the unit disk $\mathbb{D}$. Then the following two statements are equivalent:

(a) $\varphi$ has a singularity on $L$.

(b) There is a subsequence $\mathcal{N} \subset \mathbb{N}$ such that

$$\nu_{P_n} \longrightarrow \mu_L, \quad \text{as } n \to \infty, \quad n \in \mathcal{N};$$

where $P_n$ is the Bergman polynomial of degree $n$ for $G$.

In this connection, we recall the result of Andrievskii and Blatt [1, Theorem 5] mentioned in the Introduction. This result implies that if $L$ is a quasiconformal curve and if $\varphi$ satisfies

$$\|\varphi^{(k)}\|_G = \infty, \quad \text{for some } k \geq 1,$$

then (2.2) holds true. (In fact, under the stated assumptions, their result gives estimates for the discrepancy between $\nu_{P_n}$ and $\mu_L$.) Clearly, (2.3) implies that $\varphi$ has a singularity on $L$. The converse, however, is not true as the following example shows.

Example There exists a domain $G$ bounded by a $C^\infty$ Jordan curve such that $\varphi : G \to \mathbb{D}$ has a singularity on $\partial G$, but

$$\|\varphi^{(k)}\|_G < \infty \quad \text{for all } k \geq 1.$$  \hspace{1cm} (2.4)

Consider the function

$$\psi(w) = w + \varepsilon \sum_{n=1}^{\infty} 2^{-\sqrt{n}}w^n, \quad w \in \mathbb{D}.$$  \hspace{1cm} (2.5)

It is easy to see that

$$|\psi'(w) - 1| < \frac{1}{2}, \quad w \in \mathbb{D},$$
provided \( \varepsilon > 0 \) is small enough. For such \( \varepsilon \) and any \( w_1, w_2 \) in \( \mathbb{D} \), we have
\[
|\psi(w_1) - \psi(w_2)| \geq \frac{1}{2}|w_1 - w_2|,
\]
which implies that the corresponding function \( \psi \) is univalent in \( \mathbb{D} \).

Let \( G := \psi(\mathbb{D}) \), and observe that the inverse function \( \varphi := \psi^{-1} \) is a conformal mapping of \( G \) onto \( \mathbb{D} \), such that \( |\varphi'| > 2/3 \) in \( G \). If we assume that \( \varphi \) admits an analytic continuation to a neighborhood of \( z_0 := \psi(1) \in \partial G \), then \( \varphi'(z_0) \neq 0 \), and therefore \( \varphi \) will be univalent in some (possibly smaller) neighborhood of \( z_0 \). This, in turn, would imply that \( \psi \) admits an analytic continuation to a neighborhood of \( w_0 = 1 \), in contradiction to the well-known fact that \( w_0 = 1 \) is a point of singularity of the power series (2.5). Hence, \( \varphi \) has a singularity at \( z_0 \). Since \( |\psi'| > 1/2 \) in \( \mathbb{D} \), and \( |\psi^{(k)}| \) is bounded from above in \( \mathbb{D} \), for any \( k \geq 1 \), we deduce by induction that \( \varphi \) satisfies (2.4), as required.

The proof of Theorem 2.1 is a simple combination of two known facts. The first follows from Theorem III.4.1 in [11]:

**Fact A** Let \( \{q_n\} \) be a sequence of monic polynomials of respective degrees \( n = 1, 2, \ldots \) satisfying
\[
\lim_{n \to \infty} \|q_n\|_G^{1/n} = \text{cap} \ L \ (= \text{cap}(\partial G)). \tag{2.6}
\]

(i) If there exists a subsequence \( \mathcal{N} \subset \mathbb{N} \) such that
\[
\nu_{q_n} \xrightarrow{*} \mu_L \quad \text{as} \ n \to \infty, \ n \in \mathcal{N}, \tag{2.7}
\]
then there is a subsequence \( \mathcal{N}_1 \subset \mathcal{N} \) and a point \( \zeta \in G \) such that
\[
\lim_{n \to \infty, n \in \mathcal{N}_1} |q_n(\zeta)|^{1/n} = \text{cap} \ L. \tag{2.8}
\]

(ii) If (2.8) holds for a subsequence \( \mathcal{N}_1 \subset \mathbb{N} \) and for some point \( \zeta \in G \), then (2.7) holds with \( \mathcal{N} := \mathcal{N}_1 \).
The second fact is a consequence of Theorem 2.1 in [9]:

**Fact B** Given \( \zeta \in G \), let \( \varphi_\zeta \) denote the conformal map of \( G \) onto \( \mathbb{D} \) normalized by

\[
\varphi_\zeta(\zeta) = 0, \quad \varphi'_\zeta(\zeta) > 0. \tag{2.9}
\]

Then \( \varphi_\zeta \) has a singularity on \( L \) if and only if

\[
\limsup_{n \to \infty} |P_n(\zeta)|^{1/n} = 1. \tag{2.10}
\]

**Proof of Theorem 2.1.**

(a) \( \Rightarrow \) (b) Assume that \( \varphi \) has a singularity on \( L \). Then the same is true for any map \( \varphi_\zeta \), \( \zeta \in G \). By Fact B, there is a subsequence \( \mathcal{N}_1 \subset \mathbb{N} \) such that

\[
\lim_{n \to \infty, n \in \mathcal{N}_1} |P_n(\zeta)|^{1/n} = 1. \tag{2.11}
\]

Therefore, the monic polynomials

\[
q_n := \gamma_n^{-1} P_n
\]

satisfy (recall (1.5)) the relation (2.8). In view of (1.5) and (1.4) they also satisfy (2.6) and we conclude by part (ii) of Fact A that (2.7) holds, which is equivalent to the required relation (2.2).

(b) \( \Rightarrow \) (a) Assume that (2.2) holds and let \( q_n \) be as above. Then we have (2.7) and (2.6) holds as before. By part (i) of Fact A, we conclude that (2.8) holds for some \( \zeta \in G \). But this is equivalent to (2.11) which in turn leads to (2.10), this last statement following by (1.4). Hence, by Fact B, \( \varphi_\zeta \) has a singularity on \( L \) and the proof is complete.

Next we derive bounds for Bergman polynomials \( P_n \).

**Lemma 2.2** Let \( G \) be as in Theorem 2.1.
(i) If \( L \) is quasiconformal, then the Bergman polynomials \( P_n \) satisfy
\[
\|P_n\|_{\overline{\mathbb{C}}} \leq c_1 n^{c_2}, \quad n \geq 1,
\]  
where \( c_1, c_2 \) are positive constants independent of \( n \).

(ii) Let \( z_0 \in L \) and assume that there is a straight line passing through \( z_0 \) that does not divide \( G \). Then, there exist positive constants \( c_3, c_4 \) independent of \( n \) and \( z_0 \), such that
\[
|P_n(z_0)| \geq c_3 n^{-c_4}, \quad n \geq 1.
\]

**Proof.** (i) See [1, p. 360] and note that here we consider, with the notation of [1, p. 341], the unweighted case \( h(z) \equiv 1 \).

(ii) Without loss of generality we may assume that \( z_0 = 0 \) and that \( G \) lies in the right half-plane. By orthogonality,
\[
\int_G P_n(z) \frac{P_n(z) - P_n(0)}{z} dm(z) = 0.
\]

Hence,
\[
|P_n(0)| \left| \int_G \frac{P_n(z)}{z} dm(z) \right| = \left| \int_G \frac{|P_n(z)|^2}{z} dm(z) \right|.
\]

Observe that
\[
\left| \int_G \frac{P_n(z)}{z} dm(z) \right| \leq c_1 \|P_n\|_{\overline{\mathbb{C}}},
\]

for some positive constant \( c_1 = c_1(G) \), and
\[
\left| \int_G \frac{|P_n(z)|^2}{z} dm(z) \right| \geq \int_G |P_n(z)|^2 \frac{\Re z}{|z|^2} dm(z).
\]

Next, let
\[
\Delta_\varepsilon := \{ z : \frac{\Re z}{|z|^2} \geq \varepsilon \}, \quad \varepsilon > 0.
\]
(Note that $\Delta_\varepsilon$ is the closed disk \( \{ z : |z - 1/(2\varepsilon)| \leq 1/(2\varepsilon) \} \). Then, it is easy to see (a figure will help) that the area of \( G \setminus \Delta_\varepsilon \) is less than \( c_2 \varepsilon \), where \( c_2 \) depends only on the diameter of \( G \). Therefore,

\[
\int_{G \setminus \Delta_\varepsilon} |P_n(z)|^2 dm(z) \leq c_2 \varepsilon \| P_n \|^2 \| G \|
\]

so that if we choose

\[
\varepsilon = \frac{1}{2} \left( c_2 \| P_n \|^2 \| G \|^{-1} \right),
\]

(2.17)

we have

\[
\int_{G \setminus \Delta_\varepsilon} |P_n(z)|^2 dm(z) = 1 - \int_{G \setminus \Delta_\varepsilon} |P_n(z)|^2 dm(z) \geq \frac{1}{2}
\]

and, consequently, from the definition of \( \Delta_\varepsilon \),

\[
\int_{G \setminus \Delta_\varepsilon} |P_n(z)|^2 \frac{\Re z}{|z|^2} dm(z) \geq \frac{1}{2} \varepsilon. \quad (2.18)
\]

The required result then follows by combining (2.14) to (2.18) with (2.12). ■

**Corollary 2.3** Assume that \( G \) is a bounded convex domain and let \( \Phi \) be the exterior mapping defined by (1.1)--(1.2). Then there exist positive constants \( c_j, 1 \leq j \leq 4 \), independent of \( n \) and \( z \) such that

\[
c_3 n^{-c_4} \leq |P_n(z)\|\Phi(z)\|^n \leq c_1 n^{c_2}, \quad z \in \overline{\mathbb{C}} \setminus G, \ n \geq 1. \quad (2.19)
\]

In particular,

\[
c_3 n^{-c_4} \leq \gamma_n (\text{cap } L)^n \leq c_1 n^{c_2}, \quad n \geq 1. \quad (2.20)
\]

**Proof.** We note that the assumption on \( G \) implies that \( L (= \partial G) \) is quasiconformal. Furthermore, all zeros of \( P_n \) lie in \( G \). Hence the function \( P_n \Phi^{-n} \) is analytic in \( \overline{\mathbb{C}} \setminus \overline{G} \) and is continuous and non-vanishing in \( \overline{\mathbb{C}} \setminus G \). The required estimates (2.19) follow from the maximum modulus principle and Lemma 2.2. Furthermore, for \( z = \infty \), (2.19) yields (2.20); recall (1.2)--(1.3). ■
Remark 2.4 The estimate (2.20) actually holds for any region with a quasiconformal boundary as is shown in [1, Lemma 6].

3 Results for lens-shaped domains

![Diagram of a lens-shaped domain](image)

Figure 3.1

Throughout this section we shall assume that $G$ is a lens-shaped domain whose boundary $L$ consists of two orthogonal circular arcs $L_\alpha$ and $L_\beta$ that join the points $i$ and $-i$ ($L_\alpha$ being to the left of $L_\beta$) and form angles $\alpha$ and $\beta$, respectively, with the linear segment $[-i, i]$; see Figure 3.1. That is,

$$L = L_\alpha \cup L_\beta$$

and

$$\alpha + \beta = \pi/2,$$

where we assume without loss of generality that

$$\alpha \in [0, \pi/4].$$
In the special case $\alpha = 0$, the domain $G$ is the half-disk \( \{ z : |z| < 1, \Re z > 0 \} \).

Let $L'_\alpha, L'_\beta$ denote the complementary arcs of $L_\alpha, L_\beta$, so that $L_\alpha \cup L'_\alpha$ and $L_\beta \cup L'_\beta$ are circles centered, respectively, at
\[
a := \cot \alpha, \quad b := -\tan \alpha
\]  
with corresponding radii,
\[
R_\alpha := 1/\sin \alpha, \quad R_\beta := 1/\cos \alpha.
\]

For any point $z$ on $\partial G$, we use the subscript notation $z_\alpha$ and $z_\beta$ to denote the reflections of $z$ with respect to $L_\alpha$ and $L_\beta$, respectively. Hence,
\[
z_\alpha = \frac{a\overline{z} + 1}{\overline{z} - a}, \quad z_\beta = \frac{b\overline{z} + 1}{\overline{z} - b},
\]  
and since $b = -1/a$ we have
\[
z_\alpha z_\beta = -1.
\]  
In particular, when $\alpha = 0$,
\[
z_\alpha = -\overline{z}, \quad z_\beta = 1/\overline{z}.
\]  

With $G$ as above, the normalized exterior map $w = \Phi(z)$ is given, as can be easily verified, by the composition of the following three transformations:
\[
\xi(z) := e^{i(\pi/2 + \alpha)} \frac{z - i}{z + i},
\]
\[
t(\xi) := \xi^{2/3}, \quad \text{arg} \xi \in (-\pi/2, 3\pi/2],
\]
\[
w(t) := \frac{1 - \lambda t}{t - \lambda}, \quad \lambda := e^{i(\pi + 2\alpha)/3}.
\]
These lead to
\[
\Phi(z) = \left( \frac{3}{2} \sin \frac{2\alpha + \pi}{3} \right) z + O(1), \quad z \to \infty,
\]
from which we obtain
\[
\text{cap} L = \left( \frac{3}{2} \sin \frac{2\alpha + \pi}{3} \right)^{-1},
\]
for the capacity of the boundary curve $L$.  

\[\text{zeros of Bergman polynomials}\]
Figure 3.2

Let $\ell_\alpha$, $\ell_\beta$, $\ell'_\alpha$ and $\ell'_\beta$ denote the images under $\Phi$ of the four circular arcs $L_\alpha$, $L_\beta$, $L'_\alpha$ and $L'_\beta$, respectively. Then, it can be easily seen by considering the three elementary transformations (3.5)–(3.7) which constitute $\Phi$, that $\ell_\alpha$, $\ell_\beta$, $\ell'_\alpha$ and $\ell'_\beta$ are also circular arcs; see Figure 3.2.

Since $L_\alpha \perp L_\beta$, it is clear that $L'_\alpha$ is the reflection of $L_\alpha$ with respect to $L_\beta$. Therefore, by the reflection principle, the function $\Phi$ can be analytically continued across $L_\beta$, as a conformal map from $\mathbb{C} \setminus L_\alpha$ onto the exterior of the curve $\ell_\alpha \cup \ell'_\alpha$, where $\ell'_\alpha$ is the reflection of $\ell'_\alpha$ with respect to the unit circle; see Figure 3.2. This extension of $\Phi$ is given by

$$\Phi(z) := 1/\Phi(z_\beta), \quad z \in G \cup L_\beta. \quad (3.9)$$

Similarly, the formula

$$\Phi(z) := 1/\Phi(z_\alpha), \quad z \in G \cup L_\alpha, \quad (3.10)$$

gives the analytic continuation of $\Phi$ across $L_\alpha$. 
We define now the “critical arc” Γ, which plays a central role in our investigation, by
\[ \Gamma := \{ z \in \overline{G} : |\Phi(z_\alpha)| = |\Phi(z_\beta)| \}. \]  
(3.11)
We shall show in Lemma 4.1 that Γ is an analytic arc lying in G except for its endpoints \( i \) and \( -i \). Hence, Γ divides \( G \) into two domains
\[ G_\alpha := \text{int}(L_\alpha \cup \Gamma), \quad G_\beta := \text{int}(L_\beta \cup \Gamma), \]  
(3.12)
where “int” means the interior of the indicated closed curve. Obviously,
\[ |\Phi(z_\alpha)| < |\Phi(z_\beta)|, \quad z \in G_\alpha, \]  
(3.13)
with an opposite inequality for \( z \in G_\beta \).

It follows from the above considerations that the formula
\[ \Phi(z) := \begin{cases} 
1/\Phi(z_\alpha), & z \in G_\alpha \\
1/\Phi(z_\beta), & z \in G_\beta \cup L_\beta 
\end{cases} \]  
(3.14)
gives an analytic and conformal extension of \( \Phi \) to \( \mathbb{C} \setminus \Gamma \), and \( |\Phi| \) is continuous in \( \mathbb{C} \).

We now state our results on the Bergman polynomials for \( G \). Their proofs are given in Section 5.

**Theorem 3.1** Let \( \Phi \) be the normalized exterior mapping associated with \( G \) and define \( \Phi \) in \( \overline{G} \) by (3.14). Then
\[ \limsup_{n \to \infty} |P_n(z)|^{1/n} = |\Phi(z)|, \quad z \in \mathbb{C} \]  
(3.15)
Furthermore, the relation
\[ \lim_{n \to \infty} |P_n(z)|^{1/n} = |\Phi(z)| \]  
(3.16)
holds for all \( z \in \mathbb{C} \setminus G \) and also for any \( z \in G \) that is not a limit point of zeros of the \( P_n \)'s.
Corollary 3.2 In a special case $\alpha = \pi/4$, $\Gamma$ is the straight line segment $[-i, i]$. Furthermore, all zeros of $P_n$ lie on $[-i, i]$, and (3.16) holds for all $z \in \mathbb{C} \setminus [-i, i]$; see Figure 3.3.

Figure 3.3 The domain $G$ and the zeros of $P_{10}$, $P_{15}$ and $P_{20}$, for $\alpha = \pi/4$.

In order to formulate our main result on the zero distribution of Bergman polynomials we need more notation. Let $\mathcal{G}$ be an arbitrary simply-connected domain in $\mathbb{C}$ bounded by a Jordan curve $\mathcal{L}$, and let $z_0 \in \text{ext } \mathcal{L}$. (Here and in the sequel “ext” means the exterior in $\mathbb{C}$ of the indicated curve.) Consider the conformal map $\Phi(\cdot, z_0)$ of $\text{ext } \mathcal{L}$ onto $\text{ext } \mathbb{T}$ that is normalized by

$$\Phi(z_0, z_0) = \infty, \quad \Phi'(z_0, z_0) > 0. \quad (3.17)$$

Then the angular measure $d\theta/2\pi$ on $\mathbb{T}$ gives rise, in a natural way, to a unit measure on $\mathcal{L}$. The latter is called the balayage of the unit point mass $\delta_{z_0}$ onto $\mathcal{L}$, and is denoted by $\tilde{\delta}_{z_0}$ (cf. [11, Section II.4]). In particular, if $z_0 = \infty$...
then
\[ \tilde{\delta}_\infty = \mu_\mathcal{L}, \] (3.18)
where \( \mu_\mathcal{L} \) is the equilibrium measure for \( \mathcal{L} \), as defined in Section 2.

Returning now to our lens-shaped domain, we set
\[ \tilde{\Gamma}_\alpha := \Gamma \cup \Gamma_\alpha, \] (3.19)
where \( \Gamma_\alpha \) is the reflection of \( \Gamma \) with respect to \( L_\alpha \); see Figure 3.4. Note that the point \( a = \cot \alpha \) (recall (3.1)) lies outside \( \tilde{\Gamma}_\alpha \).

Figure 3.4

We can now state our main result concerning the behavior of the zeros of Bergman polynomials for lens-shaped regions.

**Theorem 3.3** Let \( \nu_{P_n} \) be the normalized counting measure of the zeros of \( P_n \). Further let \( \mu_{\tilde{\Gamma}_\alpha} \) be the equilibrium measure for \( \tilde{\Gamma}_\alpha \), and let \( \tilde{\delta}_a \) be the balayage of \( \delta_a \) onto \( \tilde{\Gamma}_\alpha \). Then
\[ \nu_{P_n} \xrightarrow{\ast} \left( \mu_{\tilde{\Gamma}_\alpha} + \tilde{\delta}_a \right) \big|_{\Gamma}, \quad n \to \infty, \] (3.20)
where \( \sigma \big|_{\Gamma} \) denotes the restriction of the measure \( \sigma \) to \( \Gamma \); see Figure 3.5.
Figure 3.5 The zeros of $P_{40}$, $P_{50}$ and $P_{60}$ and the arc $\Gamma$ for the half-disk.

Remark 3.4 In a similar way we could define $\tilde{\Gamma}_\beta := \Gamma \cup \Gamma_\beta$, where $\Gamma_\beta$ is the reflection of $\Gamma$ with respect to $L_\beta$. Then (3.20) takes the equivalent form

$$\nu_{\Gamma_\alpha} \rightarrow \left( \mu_{\tilde{\Gamma}_\beta} + \tilde{\delta}_b \right) \big|_{\Gamma},$$

where $b = -\tan \alpha$ and $\tilde{\delta}_b$ is the balayage of $\delta_b$ onto $\tilde{\Gamma}_\beta$.

4 Auxiliary results

Let $G$ be the lens-shaped domain described in Section 3. Given $\zeta \in G$, let $\varphi_\zeta : G \to \mathbb{D}$ be the interior conformal map normalized by $\varphi_\zeta(\zeta) = 0$, $\varphi_\zeta'(\zeta) > 0$. It is easy to see that $\varphi_\zeta$ has the form

$$\varphi_\zeta(z) = e^{i\theta} \left[ \left( \frac{z - i}{z + i} \right)^2 - \left( \frac{\zeta - i}{\zeta + i} \right)^2 \right] \left[ \left( \frac{z - i}{z + i} \right)^2 - e^{-i4\alpha} \left( \frac{\zeta + i}{\bar{\zeta} - i} \right)^2 \right]^{-1},$$
for a suitable $\theta \in [0, 2\pi]$. By making use of (3.1) and (3.2), we obtain after some algebra, that

$$\varphi_\zeta(z) = C(\zeta) \frac{(z - \zeta)(z \zeta + 1)}{(z - \zeta_\alpha)(z - \zeta_\beta)},$$

where

$$C(\zeta) := e^{i\theta} \left( \frac{\zeta - i}{\zeta + i} \right)^2 \frac{e^{2i\alpha}}{(\zeta \sin \alpha - \cos \alpha)(\overline{\zeta \cos \alpha + \sin \alpha}).$$

We see that $\varphi_\zeta$ is a rational function with poles at $\zeta_\alpha$ and $\zeta_\beta$ which lie in $\text{ext } L$. Thus, our $G$ falls under Case (ii) mentioned in the Introduction. Furthermore, since $a$ and $b$ also lie in $\text{ext } L$, it can be easily checked that $C(\zeta)$ is uniformly bounded in $G$. Then we obtain from (4.1), by a straightforward estimate, that for all $\zeta \in G$ and $z \in \mathbb{C}$,

$$|\varphi'_\zeta(z)| \leq c \left( \frac{1}{|z - \zeta_\alpha|^2} + \frac{1}{|z - \zeta_\beta|^2} \right),$$

where $c > 0$ is independent of $\zeta$ and $z$.

Next we describe some properties of the critical arc $\Gamma$; recall (3.11) and (3.19).

**Lemma 4.1** Let

$$\tilde{\Lambda}_\alpha := \{w : \left| \frac{w(w + A)}{Bw - 1} \right| = 1 \},$$

where

$$A = \frac{2}{\sqrt{3}} \sin \frac{\pi - 2\alpha}{3} \quad \text{and} \quad B = \frac{2}{\sqrt{3}} \sin \frac{2\alpha}{3}$$

and denote

$$\Lambda := \{w \in \tilde{\Lambda}_\alpha : |w| \leq 1\}, \quad \Lambda_\alpha := \{w \in \tilde{\Lambda}_\alpha : |w| \geq 1\}.$$ 

Then the function $\Phi$ extended by (3.9) satisfies

$$\Lambda = \Phi(\Gamma) \quad \text{and} \quad \Lambda_\alpha = \Phi(\Gamma_\alpha).$$
Hence, $\Gamma$ is an analytic arc lying in $G$ (except for its endpoints $i$ and $-i$), and $\Phi$ maps conformally the exterior of $\widetilde{\Gamma}_\alpha$ onto the exterior of $\widetilde{\Lambda}_\alpha$.

Furthermore, the two arcs $\Gamma$ and $\Gamma_\alpha$ meet each other at $i$ and $-i$ with interior angles $\pi/2$ and the arc $\Gamma$ forms angles $\pi/4$ with both $L_\alpha$ and $L_\beta$; see Figure 3.4.

We remark that $A, B$ above satisfy $|A-B| < 2$, and this condition ensures that $\widetilde{\Lambda}_\alpha$ consists of a single closed curve.

**Proof.** Let $z \in G$. Since $z_\alpha z_\beta = -1$ (see (3.3)), we obtain by using the transformations (3.5) and (3.6),

$$
\xi(z_\alpha) = -\xi(z_\beta) \quad \text{and} \quad t(\xi(z_\alpha)) = e^{i2\pi/3}t(\xi(z_\beta)).
$$

From these it follows, after some simple manipulation, that

$$
\Phi(z_\alpha) = \frac{1 + A\Phi(z_\beta)}{\Phi(z_\beta) - B} \quad \text{and} \quad \Phi(z_\beta) = \frac{B\Phi(z_\alpha) - 1}{\Phi(z_\alpha) + A}, \quad (4.7)
$$

where $A$ and $B$ are defined in (4.4). In view of (3.9), we deduce from (4.7) that

$$
\frac{\Phi(z)[\Phi(z) + A]}{B\Phi(z) - 1} = \frac{1 + A\Phi(z_\beta)}{\Phi(z_\beta)[B - \Phi(z_\beta)]} = \frac{\Phi(z_\alpha)}{\Phi(z_\beta)} = \frac{\Phi(z_\alpha)[\Phi(z_\alpha) + A]}{B\Phi(z_\alpha) - 1}.
$$

The first statement of the lemma along with (4.6) follows from the above relations and the definition of $\widetilde{\Gamma}_\alpha$.

The last statement of the lemma is an easy consequence of the way that the three transformations (3.5)–(3.7) treat angles. More precisely, since the two arcs $\Lambda$ and $\Lambda_\alpha$ meet at $-\Lambda_\alpha$ forming an exterior angle $\pi$, their preimages $\Gamma$ and $\Gamma_\alpha$ should form an exterior angle $3\pi/2$ at $i$. By symmetry, $\Gamma$ and $\Gamma_\alpha$ form at $-i$ the same exterior angle.

The very last statement of the lemma follows at once from the above, because $\Gamma$ and $\Gamma_\alpha$ are symmetric with respect to $L_\alpha$.  \[\blacksquare\]
Remark 4.2  (i) It is worth noting that $\Lambda_\alpha$ is the reflection of $\Lambda$ with respect to $\ell_\alpha$; see Figure 3.4.

(ii) For later use we remark that, in a similar way, we could consider the curve $\Gamma_\beta := \Gamma \cup \Gamma_\beta$, where $\Gamma_\beta$ is the reflection of $\Gamma$ with respect to $L_{\beta}$. Proceeding as above, we conclude that the function $\Phi$ extended by (3.10) maps conformally ext $\Gamma_\beta$ onto the exterior of the curve

$$
\tilde{\Lambda}_\beta := \{ w \colon \frac{w(w - B)}{1 + Aw} = 1 \}.
$$

Note that $\tilde{\Lambda}_\beta$ is the reflection of $\tilde{\Lambda}_\alpha$ with respect to the unit circle.

Our next task is to relate the capacity of $\Gamma_\alpha$ to the capacity of $L$. By the preceding lemma, $\Phi$ maps conformally ext $\Gamma_\alpha$ onto ext $\tilde{\Lambda}_\alpha$. Moreover (see (1.1)--(1.2))

$$
\Phi(z) = (\text{cap} \ L)^{-1} z + O(1), \quad z \to \infty.
$$

Let $F$ denote the conformal map

$$
F : \text{ext} \tilde{\Lambda}_\alpha \to \Delta = \{ w : |w| > 1 \} \quad (4.8)
$$

normalized at $\infty$ by

$$
F(w) = (\text{cap} \tilde{\Lambda}_\alpha)^{-1} w + O(1), \quad w \to \infty. \quad (4.9)
$$

Then the composition

$$
\tilde{\Phi}_\alpha := F \circ \Phi \quad (4.10)
$$

maps conformally ext $\Gamma_\alpha$ onto $\Delta$ and we obtain

$$
\text{cap} \tilde{\Gamma}_\alpha = \text{cap} L \text{ cap} \tilde{\Lambda}_\alpha. \quad (4.11)
$$

Therefore, it is sufficient for our purposes here to determine $\text{cap} \tilde{\Lambda}_\alpha$. For $\alpha \neq 0$, where $\alpha$ is the defining parameter of $G$, this is done in the following lemma.
When \( \alpha = 0 \), it follows at once from (4.3) and (4.4) that the corresponding curve \( \Lambda_0 \) is the lemniscate \( \{ w : |w(w+1)| = 1 \} \) and thus
\[
\text{cap} \Lambda_0 = 1; \quad (4.12)
\]
see e.g. [11, pp. 164–165].

**Lemma 4.3** Assume that \( \alpha \in (0, \pi/4] \) and let \( A, B \) be defined by (4.4). Then
\[
\text{cap} \Lambda_0 = B \ F \left( \frac{1}{B} \right). \quad (4.13)
\]
Furthermore,
\[
\frac{\Phi(a)}{\text{cap} L} = \frac{\Phi_\alpha(a)}{\text{cap} \Gamma_\alpha}, \quad (4.14)
\]
where, as in (3.1), \( a = \cot \alpha \).

**Proof.** A short calculation involving the transformations (3.6)-(3.7) gives \( \Phi(a) = 1/B \). Thus, \( \Phi_\alpha(a) = F(1/B) \) and the result (4.14) follows from (4.11) once (4.13) is established. The following short proof of (4.13) is due to V. Maymeskul.

It is easy to see that the function
\[
\zeta(w) := \frac{w + A}{Bw - 1}
\]
maps \( \text{ext} \Lambda_\alpha \) onto itself, \( \zeta(1/B) = \infty \text{ and } \zeta(\infty) = 1/B \). Hence, the function \( F(w)F(\zeta(w)) \) has simple poles at \( w = 1/B \) and \( w = \infty \), and it satisfies \( |F(w)F(\zeta(w))| = 1, \) for \( w \in \Lambda_\alpha \). Consequently, \( F(w)F(\zeta(w))/w\zeta(w) \) is analytic and non-vanishing in \( \text{ext} \Lambda_\alpha \), and is of modulus 1 on \( \Lambda_\alpha \). By the maximum modulus principle,
\[
w\zeta(w) = F(w)F(\zeta(w)), \quad w \in \text{ext} \Lambda_\alpha.
\]
Dividing this by \( w \) and taking the limit, as \( w \to \infty \), we obtain (4.13) in view of (4.9).
The essential idea of the proof of Theorem 3.3 is to consider the product $P_n(\zeta)P_n(\zeta_\alpha)$, where $\zeta_\alpha$ is defined as in (3.2). The next lemma provides a bound for this rational function of $\zeta$.

**Lemma 4.4** There exist positive constants $c_1$ and $c_2$ independent of $n$ and $\zeta$, such that

$$|P_n(\zeta)P_n(\zeta_\alpha)| \leq c_1 n^{c_2}, \quad \zeta \in \Gamma, \quad n \geq 1. \quad (4.15)$$

**Proof.** For quantities $A > 0$, $B > 0$, we use the notation $A \leq B$ (inequality with respect to the order) if $A \leq cB$, where $c$ depends only on $G$.

Since $\zeta_\alpha \notin G$, the application of (2.19) gives

$$|P_n(\zeta_\alpha)| \leq c_1 n^{c_2} |\Phi(\zeta_\alpha)|^n.$$

Thus, it is sufficient to show that for some $c > 0$ and $c_3 > 0$,

$$|P_n(\zeta)| \leq c n^{c_3} |\Phi(\zeta_\alpha)|^{-n}, \quad \zeta \in \Gamma, \quad n \geq 1. \quad (4.16)$$

Assume first that $|\zeta^2 + 1| < n^{-3/2}$. Then (3.2) gives $|\zeta_\alpha^2 + 1| \leq n^{-3/2}$ and this, in conjunction with the expansion of $\Phi$ at $\pm i$, implies

$$|\Phi(\zeta_\alpha)| - 1 \leq n^{-1},$$

which yields (4.16) in view of Lemma 2.2(i).

Assume now that

$$|\zeta^2 + 1| \geq n^{-3/2}.$$

The same reasoning as before yields

$$|\Phi(\zeta_\alpha)| - 1 \geq n^{-1}. \quad (4.17)$$

Furthermore, since $\Gamma$ forms a nonzero angle ($= \pi/4$) with $L_{\alpha}$, $L_{\beta}$ (see the last statement of Lemma 4.1), we have

$$d_\zeta := \text{dist}(\zeta, \partial G) \geq n^{-3/2}. \quad (4.18)$$
Using well-known arguments (the details are provided below for the reader’s convenience) one can show that, for any polynomial $Q_{n-1}$ of degree $n - 1$, there holds

$$|P_n(\zeta)| \leq \frac{c}{d_\zeta} \|\varphi'_\zeta - Q_{n-1}\|_{\overline{G}}, \tag{4.19}$$

where $c > 0$ depends only on $G$. For the moment, let us assume that (4.19) is valid. We know from (4.1) that $\varphi'_\zeta$ is a rational function with poles at $\zeta_\alpha$, $\zeta_\beta$. Since $\zeta \in \Gamma$, we have

$$|\Phi(\zeta_\alpha)| = |\Phi(\zeta_\beta)| =: R. \tag{4.20}$$

Then if we set, generically,

$$L_t := \{z \in \mathbb{C} \setminus G : |\Phi(z)| = t\}, \quad t \geq 1,$$

we see that $\varphi'_\zeta$ is analytic inside $L_R$.

Let $Q_{n-1}$ be the $(n - 1)$th section of the Faber expansion of $\varphi'_\zeta$. Since $G$ is convex, the Faber polynomials associated with $G$ are uniformly bounded on $\overline{G}$ by 2 and we can deduce (cf. [12, Theorem 2(4), p. 142]) that

$$\|\varphi'_\zeta - Q_{n-1}\|_{\overline{G}} \leq 2M_r \frac{r}{r - 1} r^{-n}, \quad 1 < r < R, \tag{4.21}$$

where

$$M_r := \max_{z \in L_r} |\varphi'_\zeta(z)|.$$

We now choose

$$r = R \left(1 - \frac{\varepsilon}{n}\right),$$

where $\varepsilon > 0$ is independent of $n$ and $\zeta$ and is small enough to ensure

$$r - 1 \geq n^{-1}. \tag{4.22}$$

(A suitable choice is any $\varepsilon$ such that $\varepsilon < c/(1 + c)$, where $c$ is a constant for which (4.17) holds.)
By using a result of Löwner on the distance between exterior level curves (see e.g. [5, p. 120]) we obtain
\[
\text{dist}(L_r, L_R) > \text{cap} L \frac{(R - r)^2}{R} \geq n^{-2}.
\]
and therefore, from (4.2),
\[
M_r \leq n^4.
\]
Then (4.21) gives
\[
\|\varphi' - Q_{n-1}\|_{\mathcal{C}} \leq n^5 R^{-n}
\]
and the result (4.16) follows by substituting the preceding relation into (4.19) and using (4.18) and (4.20).

It remains to prove (4.19). In order to do this, we recall (cf. [4, p. 33]) that
\[
\varphi'(z) = \sqrt{\frac{\pi}{K(\zeta, \zeta)}} K(z, \zeta), \tag{4.23}
\]
where \(K(z, \zeta)\) is the Bergman kernel function associated with \(G\). By the Cauchy formula for the derivative \(\varphi' \) we then obtain
\[
K(\zeta, \zeta) \leq 1/(\pi d^2). \tag{4.24}
\]
Now, by making use of the reproducing property of \(K(z, \zeta)\), the relation (4.23), the orthogonality property of \(P_n\) and, finally, the Cauchy-Schwarz inequality, we have (see also [6, p. 302])
\[
|P_n(\zeta)| = \left| \int_G K(z, \zeta) \overline{P_n(z)} dm(z) \right|
= \sqrt{\frac{K(\zeta, \zeta)}{\pi}} \left| \int_G (\varphi'(z) - Q_{n-1}(z)) \overline{P_n(z)} dm(z) \right|
\leq \sqrt{\frac{K(\zeta, \zeta)}{\pi}} \|\varphi' - Q_{n-1}\|_{L^2(G)}.
\]
The required result (actually in a stronger form) follows from the last inequality and (4.24). \[\blacksquare\]
5 Proofs of the results of Section 3

For any finite positive Borel measure $\sigma$ of compact support in $\mathbb{C}$, we define its logarithmic potential by

$$U^\sigma(z) := \int \log \frac{1}{|z - t|} d\sigma(t).$$

In particular, if $Q$ is a monic polynomial polynomial of degree $n$, then

$$U^\mu_Q(z) = -\log |Q(z)|^{1/n}.$$

Let $\mathcal{L}$ be a simple closed Jordan curve in $\mathbb{C}$. We have already defined (in Sections 2 and 3) the equilibrium measure $\mu_\mathcal{L}$ for $\mathcal{L}$ and the balayage measure $\tilde{\delta}_n$ ($z_0 \in \text{ext } \mathcal{L}$) of $\delta_n$ onto $\mathcal{L}$. It is well-known that their potentials satisfy

$$U^{\mu_\mathcal{L}}(z) = \log \frac{1}{\text{cap } \mathcal{L}}, \quad z \in \text{int } \mathcal{L} \quad (5.1)$$

and

$$U^{\tilde{\delta}_n}(z) = \log \frac{1}{|z - z_0|} + g_{\mathcal{L}}(z_0, \infty), \quad z \in \mathcal{L}, \quad (5.2)$$

where $g_{\mathcal{L}}(\cdot, \infty)$ is the Green function of ext $\mathcal{L}$ with pole at $\infty$ (cf. [11, Sections I.1, I.4, II.4]).

We now turn to the

**Proof of Theorem 3.3.** We recall from Section 2 that

$$z_\alpha = \frac{a\bar{z} + 1}{\bar{z} - a}, \quad a = \cot \alpha,$$

with $\alpha \in [0, \pi/4]$ and consider separately the two cases $\alpha \in (0, \pi/4]$ and $\alpha = 0$.

*Case (i):* $\alpha \in (0, \pi/4]$, that is, $a < \infty$.

Since the lens-shaped domain $G$ is symmetric with respect to the real axis, the Bergman polynomials of $G$

$$P_n(z) = \gamma_n z^n + \cdots, \quad \gamma_n > 0, \quad n = 0, 1, 2, \ldots,$$
have real coefficients. Therefore,
\[ P_n(z\alpha) = P_n(z^{-\alpha}) = (z - a)^{-n} P_n(a) z^n + \cdots. \]

We note that \( P_n(a) > 0 \), since otherwise \( P_n \) would change its sign on \((a, \infty)\), contradicting the fact that all the zeros of \( P_n \) lie in \( G \).

We consider the monic polynomial
\[ p_{2n}(z) := \frac{(z - a)^n P_n(z) P_n(z\alpha)}{\gamma_n P_n(a)} = z^{2n} + \cdots \]
and introduce the weight function
\[ w(z) := |z - a|^{-1/2}. \quad (5.3) \]

Then we have
\[ |w^{2n}(z)p_{2n}(z)| = \frac{1}{\gamma_n P_n(a)} |P_n(z) P_n(z\alpha)|. \quad (5.4) \]

Observe that the weighted polynomial \( w^{2n}p_{2n} \) is invariant under the transformation \( z \to z\alpha \), i.e.,
\[ |w^{2n}(z\alpha)p_{2n}(z\alpha)| = |w^{2n}(z)p_{2n}(z)| \]
and that the zeros of \( p_{2n} \) are those of \( P_n \) together with their reflections with respect to \( L_\alpha \). Since the zeros of \( P_n \) lie in \( G \), we obtain from (2.1),
\[ \nu_{p_{2n}} \big|_{\Gamma} = \frac{1}{2} \nu_{P_n} \big|_{\Gamma}. \]

It suffices, therefore, to establish the convergence
\[ \nu_{p_{2n}} \xrightarrow{*} \mu, \quad n \to \infty, \quad \text{where} \quad \mu := \frac{1}{2} \left( \mu_{\tilde{\Gamma}_\alpha} + \delta_\alpha \right). \quad (5.5) \]

Indeed, since \( \mu \) is supported on \( \tilde{\Gamma}_\alpha = \Gamma \cup \Gamma_\alpha \), (5.5) yields:
\[ \nu_{P_n} = 2 \nu_{p_{2n}} \big|_{\Gamma} \xrightarrow{*} 2\mu \big|_{\Gamma} = \left( \mu_{\tilde{\Gamma}_\alpha} + \delta_\alpha \right) \big|_{\Gamma}. \]

In order to establish (5.5) we proceed as follows:
• First, we show that there is a constant $F_w$ such that

$$U^\mu(z) + \log \frac{1}{w(z)} = F_w, \quad z \in \tilde{\Gamma}_\alpha. \quad (5.6)$$

This, in view of [11, Theorem 1.3.3], will imply that $\mu$ is the equilibrium measure associated with the weight $w$ on $\tilde{\Gamma}_\alpha$ and $F_w$ is the modified Robin constant for $w$; see [11, p. 27].

• Next, we show that

$$\lim_{n \to \infty} \left( \|w^{2n} p_{2n}\|_{\tilde{\Gamma}_\alpha} \right)^{1/2n} = \exp(-F_w) \quad (5.7)$$

and, for some $z_0 \in \text{int} \tilde{\Gamma}_\alpha$,

$$\lim_{n \to \infty} |p_{2n}(z_0)|^{1/2n} = \exp(-U^\mu(z_0)). \quad (5.8)$$

The required result (5.5) will then follow by virtue of [11, Theorem III.4.1].

To establish (5.6) we first recall from (4.10) that $\Phi_\alpha$ is the exterior mapping for $\text{ext} \tilde{\Gamma}_\alpha$. Hence, the Green function of $\text{ext} \tilde{\Gamma}_\alpha$ with pole at $\infty$ is given by $\log |\Phi_\alpha(z)|$. Since $\Phi_\alpha(a) > 0$, we have from (5.2) and (5.3) that, for $z \in \tilde{\Gamma}_\alpha$,

$$\frac{1}{2} U^\tilde{\omega}_\alpha(z) = \frac{1}{2} \log \frac{1}{|z-a|} + \frac{1}{2} \log \Phi_\alpha(a) = -\log \frac{1}{w(z)} + \frac{1}{2} \log \tilde{\Phi}_\alpha(a).$$

This, in conjunction with the definition of $\mu$ and (5.1), yields

$$U^\mu(z) + \log \frac{1}{w(z)} = \frac{1}{2} \log \frac{1}{\text{cap} \tilde{\Gamma}_\alpha} + \frac{1}{2} \log \tilde{\Phi}_\alpha(a), \quad z \in \tilde{\Gamma}_\alpha,$$

which is (5.6), with

$$F_w = \frac{1}{2} \log \frac{\tilde{\Phi}_\alpha(a)}{\text{cap} \tilde{\Gamma}_\alpha}. \quad (5.9)$$

In view of Lemma 4.3, the constant $F_w$ can be also expressed as

$$F_w = \frac{1}{2} \log \frac{\Phi(a)}{\text{cap} L}. \quad (5.10)$$
Next, by using the expression (5.4) for the weighted polynomial \( w^{2n}p_{2n} \) and utilizing the \( n \)th-root asymptotic behavior (1.5) of \( \gamma_n \), the result of Lemma 4.4 for \( |P_n(z)p_n(z_\alpha)| \) (note that \( (z_\alpha)_{\alpha} = z \)), and the lower inequality in (2.19) for \( P_n(a) \), we obtain from (5.10),

\[
\limsup_{n \to \infty} \left( \|w^{2n}p_{2n}\|_{\tilde{\Gamma}_\alpha} \right)^{1/2n} \leq \exp(-F_w). \tag{5.11}
\]

Since both functions \( U^\mu \) and \( \log(1/w) \) are harmonic in \( \text{int} \tilde{\Gamma}_\alpha \), the relation (5.6) remains valid for any \( z \in \text{int} \tilde{\Gamma}_\alpha \). In particular, it is valid for any \( z_0 \in L_\alpha \setminus \{i, -i\} \), i.e.

\[
U^\mu(z_0) + \log \frac{1}{w(z_0)} = F_w. \tag{5.12}
\]

For such \( z_0 \) we also have

\[
|w^{2n}(z_0)p_{2n}(z_0)| = \frac{|P_n(z_0)|^2}{\gamma_n P_n(a)}.
\]

By utilizing, as above, the \( n \)th-root asymptotic behavior of \( \gamma_n \), the result of Lemma 2.2(ii) for \( |P_n(z_0)| \) and the upper inequality in (2.19) for \( P_n(a) \), we obtain

\[
\liminf_{n \to \infty} \left( |w^{2n}(z_0)p_{2n}(z_0)| \right)^{1/2n} \geq \exp(-F_w), \tag{5.13}
\]

which, in view of (5.12), gives

\[
\liminf_{n \to \infty} |p_{2n}(z_0)|^{1/2n} \geq \exp(-U^\mu(z_0)). \tag{5.14}
\]

It is easy to see now that (5.11)–(5.14) yield both (5.7) and (5.8).

**Case (ii):** \( \alpha = 0 \), that is \( a = \infty \).

In this case \( z_\alpha = -\bar{z} \). Therefore, we accordingly consider

\[
p_{2n}(z) := \frac{P_n(z)P_n(-\bar{z})}{(-1)^n\gamma_n z} = z^{2n} + \ldots
\]

and take \( w(z) \equiv 1 \). Since \( \tilde{\partial}_\infty = \mu_{\Gamma_\alpha} \), we let \( \mu := \mu_{\Gamma_\alpha} \) and proceeding as above we establish (5.11)–(5.14), with

\[
F_w = \log \frac{1}{\text{cap} \bar{\Gamma}_\alpha}.
\]
This implies, as before, that

\[ \nu_{P_n} \rightarrow 2\mu_{\Gamma_n}^2 |_{\Gamma}, \]

which is (3.20) with \( a = \infty \).

**Proof of Theorem 3.1.** By (1.6), the relation (3.16) holds for \( z \in \mathbb{C} \setminus \overline{G} \) and, in view of Lemma 2.2, it also holds for \( z \in L \). Thus it remains to consider \( z \in G \). Assume, for example, that \( z \in G \cap \overline{G}_\alpha \) (recall (3.12) for the definition of \( G_\alpha \)). Since the interior map \( \varphi_z \) has poles at \( z_\alpha, z_\beta \), we obtain from (3.13) that

\[ \rho := |\Phi(z_\alpha)| \]

is the largest index for which \( \varphi_z \) is analytic in the interior of the level curve \( L_\rho := \{ \zeta \in \mathbb{C} \setminus G : |\Phi(\zeta)| = \rho \} \). According to [9, Theorem 2.1] this implies (3.15) for \( z \in G \cap \overline{G}_\alpha \). The case \( z \in G \cap \overline{G}_\beta \) can be handled in a similar way.

Now we turn our attention to establishing (3.16) for \( z \in G \setminus \Gamma \) (\( = G_\alpha \cup G_\beta \)), under the additional assumption that \( z \) is not a limit point of zeros of the \( P_n \)'s. We consider separately, again, two cases as follows.

Case (i): \( z \in G_\alpha \).

By our assumption, the measure \( \nu_{p_{2n}} \) has no mass near \( z \), provided that \( n \) is sufficiently large. Hence the weak* convergence result (5.5) gives

\[ U^{\nu_{p_{2n}}}(z) \rightarrow U^\nu(z), \quad n \rightarrow \infty, \]

i.e.,

\[ \frac{1}{2n} \log |p_{2n}(z)| = -U^{\nu_{p_{2n}}}(z) \rightarrow \log \frac{1}{w(z)} - F_w, \quad n \rightarrow \infty. \]

Next, we observe that (1.5), (1.6), and (5.10) imply that

\[ \frac{1}{2n} \log(\gamma_n P_n(a)) \rightarrow F_w, \quad n \rightarrow \infty. \]

The limit (1.6) also gives

\[ -\frac{1}{2n} \log |P_n(z_\alpha)| \rightarrow -\frac{1}{2} \log |\Phi(z_\alpha)|, \quad n \rightarrow \infty. \]
By adding up the last three convergence results and using (5.4) we arrive at
\[ \frac{1}{2n} \log |P_n(z)| \rightarrow -\frac{1}{2} \log |\Phi(z_\alpha)|, \quad n \rightarrow \infty, \]
which is the required result.

**Case (ii):** \( z \in G_\beta. \)

From Remark 4.2 the result in this case follows by interchanging the roles of \( \alpha \) and \( \beta \) in the preceding arguments.

**Proof of Corollary 3.2.** When \( \alpha = \pi/4 \), the lens-shaped domain \( G \) is symmetric with respect to the imaginary axis, in addition to the real one. It is therefore clear from the definition (3.11) that \( \Gamma \) is, in this case, the segment \([-i, i]\). In view of Theorem 3.1, it remains to show that all zeros of \( P_n \) lie on this segment.

By orthogonality, we have the relations
\[ \int_{\Gamma} P_n(z)z^k dm(z) = 0, \quad k = 0, 1, \ldots, n - 1, \]
which can be rewritten, with the help of the Green formula, as
\[ \frac{1}{2i} \int_{L_\beta} P_n(z)\overline{z}^{k+1}dz - \frac{1}{2i} \int_{L_\alpha} P_n(z)\overline{z}^{k+1}dz = 0, \quad k = 0, 1, \ldots, n - 1 (5.15) \]
(we have chosen the direction from \(-i\) to \(i\) in both integrals). Since \( z_\alpha = z \)
for \( z \in L_\alpha \), we have from (3.2),
\[ \overline{z} = \frac{z + 1}{z - 1}, \quad z \in L_\alpha \]
and similarly,
\[ \overline{z} = \frac{1 - z}{z + 1}, \quad z \in L_\beta. \]

By inserting these into (5.15) and by replacing (using the Cauchy theorem) the integrals along \( L_\alpha \) and \( L_\beta \) by integrals along \([-i, i]\), we arrive at the relations
\[ \int_{-1}^{1} P_n(iy) \left[ \left( \frac{1 - iy}{1 + iy} \right)^{k+1} - \left( \frac{1 + iy}{iy - 1} \right)^{k+1} \right] dy = 0, \quad k = 0, 1, \ldots, n - 1. \]
(5.16)
Now it is easy to see inductively, that
\[
\text{span}\left\{ \frac{1}{w^{k+1}} - (-w)^{k+1} : k = 0, 1, \ldots, n - 1 \right\} = \text{span}\left\{ \left( \frac{1}{w} - w \right)^k \left( \frac{1}{w} + w \right) : k = 0, 1, \ldots, n - 1 \right\}.
\]
Using this observation with
\[
w = \frac{1 + iy}{1 - iy},
\]
we conclude that (5.16) is equivalent to requiring that the relation
\[
\int_{-1}^1 P_n(iy) h(y) \frac{1 - y^2}{1 + y^2} dy = 0 \quad (5.17)
\]
holds true for any function \( h \) of the form
\[
h(y) = \sum_{k=0}^{n-1} c_k \left( \frac{2y}{1 + y^2} \right)^k, \quad c_k \in \mathbb{R} \quad (5.18)
\]
We recall that \( G \) is symmetric with respect to both axes. This implies that the Bergman polynomial \( P_n \) has real coefficients and is either even or odd. Hence,
\[
P_n(iy) = Q_n(y) \quad \text{or} \quad P_n(iy) = iQ_n(y),
\]
where \( Q_n \) is a real polynomial of degree \( n \). If we assume that \( Q_n \) has less than \( n \) zeros in the interval \((-1, 1)\), then it can have (at most) \( m \) (\( \leq n - 1 \)) sign changes there, say at some distinct points \( y_1, y_2, \ldots, y_m \) of \((-1, 1)\). The function
\[
h(y) := \prod_{j=1}^m \left( \frac{2y}{1 + y^2} - \frac{2y_j}{1 + y_j^2} \right),
\]
which is of the form (5.18), has simple zeros at \( y_1, y_2, \ldots, y_m \), and has no other zero in \((-1, 1)\). Therefore, our assumption implies that the function \( Q_n(y)h(y) \) is of constant sign in \((-1, 1)\), which contradicts (5.17). That is \( Q_n \), and thus \( P_n \), has exactly \( n \) simple zeros in \((-1, 1)\).
6 Discrepancy of zeros of Bergman polynomials

The asymptotic formula (1.5) for \( \gamma_n \) can be replaced by the more precise inequalities (2.20). Then proceeding as in the proof of Theorem 3.3, we obtain the following refinements of (5.11) and (5.13):

\[
0 < \frac{1}{2n} \log \|w^{2n} p_{2n}\|_{\Gamma_\alpha} + F_w \leq c_1 \frac{\log n}{n} \tag{6.1}
\]

and

\[
\frac{1}{2n} \log |w^{2n}(z_0)p_{2n}(z_0)| + F_w \geq -c_2 \frac{\log n}{n}, \quad z_0 \in \Omega_n \setminus \{-i, i\}. \tag{6.2}
\]

(The left inequality in (6.1) is a consequence of (5.6) and the maximum principle.)

Let \( g_{\Gamma_\alpha}(-, \infty) \) and \( g_{\Gamma_\alpha}(-, z_0) \) denote, respectively, the Green function of \( \text{ext} \, \Gamma_\alpha \) with pole at \( \infty \) and the Green function of \( \text{int} \, \Gamma_\alpha \) with pole at \( z_0 \).

Consider the level curves:

\[
L_n^- := \left\{ z \in \text{int} \, \Gamma_\alpha : g_{\Gamma_\alpha}(z, z_0) = \left( \frac{\log n}{n} \right)^{3/4} \right\}, \tag{6.3}
\]

\[
L_n^+ := \left\{ z \in \text{ext} \, \Gamma_\alpha : g_{\Gamma_\alpha}(z, \infty) = \left( \frac{\log n}{n} \right)^{1/4} \right\}, \tag{6.4}
\]

and let

\[
E_n := \text{ext} \, L_n^- \cap \text{int} \, L_n^+ \quad \left( \supset \, \Gamma_\alpha \right).
\]

Then, on applying Lemmas III.4.4 and III.4.5 in [11], we deduce from (6.1) and (6.2) that

\[
\nu_{p_{2n}}(\mathbb{C} \setminus E_n) \leq c \left( \frac{\log n}{n} \right)^{1/4}, \quad n = 2, 3, \ldots.
\]

The same is true, if we replace \( \nu_{p_{2n}} \) by \( \nu_P \) and \( E_n \) by \( E_n \cap G \).
Let
\[ \mu := (\mu_{\tilde{\Gamma}_\alpha} + \delta_0) \bigg|_{\Gamma} \cdot \]
Since the measure \( \mu \) is supported on \( \Gamma \) and \( \mu, \nu_{P_n} \) are unit measures, we obtain that
\[ |(\mu - \nu_{P_n}) (E_n \cap G)| \leq c \left( \frac{\log n}{n} \right)^{1/4}, \quad n = 2, 3, \ldots \quad (6.5) \]
Now consider an arc \( J \subset \Gamma \) and define two quadrilaterals based on \( J \), as follows. The first one is bounded by \( \tilde{\Gamma}_\alpha \) and \( L_n^- \), and by the level curves of the conjugate of the Green function \( g_{\tilde{\Gamma}_\alpha}(\cdot, z_0) \) that pass through the endpoints of \( J \); see Figure 6.1.

![Figure 6.1](image)

The second quadrilateral is bounded by \( \tilde{\Gamma}_\alpha \) and \( L_n^+ \), and by the level curves for the conjugate of the potential \( U^\alpha \) that pass through the endpoints of \( J \); see Figure 6.1.

Let \( A_n(J) \) be the union of these two quadrilaterals. The following “discrepancy result” can be proved:

**Result 6.1** Let \( J \) be a subarc of \( \Gamma \). Then, for any \( n = 2, 3, \ldots \), there exists a constant \( c > 0 \) independent of \( J \) and \( n \) such that
\[ |(\mu - \nu_{P_n}) (A_n(J))| \leq c \left( \frac{\log n}{n} \right)^{1/4}. \quad (6.6) \]
This result can be proved by adopting the method in [1], but the details are tedious, so we omit them. In this connection we mention a recent result of Andrievskii, Pritsker and Varga. For an arbitrary convex domain $G$, they proved in [2] that if one replaces $\nu_{P_n}$ by its balayage $\tilde{\nu}_{P_n}$ onto $L = \partial G$, then the relation

\[ |(\mu_L - \tilde{\nu}_{P_n})(J)| \leq c \left( \frac{\log n}{n} \right)^{1/2}, \quad n = 2, 3, \ldots, \]

holds for any $J \subset L$, with $c$ independent of $J$ and $n$; recall that $\mu_L$ denotes the equilibrium measure of $L$.

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**References**


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