Zero Distribution of Bergman Orthogonal Polynomials for Certain Planar Domains

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Abstract

Let G be a simply-connected domain in the complex plane bounded by a closed Jordan curve L, and let P_n, $n \geq 0$, be polynomials of respective degrees $n = 0, 1, \ldots$ that are orthonormal in G with respect to the area measure (the so-called Bergman polynomials). Let φ be a conformal map of G onto the unit disk. We characterize in terms of the asymptotic behavior of the zeros of P_n's the case when φ has a singularity on L. To investigate the opposite case we consider a special class of lens-shaped domains G that are bounded by two orthogonal circular arcs. Utilizing the theory of logarithmic potentials with external fields, we show that the limiting distribution of the zeros of the P_n's for such lens domains is supported on a Jordan arc joining the two vertices of G. We determine this arc along with the distribution function.

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1 Introduction

Let G be a bounded simply-connected domain in the complex plane $\mathbb{C}$, whose boundary $L := \partial G$ is a Jordan curve and let $\{P_n\}_{n=0}^{\infty}$ denote the sequence of Bergman polynomials of G. This is defined as the sequence

$$P_n(z) = \gamma_n z^n + \cdots, \quad \gamma_n > 0, \quad n = 0, 1, 2, \ldots,$$

of polynomials that are orthonormal with respect to the inner product

$$(f, g) := \int_G f(z)\overline{g(z)}dm(z),$$

where dm stands for the 2-dimensional Lebesgue measure.

Bergman polynomials form a complete orthonormal system for the corresponding Hilbert space $L_2(G)$ and they provide a convenient representation for its reproducing kernel function. The latter is intimately related to the derivative of a conformal map of G onto the unit disk $\mathbb{D} := \{w : |w| < 1\}$. Consequently, Bergman polynomials are useful for the construction of approximations to the conformal map (via the so-called Bieberbach polynomials); see e.g. [4, pp. 34–35].

Let $\Omega := \overline{\mathbb{C}} \setminus \overline{G}$ and $\Delta := \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ denote, respectively, the exterior (in $\overline{\mathbb{C}}$) of $\overline{G}$ and $\overline{\mathbb{D}}$. Then the exterior conformal map Φ associated with G is the conformal map

$$\Phi : \Omega \to \Delta, \tag{1.1}$$

normalized so that

$$\Phi(z) = cz + \mathcal{O}(1), \quad z \to \infty, \quad c > 0. \tag{1.2}$$

We note that the (logarithmic) capacity $\operatorname{cap} L$ of L (or $\overline{G}$) is given by

$$\operatorname{cap} L = 1/c. \tag{1.3}$$

It is well-known that (cf. [9, Lemma 4.3])

$$\lim_{n\to\infty} \|P_n\|_{G}^{1/n} = 1. \tag{1.4}$$

(Here and in the sequel $\|\cdot\|$ means the uniform norm on the subscripted set.) Furthermore, (1.4) implies that

$$\lim_{n\to\infty} \gamma_n^{1/n} = \frac{1}{\operatorname{cap} L} \tag{1.5}$$

and

$$\lim_{n\to\infty} |P_n(z)|^{1/n} = |\Phi(z)|, \tag{1.6}$$

locally uniformly in $\overline{\mathbb{C}} \setminus \mathrm{Co}(\overline{G})$, where $\mathrm{Co}(\overline{G})$ denotes the convex hull of $\overline{G}$; see [13, Theorem 3.1.1].

If G is convex, the limit (1.6) holds for all $z \in \overline{\mathbb{C}} \setminus \overline{G}$. Moreover, in this case all zeros of P_n lie in G (cf. [10, p. 366]) and, as we shall show in Corollary 2.3 below, there are positive constants c_1 and c_2 such that for all $z \in \overline{\mathbb{C}} \setminus G$ and all n,

$$|P_n(z)||\Phi(z)|^{-n} \geq c_1 n^{-c_2}. \tag{1.7}$$

A matching upper bound of the form $c_3 n^{c_4}$ follows from [1, p. 360], and these estimates show that a stronger form of (1.6) holds for all $z \in \overline{\mathbb{C}} \setminus G$, at least for convex G.

The behavior of the P_n's inside G is much more delicate and depends, in particular, on the analytic properties of an *interior* conformal mapping φ of G onto $\mathbb{D}$. We shall consider the following two cases separately:

Case (i) φ has a singularity on L ($= \partial G$);

Case (ii) φ has no singularities on L (that is, φ can be extended analytically to a domain $G_0 \supset \overline{G}$).

Note that the above splitting is independent of the choice of φ, since any two conformal mappings of G onto $\mathbb{D}$ are related by a Möbius transformation. As we show in Theorem 2.1, Case (i) can be characterized in terms of the asymptotic behavior of the zeros of Bergman polynomials. It follows, in particular, that in this case *every* point of L is a limit point of zeros of the P_n's. Under additional assumptions on G, Case (i) has been illuminated by the work of Andrievskii and Blatt [1], who have established Erdős-Turán type theorems (Theorem 5 in [1]) for the distribution of zeros of the P_n's.

Case (ii) is more complicated, and different situations may arise. For example, if L is an analytic curve, then the exterior map Φ can also be extended to a larger domain $\Omega_0 \supset (\overline{\mathbb{C}} \setminus G)$, and it turns out (cf. [4, pp. 12–13]) that the zeros of the P_n's have no limit points in Ω_0 (in particular, on L). On the other hand, if G is a square or an equilateral triangle, then all the zeros of the P_n's must lie on the segments that join the vertices of G with its center, and every point of these segments attracts the zeros of the P_n's. This fact was conjectured by Eiermann and Stahl [3], and has recently been proved in [8].

Regarding the Case (ii), one goal of this paper is to consider a special class of domains for which the location of zeros of Bergman polynomials is much less transparent. Let G be a lens-shaped domain bounded by two circular arcs that are orthogonal to each other. We show (in Theorem 3.3) that there is a "critical arc" Γ in G, which joins the two vertices of L, such that the limiting distribution of the zeros of the P_n's is given by a certain measure supported on Γ. This is in contrast to the behavior of the zeros of Faber polynomials for such regions (see [7]), which is somewhat surprising since many results concerning Bergman polynomials are obtained via comparison with Faber polynomials.

The paper is organized as follows: In the next section we address Case (i) and we also establish there the estimate (1.7). Our main results concerning lens-shaped domains are formulated in Section 3 and their proofs appear in Sections 4, 5. A strengthened version of Theorem 3.3 (the discrepancy of zeros) is discussed in Section 6.

2 Some general results

We begin with some definitions that are needed for the statements of our results. Let Q be a polynomial of degree n with zeros $z_1, z_2, \dots, z_n$. The *normalized counting measure of the zeros* of Q is defined by

$$\nu_Q := \frac{1}{n}\sum_{k=1}^{n}\delta_{z_k}, \tag{2.1}$$

where δ_z denotes the unit point mass at the point z. In other words, for any subset A of $\mathbb{C}$,

$$\nu_Q(A) = \frac{\text{number of zeros of } Q \text{ in } A}{n}.$$

Next, given a sequence $\{\sigma_n\}$ of Borel measures, we say that $\{\sigma_n\}$ *converges in the weak* sense* to a measure σ, symbolically $\sigma_n \stackrel{*}{\longrightarrow} \sigma$, if

$$\int f d\sigma_n \longrightarrow \int f d\sigma, \quad n \to \infty,$$

for every function f continuous on $\overline{\mathbb{C}}$.

The exterior mapping function defined by (1.1) and (1.2) can be naturally extended to a homeomorphism between the corresponding closed domains, so that

$$\Phi(L) = \mathbb{T} := \{w : |w| = 1\}.$$

Then the normalized angular measure $d\theta/2\pi$ on $\mathbb{T}$ gives rise, in a natural way, to a unit measure μ_L on L. Namely, for any Borel set $A \subset L$,

$$\mu_L(A) := \frac{1}{2\pi}\int_{\Phi(A)} d\theta.$$

This measure is called the *equilibrium measure for* L (or for $\overline{G}$).

We can now formulate our first result whose proof is given later in this section.

Theorem 2.1 *Let $G \subset \mathbb{C}$ be a simply connected domain bounded by a closed Jordan curve L, and let φ be a conformal map of G onto the unit disk $\mathbb{D}$. Then the following two statements are equivalent:*

(a) *φ has a singularity on L.*

(b) *There is a subsequence $\mathcal{N} \subset \mathbb{N}$ such that*

$$\nu_{P_n} \xrightarrow{*} \mu_L, \quad \text{as } n \to \infty,\ n \in \mathcal{N}, \tag{2.2}$$

where P_n is the Bergman polynomial of degree n for G.

In this connection, we recall the result of Andrievskii and Blatt [1, Theorem 5] mentioned in the Introduction. This result implies that if L is a quasiconformal curve and if φ satisfies

$$\|\varphi^{(k)}\|_G = \infty, \quad \text{for some } k \geq 1, \tag{2.3}$$

then (2.2) holds true. (In fact, under the stated assumptions, their result gives estimates for the discrepancy between ν_{P_n} and μ_L.) Clearly, (2.3) implies that φ has a singularity on L. The converse, however, is not true as the following example shows.

Example *There exists a domain G bounded by a C^∞ Jordan curve such that $\varphi : G \to \mathbb{D}$ has a singularity on ∂G, but*

$$\|\varphi^{(k)}\|_G < \infty \quad \text{for all } k \geq 1. \tag{2.4}$$

Consider the function

$$\psi(w) = w + \varepsilon \sum_{n=1}^{\infty} 2^{-\sqrt{n}} w^n, \quad w \in \mathbb{D}. \tag{2.5}$$

It is easy to see that

$$|\psi'(w) - 1| < \frac{1}{2}, \quad w \in \mathbb{D},$$

provided $\varepsilon > 0$ is small enough. For such ε and any w_1, w_2 in $\mathbb{D}$, we have

$$|\psi(w_1) - \psi(w_2)| \geq \frac{1}{2}|w_1 - w_2|,$$

which implies that the corresponding function ψ is univalent in $\mathbb{D}$.

Let $G := \psi(\mathbb{D})$, and observe that the inverse function $\varphi := \psi^{-1}$ is a conformal mapping of G onto $\mathbb{D}$, such that $|\varphi'| > 2/3$ in G. If we assume that φ admits an analytic continuation to a neighborhood of $z_0 := \psi(1) \in \partial G$, then $\varphi'(z_0) \neq 0$, and therefore φ will be univalent in some (possibly smaller) neighborhood of z_0. This, in turn, would imply that ψ admits an analytic continuation to a neighborhood of $w_0 = 1$, in contradiction to the well-known fact that $w_0 = 1$ is a point of singularity of the power series (2.5). Hence, φ has a singularity at z_0. Since $|\psi'| > 1/2$ in $\mathbb{D}$, and $|\psi^{(k)}|$ is bounded from above in $\mathbb{D}$, for any $k \geq 1$, we deduce by induction that φ satisfies (2.4), as required. ■

The proof of Theorem 2.1 is a simple combination of two known facts. The first follows from Theorem III.4.1 in [11]:

Fact A *Let $\{q_n\}$ be a sequence of monic polynomials of respective degrees $n = 1, 2, \ldots$ satisfying*

$$\lim_{n\to\infty} \|q_n\|_G^{1/n} = \operatorname{cap} L \ (= \operatorname{cap}(\partial \mathrm{G})). \tag{2.6}$$

(i) *If there exists a subsequence $\mathcal{N} \subset \mathbb{N}$ such that*

$$\nu_{q_n} \xrightarrow{*} \mu_L \quad \text{as } n \to \infty,\ n \in \mathcal{N}, \tag{2.7}$$

then there is a subsequence $\mathcal{N}_1 \subset \mathcal{N}$ and a point $\zeta \in G$ such that

$$\lim_{n\to\infty,\, n\in\mathcal{N}_1} |q_n(\zeta)|^{1/n} = \operatorname{cap} L. \tag{2.8}$$

(ii) *If* (2.8) *holds for a subsequence $\mathcal{N}_1 \subset \mathbb{N}$ and for some point $\zeta \in G$, then* (2.7) *holds with $\mathcal{N} := \mathcal{N}_1$.*

The second fact is a consequence of Theorem 2.1 in [9]:

Fact B *Given $\zeta \in G$, let φ_ζ denote the conformal map of G onto $\mathbb{D}$ normalized by*

$$\varphi_\zeta(\zeta) = 0, \quad \varphi'_\zeta(\zeta) > 0. \tag{2.9}$$

Then φ_ζ has a singularity on L if and only if

$$\limsup_{n\to\infty} |P_n(\zeta)|^{1/n} = 1. \tag{2.10}$$

Proof of Theorem 2.1.

(a)⇒(b) Assume that φ has a singularity on L. Then the same is true for any map φ_ζ, $\zeta \in G$. By Fact B, there is a subsequence $\mathcal{N}_1 \subset \mathbb{N}$ such that

$$\lim_{n\to\infty, n\in\mathcal{N}_1} |P_n(\zeta)|^{1/n} = 1. \tag{2.11}$$

Therefore, the monic polynomials

$$q_n := \gamma_n^{-1} P_n$$

satisfy (recall (1.5)) the relation (2.8). In view of (1.5) and (1.4) they also satisfy (2.6) and we conclude by part (ii) of Fact A that (2.7) holds, which is equivalent to the required relation (2.2).

(b)⇒(a) Assume that (2.2) holds and let q_n be as above. Then we have (2.7) and (2.6) holds as before. By part (i) of Fact A, we conclude that (2.8) holds for some $\zeta \in G$. But this is equivalent to (2.11) which in turn leads to (2.10), this last statement following by (1.4). Hence, by Fact B, φ_ζ has a singularity on L and the proof is complete. ∎

Next we derive bounds for Bergman polynomials P_n.

Lemma 2.2 *Let G be as in Theorem* 2.1.

(i) *If L is quasiconformal, then the Bergman polynomials P_n satisfy*

$$\|P_n\|_{\overline{G}} \leq c_1 n^{c_2}, \quad n \geq 1, \tag{2.12}$$

where c_1, c_2 are positive constants independent of n.

(ii) *Let $z_0 \in L$ and assume that there is a straight line passing through z_0 that does not divide G. Then, there exist positive constants c_3, c_4 independent of n and z_0, such that*

$$|P_n(z_0)| \geq c_3\, n^{-c_4}, \quad n \geq 1. \tag{2.13}$$

Proof. (i) See [1, p. 360] and note that here we consider, with the notation of [1, p. 341], the unweighted case $h(z) \equiv 1$.
(ii) Without loss of generality we may assume that $z_0 = 0$ and that G lies in the right half-plane. By orthogonality,

$$\int_G P_n(z) \frac{\overline{P_n(z)} - \overline{P_n(0)}}{\overline{z}} dm(z) = 0.$$

Hence,

$$|P_n(0)| \left| \int_G \frac{P_n(z)}{\overline{z}} dm(z) \right| = \left| \int_G \frac{|P_n(z)|^2}{\overline{z}} dm(z) \right|. \tag{2.14}$$

Observe that

$$\left| \int_G \frac{P_n(z)}{\overline{z}} dm(z) \right| \leq c_1 \|P_n\|_{\overline{G}}, \tag{2.15}$$

for some positive constant $c_1 = c_1(G)$, and

$$\left| \int_G \frac{|P_n(z)|^2}{\overline{z}} dm(z) \right| \geq \int_G |P_n(z)|^2 \frac{\Re z}{|z|^2} dm(z). \tag{2.16}$$

Next, let

$$\Delta_\varepsilon := \{z : \frac{\Re z}{|z|^2} \geq \varepsilon\}, \quad \varepsilon > 0.$$

(Note that Δ_ε is the closed disk $\{z : |z-1/(2\varepsilon)| \le 1/(2\varepsilon)\}$.) Then, it is easy to see (a figure will help) that the area of $G \setminus \Delta_\varepsilon$ is less than $c_2\,\varepsilon$, where c_2 depends only on the diameter of G. Therefore,

$$\int_{G\setminus\Delta_\varepsilon} |P_n(z)|^2 dm(z) \le c_2\,\varepsilon\,\|P_n\|_G^2,$$

so that if we choose

$$\varepsilon = \frac{1}{2}\left(c_2\,\|P_n\|_G^2\right)^{-1}, \tag{2.17}$$

we have

$$\int_{G\cap\Delta_\varepsilon} |P_n(z)|^2 dm(z) = 1 - \int_{G\setminus\Delta_\varepsilon} |P_n(z)|^2 dm(z) \ge \frac{1}{2}$$

and, consequently, from the definition of Δ_ε,

$$\int_{G\cap\Delta_\varepsilon} |P_n(z)|^2 \frac{\Re z}{|z|^2} dm(z) \ge \frac{1}{2}\varepsilon. \tag{2.18}$$

The required result then follows by combining (2.14) to (2.18) with (2.12). ■

Corollary 2.3 *Assume that G is a bounded convex domain and let Φ be the exterior mapping defined by* (1.1)–(1.2)*. Then there exist positive constants c_j, $1 \le j \le 4$, independent of n and z such that*

$$c_3 n^{-c_4} \le |P_n(z)|\,|\Phi(z)|^{-n} \le c_1 n^{c_2}, \quad z \in \overline{\mathbb{C}} \setminus G,\ n \ge 1. \tag{2.19}$$

In particular,

$$c_3 n^{-c_4} \le \gamma_n (\operatorname{cap} L)^n \le c_1 n^{c_2}, \quad n \ge 1. \tag{2.20}$$

Proof. We note that the assumption on G implies that L $(=\partial G)$ is quasiconformal. Furthermore, all zeros of P_n lie in G. Hence the function $P_n \Phi^{-n}$ is analytic in $\overline{\mathbb{C}} \setminus \overline{G}$ and is continuous and non-vanishing in $\overline{\mathbb{C}} \setminus G$. The required estimates (2.19) follow from the maximum modulus principle and Lemma 2.2. Furthermore, for $z = \infty$, (2.19) yields (2.20); recall (1.2)–(1.3). ■

Remark 2.4 The estimate (2.20) actually holds for any region with a quasiconformal boundary as is shown in [1, Lemma 6].

3 Results for lens-shaped domains

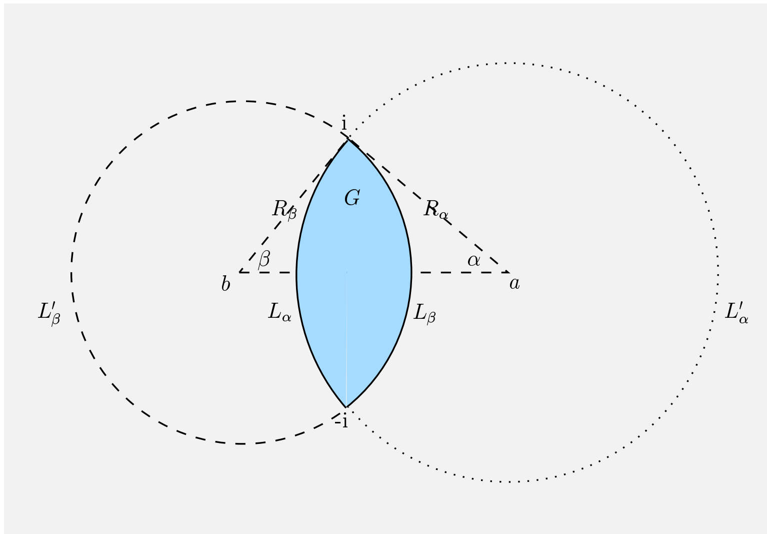

Figure 3.1

Throughout this section we shall assume that G is a lens-shaped domain whose boundary L consists of two orthogonal circular arcs L_α and L_β that join the points i and $-i$ (L_α being to the left of L_β) and form angles α and β, respectively, with the linear segment $[-i, i]$; see Figure 3.1. That is,

$$L = L_\alpha \cup L_\beta$$

and

$$\alpha + \beta = \pi/2,$$

where we assume without loss of generality that

$$\alpha \in [0, \pi/4].$$

In the special case $\alpha = 0$, the domain G is the half-disk $\{z : |z| < 1, \Re z > 0\}$.

Let L'_α, L'_β denote the complementary arcs of L_α, L_β, so that $L_\alpha \cup L'_\alpha$ and $L_\beta \cup L'_\beta$ are circles centered, respectively, at

$$a := \cot\alpha, \qquad b := -\tan\alpha \tag{3.1}$$

with corresponding radii,

$$R_\alpha := 1/\sin\alpha, \qquad R_\beta := 1/\cos\alpha.$$

For any point z on $\overline{G}$, we use the subscript notation z_α and z_β to denote the reflections of z with respect to L_α and L_β, respectively. Hence,

$$z_\alpha = \frac{a\overline{z}+1}{\overline{z}-a}, \quad z_\beta = \frac{b\overline{z}+1}{\overline{z}-b}, \tag{3.2}$$

and since $b = -1/a$ we have

$$z_\alpha z_\beta = -1. \tag{3.3}$$

In particular, when $\alpha = 0$,

$$z_\alpha = -\overline{z}, \quad z_\beta = 1/\overline{z}. \tag{3.4}$$

With G as above, the normalized exterior map $w = \Phi(z)$ is given, as can be easily verified, by the composition of the following three transformations:

$$\xi(z) := e^{i(\pi/2+\alpha)}\,\frac{z-i}{z+i}, \tag{3.5}$$

$$t(\xi) := \xi^{2/3}, \quad \arg\xi \in (-\pi/2, 3\pi/2], \tag{3.6}$$

$$w(t) := \frac{1-\lambda_\alpha t}{t-\lambda_\alpha}, \quad \lambda_\alpha := e^{i(\pi+2\alpha)/3}. \tag{3.7}$$

These lead to

$$\Phi(z) = \left(\frac{3}{2}\sin\frac{2\alpha+\pi}{3}\right) z + \mathcal{O}(1), \quad z\to\infty,$$

from which we obtain

$$\mathrm{cap}L = \left(\frac{3}{2}\sin\frac{2\alpha+\pi}{3}\right)^{-1}, \tag{3.8}$$

for the capacity of the boundary curve L.

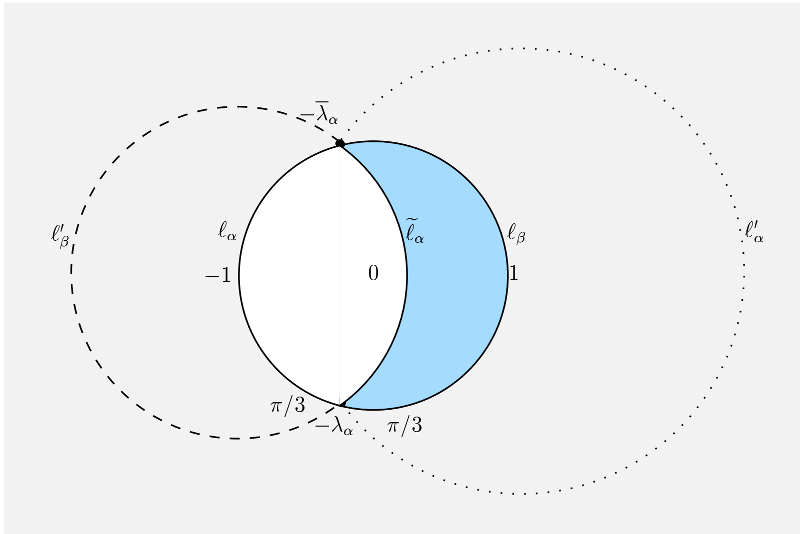

Figure 3.2

Let ℓ_α, ℓ_β, ℓ'_α and ℓ'_β denote the images under Φ of the four circular arcs L_α, L_β, L'_α and L'_β, respectively. Then, it can be easily seen by considering the three elementary transformations (3.5)–(3.7) which constitute Φ, that ℓ_α, ℓ_β, ℓ'_α and ℓ'_β are also circular arcs; see Figure 3.2.

Since $L_\alpha \perp L_\beta$, it is clear that L'_α is the reflection of L_α with respect to L_β. Therefore, by the reflection principle, the function Φ can be analytically continued across L_β, as a conformal map from $\overline{\mathbb{C}} \setminus L_\alpha$ onto the exterior of the curve $\ell_\alpha \cup \widetilde{\ell}_\alpha$, where $\widetilde{\ell}_\alpha$ is the reflection of ℓ'_α with respect to the unit circle; see Figure 3.2. This extension of Φ is given by

$$\Phi(z) := 1/\overline{\Phi}(z_\beta), \quad z \in G \cup L_\beta. \tag{3.9}$$

Similarly, the formula

$$\Phi(z) := 1/\overline{\Phi}(z_\alpha), \quad z \in G \cup L_\alpha, \tag{3.10}$$

gives the analytic continuation of Φ across L_α.

We define now the "critical arc" Γ, which plays a central role in our investigation, by

$$\Gamma := \{z \in \overline{G} : |\Phi(z_\alpha)| = |\Phi(z_\beta)|\}. \tag{3.11}$$

We shall show in Lemma 4.1 that Γ is an analytic arc lying in G except for its endpoints i and $-i$. Hence, Γ divides G into two domains

$$G_\alpha := \operatorname{int}(L_\alpha \cup \Gamma), \quad G_\beta := \operatorname{int}(L_\beta \cup \Gamma), \tag{3.12}$$

where "int" means the interior of the indicated closed curve. Obviously,

$$|\Phi(z_\alpha)| < |\Phi(z_\beta)|, \quad z \in G_\alpha, \tag{3.13}$$

with an opposite inequality for $z \in G_\beta$.

It follows from the above considerations that the formula

$$\Phi(z) := \begin{cases} 1/\overline{\Phi}(z_\alpha), & z \in \overline{G}_\alpha \\ 1/\overline{\Phi}(z_\beta), & z \in G_\beta \cup L_\beta \end{cases} \tag{3.14}$$

gives an analytic and conformal extension of Φ to $\mathbb{C}\setminus\Gamma$, and $|\Phi|$ is continuous in $\mathbb{C}$.

We now state our results on the Bergman polynomials for G. Their proofs are given in Section 5.

Theorem 3.1 *Let Φ be the normalized exterior mapping associated with G and define Φ in $\overline{G}$ by* (3.14)*. Then*

$$\limsup_{n\to\infty} |P_n(z)|^{1/n} = |\Phi(z)|, \quad z \in \mathbb{C}. \tag{3.15}$$

Furthermore, the relation

$$\lim_{n\to\infty} |P_n(z)|^{1/n} = |\Phi(z)| \tag{3.16}$$

holds for all $z \in \mathbb{C} \setminus G$ and also for any $z \in G$ that is not a limit point of zeros of the P_n's.

Corollary 3.2 *In a special case $\alpha = \pi/4$, Γ is the straight line segment $[-i, i]$. Furthermore, all zeros of P_n lie on $[-i, i]$, and* (3.16) *holds for all $z \in \mathbb{C} \setminus [-i, i]$; see Figure* 3.3.

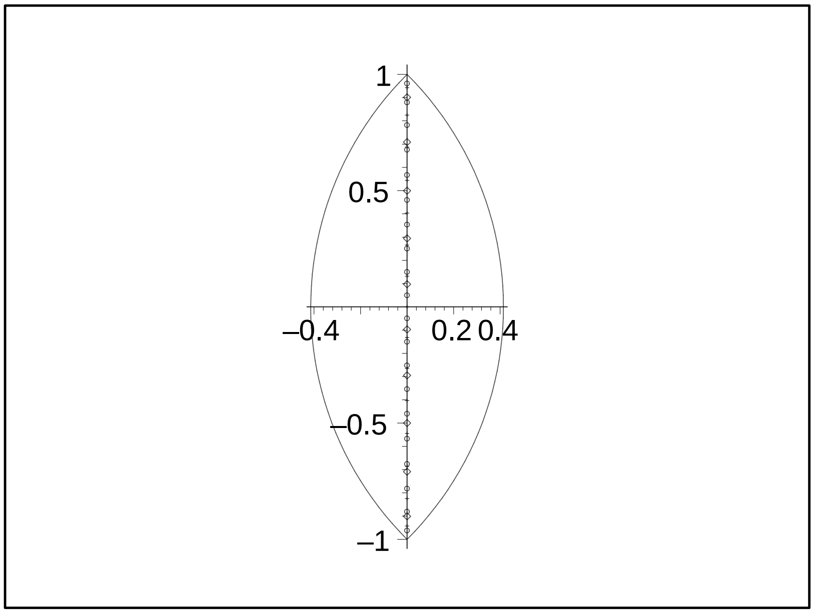

Figure 3.3 The domain G and the zeros of P_{10}, P_{15} and P_{20}, for $\alpha = \pi/4$.

In order to formulate our main result on the zero distribution of Bergman polynomials we need more notation. Let $\mathcal{G}$ be an arbitrary simply-connected domain in $\mathbb{C}$ bounded by a Jordan curve $\mathcal{L}$, and let $z_0 \in \operatorname{ext} \mathcal{L}$. (Here and in the sequel "ext" means the exterior in $\overline{\mathbb{C}}$ of the indicated curve.) Consider the conformal map $\Phi(\cdot, z_0)$ of $\operatorname{ext} \mathcal{L}$ onto $\operatorname{ext} \mathbb{T}$ that is normalized by

$$\Phi(z_0, z_0) = \infty, \quad \Phi'(z_0, z_0) > 0. \tag{3.17}$$

Then the angular measure $d\theta/2\pi$ on $\mathbb{T}$ gives rise, in a natural way, to a unit measure on $\mathcal{L}$. The latter is called the *balayage* of the unit point mass δ_{z_0} onto $\mathcal{L}$, and is denoted by $\widehat{\delta}_{z_0}$ (cf. [11, Section II.4]). In particular, if $z_0 = \infty$

then

$$\widehat{\delta}_\infty = \mu_{\mathcal{L}}, \tag{3.18}$$

where $\mu_{\mathcal{L}}$ is the equilibrium measure for $\mathcal{L}$, as defined in Section 2.

Returning now to our lens-shaped domain, we set

$$\widehat{\Gamma}_\alpha := \Gamma \cup \Gamma_\alpha, \tag{3.19}$$

where Γ_α is the reflection of Γ with respect to L_α; see Figure 3.4. Note that the point $a = \cot\alpha$ (recall (3.1)) lies outside $\widehat{\Gamma}_\alpha$.

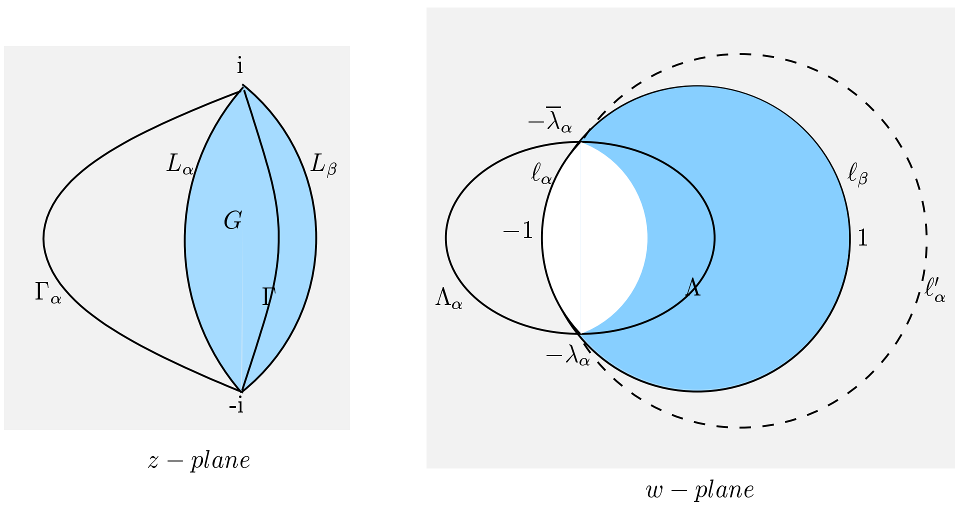

Figure 3.4

We can now state our main result concerning the behavior of the zeros of Bergman polynomials for lens-shaped regions.

Theorem 3.3 *Let ν_{P_n} be the normalized counting measure of the zeros of P_n. Further let $\mu_{\widehat{\Gamma}_\alpha}$ be the equilibrium measure for $\widehat{\Gamma}_\alpha$, and let $\widehat{\delta}_a$ be the balayage of δ_a onto $\widehat{\Gamma}_\alpha$. Then*

$$\nu_{P_n} \xrightarrow{*} \left(\mu_{\widehat{\Gamma}_\alpha} + \widehat{\delta}_a\right)\Big|_\Gamma, \quad n \to \infty, \tag{3.20}$$

where $\sigma\big|_\Gamma$ denotes the restriction of the measure σ to Γ; see Figure 3.5.

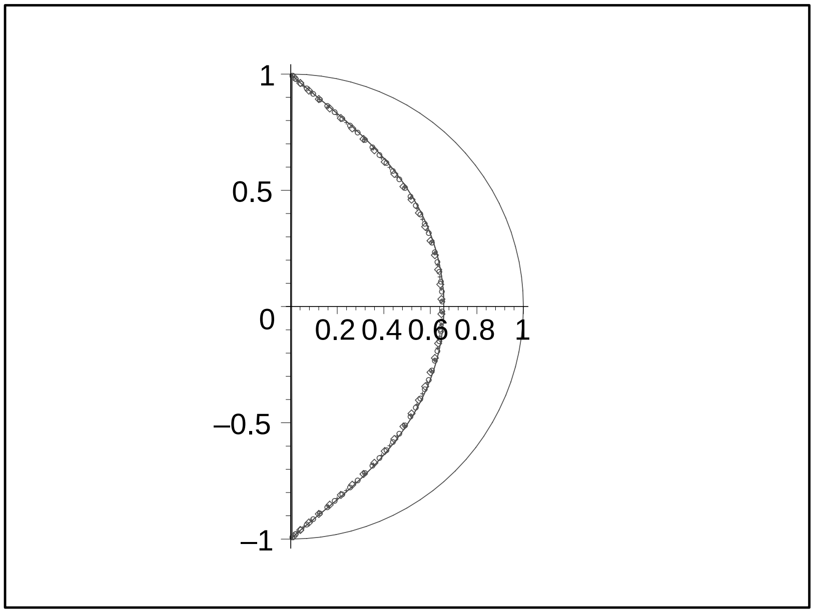

Figure 3.5 The zeros of P_{40}, P_{50} and P_{60} and the arc Γ for the half-disk.

Remark 3.4 In a similar way we could define $\widehat{\Gamma}_\beta := \Gamma \cup \Gamma_\beta$, where Γ_β is the reflection of Γ with respect to L_β. Then (3.20) takes the equivalent form

$$\nu_{P_n} \xrightarrow{*} \left(\mu_{\widehat{\Gamma}_\beta} + \widehat{\delta}_b\right)\Big|_{\Gamma},$$

where $b = -\tan\alpha$ and $\widehat{\delta}_b$ is the balayage of δ_b onto $\widehat{\Gamma}_\beta$.

4 Auxiliary results

Let G be the lens-shaped domain described in Section 3. Given $\zeta \in G$, let $\varphi_\zeta : G \to \mathbb{D}$ be the interior conformal map normalized by $\varphi_\zeta(\zeta) = 0$, $\varphi'_\zeta(\zeta) > 0$. It is easy to see that φ_ζ has the form

$$\varphi_\zeta(z) = e^{i\theta}\left[\left(\frac{z-i}{z+i}\right)^2 - \left(\frac{\zeta-i}{\zeta+i}\right)^2\right]\left[\left(\frac{z-i}{z+i}\right)^2 - e^{-i4\alpha}\left(\frac{\overline{\zeta}+i}{\overline{\zeta}-i}\right)^2\right]^{-1},$$

for a suitable $\theta \in [0, 2\pi]$. By making use of (3.1) and (3.2), we obtain after some algebra, that

$$\varphi_\zeta(z) = C(\zeta)\frac{(z-\zeta)(z\zeta+1)}{(z-\zeta_\alpha)(z-\zeta_\beta)}, \tag{4.1}$$

where

$$C(\zeta) := e^{i\theta}\left(\frac{\overline{\zeta}-i}{\zeta+i}\right)^2 \frac{e^{2i\alpha}}{(\overline{\zeta}\sin\alpha-\cos\alpha)(\overline{\zeta}\cos\alpha+\sin\alpha)}.$$

We see that φ_ζ is a rational function with poles at ζ_α and ζ_β which lie in $\operatorname{ext} L$. Thus, our G falls under Case (ii) mentioned in the Introduction. Furthermore, since a and b also lie in $\operatorname{ext} L$, it can be easily checked that $C(\zeta)$ is uniformly bounded in G. Then we obtain from (4.1), by a straightforward estimate, that for all $\zeta \in G$ and $z \in \mathbb{C}$,

$$|\varphi'_\zeta(z)| \le c\left(\frac{1}{|z-\zeta_\alpha|^2} + \frac{1}{|z-\zeta_\beta|^2}\right), \tag{4.2}$$

where $c > 0$ is independent of ζ and z.

Next we describe some properties of the critical arc Γ; recall (3.11) and (3.19).

Lemma 4.1 *Let*

$$\widehat{\Lambda}_\alpha := \{w : \left|\frac{w(w+A)}{Bw-1}\right| = 1\}, \tag{4.3}$$

where

$$A = \frac{2}{\sqrt{3}}\sin\frac{\pi-2\alpha}{3} \quad \textit{and} \quad B = \frac{2}{\sqrt{3}}\sin\frac{2\alpha}{3} \tag{4.4}$$

and denote

$$\Lambda := \{w \in \widehat{\Lambda}_\alpha : |w| \le 1\}, \quad \Lambda_\alpha := \{w \in \widehat{\Lambda}_\alpha : |w| \ge 1\}. \tag{4.5}$$

Then the function Φ *extended by* (3.9) *satisfies*

$$\Lambda = \Phi(\Gamma) \quad \textit{and} \quad \Lambda_\alpha = \Phi(\Gamma_\alpha). \tag{4.6}$$

Hence, Γ is an analytic arc lying in G (except for its endpoints i and $-i$), and Φ maps conformally the exterior of $\widehat{\Gamma}_\alpha$ onto the exterior of $\widehat{\Lambda}_\alpha$.

Furthermore, the two arcs Γ and Γ_α meet each other at i and $-i$ with interior angles $\pi/2$ and the arc Γ forms angles $\pi/4$ with both L_α and L_β; see Figure 3.4.

We remark that A, B above satisfy $|A-B|<2$, and this condition ensures that $\widehat{\Lambda}_\alpha$ consists of a single closed curve.

Proof. Let $z \in G$. Since $z_\alpha z_\beta = -1$ (see (3.3)), we obtain by using the transformations (3.5) and (3.6),

$$\xi(z_\alpha) = -\xi(z_\beta) \quad \text{and} \quad t(\xi(z_\alpha)) = e^{i2\pi/3} t(\xi(z_\beta)).$$

From these it follows, after some simple manipulation, that

$$\Phi(z_\alpha) = -\frac{1+A\Phi(z_\beta)}{\Phi(z_\beta)-B} \quad \text{and} \quad \Phi(z_\beta) = \frac{B\Phi(z_\alpha)-1}{\Phi(z_\alpha)+A}, \tag{4.7}$$

where A and B are defined in (4.4). In view of (3.9), we deduce from (4.7) that

$$\begin{aligned} \frac{\Phi(z)[\Phi(z)+A]}{B\Phi(z)-1} &= \frac{1+A\overline{\Phi}(z_\beta)}{\overline{\Phi}(z_\beta)[B-\overline{\Phi}(z_\beta)]} = \frac{\overline{\Phi}(z_\alpha)}{\overline{\Phi}(z_\beta)} \\ &= \frac{\overline{\Phi}(z_\alpha)[\overline{\Phi}(z_\alpha)+A]}{B\overline{\Phi}(z_\alpha)-1}. \end{aligned}$$

The first statement of the lemma along with (4.6) follows from the above relations and the definition of $\widehat{\Gamma}_\alpha$.

The last statement of the lemma is an easy consequence of the way that the three transformations (3.5)–(3.7) treat angles. More precisely, since the two arcs Λ and Λ_α meet at $-\overline{\lambda}_\alpha$ forming an exterior angle π, their preimages Γ and Γ_α should form an exterior angle $3\pi/2$ at i. By symmetry, Γ and Γ_α form at $-i$ the same exterior angle.

The very last statement of the lemma follows at once from the above, because Γ and Γ_α are symmetric with respect to L_α. ∎

Remark 4.2 (i) It is worth noting that Λ_α is the reflection of Λ with respect to ℓ_α; see Figure 3.4.

(ii) For later use we remark that, in a similar way, we could consider the curve $\widehat{\Gamma}_\beta := \Gamma \cup \Gamma_\beta$, where Γ_β is the reflection of Γ with respect to L_β. Proceeding as above, we conclude that the function Φ extended by (3.10) maps conformally $\operatorname{ext}\widehat{\Gamma}_\beta$ onto the exterior of the curve

$$\widehat{\Lambda}_\beta := \{w : \left|\frac{w(w-B)}{1+Aw}\right| = 1\}.$$

Note that $\widehat{\Lambda}_\beta$ is the reflection of $\widehat{\Lambda}_\alpha$ with respect to the unit circle.

Our next task is to relate the capacity of $\widehat{\Gamma}_\alpha$ to the capacity of L. By the preceding lemma, Φ maps conformally $\operatorname{ext}\widehat{\Gamma}_\alpha$ onto $\operatorname{ext}\widehat{\Lambda}_\alpha$. Moreover (see (1.1)–(1.2))

$$\Phi(z) = (\operatorname{cap} L)^{-1} z + \mathcal{O}(1), \quad z \to \infty.$$

Let F denote the conformal map

$$F : \operatorname{ext}\widehat{\Lambda}_\alpha \to \Delta = \{w : |w| > 1\} \tag{4.8}$$

normalized at ∞ by

$$F(w) = (\operatorname{cap}\widehat{\Lambda}_\alpha)^{-1} w + \mathcal{O}(1), \quad w \to \infty. \tag{4.9}$$

Then the composition

$$\widehat{\Phi}_\alpha := F \circ \Phi \tag{4.10}$$

maps conformally $\operatorname{ext}\widehat{\Gamma}_\alpha$ onto Δ and we obtain

$$\operatorname{cap}\widehat{\Gamma}_\alpha = \operatorname{cap} L \operatorname{cap}\widehat{\Lambda}_\alpha. \tag{4.11}$$

Therefore, it is sufficient for our purposes here to determine $\operatorname{cap}\widehat{\Lambda}_\alpha$. For $\alpha \neq 0$, where α is the defining parameter of G, this is done in the following lemma.

When $\alpha = 0$, it follows at once from (4.3) and (4.4) that the corresponding curve $\widehat{\Lambda}_0$ is the lemniscate $\{w : |w(w+1)| = 1\}$ and thus

$$\operatorname{cap} \widehat{\Lambda}_0 = 1; \tag{4.12}$$

see e.g. [11, pp. 164–165].

Lemma 4.3 *Assume that* $\alpha \in (0, \pi/4]$ *and let* A*,* B *be defined by* (4.4)*. Then*

$$\operatorname{cap} \widehat{\Lambda}_\alpha = B\, F\left(\frac{1}{B}\right). \tag{4.13}$$

Furthermore,

$$\frac{\Phi(a)}{\operatorname{cap} L} = \frac{\widehat{\Phi}_\alpha(a)}{\operatorname{cap} \widehat{\Gamma}_\alpha}, \tag{4.14}$$

where, as in (3.1)*,* $a = \cot\alpha$*.*

Proof. A short calculation involving the transformations (3.6)–(3.7) gives $\Phi(a) = 1/B$. Thus, $\widehat{\Phi}_\alpha(a) = F(1/B)$ and the result (4.14) follows from (4.11) once (4.13) is established. The following short proof of (4.13) is due to V. Maymeskul.

It is easy to see that the function

$$\zeta(w) := \frac{w+A}{Bw-1}$$

maps $\operatorname{ext} \widehat{\Lambda}_\alpha$ onto itself, $\zeta(1/B) = \infty$ and $\zeta(\infty) = 1/B$. Hence, the function $F(w)F(\zeta(w))$ has simple poles at $w = 1/B$ and $w = \infty$, and it satisfies $|F(w)F(\zeta(w))| = 1$, for $w \in \widehat{\Lambda}_\alpha$. Consequently, $F(w)F(\zeta(w))/w\zeta(w)$ is analytic and non-vanishing in $\operatorname{ext} \widehat{\Lambda}_\alpha$, and is of modulus 1 on $\widehat{\Lambda}_\alpha$. By the maximum modulus principle,

$$w\zeta(w) = F(w)F(\zeta(w)), \quad w \in \operatorname{ext} \widehat{\Lambda}_\alpha.$$

Dividing this by w and taking the limit, as $w \to \infty$, we obtain (4.13) in view of (4.9). ∎

The essential idea of the proof of Theorem 3.3 is to consider the product $P_n(\zeta)P_n(\zeta_\alpha)$, where ζ_α is defined as in (3.2). The next lemma provides a bound for this rational function of ζ.

Lemma 4.4 *There exist positive constants c_1 and c_2 independent of n and ζ, such that*

$$|P_n(\zeta)P_n(\zeta_\alpha)| \le c_1\, n^{c_2}, \quad \zeta \in \Gamma, \quad n \ge 1. \tag{4.15}$$

Proof. For quantities $\mathcal{A} > 0$, $\mathcal{B} > 0$, we use the notation $\mathcal{A} \preceq \mathcal{B}$ (inequality with respect to the order) if $\mathcal{A} \le c\mathcal{B}$, where c depends only on G.

Since $\zeta_\alpha \notin G$, the application of (2.19) gives

$$|P_n(\zeta_\alpha)| \le c_1\, n^{c_2} |\Phi(\zeta_\alpha)|^n.$$

Thus, it is sufficient to show that for some $c > 0$ and $c_3 > 0$,

$$|P_n(\zeta)| \le c\, n^{c_3}\, |\Phi(\zeta_\alpha)|^{-n}, \quad \zeta \in \Gamma, \quad n \ge 1. \tag{4.16}$$

Assume first that $|\zeta^2 + 1| < n^{-3/2}$. Then (3.2) gives $|\zeta_\alpha^2 + 1| \preceq n^{-3/2}$ and this, in conjunction with the expansion of Φ at $\pm i$, implies

$$|\Phi(\zeta_\alpha)| - 1 \preceq n^{-1},$$

which yields (4.16) in view of Lemma 2.2(i).

Assume now that

$$|\zeta^2 + 1| \ge n^{-3/2}.$$

The same reasoning as before yields

$$|\Phi(\zeta_\alpha)| - 1 \succeq n^{-1}. \tag{4.17}$$

Furthermore, since Γ forms a nonzero angle ($= \pi/4$) with L_α, L_β (see the last statement of Lemma 4.1), we have

$$d_\zeta := \operatorname{dist}(\zeta, \partial G) \succeq n^{-3/2}. \tag{4.18}$$

Using well-known arguments (the details are provided below for the reader's convenience) one can show that, for any polynomial Q_{n-1} of degree $n-1$, there holds

$$|P_n(\zeta)| \leq \frac{c}{d_\zeta}\|\varphi'_\zeta - Q_{n-1}\|_{\overline{G}}, \tag{4.19}$$

where $c > 0$ depends only on G. For the moment, let us assume that (4.19) is valid. We know from (4.1) that φ'_ζ is a rational function with poles at ζ_α, ζ_β. Since $\zeta \in \Gamma$, we have

$$|\Phi(\zeta_\alpha)| = |\Phi(\zeta_\beta)| =: R. \tag{4.20}$$

Then if we set, generically,

$$L_t := \{z \in \mathbb{C} \setminus G : |\Phi(z)| = t\}, \quad t \geq 1,$$

we see that φ'_ζ is analytic inside L_R.

Let Q_{n-1} be the $(n-1)$th section of the Faber expansion of φ'_ζ. Since G is convex, the Faber polynomials associated with G are uniformly bounded on $\overline{G}$ by 2 and we can deduce (cf. [12, Theorem 2(4), p. 142]) that

$$\|\varphi'_\zeta - Q_{n-1}\|_{\overline{G}} \leq 2M_r \frac{r}{r-1} r^{-n}, \quad 1 < r < R, \tag{4.21}$$

where

$$M_r := \max_{z \in L_r} |\varphi'_\zeta(z)|.$$

We now choose

$$r = R\left(1 - \frac{\varepsilon}{n}\right),$$

where $\varepsilon > 0$ is independent of n and ζ and is small enough to ensure

$$r - 1 \succeq n^{-1}. \tag{4.22}$$

(A suitable choice is any ε such that $\varepsilon < c/(1+c)$, where c is a constant for which (4.17) holds.)

By using a result of Löwner on the distance between exterior level curves (see e.g. [5, p. 120]) we obtain

$$\operatorname{dist}(L_r, L_R) > \operatorname{cap} L \frac{(R-r)^2}{R} \succeq n^{-2}.$$

and therefore, from (4.2),

$$M_r \preceq n^4.$$

Then (4.21) gives

$$\|\varphi'_\zeta - Q_{n-1}\|_{\overline{G}} \preceq n^5 \, R^{-n}$$

and the result (4.16) follows by substituting the preceding relation into (4.19) and using (4.18) and (4.20).

It remains to prove (4.19). In order to do this, we recall (cf. [4, p. 33]) that

$$\varphi'_\zeta(z) = \sqrt{\frac{\pi}{K(\zeta,\zeta)}}\, K(z,\zeta), \tag{4.23}$$

where $K(z,\zeta)$ is the Bergman kernel function associated with G. By the Cauchy formula for the derivative φ'_ζ we then obtain

$$K(\zeta,\zeta) \leq 1/(\pi d_\zeta^2). \tag{4.24}$$

Now, by making use of the reproducing property of $K(z,\zeta)$, the relation (4.23), the orthogonality property of P_n and, finally, the Cauchy-Schwarz inequality, we have (see also [6, p. 302])

$$\begin{aligned}
|P_n(\zeta)| &= \left| \int_G K(z,\zeta)\overline{P_n(z)}dm(z) \right| \\
&= \sqrt{\frac{K(\zeta,\zeta)}{\pi}} \left| \int_G \big(\varphi'_\zeta(z) - Q_{n-1}(z)\big)\, \overline{P_n(z)}dm(z) \right| \\
&\leq \sqrt{\frac{K(\zeta,\zeta)}{\pi}} \, \|\varphi'_\zeta - Q_{n-1}\|_{L^2(G)}.
\end{aligned}$$

The required result (actually in a stronger form) follows from the last inequality and (4.24). ■

5 Proofs of the results of Section 3

For any finite positive Borel measure σ of compact support in $\mathbb{C}$, we define its *logarithmic potential* by

$$U^{\sigma}(z) := \int \log \frac{1}{|z-t|} d\sigma(t).$$

In particular, if Q is a monic polynomial polynomial of degree n, then

$$U^{\nu_Q}(z) = -\log |Q(z)|^{1/n}.$$

Let $\mathcal{L}$ be a simple closed Jordan curve in $\mathbb{C}$. We have already defined (in Sections 2 and 3) the equilibrium measure $\mu_{\mathcal{L}}$ for $\mathcal{L}$ and the balayage measure $\widehat{\delta}_{z_0}$ ($z_0 \in \operatorname{ext}\mathcal{L}$) of δ_{z_0} onto $\mathcal{L}$. It is well-known that their potentials satisfy

$$U^{\mu_{\mathcal{L}}}(z) = \log \frac{1}{\operatorname{cap}\mathcal{L}}, \quad z \in \overline{\operatorname{int}\mathcal{L}} \tag{5.1}$$

and

$$U^{\widehat{\delta}_{z_0}}(z) = \log \frac{1}{|z-z_0|} + g_{\mathcal{L}}(z_0,\infty), \quad z \in \mathcal{L}, \tag{5.2}$$

where $g_{\mathcal{L}}(\cdot,\infty)$ is the Green function of $\operatorname{ext}\mathcal{L}$ with pole at ∞ (cf. [11, Sections I.1, I.4, II.4]).

We now turn to the

Proof of Theorem 3.3. We recall from Section 2 that

$$z_\alpha = \frac{a\overline{z}+1}{\overline{z}-a}, \quad a = \cot\alpha,$$

with $\alpha \in [0,\pi/4]$ and consider separately the two cases $\alpha \in (0,\pi/4]$ and $\alpha = 0$.

Case (i): $\alpha \in (0,\pi/4]$, that is, $a < \infty$.

Since the lens-shaped domain G is symmetric with respect to the real axis, the Bergman polynomials of G

$$P_n(z) = \gamma_n z^n + \cdots, \quad \gamma_n > 0, \quad n = 0,1,2,\ldots,$$

have real coefficients. Therefore,

$$\overline{P_n(z_\alpha)} = P_n(\overline{z_\alpha}) = (z-a)^{-n}\{P_n(a)z^n + \cdots\}.$$

We note that $P_n(a) > 0$, since otherwise P_n would change its sign on (a, ∞), contradicting the fact that all the zeros of P_n lie in G.

We consider the monic polynomial

$$p_{2n}(z) := \frac{(z-a)^n P_n(z)\overline{P_n(z_\alpha)}}{\gamma_n P_n(a)} = z^{2n} + \cdots$$

and introduce the weight function

$$w(z) := |z-a|^{-1/2}. \tag{5.3}$$

Then we have

$$|w^{2n}(z)p_{2n}(z)| = \frac{1}{\gamma_n P_n(a)}|P_n(z)\overline{P_n(z_\alpha)}|. \tag{5.4}$$

Observe that the weighted polynomial $w^{2n}p_{2n}$ is invariant under the transformation $z \to z_\alpha$, i.e.,

$$|w^{2n}(z_\alpha)p_{2n}(z_\alpha)| = |w^{2n}(z)p_{2n}(z)|$$

and that the zeros of p_{2n} are those of P_n together with their reflections with respect to L_α. Since the zeros of P_n lie in G, we obtain from (2.1),

$$\nu_{p_{2n}}\big|_{\overline{G}} = \frac{1}{2}\,\nu_{P_n}.$$

It suffices, therefore, to establish the convergence

$$\nu_{p_{2n}} \xrightarrow{*} \mu, \quad n \to \infty, \quad \text{where} \quad \mu := \frac{1}{2}\left(\mu_{\widehat{\Gamma}_\alpha} + \widehat{\delta}_a\right). \tag{5.5}$$

Indeed, since μ is supported on $\widehat{\Gamma}_\alpha = \Gamma \cup \Gamma_\alpha$, (5.5) yields:

$$\nu_{P_n} = 2\,\nu_{p_{2n}}\big|_{\overline{G}} \xrightarrow{*} 2\mu\big|_{\overline{G}} = \left(\mu_{\widehat{\Gamma}_\alpha} + \widehat{\delta}_a\right)\big|_{\Gamma}.$$

In order to establish (5.5) we proceed as follows:

- First, we show that there is a constant F_w such that

$$U^\mu(z) + \log \frac{1}{w(z)} = F_w, \quad z \in \widehat{\Gamma}_\alpha. \tag{5.6}$$

  This, in view of [11, Theorem I.3.3], will imply that μ is the equilibrium measure associated with the weight w on $\widehat{\Gamma}_\alpha$ and F_w is the modified Robin constant for w; see [11, p. 27].

- Next, we show that

$$\lim_{n\to\infty} \left(\|w^{2n} p_{2n}\|_{\widehat{\Gamma}_\alpha}\right)^{1/2n} = \exp(-F_w) \tag{5.7}$$

  and, for some $z_0 \in \operatorname{int} \widehat{\Gamma}_\alpha$,

$$\lim_{n\to\infty} |p_{2n}(z_0)|^{1/2n} = \exp(-U^\mu(z_0)). \tag{5.8}$$

  The required result (5.5) will then follow by virtue of [11, Theorem III.4.1].

To establish (5.6) we first recall from (4.10) that $\widehat{\Phi}_\alpha$ is the exterior mapping for $\operatorname{ext} \widehat{\Gamma}_\alpha$. Hence, the Green function of $\operatorname{ext} \widehat{\Gamma}_\alpha$ with pole at ∞ is given by $\log |\widehat{\Phi}_\alpha(z)|$. Since $\widehat{\Phi}_\alpha(a) > 0$, we have from (5.2) and (5.3) that, for $z \in \widehat{\Gamma}_\alpha$,

$$\frac{1}{2} U^{\widehat{\delta_\alpha}}(z) = \frac{1}{2} \log \frac{1}{|z-a|} + \frac{1}{2} \log \widehat{\Phi}_\alpha(a) = -\log \frac{1}{w(z)} + \frac{1}{2} \log \widehat{\Phi}_\alpha(a).$$

This, in conjunction with the definition of μ and (5.1), yields

$$U^\mu(z) + \log \frac{1}{w(z)} = \frac{1}{2} \log \frac{1}{\operatorname{cap} \widehat{\Gamma}_\alpha} + \frac{1}{2} \log \widehat{\Phi}_\alpha(a), \quad z \in \widehat{\Gamma}_\alpha,$$

which is (5.6), with

$$F_w = \frac{1}{2} \log \frac{\widehat{\Phi}_\alpha(a)}{\operatorname{cap} \widehat{\Gamma}_\alpha}. \tag{5.9}$$

In view of Lemma 4.3, the constant F_w can be also expressed as

$$F_w = \frac{1}{2} \log \frac{\Phi(a)}{\operatorname{cap} L}. \tag{5.10}$$

Next, by using the expression (5.4) for the weighted polynomial $w^{2n}p_{2n}$ and utilizing the nth-root asymptotic behavior (1.5) of γ_n, the result of Lemma 4.4 for $|P_n(z)P_n(z_\alpha)|$ (note that $(z_\alpha)_\alpha = z$), and the lower inequality in (2.19) for $P_n(a)$, we obtain from (5.10),

$$\limsup_{n\to\infty} \left(\|w^{2n}p_{2n}\|_{\widehat{\Gamma}_\alpha}\right)^{1/2n} \leq \exp(-F_w). \tag{5.11}$$

Since both functions U^μ and $\log(1/w)$ are harmonic in $\operatorname{int}\widehat{\Gamma}_\alpha$, the relation (5.6) remains valid for any $z \in \operatorname{int}\widehat{\Gamma}_\alpha$. In particular, it is valid for any $z_0 \in L_\alpha \setminus \{i, -i\}$, i.e.

$$U^\mu(z_0) + \log \frac{1}{w(z_0)} = F_w. \tag{5.12}$$

For such z_0 we also have

$$|w^{2n}(z_0)p_{2n}(z_0)| = \frac{|P_n(z_0)|^2}{\gamma_n P_n(a)}.$$

By utilizing, as above, the nth-root asymptotic behavior of γ_n, the result of Lemma 2.2(ii) for $|P_n(z_0)|$ and the upper inequality in (2.19) for $P_n(a)$, we obtain

$$\liminf_{n\to\infty} \left(|w^{2n}(z_0)p_{2n}(z_0)|\right)^{1/2n} \geq \exp(-F_w), \tag{5.13}$$

which, in view of (5.12), gives

$$\liminf_{n\to\infty} |p_{2n}(z_0)|^{1/2n} \geq \exp(-U^\mu(z_0)). \tag{5.14}$$

It is easy to see now that (5.11)–(5.14) yield both (5.7) and (5.8).

Case (ii): $\alpha = 0$, that is $a = \infty$.

In this case $z_\alpha = -\overline{z}$. Therefore, we accordingly consider

$$p_{2n}(z) := \frac{P_n(z)P_n(-z)}{(-1)^n\gamma_n^2} = z^{2n} + \cdots$$

and take $w(z) \equiv 1$. Since $\widehat{\delta}_\infty = \mu_{\widehat{\Gamma}_\alpha}$, we let $\mu := \mu_{\widehat{\Gamma}_\alpha}$ and proceeding as above we establish (5.11)–(5.14), with

$$F_w = \log \frac{1}{\operatorname{cap}\widehat{\Gamma}_\alpha}.$$

This implies, as before, that

$$\nu_{P_n} \xrightarrow{*} 2\mu_{\widehat{\Gamma}_\alpha}\big|_\Gamma ,$$

which is (3.20) with $a = \infty$. ■

Proof of Theorem 3.1. By (1.6), the relation (3.16) holds for $z \in \mathbb{C} \setminus \overline{G}$ and, in view of Lemma 2.2, it also holds for $z \in L$. Thus it remains to consider $z \in G$. Assume, for example, that $z \in G \cap \overline{G_\alpha}$ (recall (3.12) for the definition of G_α). Since the interior map φ_z has poles at z_α, z_β, we obtain from (3.13) that

$$\rho := |\Phi(z_\alpha)|$$

is the largest index for which φ_z is analytic in the interior of the level curve $L_\rho := \{\zeta \in \mathbb{C} \setminus G : |\Phi(\zeta)| = \rho\}$. According to [9, Theorem 2.1] this implies (3.15) for $z \in G \cap \overline{G_\alpha}$. The case $z \in G \cap \overline{G_\beta}$ can be handled in a similar way.

Now we turn our attention to establishing (3.16) for $z \in G \setminus \Gamma$ $(= G_\alpha \cup G_\beta)$, under the additional assumption that z is not a limit point of zeros of the P_n's. We consider separately, again, two cases as follows.

Case (i): $z \in G_\alpha$.

By our assumption, the measure $\nu_{p_{2n}}$ has no mass near z, provided that n is sufficiently large. Hence the weak* convergence result (5.5) gives

$$U^{\nu_{p_{2n}}}(z) \to U^\mu(z), \quad n \to \infty,$$

i.e.,

$$\frac{1}{2n} \log |p_{2n}(z)| = -U^{\nu_{p_{2n}}}(z) \to \log \frac{1}{w(z)} - F_w, \quad n \to \infty.$$

Next, we observe that (1.5), (1.6), and (5.10) imply that

$$\frac{1}{2n} \log(\gamma_n P_n(a)) \to F_w, \quad n \to \infty.$$

The limit (1.6) also gives

$$-\frac{1}{2n} \log |P_n(z_\alpha)| \to -\frac{1}{2} \log |\Phi(z_\alpha)|, \quad n \to \infty.$$

By adding up the last three convergence results and using (5.4) we arrive at

$$\frac{1}{2n}\log|P_n(z)| \to -\frac{1}{2}\log|\Phi(z_\alpha)|, \quad n\to\infty,$$

which is the required result.

Case (ii): $z \in G_\beta$.

From Remark 4.2 the result in this case follows by interchanging the roles of α and β in the preceding arguments. ■

Proof of Corollary 3.2. When $\alpha = \pi/4$, the lens-shaped domain G is symmetric with respect to the imaginary axis, in addition to the real one. It is therefore clear from the definition (3.11) that Γ is, in this case, the segment $[-i, i]$. In view of Theorem 3.1, it remains to show that all zeros of P_n lie on this segment.

By orthogonality, we have the relations

$$\int_G P_n(z)\overline{z}^k dm(z) = 0, \quad k = 0, 1, \ldots, n-1,$$

which can be rewritten, with the help of the Green formula, as

$$\frac{1}{2i}\int_{L_\beta} P_n(z)\overline{z}^{k+1}dz - \frac{1}{2i}\int_{L_\alpha} P_n(z)\overline{z}^{k+1}dz = 0, \quad k = 0, 1, \ldots, n-1 \quad (5.15)$$

(we have chosen the direction from $-i$ to i in both integrals). Since $z_\alpha = z$ for $z \in L_\alpha$, we have from (3.2),

$$\overline{z} = \frac{z+1}{z-1}, \quad z \in L_\alpha$$

and similarly,

$$\overline{z} = \frac{1-z}{z+1}, \quad z \in L_\beta.$$

By inserting these into (5.15) and by replacing (using the Cauchy theorem) the integrals along L_α and L_β by integrals along $[-i, i]$, we arrive at the relations

$$\int_{-1}^{1} P_n(iy)\left[\left(\frac{1-iy}{1+iy}\right)^{k+1} - \left(\frac{1+iy}{iy-1}\right)^{k+1}\right] dy = 0, \quad k = 0, 1, \ldots, n-1. \quad (5.16)$$

Now it is easy to see inductively, that

$$\begin{aligned}&\operatorname{span}\Big\{\frac{1}{w^{k+1}}-(-w)^{k+1}:\ k=0,1,\ldots,n-1\Big\}\\ &=\operatorname{span}\Big\{\left(\frac{1}{w}-w\right)^k\left(\frac{1}{w}+w\right):\ k=0,1,\ldots,n-1\Big\}.\end{aligned}$$

Using this observation with

$$w=\frac{1+iy}{1-iy},$$

we conclude that (5.16) is equivalent to requiring that the relation

$$\int_{-1}^{1} P_n(iy)\,h(y)\frac{1-y^2}{1+y^2}dy=0 \tag{5.17}$$

holds true for any function h of the form

$$h(y)=\sum_{k=0}^{n-1}c_k\left(\frac{2y}{1+y^2}\right)^k,\qquad c_k\in\mathbb{R}. \tag{5.18}$$

We recall that G is symmetric with respect to both axes. This implies that the Bergman polynomial P_n has real coefficients and is either even or odd. Hence,

$$P_n(iy)=Q_n(y)\quad\text{or}\quad P_n(iy)=iQ_n(y),$$

where Q_n is a real polynomial of degree n. If we assume that Q_n has less than n zeros in the interval $(-1,1)$, then it can have (at most) m $(\le n-1)$ sign changes there, say at some distinct points y_1, $y_2,\ldots$, y_m of $(-1,1)$. The function

$$h(y):=\prod_{j=1}^{m}\left(\frac{2y}{1+y^2}-\frac{2y_j}{1+y_j^2}\right),$$

which is of the form (5.18), has simple zeros at y_1, $y_2,\ldots$, y_m, and has no other zero in $(-1,1)$. Therefore, our assumption implies that the function $Q_n(y)h(y)$ is of constant sign in $(-1,1)$, which contradicts (5.17). That is Q_n, and thus P_n, has exactly n simple zeros in $(-1,1)$. ∎

6 Discrepancy of zeros of Bergman polynomials

The asymptotic formula (1.5) for γ_n can be replaced by the more precise inequalities (2.20). Then proceeding as in the proof of Theorem 3.3, we obtain the following refinements of (5.11) and (5.13):

$$0 < \frac{1}{2n} \log \|w^{2n} p_{2n}\|_{\widehat{\Gamma}_\alpha} + F_w \leq c_1 \frac{\log n}{n} \tag{6.1}$$

and

$$\frac{1}{2n} \log |w^{2n}(z_0) p_{2n}(z_0)| + F_w \geq -c_2 \frac{\log n}{n}, \quad z_0 \in L_\alpha \setminus \{-i, i\}. \tag{6.2}$$

(The left inequality in (6.1) is a consequence of (5.6) and the maximum principle.)

Let $g_{\widehat{\Gamma}_\alpha}(\cdot, \infty)$ and $g_{\widehat{\Gamma}_\alpha}(\cdot, z_0)$ denote, respectively, the Green function of $\operatorname{ext} \widehat{\Gamma}_\alpha$ with pole at ∞ and the Green function of $\operatorname{int} \widehat{\Gamma}_\alpha$ with pole at z_0.

Consider the level curves:

$$L_n^- := \left\{ z \in \operatorname{int} \widehat{\Gamma}_\alpha : g_{\widehat{\Gamma}_\alpha}(z, z_0) = \left(\frac{\log n}{n} \right)^{3/4} \right\}, \tag{6.3}$$

$$L_n^+ := \left\{ z \in \operatorname{ext} \widehat{\Gamma}_\alpha : g_{\widehat{\Gamma}_\alpha}(z, \infty) = \left(\frac{\log n}{n} \right)^{1/4} \right\}, \tag{6.4}$$

and let

$$E_n := \operatorname{ext} L_n^- \cap \operatorname{int} L_n^+ \qquad \left(\supset \widehat{\Gamma}_\alpha \right).$$

Then, on applying Lemmas III.4.4 and III.4.5 in [11], we deduce from (6.1) and (6.2) that

$$\nu_{p_{2n}}(\mathbb{C} \setminus E_n) \leq c \left(\frac{\log n}{n} \right)^{1/4}, \quad n = 2, 3, \ldots.$$

The same is true, if we replace $\nu_{p_{2n}}$ by ν_{P_n} and E_n by $E_n \cap G$.

Let

$$\mu := \left(\mu_{\widehat{\Gamma}_\alpha} + \widehat{\delta}_a\right)\Big|_{\Gamma}.$$

Since the measure μ is supported on Γ and μ, ν_{P_n} are unit measures, we obtain that

$$|(\mu - \nu_{P_n})(E_n \cap G)| \leq c\left(\frac{\log n}{n}\right)^{1/4}, \quad n = 2, 3, \ldots. \tag{6.5}$$

Now consider an arc $J \subset \Gamma$ and define two quadrilaterals based on J, as follows. The first one is bounded by $\widehat{\Gamma}_\alpha$ and L_n^-, and by the level curves of the conjugate of the Green function $g_{\widehat{\Gamma}_\alpha}(\cdot, z_0)$ that pass through the endpoints of J; see Figure 6.1.

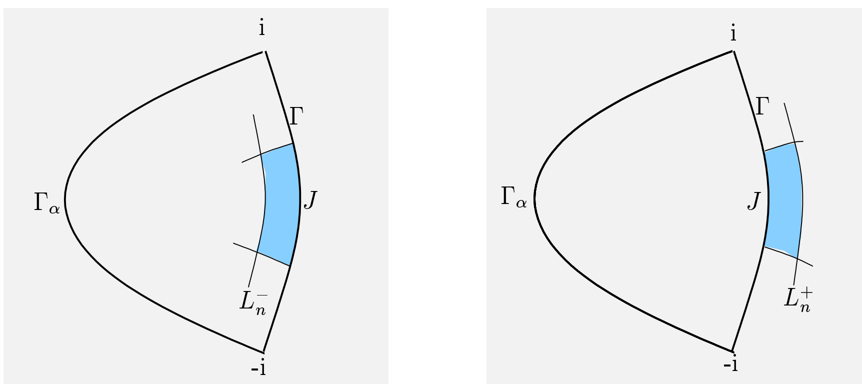

Figure 6.1

The second quadrilateral is bounded by $\widehat{\Gamma}_\alpha$ and L_n^+, and by the level curves for the conjugate of the potential U^μ that pass through the endpoints of J; see Figure 6.1.

Let $A_n(J)$ be the union of these two quadrilaterals. The following "discrepancy result" can be proved:

Result 6.1 *Let J be a subarc of Γ. Then, for any $n = 2, 3, \ldots$, there exists a constant $c > 0$ independent of J and n such that*

$$|(\mu - \nu_{P_n})(A_n(J))| \leq c\left(\frac{\log n}{n}\right)^{1/4}. \tag{6.6}$$

This result can be proved by adopting the method in [1], but the details are tedious, so we omit them. In this connection we mention a recent result of Andrievskii, Pritsker and Varga. For an *arbitrary* convex domain G, they proved in [2] that if one replaces ν_{P_n} by its balayage $\widehat{\nu}_{P_n}$ onto $L = \partial G$, then the relation

$$|(\mu_L - \widehat{\nu}_{P_n})(J)| \leq c \left(\frac{\log n}{n}\right)^{1/2}, \quad n = 2, 3, \ldots,$$

holds for any $J \subset L$, with c independent of J and n; recall that μ_L denotes the equilibrium measure of L.

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