Journal of Approximation Theory **118**, 131–151 (2002)
doi:10.1006/jath.2002.3705

Asymptotic Properties of Heine–Stieltjes and Van Vleck Polynomials

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Communicated by Arno B. J. Kuijlaars

Received February 14, 2002; accepted in revised form April 1, 2002

We study the asymptotic behavior of the zeros of polynomial solutions of a class of generalized Lamé differential equations, when their coefficients satisfy certain asymptotic conditions. The limit distribution is described by an equilibrium measure in the presence of an external field, generated by charges at the singular points of the equation. Moreover, a case of non-positive charges is considered, which leads to an equilibrium with a non-convex external field.

Key Words: Heine–Stieltjes polynomials; Van Vleck polynomials; generalized Lamé differential equation; zero asymptotics; electrostatics; logarithmic potential; equilibrium distribution.

1. HEINE–STIELTJES AND VAN VLECK POLYNOMIALS

Let $\mathbb{P}_n$ stand for the class of all algebraic polynomials of degree at most n, and $\mathbb{P} = \bigcup_{n\geqslant 0} \mathbb{P}_n$. The *generalized Lamé differential* equation (in algebraic form) is

$$A(x)E''(x) + B(x)E'(x) - C(x)E(x) = 0, \tag{1}$$

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0021-9045/02 $35.00

where A, B are polynomials of degree $p+1$, p, respectively, and $C \in \mathbb{P}_{p-1}$. The case $p = 1$ corresponds to the hypergeometric differential equation, and $p = 2$, to the Heun's equation (see [19]).

Heine [11] proved that for every $N \in \mathbb{N}$, there exists at most

$$\sigma(N) = \binom{N+p-1}{N}$$

different polynomials C in (1) such that this equation admits a polynomial solution $y \in \mathbb{P}_N$. These coefficients C are called Van Vleck polynomials, and the corresponding polynomial solutions E are known as Heine–Stieltjes polynomials.

In fact, Stieltjes studied the problem in the following particular setting. The zeros a_i of A are assumed to be simple and real, so that without loss of generality we may take

$$-1 = a_0 < a_1 < \cdots < a_p = 1 \tag{2}$$

and A monic. Moreover, it is assumed that

$$\frac{B(x)}{A(x)} = \sum_{i=0}^{p} \frac{\rho_i}{x - a_i}, \qquad \rho_i > 0, \quad i = 0, \ldots, p, \tag{3}$$

(this is equivalent to the assumption that the zeros of A alternate with those of B and that the leading coefficient of B is positive). The case $\rho_0 = \cdots = \rho_p = 1/2$ corresponds to the classical Lamé equation (in algebraic form).

Stieltjes proved in [28] (see also [29, Theorem 6.8]) that for each $N \in \mathbb{N}$ there are exactly $\sigma(N)$ different Van Vleck polynomials of degree $p-1$ and the same number of corresponding Heine–Stieltjes polynomials of degree N, given by all possible ways how the N zeros of E can be distributed in the p open intervals defined by the zeros a_i of A. This allows a vector parametrization in the class of Van Vleck and Heine–Stieltjes polynomials.

With every $P \in \mathbb{P}$ we associate its zero-counting measure, $\nu(P)$,

$$\nu(P) = \sum_{P(x)=0} \delta_x,$$

where the zeros are counted according to their multiplicity. Given a vector $\boldsymbol{n} = (n_1, \ldots, n_p)$, we denote by $E_{\boldsymbol{n}} \in \mathbb{P}_N$, $N = n_1 + \cdots + n_p$, the unique (up to a constant factor) Heine–Stieltjes polynomial, and by $C_{\boldsymbol{n}} \in \mathbb{P}_{p-1}$ the unique Van Vleck polynomial, such that

$$\int_{a_{i-1}}^{a_i} d\nu(E_{\boldsymbol{n}}) = n_i, \qquad i = 1, \ldots, p.$$

Stieltjes [28] gave also the following characterization of the zeros of $E_{\boldsymbol{n}}$: they are in the position of the electrostatic equilibrium in the field generated by the positive charges $\rho_i/2$ at a_i, if the interaction obeys the logarithmic law. In other words, the zeros

$$a_0 < \zeta_1 < \cdots < \zeta_{n_1} < a_1 < \zeta_{n_1+1} < \cdots < \zeta_{n_1+n_2} < a_2 < \cdots < \zeta_N < a_p \tag{4}$$

of $E_{\boldsymbol{n}}$ minimize the discrete energy

$$\sum_{1 \leqslant i < j \leqslant N} \ln \frac{1}{|\zeta_i - \zeta_j|} + \sum_{j=0}^{p} \frac{\rho_j}{2} \sum_{i=1}^{N} \ln \frac{1}{|\zeta_i - a_j|}, \tag{5}$$

among all the N point distributions satisfying (4).

Further generalizations of the work of Heine and Stieltjes followed several paths; we will mention only some of them. First, under assumptions (2)–(3) Van Vleck [30] and Bôcher [5] proved that the zeros of C belong to $[a_0, a_p]$. A refinement of this result is due to a series of works of Shah [22–25]. Furthermore, Pólya [18] showed that for complex a_i under assumption (3) the zeros of E are located in the convex hull of the zeros of A. Marden [15], and later, Al-Rashed, Alam and Zaheer (see [1, 2, 32, 33]) established further results on location of the zeros of the Heine–Stieltjes polynomials under weaker conditions on the coefficients A and B of (1). An electrostatic interpretation of these zeros in cases when some residues ρ_i in (3) are negative has been studied by Grünbaum [10], and Dimitrov and Van Assche [6]. A general approach to the electrostatic interpretation of the zeros of orthogonal polynomials was proposed recently by Ismail [13].

An orthogonality property of the solutions of hypergeometric differential equations ($p = 1$) is a well-known fact (see, e.g., [17]). The orthogonality of products of different Heine–Stieltjes polynomials in the Cartesian product space was proved by Germanski [7] and rediscovered recently by Volkmer [31] (whose paper goes beyond this orthogonality); for the case of the Heun differential equation ($p = 2$), this fact was established by Arscott [3] and Sleeman [26] (see also [4,19, Section A.5.3]).

Nevertheless, nothing has been published about the zero asymptotics of the Heine–Stieltjes and Van Vleck polynomials for large values of parameter N. This is rather surprising, taking into account that the necessary machinery existed for several decades.

The object of this paper is to study the asymptotic behavior of the zeros of $E_{\boldsymbol{n}}$ and $C_{\boldsymbol{n}}$ when $N = n_1 + \cdots + n_p \to \infty$ in such a way that

$$\lim_{N \to \infty} \frac{n_i}{N} = \theta_i, \qquad i = 1, \ldots, p. \tag{6}$$

We also allow that the polynomial coefficient $B = B_{\boldsymbol{n}}$ in (1) depends on $\boldsymbol{n}$ in such a way that the limit

$$\lim_{N\to\infty} \frac{B_{\boldsymbol{n}}}{N} = B \tag{7}$$

exists and satisfies (3). In other words,

$$\frac{B_{\boldsymbol{n}}(x)}{A(x)} = \sum_{j=0}^{p} \frac{\rho_{j,\boldsymbol{n}}}{x - a_j}, \quad \rho_{j,\boldsymbol{n}} > 0 \quad \text{and} \quad \lim_{N\to\infty} \frac{\rho_{j,\boldsymbol{n}}}{N} = \rho_j \geqslant 0. \tag{8}$$

The asymptotics for $E_{\boldsymbol{n}}$ is understood in the sense of weak-* convergence. Namely, we describe the limit of the sequence of normalized counting measures $\nu(E_{\boldsymbol{n}})/N$ under assumptions (6), (8) in terms of the solution of a certain extremal problem for vector logarithmic potentials. The main results are stated in Section 2, their proofs are presented in Section 3, and particular cases are discussed in Section 4.

Our method is applicable also when not all the residues ρ_j are positive, but we still have electrostatic equilibrium. This is a situation described by Dimitrov and Van Assche [6], and in Section 5 we derive the asymptotics for the corresponding Heine–Stieltjes and Van Vleck polynomials. This situation yields to an equilibrium problem in a non-convex external field.

2. VECTOR EQUILIBRIUM PROBLEM AND ZERO DISTRIBUTION

If μ is a finite and compactly supported Borel measure on the complex plane $\mathbb{C}$, we denote by $\operatorname{supp}(\mu)$ its support, by

$$V(\mu; z) = \int \ln \frac{1}{|z - t|} \, d\mu(t)$$

its logarithmic potential, and by

$$I(\mu) = \int \int \ln \frac{1}{|z - t|} \, d\mu(t) \, d\mu(z)$$

its logarithmic energy.

A function $w : [-1, 1] \to \mathbb{R}_+$ is an *admissible weight* on $[-1, 1]$ if w is upper-semicontinuous and the set $\{x \in [-1, 1] : w(x) > 0\}$ has positive logarithmic capacity (for basic definitions, see, e.g., [20, Section I.1] or [27, Appendix]). The (admissible) *external field* φ on $[-1, 1]$ is defined by

$$w(x) = e^{-\varphi(x)}, \qquad x \in [-1, 1],$$

and the weighted energy $I_\varphi(\mu)$ of a Borel measure μ on $[-1,1]$, by

$$I_\varphi(\mu) = I(\mu) + 2\int \varphi\, d\mu.$$

Let $\mathcal{N}$ be the standard simplex in $\mathbb{R}^{p-1}$,

$$\mathcal{N} = \left\{ \boldsymbol{\theta} = (\theta_1, \ldots, \theta_p) : \theta_i \geqslant 0,\ i = 1, \ldots, p,\ \text{and}\ \sum_{i=1}^{p} \theta_i = 1 \right\}.$$

For $\boldsymbol{\theta} \in \mathcal{N}$ denote by $\mathcal{M}(\boldsymbol{\theta})$ the class of all unit Borel measures μ on $[-1,1]$ such that[2]

$$\int_{a_{i-1}}^{a_i} d\mu = \theta_i, \qquad i = 1, \ldots, p.$$

Given $\boldsymbol{\theta} \in \mathcal{N}$ we can consider the problem of minimization of the weighted energy $I_\varphi(\mu)$ in the class $\mathcal{M}(\boldsymbol{\theta})$. In fact, this is a particular instance of the vector-valued equilibrium problem for the vector potentials: the restriction of the solution μ to a particular subinterval $[a_{i-1}, a_i]$ solves the equilibrium problem in the presence of the external field jointly generated by φ and the potential of the remaining part of μ. Thus, the following lemma is a direct consequence of the well-known results (see [8,20, Theorem VIII.1.4]):

Lemma 1. *Let φ be an admissible weight. For every $\boldsymbol{\theta} \in \mathcal{N}$ there exists a unique $\mu_{\boldsymbol{\theta}} \in \mathcal{M}(\boldsymbol{\theta})$ (the equilibrium measure) such that*

$$I_\varphi(\mu_{\boldsymbol{\theta}}) \leqslant I_\varphi(\mu), \quad \text{for every } \mu \in \mathcal{M}(\boldsymbol{\theta}).$$

Moreover, $\mu_{\boldsymbol{\theta}}$ is characterized by the following property: for $i = 1, \ldots, p$,

$$\min_{x \in [a_{i-1}, a_i]} (V(\mu_{\boldsymbol{\theta}}; x) + \varphi(x)) = V(\mu_{\boldsymbol{\theta}}; x) + \varphi(x), \quad x \in \operatorname{supp}(\mu_{\boldsymbol{\theta}}) \cap [a_{i-1}, a_i]. \tag{9}$$

Finding the explicit solution for a given equilibrium problem is in general a formidable task. In the case we are interested in, we can describe the equilibrium measure $\mu_{\boldsymbol{\theta}}$ as follows: Let B and ρ_i be given in (7) and (8). Then

$$\varphi(x) = -\sum_{j=0}^{p} \frac{\rho_j}{2} \ln |x - a_j| \tag{10}$$

defines an admissible external field on $[-1,1]$.

[2] We remark that the conditions on $\mathcal{M}(\boldsymbol{\theta})$ imply that $\mu \in \mathcal{M}(\boldsymbol{\theta})$ have no mass points at a_i's.

We make the following convention: if H is an analytic and single-valued function in $\mathbb{C}\setminus[-1,1]$, we understand by $H(x)$ for $x \in (-1,1)$ the boundary values of H from the upper half plane. Let us also denote

$$\eta = 1 + \sum_{j=0}^{p} \frac{\rho_j}{2}. \tag{11}$$

Then, we have the following result which is proved in Section 3:

THEOREM 1. *Let $Q \in \mathbb{P}_{2p}$ be a polynomial of the form*

$$Q(z) = \eta^2 \prod_{j=1}^{2p} (z - \alpha_j), \tag{12}$$

with $[\alpha_{2j-1}, \alpha_{2j}] \subset [a_{j-1}, a_j]$ for $j = 1, \ldots, p$, and let

$$K = [\alpha_1, \alpha_2] \cup \cdots \cup [\alpha_{2p-1}, \alpha_{2p}].$$

In $\mathbb{C}\setminus K$ we fix the single-valued branch of $\sqrt{Q}$ by

$$\lim_{z\to\infty} \frac{\sqrt{Q(z)}}{z^p} = \eta. \tag{13}$$

If conditions (6) *and* (8) *are fulfilled, then given $\boldsymbol{\theta}$ and φ, there exists a unique $Q = Q_{\boldsymbol{\theta}}$ as above, determined by the following conditions:*[3]

$$\sqrt{Q_{\boldsymbol{\theta}}(a_j)} = \frac{\rho_j}{2} A'(a_j), \qquad j = 0, \ldots, p, \tag{14}$$

$$\int_{\alpha_{2j-1}}^{\alpha_{2j}} \frac{\sqrt{Q_{\boldsymbol{\theta}}(x)}}{A(x)}\, dx = -\pi i\, \theta_j, \qquad j = 1, \ldots, p-1. \tag{15}$$

Then the equilibrium measure $\mu_{\boldsymbol{\theta}}$ is absolutely continuous with respect to the Lebesgue measure,

$$\operatorname{supp}(\mu_{\boldsymbol{\theta}}) = K = \overline{\{x \in \mathbb{R} : Q_{\boldsymbol{\theta}}(x) < 0\}},$$

$$\mu'_{\boldsymbol{\theta}}(x) = -\frac{1}{\pi i} \frac{\sqrt{Q_{\boldsymbol{\theta}}}(x)}{A(x)}, \qquad x \in K \tag{16}$$

[3] In particular, if $\theta_j = 0$, then $\alpha_{2j-1} = \alpha_{2j}$.

and

$$\int \frac{d\mu_{\boldsymbol{\theta}}(t)}{z-t} = H(z) = -\frac{B(z)}{2A(z)} + \frac{\sqrt{Q_{\boldsymbol{\theta}}}(z)}{A(z)}, \qquad z \notin K. \tag{17}$$

We consider particularly the case $B \equiv 0$, that is,

$$\rho_0 = \cdots = \rho_p = 0,$$

which appears, for example, when $B_{\boldsymbol{n}}$ does not depend on $\boldsymbol{n}$.

COROLLARY 1. *Let* $B \equiv 0$. *There exist* $p-1$ *points*

$$-1 \leqslant \beta_1 \leqslant \cdots \leqslant \beta_{p-1} \leqslant 1$$

uniquely determined by the following system of equations:

$$\operatorname{Im} \int_{a_{j-1}}^{a_j} H_{\boldsymbol{\theta}}(x)\, dx = -\pi\, \theta_j, \qquad j = 1, \dots, p-1, \tag{18}$$

where

$$H_{\boldsymbol{\theta}}(x) := \sqrt{\frac{R_{\boldsymbol{\theta}}(x)}{A(x)}}, \qquad R_{\boldsymbol{\theta}}(x) = \prod_{j=1}^{p-1} (x - \beta_j). \tag{19}$$

If we introduce the counting function

$$Z(x) := [\nu(A) - \nu(R_{\boldsymbol{\theta}})]((-\infty, x]),$$

then

$$\operatorname{supp}(\mu_{\boldsymbol{\theta}}) = \overline{\{x \in \mathbb{R} : Z(x) = 1\}}. \tag{20}$$

The support $\operatorname{supp}(\mu_{\boldsymbol{\theta}})$ *of* $\mu_{\boldsymbol{\theta}}$ *consists of at most* $p-1$ *disjoint intervals in* $[-1, 1]$.

Furthermore, $\mu_{\boldsymbol{\theta}}$ *is an absolutely continuous measure*,

$$\mu'_{\boldsymbol{\theta}}(x) = -\frac{1}{\pi i} H_{\boldsymbol{\theta}}(x) = \frac{1}{\pi} |H_{\boldsymbol{\theta}}(x)|, \qquad x \in \operatorname{supp}(\mu_{\boldsymbol{\theta}}), \tag{21}$$

and for $z \notin \operatorname{supp}(\mu_{\boldsymbol{\theta}})$,

$$\int \frac{d\mu_{\boldsymbol{\theta}}(t)}{z-t} = H_{\boldsymbol{\theta}}(z). \tag{22}$$

Finally, we establish the following relation between the equilibrium problem described above and the distribution of zeros of Van Vleck and Stieltjes polynomials.

Let us denote the weak-* convergence of a sequence of measures ν_n on $[-1,1]$ to a measure ν by $\nu_n \to \nu$, meaning that

$$\int f\, d\nu_n \to \int f\, d\nu, \qquad \forall f \in C[-1,1].$$

THEOREM 2. *Assume that* $E_{\boldsymbol{n}}$ *and* $C_{\boldsymbol{n}}$ *are as above and* (6), (8) *hold. If* $\mu_{\boldsymbol{\theta}}$ *and* $Q_{\boldsymbol{\theta}}$ *are as in Theorem* 1, *then for all* $z \in \mathbb{C}$,

$$\lim_{N\to\infty} \frac{C_{\boldsymbol{n}}(z)}{N^2} = C(z) = \frac{Q_{\boldsymbol{\theta}} - (B/2)^2}{A}(z); \tag{23}$$

in particular, the zeros of Van Vleck polynomials $C_{\boldsymbol{n}}$ *converge to those of* C.

Furthermore,

$$\nu_{\boldsymbol{n}} := \frac{\nu(E_{\boldsymbol{n}})}{N} \to \mu_{\boldsymbol{\theta}}. \tag{24}$$

Consequently, if the $E_{\boldsymbol{n}}$*'s are normalized to be monic, then*

$$\lim_{\boldsymbol{n}} |E_{\boldsymbol{n}}(z)|^{1/N} = \exp(-V(\mu_{\boldsymbol{\theta}}; z)) = \left| z \exp\left[\int_\infty^z \left(H(z) - \frac{1}{z} \right) dz \right] \right|, \tag{25}$$

uniformly on compact subsets of $\mathbb{C} \setminus [-1,1]$.

3. PROOF OF THE MAIN RESULTS

First of all, Eq. (24) is a consequence of the electrostatic interpretation of the zeros of E and we establish it using a modification of the proof of the asymptotic behavior of the weighted Fekete points (see [20, Theorem 1.3, Section III.1]). Indeed, let

$$\varphi_{\boldsymbol{n}}(x) = -\sum_{i=0}^{p} \frac{\rho_{i,\boldsymbol{n}}}{2} \ln |x - a_i|$$

be the external field generated by the positive charges at the zeros of A. According to the electrostatic interpretation given by Stieltjes, if we define

$$\delta_N := \int \int_{x \neq y} \ln \frac{1}{|x-y|}\, d\nu(E_{\boldsymbol{n}})(x)\, d\nu(E_{\boldsymbol{n}})(y) + 2 \int \varphi_{\boldsymbol{n}}\, d\nu(E_{\boldsymbol{n}}), \tag{26}$$

then, for any N distinct points $-1<z_1<\cdots<z_N<1$ such that exactly n_i of them belong to (a_{i-1}, a_i), $i=1,\dots,p$, we have that

$$\delta_N \leqslant -\sum_{i\neq j} \ln|z_i - z_j| + 2\sum_{i=0}^{N} \varphi_{\boldsymbol{n}}(z_i).$$

Integrating this inequality with respect to $d\mu_{\boldsymbol{\theta}}(z_1)\dots d\mu_{\boldsymbol{\theta}}(z_N)$, we get that

$$\delta_N \leqslant N(N-1)I(\mu_{\boldsymbol{\theta}}) + 2N\int \varphi_{\boldsymbol{n}}\, d\mu_{\boldsymbol{\theta}}. \tag{27}$$

On the other hand, for $\varepsilon>0$ define

$$K_\varepsilon(x,y) = \min\{-\ln|x-y|, -\ln\varepsilon\}.$$

For every fixed $\varepsilon>0$ we have

$$\begin{aligned}
&\int\int K_\varepsilon(x,y)\, d\nu(E_{\boldsymbol{n}})(x)\, d\nu(E_{\boldsymbol{n}})(y) + 2\int \varphi_{\boldsymbol{n}}\, d\nu(E_{\boldsymbol{n}}) \\
&\quad = \int\int_{x\neq y} K_\varepsilon(x,y)\, d\nu(E_{\boldsymbol{n}})(x)\, d\nu(E_{\boldsymbol{n}})(y) + 2\int \varphi_{\boldsymbol{n}}\, d\nu(E_{\boldsymbol{n}}) - N\ln\varepsilon \\
&\quad \leqslant \delta_N - N\ln\varepsilon \leqslant N(N-1)I(\mu_{\boldsymbol{\theta}}) + 2N\int \varphi_{\boldsymbol{n}}\, d\mu_{\boldsymbol{\theta}} - N\ln\varepsilon,
\end{aligned}$$

where we have used (27). By compactness of the sequence $\nu_{\boldsymbol{n}}$, we may take a subsequence Λ of the indices $\boldsymbol{n}$ such that $\nu_{\boldsymbol{n}}$, $\boldsymbol{n}\in\Lambda$, converges (in the weak-* topology) to a measure ν supported on $[-1,1]$. Dividing by N^2 and taking limits, we get

$$\int\int K_\varepsilon(x,y)\, d\nu(x)\, d\nu(y) + 2\int \varphi\, d\nu(x) \leqslant I_\varphi(\mu_{\boldsymbol{\theta}}),$$

where φ is given by (10). Taking now $\varepsilon\to 0$ we see that

$$I_\varphi(\nu) \leqslant I_\varphi(\mu_{\boldsymbol{\theta}}).$$

From the uniqueness of the extremal measure $\mu_{\boldsymbol{\theta}}$, it follows that $\nu=\mu_{\boldsymbol{\theta}}$ and we get (24). This fact will help us in describing the equilibrium measure $\mu_{\boldsymbol{\theta}}$.

Indeed, we can rewrite the differential equation (1) in terms of the function $h=E'/E$, reducing it to a Riccati equation (see, e.g., [14, I.4.9; 21] or [34, Section 86]):

$$A(x)(h^2(x)+h'(x)) + B_{\boldsymbol{n}}(x)h(x) - C_{\boldsymbol{n}}(x) = 0. \tag{28}$$

In particular, if $E = E_n$, we have that

$$h_n(x) := \frac{E_n'(x)}{E_n(x)} = \int \frac{d\nu(E_n)(t)}{x-t} = N \int \frac{d\nu_n(t)}{x-t}.$$

By (24),

$$h_n(x)/N \to H(x) = \int \frac{d\mu_\theta(t)}{x-t},$$

locally uniformly in $\mathbb{C}\setminus[-1,1]$. If we rewrite (28) as

$$A(x)\left(\frac{h_n^2(x)}{N^2} + \frac{h_n'(x)}{N^2}\right) + \frac{B_n(x)}{N}\frac{h_n(x)}{N} = \frac{C_n(x)}{N^2},$$

we see that the left-hand side converges along the chosen subsequence to the function $AH^2 + BH$. Thus, the right-hand side also converges locally uniformly in $\mathbb{C}\setminus[-1,1]$, which proves the existence of the limit in (23).

Denoting by C the limit of C_n/N^2, we readily see that

$$H(z) = -\frac{B(z)}{2A(z)} + \frac{\sqrt{Q(z)}}{A(z)}, \qquad Q = \left(\frac{B}{2}\right)^2 + AC, \tag{29}$$

(compare with (17)).

The behavior of $\sqrt{Q}$ at $z = \infty$ is determined by the fact that

$$\lim_{z\to\infty} zH(z) = \lim_{z\to\infty} z \int \frac{d\mu_\theta(t)}{z-t}\, dt = 1.$$

Indeed, by (8),

$$\lim_{z\to\infty} \frac{zB(z)}{A(z)} = \sum_{j=0}^{p} \rho_j,$$

so that we must take in (29)

$$\lim_{z\to\infty} \frac{z\sqrt{Q(z)}}{A(z)} = \lim_{z\to\infty} \frac{\sqrt{Q(z)}}{z^p} = \eta,$$

as in (13).

We can recover the measure μ_θ using either the well-known Stieltjes–Perron inversion formula or the Sokhotski–Plemelj theorem (see, e.g., [12, Section 14.1]). Thus, from (29) we get that

$$\mu_\theta'(x) = -\frac{1}{\pi i}\frac{\sqrt{Q(x)}}{A(x)}.$$

Since μ_θ is a positive measure on $[-1,1]$, its support K will be the closure of the set

$$\mathscr{K} := \left\{ x \in [-1,1] : \frac{\sqrt{Q(x)}}{iA(x)} < 0 \right\}. \tag{30}$$

On each subinterval $[a_{j-1}, a_j]$ the external field (10) is a convex function, thus (see, e.g., [20. Section IV.1]), $\mathrm{supp}(\mu_\theta) \cup [a_{j-1}, a_j]$ will be connected. In other words,

$$K = [\alpha_1, \alpha_2] \cup \cdots \cup [\alpha_{2p-1}, \alpha_{2p}]$$

for $[\alpha_{2j-1}, \alpha_{2j}] \subset [a_{j-1}, a_j]$. By (30),

$$Q(\alpha_j) = 0, \qquad j = 1, \ldots, 2p,$$

and from (13) we deduce (12).

Let us establish the necessary conditions on Q (later we will see that they are also sufficient). First, in our situation equations (15) are equivalent to the fact that $\mu_\theta \in \mathscr{M}(\boldsymbol{\theta})$.

Furthermore, if for $j \in \{0, 1, \ldots, p\}$, $\rho_j > 0$, then $a_j \notin K$. In such a case, H must be holomorphic in a neighborhood of a_j, so that

$$\operatorname*{res}_{z=a_j} H(z) = 0,$$

which renders (14) for $\rho_j > 0$.

Let $\mathscr{F}$ be an analytic multivalued function in $\mathbb{C} \setminus K$ such that

$$\mathrm{Re}\, \mathscr{F}(z) = V(\mu_\theta; z) + \varphi(z), \qquad z \in \mathbb{C} \setminus K.$$

Then

$$\mathscr{F}'(z) = -H(z) - \frac{B(z)}{2A(z)} = -\frac{\sqrt{Q(z)}}{A(z)}.$$

Thus,

$$V(\mu_\theta; z) + \varphi(z) = -\mathrm{Re}\left(\int^z \frac{\sqrt{Q(z)}}{A(z)}\, dz \right) + \text{const}. \tag{31}$$

Taking into account (9), $V(\mu_\theta; z) + \varphi(z)$ is bounded at a_j if and only if $\rho_j = 0$, which by (31) is equivalent to $Q(a_j) = 0$. This establishes (14) for the remaining case.

Finally, Eq. (25) is an immediate consequence of (24), (31), and the fact that for monic $E_{\boldsymbol{n}}$,

$$\lim_{z \to \infty} |E_{\boldsymbol{n}}(z)/z^N| = 1.$$

It remains to establish the uniqueness of Q, for which it is sufficient to show that conditions above characterize the equilibrium measure (the uniqueness of the latter does the rest).

Assume that we have constructed a polynomial $Q = Q_{\boldsymbol{\theta}}$ satisfying (12)–(15). Then

$$K = [\alpha_1, \alpha_2] \cup \cdots \cup [\alpha_{2p-1}, \alpha_{2p}] = \overline{\mathscr{K}},$$

where $\mathscr{K}$ is given in (30). Consequently, the function on the right-hand side of (16) is positive on K and non-positive on $\mathbb{R} \setminus K$. Thus, (16) defines a positive absolutely continuous measure on K, which, according to (15), belongs to $\mathscr{M}(\boldsymbol{\theta})$. Furthermore, by the interlacing property of a_j's and α_j's, for $j = 1, \ldots, p$,

$$\frac{\sqrt{Q(x)}}{A(x)} \begin{cases} > 0 & \text{for } \alpha_{2j-1} < x < a_j, \\ < 0 & \text{for } a_{j-1} < x < \alpha_{2j-1}. \end{cases}$$

Taking into account the expression in (31) we deduce that for each $j = 1, \ldots, p$,

$$V(\mu_{\boldsymbol{\theta}}; z) + \varphi(z) \begin{cases} c_j = \text{const} & \text{for } x \in [\alpha_{2j-1}, \alpha_{2j}], \\ > c_j & \text{for } x \in [a_{j-1}, a_j] \setminus [\alpha_{2j-1}, \alpha_{2j}], \end{cases}$$

which, by (9), characterizes the equilibrium measure of Lemma 1.

Let us switch now to the proof of the Corollary, when $\eta = 1$. First of all, by (14), $p+1$ zeros of Q coincide now with $a_0, \ldots, a_p$, so that by (12),

$$Q(z) = A(z) R_{\boldsymbol{\theta}}(z), \qquad R_{\boldsymbol{\theta}}(z) = \prod_{j=1}^{p-1} (z - \beta_j), \quad \beta_j \in [-1, 1].$$

Denote

$$H_{\boldsymbol{\theta}}(z) := \frac{\sqrt{Q(z)}}{A(z)} = \sqrt{\frac{R_{\boldsymbol{\theta}}(z)}{A(z)}},$$

taking, by (13),

$$\lim_{z \to \infty} z H_{\boldsymbol{\theta}}(z) = 1.$$

Then (15) reduces to (18), where we have used the fact that for each j, $\operatorname{supp}(\mu_{\boldsymbol{\theta}}) \cap [a_{j-1}, a_j]$ is connected. Moreover, (16) and (17) reduce to (21) and (22), respectively.

It remains only to prove (20). This is a consequence of the fact that $\text{supp}(\mu_{\boldsymbol{\theta}})$ is the closure of

$$\mathscr{K} := \{x \in [-1, 1] : iH_{\boldsymbol{\theta}}(x) > 0\}$$

(where we follow our convention of taking the limit values from the upper half plane). Indeed, by the selection of the branch of $H_{\boldsymbol{\theta}}$,

$$\arg H_{\boldsymbol{\theta}}(x) = -\pi, \qquad x < 0.$$

Taking into account the form of $H_{\boldsymbol{\theta}}$, it is easy to verify that

$$\arg H_{\boldsymbol{\theta}}(x) = \frac{\pi}{2}(Z(x) - 2), \qquad x \in \mathbb{R} \backslash (\{a_0, \ldots, a_p\} \cup \{\beta_1, \ldots, \beta_{p-1}\}).$$

From the definition of $\mathscr{K}$ we get (20). This relation shows that at least one endpoint of each connected component of $\text{supp}(\mu_{\boldsymbol{\theta}})$ belongs to $\{a_0, \ldots, a_p\}$.

4. SOME SPECIAL CASES

In this section we consider some important particular cases of the previous theorems.

Obviously, the simplest situation is when $p = 1$, which corresponds to an hypergeometric equation. To be more precise, Eq. (1) in this case is the differential equation for Jacobi polynomials. The zero distribution of the Jacobi polynomials with varying weights has been studied before (see, e.g., [20, Sections IV.1 and IV.5]). In particular, now the only condition on the measure, (14), reduces to

$$\sqrt{(1 + \alpha_1)(1 + \alpha_2)} = \frac{2\rho_0}{2 + \rho_0 + \rho_1}, \qquad \sqrt{(1 - \alpha_1)(1 - \alpha_2)} = \frac{2\rho_1}{2 + \rho_0 + \rho_1},$$

which coincides with the equation on the endpoints of the support given in [20, Example IV.1.17]. Moreover, Eq. (16) corresponds to formula (IV.5.8) in the same monograph.

The case $p = 2$ corresponds to the well-known and thoroughly studied Heun equation [19]. Now we have two intervals, $[-1, a_1]$, $[a_1, 1]$, and respective constants, $\theta_1, \theta_2 = 1 - \theta_1$ (cf. (6)), and conditions (14) and (15) yield equations involving elliptic integrals. In particular, when $B \equiv 0$, we obtain that

$$\text{supp}(\mu_{\boldsymbol{\theta}}) = [-1, \alpha] \cup [\beta, 1].$$

If

$$\theta_1 \leqslant \frac{1}{\pi}\int_{-1}^{a_1} \frac{1}{\sqrt{1-x^2}}\,dx = \frac{1}{2} + \frac{1}{\pi}\arcsin(a_1), \tag{32}$$

then $\alpha \in [-1, a_1]$ and $\beta = a_1$; otherwise, $\alpha = a_1$ and $\beta \in [a_1, 1]$. Moreover, under condition (32) the endpoint α is obtained from Eq. (18), which takes the form

$$\int_{-1}^{\alpha} \sqrt{\frac{\alpha - x}{(x+1)(a_1 - x)(1-x)}}\,dx = \pi\theta_1.$$

After some cumbersome computation, it can be rewritten in terms of standard elliptic integrals (see [9, Section 3.167]) as

$$2\sqrt{2}\,(m\,\Pi(1-m,k) - K(k)) = \pi\theta_1\,\sqrt{(1+\,a_1)m}, \tag{33}$$

where

$$0 < k = \sqrt{\frac{(1-a_1)(1+\alpha)}{(1+a_1)(1-\alpha)}} < 1, \qquad m = \frac{2}{1-\alpha} > 1$$

and

$$K(k) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}, \qquad \Pi(m,k) = \int_0^{\pi/2} \frac{d\phi}{(1-m\sin^2\phi)\sqrt{1-k^2\sin^2\phi}}$$

are the elliptic integrals of the first and third kinds in the Legendre normal form.

Equation (33) can be presented in another equivalent way as

$$\sqrt{2(1-a_1^2)}\int_0^{k^2} \frac{K(\sqrt{u})}{(1-a_1+(1+a_1)u)^{3/2}}\,du = \pi\theta_1, \tag{34}$$

which is suitable for differentiation. Thus, combining (33) and (34) we can easily apply the Newton method in order to find the endpoint α (or equivalently, the modulus k) corresponding to a value of a parameter θ_1. For instance, the following iteration starting from $k_0 = 0.25$,

$$k_{j+1} = k_j - \left(\frac{2\sqrt{2}\,(m_j\,\Pi(1-m_j,k_j) - K(k_j))}{\sqrt{(1+\,a_1)m_j}} - \pi\theta_1\right)D_j$$

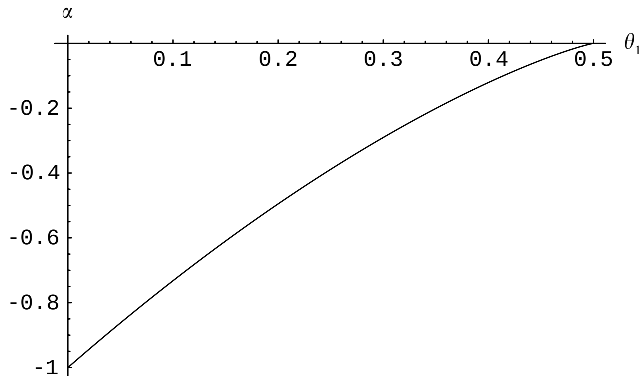

FIG. 1. Relation between $0 \leqslant \theta_1 \leqslant 1/2$ and α (the free endpoint of the support) for $a_1 = 0$.

with

$$m_j = 1 + \frac{(1 + a_1)}{1 - a_1} k_j^2, \qquad D_j = \frac{m_j^{3/2}(1 - a_1)}{2\sqrt{2(1 + a_1)}\, k_j\, K(k_j)},$$

provides a quadratic convergence to the value of k, corresponding to the given θ_1. It remains to take

$$\alpha = \frac{a_1 - 1 + (1 + a_1)k^2}{1 - a_1 + (1 + a_1)k^2}.$$

In this way we computed the relation between θ_1 $(0 \leqslant \theta_1 \leqslant 1/2)$ and α (the free endpoint of the support) for $a_1 = 0$ and $B \equiv 0$, presented in Fig. 1.

Finally, inequality (32) is a consequence of the following general observation, based on the uniqueness of the equilibrium measure:

PROPOSITION 1. *The zeros of the Heine–Stieltjes polynomials subject to conditions* (2), (6) *and* (7) *are dense on* $[-1, 1]$ *if and only if* $B \equiv 0$ *and*

$$\arcsin(a_j) - \arcsin(a_{j-1}) = \pi\, \theta_j, \qquad j = 1, \ldots, p - 1.$$

5. NEGATIVE RESIDUES

In contrast to the classical results cited in Section 1, the case when the residues ρ_j are allowed to take negative values has not been thoroughly studied. In this case, even the existence and unicity of both Van Vleck and

Heine–Stieltjes polynomials is not a trivial question. Some situations when this existence and uniqueness are guaranteed have been studied by Dimitrov and Van Assche [6]; namely, they considered the case $p = 3$ and the signs of ρ_j's distributed as in Fig. 2.

As was shown in [6], in this case for every sufficiently large $N \in \mathbb{N}$ there exists a unique pair (C_N, E_N) of, respectively, Van Vleck and Heine–Stieltjes polynomials with $\deg E_N = N$. All zeros of E_N belong to the interval enclosed by a_j's with $\rho_j > 0$, and they are in the equilibrium position, given by the absolute minimum of the discrete energy (5).

Thus, we can apply the methods above in order to find the asymptotic distribution of these zeros. Observe that the electrostatic interpretation yields the extremal problem (9) with a non-convex external field, and the connectedness of the support of the equilibrium measure is no longer guaranteed. Nevertheless, the differential equation (1) contains additional information which allows to obtain the Stieltjes transform of the limit distribution. Once again, it will be described by a polynomial Q as in (12), except that now some of the zeros will leave $[-1, 1]$.

We consider the asymptotics with conditions such as in (8). Since all the zeros of the Heine–Stieltjes polynomials belong now to the same interval, it is sufficient to introduce the scalar index N. Thus, we assume that $-1 = a_0 < a_1 < a_2 < a_3 = 1$,

$$\frac{B_N(x)}{A(x)} = \sum_{j=0}^{3} \frac{\rho_{j,N}}{x - a_j}, \qquad \lim_{N\to\infty} \frac{\rho_{j,N}}{N} = \rho_j, \tag{35}$$

and restrict our attention to the situation described in [6] (up to a misprint) when the existence and unicity are guaranteed. Namely, for a sufficiently large N, let the coefficients $\rho_{j,N}$ have the signs according to one of the following cases (depicted in Fig. 2):

$$\rho_{0,N}, \rho_{1,N} < 0, \qquad \rho_{2,N}, \rho_{3,N} > 0, \quad N \text{ arbitrary}$$
$$(\text{then, } \ \rho_0, \rho_1 \leqslant 0 \ \text{ and } \ \rho_2, \rho_3 \geqslant 0); \tag{C.1}$$

$$\rho_{0,N}, \rho_{3,N} < 0, \qquad \rho_{1,N}, \rho_{2,N} > 0, \quad N > 1 - \textstyle\sum_{j=0}^{3} \rho_{j,N},$$
$$(\text{then, } \ \rho_0, \rho_3 \leqslant 0 \ \text{ and } \ \rho_1, \rho_2 \geqslant 0). \tag{C.2}$$

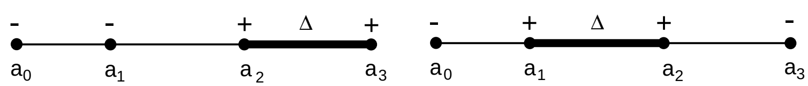

FIG. 2. Cases (C.1) (left) and (C.2) (right).

We assume initially that $\eta \neq 0$, where η was defined in (11); in the situation (C.2) we have necessarily $\eta \geqslant 1/2$. Let us denote by Δ the interval $[a_{j-1}, a_j]$, determined by the positive residues. The following result holds:

THEOREM 3. *Assume that either case* (C.1) *with* $\eta \neq 0$ *or case* (C.2) *with* $\eta > 0$ *holds. Let* Q *be a polynomial of the form*

$$Q(z) = \eta^2 (z - \alpha_1)^2 (z - \alpha_2)^2 (z - \beta_1)(z - \beta_2), \tag{36}$$

with $\alpha_1, \alpha_2 \in \mathbb{R}$, $[\beta_1, \beta_2] \subset \Delta$, *and let*

$$\lim_{z \to \infty} \frac{\sqrt{Q(z)}}{z^3} = \eta. \tag{37}$$

Then there exists a unique Q *of this type, determined by the following system of equations*:

$$\sqrt{Q(a_j)} = \frac{\rho_j}{2} A'(a_j), \qquad j = 0, \ldots, 3. \tag{38}$$

The relative position of the zeros of Q *is represented in* Fig. 3.

The equilibrium unit measure μ *on* Δ *under the external field* φ *given in* (10), *is absolutely continuous with respect to the Lebesgue measure*, $\text{supp}(\mu) = [\beta_1, \beta_2]$,

$$\mu'(x) = -\frac{1}{\pi i} \frac{\sqrt{Q(x)}}{A(x)}, \qquad x \in \text{supp}(\mu) \tag{39}$$

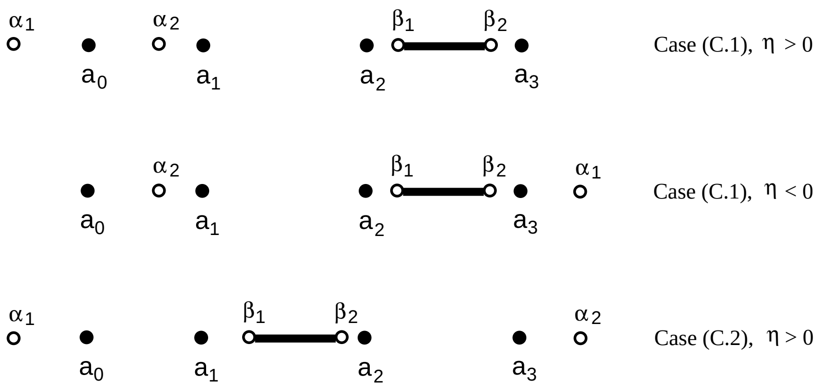

FIG. 3. Zeros of Q and the support of μ.

and

$$\int \frac{d\mu(t)}{z-t} = H(z) = -\frac{B(z)}{2A(z)} + \frac{\sqrt{Q(z)}}{A(z)}, \qquad z \notin \operatorname{supp}(\mu). \tag{40}$$

For all $z \in \mathbb{C}$,

$$\lim_{N\to\infty} \frac{C_N(z)}{N^2} = C(z) = \frac{Q-(B/2)^2}{A}(z); \tag{41}$$

in particular, the zeros of Van Vleck polynomials C_N *converge to those of* C.

Furthermore,

$$\frac{\nu(E_N)}{N} \to \mu. \tag{42}$$

Consequently, if the E_N*'s are normalized to be monic, then*

$$\lim_{N\to\infty} |E_N(z)|^{1/N} = \exp(-V(\mu; z)) = \left| z \exp\left[\int_\infty^z \left(H(z) - \frac{1}{z} \right) dz \right] \right|, \tag{43}$$

uniformly on compact subsets of $\mathbb{C}\setminus[-1,1]$.

Proof. As it was observed above, from the electrostatic interpretation of zeros, derived in [6], we obtain (42), where μ is the equilibrium measure. In the notation of Section 2, $\mu = \mu_{\boldsymbol{\theta}} \in \mathcal{M}(\boldsymbol{\theta})$, where $\boldsymbol{\theta} = (0,0,1)$ (in case (C.1)), or $\boldsymbol{\theta} = (0,1,0)$ (in case (C.2)). Furthermore, from the differential equation we obtain that the limit in the left-hand side of (41) exists, which defines the polynomial Q. This yields expression (40) of the Stieltjes transform of μ, from which (39) immediately follows.

On the other hand, Eqs. (14) and (15) on Q reduce now to (38). Thus, it remains to show that Q is of the form (36). Observe that now the external field φ in (10) is no longer convex, and the connectedness of the support of μ is not trivial.

In analogy with Corollary 1, let us introduce the counting function

$$Z(x) = \nu(Q)((-\infty, x]) = \text{number of zeros of } Q \text{ in } (-\infty, x]. \tag{44}$$

Then by (37),

$$\arg(\sqrt{Q(x)}) = -\arg(\eta) - \frac{\pi}{2} Z(x), \qquad \text{if } x \in \mathbb{R}, \quad Q(x) \neq 0,$$

and

$$\lim_{x\to-\infty} Z(x) = 0, \qquad \lim_{x\to+\infty} Z(x) \equiv 2 \bmod 4.$$

Assume that $\rho_j \neq 0,\ j = 0, \ldots, 3$. By Eq. (38),

$$\sqrt{Q(a_j)} \begin{cases} > 0 & \text{for } j = 0, 3, \\ < 0 & \text{for } j = 1, 2, \end{cases} \quad \text{in case (C.1)},$$

and

$$\sqrt{Q(a_j)} \begin{cases} > 0 & \text{for } j = 0, 1, \\ < 0 & \text{for } j = 2, 3, \end{cases} \quad \text{in case (C.2)}.$$

Consider first case (C.1), and let $\eta > 0$. Then we have

$$Z(a_j) \equiv 2 \bmod 4, \ \text{ for } j = 0, 3, \quad \text{and} \quad Z(a_j) \equiv 0 \bmod 4, \ \text{ for } j = 1, 2.$$

Since Z is an integer-valued increasing function, and taking into account the behavior at $\pm\infty$, we see that in this case necessarily

$$Z(a_0) = 2, \quad Z(a_1) = 4, \quad Z(a_2) = 4, \quad Z(a_3) = 6, \quad \lim_{x \to +\infty} Z(x) = 6.$$

This means that all the zeros of Q are real, two of them belong to $(-\infty, a_0)$, other two, to (a_0, a_1), and the last pair, to (a_2, a_3). By (40), Q cannot have simple zeros in $\mathbb{R} \setminus \Delta$. Thus, the zeros in $(-\infty, a_0)$ and (a_0, a_1) are double, and Q is of the form (36), where

$$\alpha_1 \in (-\infty, a_0] \quad \text{and} \quad \alpha_2 \in [a_0, a_1].$$

Analogously, when $\eta < 0$, we obtain that

$$Z(a_0) = 0, \quad Z(a_1) = Z(a_2) = 2, \quad Z(a_3) = 4, \quad \lim_{x \to +\infty} Z(x) = 6,$$

and Q is of the form (36) with

$$\alpha_1 \in [a_3, +\infty) \quad \text{and} \quad \alpha_2 \in [a_0, a_1].$$

Finally, in case (C.2) with $\eta > 0$ we have

$$Z(a_j) \equiv 2 \bmod 4, \ \text{ for } j = 0, 1, \quad \text{and} \quad Z(a_j) \equiv 0 \bmod 4, \ \text{ for } j = 2, 3,$$

so that

$$Z(a_0) = Z(a_1) = 2, \quad Z(a_2) = Z(a_3) = 4, \quad \lim_{x \to +\infty} Z(x) = 6.$$

Thus, Q is of the form (36) with

$$\alpha_1 \in (-\infty, a_0] \quad \text{and} \quad \alpha_2 \in [a_3, +\infty).$$

If one or more $\rho_j = 0$, then the corresponding α's coincide with a_j, and the conclusions above remain valid. ∎

ACKNOWLEDGMENTS

The research of the first author was conducted while visiting the University of South Florida. His work was partially supported by INTAS Project 2000-272, by a research grant from Ministerio de Ciencia y Tecnología (MCYT) of Spain, Project Code BFM2001-3878-C02-02, and by Junta de Andalucía, Grupo de Investigación FQM 0229. The second author was partially supported by the US National Science Foundation under Grant DMS-0100737.

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