Note on d-extremal configurations for the sphere in $\mathbb{R}^{d+1}$

M. Götz and E.B. Saff†

Abstract. It is shown that d-extremal configurations of N points on the unit sphere in $\mathbb{R}^{d+1}$, i.e., points minimizing energy with respect to the Riesz kernel $|x-y|^{-d}$, are asymptotically equidistributed as $N \to \infty$.

Consider the unit sphere $S^d := \{x \in \mathbb{R}^{d+1} : |x| = 1\}$ in $\mathbb{R}^{d+1}$, $d \geq 2$, and denote by σ the surface measures on S^d, normalized to have total mass 1. For given $s > 0$, the discrete s-energy of a system of N points $\omega_N = \{x_1, \ldots, x_N\} \subset S^d$ is given by

$$E_d(s, \omega_N) := \sum_{1 \leq i < j \leq N} \frac{1}{|x_i - x_j|^s} .$$

The paper [2] contains asymptotics of the *minimal* s*-energy*

$$\mathcal{E}_d(s, N) := \inf_{\omega_N} E_d(s, \omega_N) ,$$

where the infimum is taken over all N-point sets $\omega_N \subset S^d$ (see also [3], [4], [5]). Any set $\omega_N^* = \{x_1^{(N)}, \ldots, x_N^{(N)}\} \subset S^d$, for which this infimum is attained is called an s*-extremal configuration.* It is well-known that under the assumption $s < d$, each sequence of such s-extremal configurations is asymptotically equidistributed in the sense that the normalized discrete measures $\mu_{\omega_N^*}$ associating mass $1/N$ with each point $x_i^{(N)}$ converge to σ in the weak-star topology:

$$\lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{n} f(x_i^{(N)}) = \int f \, d\sigma \qquad (f \in C(S^d)) . \tag{1}$$

The key tool for a proof of this relation is a comparison of the normalized energy $\frac{2}{N(N-1)} \mathcal{E}_d(s, N)$ with the s-energy

$$\iint \frac{1}{|x-y|^s} \, d\sigma(x) \, d\sigma(y) \tag{2}$$

†The research of this author was supported, in part, by the U.S. National Science Foundation research grant DMS 9801677.

of the measure σ. However, for $s \geq d$ the integral in (2) becomes infinite, and the conventional methods fail. In this note we will show that the weak-star convergence (1) does also hold in the case $s = d$:

Theorem. *Each sequence* $(\omega_N^*)_{N\geq 2}$ *of* d*-extremal configurations on* S^d *is equidistributed in the sense of* (1).

The proof of this result is based on the following asymptotic behavior of the d-energy.

Theorem A ([2, Th.3]). *The minimal* d*-energy satisfies*

$$\lim_{N\to\infty} (N^2 \log N)^{-1} \mathcal{E}_d(d, N) = \frac{1}{2d}\gamma_d ,$$

where

$$\gamma_d := \frac{\Gamma((d+1)/2)}{\Gamma(d/2)\,\Gamma(1/2)} .$$

Proof of the Theorem. For $t > 0$ and $z \in S^d$, denote by $C(z,t)$ the intersection of the closed ball of radius t centered at z with the sphere S^d. Such sets are called spherical caps. They can also be viewed as the intersection of the sphere with some closed half-space in $\mathbb{R}^{d+1}$. We remark that (see [2, (3.7)])

$$\int_{S^d\setminus C(z,t)} |z-y|^{-d}\, d\sigma(y) = -\gamma_d \log t + \mathcal{O}(1) \quad \text{as } t \to 0 . \tag{3}$$

Let $\omega_N^* = \{x_1^{(N)}, \ldots, x_N^{(N)}\} \subset S^d$ be a set of d–extremal points. Define

$$U_i(x) := \sum_{\substack{j=1\\ j\neq i}}^{N} |x - x_j^{(N)}|^{-d} \qquad (x \in S^d) .$$

Then

$$\mathcal{E}_d(d, N) = E_d(d, \omega_N^*) = \frac{1}{2}\sum_{i=1}^{N} U_i(x_i^{(N)}) , \tag{4}$$

and since ω_N^* is a d-extremal configuration,

$$U_i(x_i^{(N)}) \leq U_i(x) \qquad (x \in S^d) . \tag{5}$$

For the moment, fix $r > 0$ (sufficiently small), and set

$$D_i(r) := S^d \setminus C(x_i^{(N)}, rN^{-1/d}) \,, \qquad D(r) := \bigcap_{i=1}^{N} D_i(r) \,.$$

Assume, contrary to the assertion of the Theorem, that the measures $\mu_{\omega_N^*}$ do not converge to σ in the weak-star topology. Then, by Helly's selection theorem, there exists some unit measure $\mu \neq \sigma$ on S^d, which is the weak–star limit of the measures $\mu_{\omega_N^*}$ along some subsequence $\Lambda \subset \mathbb{N}$. By the Cramér–Wold theorem (cf. [1]), which states that a probability measure on Euclidean space is uniquely determined by the values it gives to halfspaces, there is a closed spherical cap C such that $\mu(C) < \sigma(C)$. Thus, one can find an $\varepsilon > 0$ such that (after possibly passing to another subsequence) the cardinality of $\omega_N^* \cap C$ satisfies

$$\#(\omega_N^* \cap C) \leq N\,(\sigma(C) - \varepsilon) \qquad (N \in \Lambda) \,. \tag{6}$$

Now, choose a second spherical cap $C' \subset C$ such that

$$\frac{\sigma(C) - \varepsilon}{\sigma(C')} < 1 \,, \qquad \rho := \operatorname{dist}(C', S^d \setminus C) > 0 \,.$$

Taking into account (3) and (6), an integration over $C' \cap D(r)$ yields

$$\begin{aligned}
\int_{C' \cap D(r)} U_i(x)\, d\sigma(x) &\leq \sum_{\substack{j=1 \\ j \neq i}}^{N} \int_{C' \cap D_j(r)} |x - x_j^{(N)}|^{-d} d\sigma(x) \\
&\leq \sum_{x_j^{(N)} \in C} \gamma_d[-\log(rN^{-1/d})] + \sum_{x_j^{(N)} \notin C} \frac{1}{\rho^d} + \mathcal{O}(N) \\
&\leq (\sigma(C) - \varepsilon) N \gamma_d[-\log(rN^{-1/d})] + \mathcal{O}(N) \,.
\end{aligned}$$

Consequently, by (4) and (5),

$$\mathcal{E}_d(d, N) \leq \frac{\sigma(C) - \varepsilon}{\sigma(C' \cap D(r))} \frac{\gamma_d}{2d} N^2 \log N + \log \frac{1}{r}\, \mathcal{O}(N^2) \,. \tag{7}$$

On the other hand,

$$\sigma(C' \cap D(r)) \geq \sigma(C') - \sum_{i=1}^{N} \sigma\left(C\left(x_i^{(N)}, rN^{-1/d}\right)\right) \geq \sigma(C') - \gamma_d\, r^d/d \,,$$

where we used the fact that $\sigma(C(x,t)) \leq \gamma_d t^d/d$ for all $x \in S^d$. Thus, we may choose $r = r(d)$ so small that

$$\frac{\sigma(C) - \varepsilon}{\sigma(C' \cap D(r))} < 1 .$$

Inserting this estimate into (7) yields a contradiction to the asymptotic behavior of the minimal d-energy according to Theorem A. ∎

References

[1] H. Cramér and H. Wold, *Some theorems on distribution functions.* J. London Math. Soc. **11** (1936), pp. 290–294.

[2] A.B.J. Kuijlaars and E.B. Saff, *Asymptotics for Minimal Discrete Energy on the Sphere.* Trans. Am. Math. Soc. **350(2)** (1998), pp. 523–538.

[3] E.A. Rakhmanov, E.B. Saff, and Y.M. Zhou, *Electrons on the Sphere.* In: Computational Methods and Function Theory (R.M. Ali, St. Ruscheweyh, and E.B. Saff, eds.) World Scientific (1995), pp. 111–127.

[4] E.A. Rakhmanov, E.B. Saff, and Y.M. Zhou, *Minimal Discrete Energy on the Sphere.* Math. Research Letters, **1** (1994), pp. 647–662.

[5] E.B. Saff and A.B.J. Kuijlaars, *Distributing Many Points on a Sphere.* Math. Intelligencer **19** (1997), pp. 5–11.

M. Götz
Kath. Univ. Eichstätt-Ingolstadt
Ostenstraße 26
85071 Eichstätt
Deutschland
email: `Mario.Goetz@ku-eichstaett.de`

E.B. Saff
Inst. for Constructive Mathematics
Department of Mathematics
University of South Florida
4202 East Fowler Avenue
Tampa, FL 33620, U.S.A.
email: `esaff@math.usf.edu`