Convergence of Padé Approximants to $e^{-z}$ on Unbounded Sets

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1. Introduction

The basic aims of this paper are to study the convergence in the uniform norm of particular Padé approximants to $e^{-z}$ on certain unbounded sets in the complex plane. After some preliminary results are developed in Section 2, we consider in Section 3 the convergence of Padé approximants to $e^{-z}$ on the ray $\{z = x + iy : x \geq 0, y = 0\}$. In Theorem 3.1, we give a necessary and sufficient condition for the uniform convergence of a sequence of Padé approximants to $e^{-z}$ on this set, while in Theorem 3.2, we give a sufficient condition for the geometric convergence of a sequence of Padé approximants to $e^{-z}$ on this set. Also, an application of these results to the problem of constrained Chebyshev rational approximations to $e^{-z}$ on $[0, +\infty)$ is included in this section.

In Section 4, the geometric convergence of the particular Padé approximants $\{R_{0,n}(z)\}_{n=0}^\infty$ to $e^{-z}$ on unbounded parabolic-like sets in the complex plane is derived in Theorem 4.1, while in Theorem 4.3, it is shown that the particular Padé approximants $\{R_{n-1,n}(z)\}_{n=1}^\infty$ and $\{R_{n-2,n}(z)\}_{n=2}^\infty$ converge uniformly to $e^{-z}$ on the sectors $S_\delta = \{z = re^{i\theta} : |\theta| \leq (\pi/2) - \delta\}$, for any $0 < \delta < (\pi/2)$.

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2. Notation and Preliminary Results

We shall make of the following notation. Let $\pi_m$ denote the set of all complex polynomials in the variable $z$ having degree at most $m$, and let $\pi_{\nu, n}$ denote the set of all complex rational functions $r_{\nu, n}(z)$ of the form

$$r_{\nu, n}(z) = \frac{q_{\nu, n}(z)}{p_{\nu, n}(z)}$$

where $q_{\nu, n} \in \pi_{\nu}$, $p_{\nu, n} \in \pi_n$, $p_{\nu, n}(0) = 1$.

Then, given any function $f(z) = \sum_{k=0}^{\infty} a_k z^k$ analytic in a neighborhood of $z = 0$, and given any nonnegative integers $\nu$ and $n$, the $(\nu, n)$-th Padé approximant to $f(z)$ is defined as that element $R_{\nu, n} \in \pi_{\nu, n}$ for which the following expression,

$$f(z) - R_{\nu, n}(z) = O(|z|^m) \text{ as } |z| \to 0,$$  \hspace{1cm} (2.1)

is valid with the largest integer $m$. In the case that $f(z) = e^{-z}$, the $(\nu, n)$-th Padé approximant $R_{\nu, n}(z) \equiv Q_{\nu, n}(z)/P_{\nu, n}(z)$ of $e^{-z}$ is explicitly given by (cf. [12, p. 433; 15, p. 269])

$$Q_{\nu, n}(z) \equiv \sum_{k=0}^{\nu} \frac{(\nu + n - k)! \nu! (-z)^k}{(\nu + n)! k! (\nu - k)!},$$  \hspace{1cm} (2.2)

and

$$P_{\nu, n}(z) \equiv \sum_{k=0}^{n} \frac{(\nu + n - k)! n! z^k}{(\nu + n)! k! (n - k)!},$$  \hspace{1cm} (2.3)

It is further known that (cf. [11; 12, p. 436; 14]), for finite $z$,

$$e_{\nu, n}(z) \equiv R_{\nu, n}(z) - e^{-z} = \frac{(-1)^{\nu} z^{n+\nu+1}}{(n + \nu)! e^z P_{\nu, n}(z)} \int_0^1 e^{t z} t^n (1 - t)^n \, dt.$$  \hspace{1cm} (2.4)

In particular, this expression shows that (2.1) is always valid with

$$m = n + \nu + 1,$$

when $f(z) = e^{-z}$. Moreover, as $P_{\nu, n}(x) \geq 1$ from (2.3) for all $x \geq 0$, it also follows from (2.4) that the error, $e_{\nu, n}(x)$, for the $(\nu, n)$-th Padé approximant to $e^{-x}$, is of one sign for all $x \geq 0$. It is convenient to define the numbers $\eta_{\nu, n}$ as

$$\eta_{\nu, n} = \sup\{|e_{\nu, n}(x)| : x \geq 0\} = \|R_{\nu, n} - e^{-x}\|_{L_\infty[0, \infty]}.$$  \hspace{1cm} (2.5)

We begin with

**Proposition 2.1.** $\eta_{\nu, n} = 1$ for all integers $\nu \geq 0$. 

Proof. \ First, assume $\nu = 2j$, $j \geq 0$. From (2.4), $\varepsilon_{2j, 2j}(x) \geq 0$ for all $x \geq 0$. Next, it is clear upon comparing coefficients in (2.2)-(2.3) that $R_{2j, 2j}(x) \leq 1$ for all $x \geq 0$, with $R_{2j, 2j}(x) \to 1$ as $x \to +\infty$. Hence,

$$0 \leq \varepsilon_{2j, 2j}(x) = R_{2j, 2j}(x) - e^{-x} \leq 1 - e^{-x} \leq 1 \text{ for all } x \geq 0,$$

with $\varepsilon_{2j, 2j}(x) \to 1$ as $x \to +\infty$. Thus it follows that $\eta_{2j, 2j} = 1$.

Assuming $\nu = 2j + 1$, $j \geq 0$, note that $Q_{2j+1, 2j+1}(x) = P_{2j+1, 2j+1}(-x)$ for any real $x$. Thus, we can write $R_{2j+1, 2j+1}(x) = P_{2j+1, 2j+1}(-x)/P_{2j+1, 2j+1}(x)$, or equivalently,

$$R_{2j+1, 2j+1}(x) = \frac{P_{2j+1, 2j+1}(x) + P_{2j+1, 2j+1}(-x)}{P_{2j+1, 2j+1}(x)} - 1 \text{ for all } x.$$

But from (2.3), $\{P_{2j+1, 2j+1}(x) + P_{2j+1, 2j+1}(-x)\} = \tilde{P}_{2j}(x)$, a polynomial of degree $2j$, is a positive sum of even powers of $x$ with constant term 2, so that trivially $\tilde{P}_{2j}(x) \geq 1$ for all $x \geq 0$. Next, by comparing coefficients, it is easy to verify from (2.3) that $P_{2j+1, 2j+1}(x) \leq e^x$ for all $x \geq 0$. Thus, from (2.4),

$$0 \leq -\varepsilon_{2j+1, 2j+1}(x) = e^{-x} - R_{2j+1, 2j+1}(x) = e^{-x} - \frac{\tilde{P}_{2j}(x)}{P_{2j+1, 2j+1}(x)} + 1$$

$$\leq e^{-x} - \frac{1}{e^x} + 1 = 1, \text{ for } x \geq 0,$$

with $-\varepsilon_{2j+1, 2j+1}(x) \to 1$ as $x \to \infty$. Thus, $\eta_{2j+1, 2j+1} = 1$. Q.E.D.

We now state an identity in (2.6), which can be obtained by directly appealing to the definitions of $Q_{\nu, n}$ and $P_{\nu, n}$ in (2.2) and (2.3).

**Lemma 2.2.** For any $\nu \geq 0$, $n \geq 1$,

$$\frac{d}{dx} [e^x Q_{\nu, n}(x) - P_{\nu, n}(x)] = \frac{n}{n+\nu} [e^x Q_{\nu, n-1}(x) - P_{\nu, n-1}(x)].$$

(2.6)

With these results and with the definition of $\eta_{\nu, n}$ in (2.5), we now prove

**Theorem 2.3.** For any nonnegative integers $\nu$ and $n$ with $n > \nu$,

$$\eta_{\nu, n} \leq \frac{n}{(2n + \nu)} \eta_{\nu, n-1}. \quad \text{(2.7)}$$

Thus,

$$\eta_{\nu, n} \leq \prod_{j=1}^{n-\nu} \left( \frac{\nu + j}{3\nu + 2j} \right) \leq \frac{1}{2^{n-\nu}}, \quad \text{for all } 0 \leq \nu < n. \quad \text{(2.8)}$$
Proof. Using (2.4), we see that \((-1)^\nu \varepsilon_{\nu,n}(x) \geq 0\) for all \(x \geq 0\). Because \(n\) exceeds \(\nu\) by hypothesis then \(\varepsilon_{\nu,n}(x) \to 0\) as \(x \to +\infty\). Hence, there exists a \(\xi > 0\) for which \((-1)^\nu \varepsilon_{\nu,n}(\xi) = \eta_{\nu,n}\), and \(\varepsilon_{\nu,n}(\xi) = 0\). Now, from (2.4), we can write

\[
(-1)^\nu \varepsilon_{\nu,n}(x) \cdot e^x \cdot P_{\nu,n}(x) = (-1)^\nu \{e^x Q_{\nu,n}(x) - P_{\nu,n}(x)\}.
\]

Thus, on differentiating the above expression and evaluating the result at \(x = \xi\), we obtain since \(\varepsilon_{\nu,n}(\xi) = 0\) that

\[
(-1)^\nu \varepsilon_{\nu,n}(\xi) e^\xi [P_{\nu,n}(\xi) + P'_{\nu,n}(\xi)] = (-1)^\nu \frac{d}{dx} [e^x Q_{\nu,n}(x) - P_{\nu,n}(x)]_{x=\xi}.
\]

Hence, from (2.6) of Lemma 2.2,

\[
(-1)^\nu \varepsilon_{\nu,n}(\xi) e^\xi [P_{\nu,n}(\xi) + P'_{\nu,n}(\xi)] = \left(\frac{-1}{n+n}\right)^\nu \frac{n}{n+n} [e^x Q_{\nu,n-1}(\xi) - P_{\nu,n-1}(\xi)]. \tag{2.9}
\]

Now, from (2.3), it is easy to verify that \(P_{\nu,n}(x) = (n/(n+n)) P_{\nu,n-1}(x)\) for all \(x\), and that \(P_{\nu,n}(x) \geq P_{\nu,n-1}(x)\) for all \(x \geq 0\). Thus,

\[
(-1)^\nu \varepsilon_{\nu,n}(\xi) e^\xi [P_{\nu,n}(\xi) + P'_{\nu,n}(\xi)] \geq \left(\frac{2n+n}{n+n}\right) (-1)^\nu \varepsilon_{\nu,n}(\xi) e^\xi P_{\nu,n-1}(\xi).
\]

Using (2.9), this implies that

\[
\left(\frac{-1}{n+n} \right)^\nu \frac{n}{n+n} [e^\xi Q_{\nu,n-1}(\xi) - P_{\nu,n-1}(\xi)] \geq \left(\frac{2n+n}{n+n}\right) (-1)^\nu \varepsilon_{\nu,n}(\xi) e^\xi P_{\nu,n-1}(\xi),
\]

or

\[
0 \leq (-1)^\nu \varepsilon_{\nu,n}(\xi) \leq \left(\frac{-1}{2n+n}\right)^\nu \frac{n}{n+n} [R_{\nu,n-1}(\xi) - e^{-\xi}] = \left(\frac{-1}{2n+n}\right)^\nu \varepsilon_{\nu,n-1}(\xi).
\]

Since \((-1)^\nu \varepsilon_{\nu,n}(\xi) = \eta_{\nu,n}\) and since \((-1)^\nu \varepsilon_{\nu,n-1}(\xi) \leq \eta_{\nu,n-1}\), then

\[
\eta_{\nu,n} \leq \left(\frac{n}{2n+n}\right) \eta_{\nu,n-1},
\]

the desired result of (2.7). By induction on the above inequality, it follows that

\[
\eta_{\nu,n} \leq \prod_{j=1}^{n-\nu} \left(\frac{\nu+j}{3\nu+2j}\right) \eta_{\nu,n} = \prod_{j=1}^{n-\nu} \left(\frac{\nu+j}{3\nu+2j}\right), \tag{2.10}
\]

the last expression following from Proposition 2.1. But as each term in the above product is at most \(\frac{1}{3}\), then we obtain the desired result of (2.8).

Q.E.D.
We remark that the inequality of (2.8) reduces in the case \( \nu = 0 \) to
\[
\eta_{0,n} \leq 1/2^n,
\]
which was first established in Cody, Meinardus, and Varga [2].
For the special case \( \nu = n - 1 \), we note that the inequality of (2.7), coupled with Proposition 2.1, gives simply
\[
\eta_{n-1,n} \leq \left( \frac{n}{3n-1} \right), \quad \text{for all} \quad n \geq 1,
\]
which implies only the boundedness of the sequence \( \{ \eta_{n-1,n} \}_{n=1}^\infty \). Actually, for our later use in Theorem 3.1, we need that \( \eta_{n-1,n} \) tends to zero as \( n \to \infty \), but we prove the following stronger result.

**Proposition 2.4.** There exist positive constants \( A_1 \) and \( A_2 \) such that
\[
\frac{A_1}{n} \leq \eta_{n-1,n} \leq \frac{A_2 \ln n}{n} \quad \text{for all} \quad n > 1. \tag{2.11}
\]

**Proof.** With definitions in (2.2) and (2.3), it is easy to show, by comparing coefficients, that
\[
| R_{n-1,n}(x) | = \left| \frac{Q_{n-1,n}(x)}{P_{n-1,n}(x)} \right| \leq \frac{Q_{n-1,n}(-x)}{P_{n-1,n}(x)} \leq \left( 1 + \frac{x}{2n-1} \right)^{-1}
\]
for all \( x \geq 0, n \geq 1 \). Thus, from (2.4),
\[
| \epsilon_{n-1,n}(x) | \leq | R_{n-1,n}(x) | + e^{-x} \leq e^{-x} + \left( 1 + \frac{x}{2n-1} \right)^{-1}, \quad x \geq 0. \tag{2.12}
\]
On the other hand, the integral representation in (2.4) gives us that
\[
| \epsilon_{n-1,n}(x) | = \frac{x^{2n}}{(2n-1)!} e^x P_{n-1,n}(x) \int_0^1 e^{tx} t^{n-1} (1-t)^n \, dt, \quad x \geq 0,
\]
which can be written in the form
\[
| \epsilon_{n-1,n}(x) | = \frac{x^{2n}}{(2n-1)!} P_{n-1,n}(x) \int_0^1 e^{-tx} t^{n} (1-t)^{n-1} \, dt, \quad x \geq 0.
\]
A simple calculation shows that the above integrand, considered as a function of \( t \in [0, 1] \) is maximized when \( t = u_n(x) \), where
\[
0 < u_n(x) \equiv \frac{2n}{(2n-1 + x) + ((2n-1 + x)^2 - 4nx)^{1/2}} < 1, \quad n > 1. \tag{2.13}
\]
Thus, $|\epsilon_{n-1, n}(x)|$ can be bounded above by

$$|\epsilon_{n-1, n}(x)| \leq \frac{x^{2n}(u_n(x))^n (1 - u_n(x))^{n-1}}{(2n - 1)! e^{x \cdot u_n(x)} P_{n-1, n}(x)}, \quad x \geq 0.$$ 

Next, it follows from (2.3) that $P_{n-1, n}(x) \geq ((n - 1)! x^n/(2n - 1)!)$ for all $x \geq 0$, so that

$$|\epsilon_{n-1, n}(x)| \leq \frac{[x \cdot u_n(x)]^n (1 - u_n(x))^{n-1}}{(n - 1)! e^{x \cdot u_n(x)}}, \quad x \geq 0,$$

and since $e^{x \cdot u_n(x)} \geq (x \cdot u_n(x))^n/n!$ and since $e^{-u_n(x)} \geq 1 - u_n(x) > 0$, then the above inequality implies that

$$|\epsilon_{n-1, n}(x)| \leq ne^{-(n-1) \cdot u_n(x)}, \quad x \geq 0. \quad (2.14)$$

Consequently, from (2.12) and (2.14),

$$|\epsilon_{n-1, n}(x)| \leq \min \left\{ ne^{-(n-1) \cdot u_n(x)}; e^{-x} + \left(1 + \frac{x}{2n - 1}\right)^{-1} \right\}, \quad x \geq 0. \quad (2.15)$$

Now, let $\alpha_n = n^2/(6 \ln n), n > 1$. For all $x \geq \alpha_n$, it is clear that

$$e^{-x} + \left(1 + \frac{x}{2n - 1}\right)^{-1} \leq e^{-\alpha_n} + \left(1 + \frac{\alpha_n}{2n - 1}\right)^{-1} \leq A \frac{\ln n}{n}, \quad x \geq \alpha_n, \quad (2.16)$$

for some positive constant $A$ independent of $n$. Next, using (2.13), for $0 \leq x \leq \alpha_n$,

$$u_n(x) \geq \frac{2n}{2(2n - 1) + x} \geq \frac{n}{2n - 1 + \alpha_n} \geq \frac{3 \ln n}{n - 1},$$

for all $n$ sufficiently large. Hence,

$$ne^{-(n-1) \cdot u_n(x)} \leq ne^{-3 \ln n} = \frac{1}{n^2} \quad \text{for} \quad 0 \leq x \leq \alpha_n. \quad (2.17)$$

Consequently, using (2.15)–(2.17) and the definition of $\eta_{e, n}$ in (2.5), then

$$\eta_{n-1, n} \leq A_2(\ln n)/n$$

for all $n > 1$.

To obtain the first inequality of (2.11), we first write $P_{n-1, n}(x)$ in the form

$$P_{n-1, n}(x) = \frac{n!}{(2n - 1)!} x^n \sum_{m=0}^{n} \frac{(n - 1 + m)!}{(n - m)! m! x^m}, \quad x \neq 0.$$
Because

\[
\frac{(n - 1 + m)!}{(n - m)! m!} = \frac{n(n^2 - 1) \cdots [n^2 - (m - 1)^2]}{m!} \leq \frac{n^{2m-1}}{m!}, \quad 0 \leq m \leq n,
\]

the above sum can be bounded above by

\[
P_{n-1,n}(x) \leq \frac{n! x^n}{n \cdot (2n - 1)!} \sum_{m=0}^{n} \frac{(n^2/x)^m}{m!} < \frac{n! x^n}{n(2n - 1)!} e^{n^2/x}, \quad x > 0.
\]

Now, let \( x = 2n^2 \). Since \( e^{1/2} < 2 \), this implies that

\[
P_{n-1,n}(2n^2) \leq \frac{n! 2^{2n+1}2^{n-1}}{(2n - 1)!}.
\] (2.18)

To obtain a similar lower bound for \( Q_{n-1,n}(x) \), we first write \( Q_{n-1,n}(x) \) in the form

\[
(-1)^{n-1} Q_{n-1,n}(x) = \frac{(n - 1)! x^{n-1}}{(2n - 1)!} \sum_{m=0}^{n-1} \frac{(n + m)!}{m! (n - 1 - m)!} x^m, \quad x \neq 0.
\]

For \( x = 2n^2 \), the above sum is an alternating sum with strictly decreasing terms, so that \((-1)^{n-1} Q_{n-1,n}(2n^2)\) exceeds the sum of the first two terms:

\[
(-1)^{n-1} Q_{n-1,n}(2n^2) > \frac{(n - 1)! (2n^2)^{n-1} \cdot (n^2 + 1)}{2n \cdot (2n - 1)!}.
\] (2.19)

Thus, from (2.18) and (2.19),

\[
\frac{(-1)^{n-1} Q_{n-1,n}(2n^2)}{P_{n-1,n}(2n^2)} > \frac{(n^2 + 1)}{8n^3},
\]

so that \((-1)^{n-1} \epsilon_{n-1,n}(2n^2) > (n^2 + 1)/8n^3 - (-1)^{n-1} e^{-2n^2}\). It is thus clear that

\[
(-1)^{n-1} \epsilon_{n-1,n}(2n^2) \geq \frac{A}{n},
\]

which implies the first inequality of (2.11).

Q.E.D.

3. The Convergence of Padé Approximants to \( e^{-x} \) on \([0, +\infty)\)

Based on the results of the previous section, we now establish the convergence of particular Padé approximants to \( e^{-x} \) on the infinite segment \([0, +\infty)\). Actually, we are interested in two kinds of convergence, namely, the uniform convergence and, more particularly, the geometric convergence.
of sequences of Padé approximants to $e^{-x}$ on $[0, +\infty)$. We first treat uniform convergence in

**Theorem 3.1.** The sequence $\{R_{\nu(n),n}\}_{n=1}^{\infty}$ of Padé approximants converges uniformly to $e^{-x}$ on $[0, \infty)$ if and only if $\nu(n) < n$ for all $n$ sufficiently large.

**Proof.** Assume first that $\nu(n) < n$ for all $n \geq n_0$. From (2.7) and (2.8), we have that

$$\eta_{\nu(n),n} \leq \eta_{\nu(n),\nu(n)+1}, \quad n \geq n_0, \tag{3.1}$$

and that

$$\eta_{\nu(n),n} \leq \frac{1}{2^{n-\nu(n)}}, \quad n \geq n_0. \tag{3.2}$$

How, given any $\epsilon > 0$, there is, from Proposition 2.4, an $n_1(\epsilon)$ such that $\eta_{n,n+1} < \epsilon$ for all $n > n_1(\epsilon)$. We may assume that $n_1(\epsilon) \geq n_0$. Next, choose $n_2(\epsilon) > n_1(\epsilon)$ such that $2^{-n_2(\epsilon)+n_1(\epsilon)} < \epsilon$. Consider then any $n \geq n_2(\epsilon)$. If $0 \leq \nu(n) \leq n_1(\epsilon)$, then using (3.2),

$$\eta_{\nu(n),n} \leq \frac{1}{2^{n-\nu(n)}} \leq \frac{1}{2^{n_2(\epsilon)-n_1(\epsilon)}} < \epsilon.$$

On the other hand, suppose that $n_1(\epsilon) < \nu(n) \leq n - 1$. With the inequality of (3.1) and the fact that $\eta_{n,n+1} < \epsilon$ for all $n > n_1(\epsilon)$, then $\eta_{\nu(n),n} < \epsilon$. Thus, for any $n \geq n_2(\epsilon)$ and for any $\nu(n)$ with $\nu(n) < n$, we have that $\eta_{\nu(n),n} < \epsilon$.

Conversely, assume that $\{\eta_{\nu(n),n}\}_{n=1}^{\infty}$ converges to zero as $n \to \infty$. Since $\eta_{\nu,n}$ is finite only if $\nu \leq n$, and since $\nu_{\nu,\nu} = 1$ for all $\nu \geq 0$ from Proposition 2.1, then evidently $\nu(n) < n$ for all $n$ sufficiently large. Q.E.D.

To establish a sufficient condition for the geometric convergence of certain Padé approximants to $e^{-x}$ on $[0, +\infty)$, we need only use (2.8) of Theorem 2.3 to prove

**Theorem 3.2.** If $\limsup_{n \to \infty} (\prod_{j=1}^{n-\nu(n)} (\nu(n) + j)/(3\nu(n) + 2j))^{1/n} = \alpha < 1$, then the sequence of Padé approximants $\{R_{\nu(n),n}(x)\}_{n=1}^{\infty}$ converges geometrically in the uniform norm to $e^{-x}$ on $[0, +\infty)$, i.e.,

$$\limsup_{n \to \infty} (\eta_{\nu(n),n})^{1/n} \leq \alpha < 1. \tag{3.3}$$

As a special case, if $\limsup_{n \to \infty} (\nu(n)/n) = \beta < 1$, then

$$\limsup_{n \to \infty} (\eta_{\nu(n),n})^{1/n} \leq \frac{1}{2^{1-\beta}} < 1. \tag{3.4}$$
While the result of Theorem 3.2 establishes a sufficient condition for the geometric convergence of the Padé approximants \( \{R_{x(n), n(x)}\}_{n=1}^{\infty} \) to \( e^{-x} \) on \([0, +\infty)\), it is not known whether this condition is also necessary. On the other hand, from the lower bound in Proposition 2.4, i.e.,

\[
\frac{A_1}{n} \leq \eta_{n-1, n}, \quad \text{for all } n \geq 1,
\]

it is clear that the particular Padé approximants \( \{R_{x(n), n(x)}\}_{n=1}^{\infty} \) with \( \nu(n) = n - 1 \), for which \( \limsup_{n \to \infty} (\nu(n)/n) = 1 \), cannot possess geometric convergence to \( e^{-x} \) on \([0, +\infty)\). More generally, it can be shown that no sequence of the form \( \{R_{\eta_{-\mu, \mu}, n} \}_{n=1}^{\infty}, \mu \geq 1 \) fixed, converges geometrically on this ray.

It is interesting to note that the result of Theorem 3.2 has applications to the problem of constrained Chebyshev rational approximations to \( e^{-x} \) on \([0, +\infty)\). We use the following notation. Let \( \hat{\mathcal{P}}_m \) be the set of all real polynomials of degree at most \( m \), let \( \hat{\mathcal{P}}_{v, n} \) be the set of all real rational functions \( r_{v, n}(x) \) of the form \( r_{v, n}(x) = q_{v, n}(x)/p_{v, n}(x) \), where \( q_{v, n} \in \hat{\mathcal{P}}_v \), and \( p_{v, n} \in \hat{\mathcal{P}}_n \), and \( p_{v, n}(0) = 1 \), and, for any nonnegative integer \( k \) with \( 0 \leq k \leq n + \nu + 1 \), let \( \hat{\mathcal{P}}_{v, n}^{(k)} \) be the subset of those \( r_{v, n} \) in \( \hat{\mathcal{P}}_{v, n} \) for which

\[
e^{-x} - r_{v, n}(x) = O(\lvert x \rvert^k), \quad x \text{ real}, \lvert x \rvert \to 0.
\]

Then, for any nonnegative integers \( n, \nu, \) and \( k \) with \( 0 \leq \nu \leq n \) and with \( 0 \leq k \leq n + \nu + 1 \), the constrained Chebyshev constants \( \lambda_{v, n}^{(k)} \) for \( e^{-x} \) on \([0, +\infty)\) are defined as

\[
\lambda_{v, n}^{(k)} = \inf \{ \sup_{0 \leq x < \infty} \lvert e^{-x} - r_{v, n}(x) \rvert : r_{v, n} \in \hat{\mathcal{P}}_{v, n}^{(k)} \}.
\]

(3.5)

For the special case \( k = 0 \), these (unconstrained) Chebyshev constants for \( e^{-x} \) have been studied in [2], Newman [8], and Schönhage [13]. Note that because the \((\nu, n)\)-the Padé approximant \( R_{v, n} \) is real, i.e., \( R_{v, n} \in \hat{\mathcal{P}}_{v, n} \), the special case \( k = n + \nu + 1 \) is, from (2.5) such that \( \lambda_{v, n}^{(n+\nu+1)} = \eta_{v, n} \).

Recently, J. D. Lawson [5] has considered the particular constrained Chebyshev constants \( \lambda_{n, n}^{(n+1)} \) for \( e^{-x} \) on \([0, +\infty)\), and, from his computed values of \( \lambda_{n, n}^{(n+1)} \) for \( 2 \leq n \leq 5 \), one would naturally suspect the geometric convergence of these constants to zero. That this is theoretically so can be seen to be a special case of

**Theorem 3.3.** Assume that the sequence of nonnegative integers

\[
\{k(n)\}_{n=0}^{\infty},
\]

satisfying \( 0 \leq k(n) \leq 2n + 1 \) for every \( n \geq 0 \), has the property that

\[
\limsup_{n \to \infty} \left( \frac{k(n) - (n + 1)}{n} \right) = \alpha < 1,
\]

(3.6)

and define \( \delta = \max(0, \alpha) \). Then, for any sequence of nonnegative integers \( \{m(n)\}_{n=0}^{\infty} \) satisfying \( \max\{0; k(n) - (n + 1)\} \leq m(n) \leq n \),

\[
\frac{1}{1280} \leq \liminf_{n \to \infty} \{\lambda_{m(n), n}^{(k(n))}\}^{1/n} \leq \limsup_{n \to \infty} \{\lambda_{m(n), n}^{(k(n))}\}^{1/n} \leq \frac{1}{2^{1-\delta}}.
\]

(3.7)

Proof. First, set \( \nu(n) = \max\{0; k(n) - (n + 1)\} \), so that \( 0 \leq \nu(n) \leq n \). From the integral representation in (2.4), the Padé approximant \( R_{\nu(n), n}(x) \) to \( e^{-x} \) evidently satisfies

\[
| R_{\nu(n), n}(x) - e^{-x} | = \mathcal{O}(|x|^{n+\nu(n)+1}), \quad |x| \to 0,
\]

for each \( n \geq 0 \), and hence, from the definition of \( \nu(n) \),

\[
| R_{\nu(n), n}(x) - e^{-x} | = \mathcal{O}(|x|^{k(n)}), \quad |x| \to 0,
\]

for each \( n \geq 0 \). Thus, \( R_{\nu(n), n} \in \hat{\mathcal{P}}^{(k(n))}_{\nu(n), n} \subset \hat{\mathcal{P}}^{(k(n))}_{m(n), n} \) for any integer \( m(n) \) with \( \nu(n) \leq m(n) \leq n \). Hence, by definition,

\[
\lambda_{m(n), n}^{(k(n))} \leq \eta_{\nu(n), n}, \quad \text{for each } n \geq 0.
\]

But using (3.4), we have that

\[
\{\lambda_{m(n), n}^{(k(n))}\}^{1/n} \leq \frac{1}{2^{1-(\nu(n)/n)}},
\]

so that applying the hypothesis of (3.6) establishes the last inequality of (3.7). On the other hand, Newman [8] has shown that for any polynomials \( p, q \in \hat{\mathcal{P}}_{n} \),

\[
\sup_{0 \leq x < \infty} \left| e^{-x} - \frac{p(x)}{q(x)} \right| > \frac{1}{(1280)^{n+1}},
\]

which establishes the first inequality of (3.7). Q.E.D.

We remark that a stronger result, analogous to (3.7), can be similarly established from the inequality (2.10).
4. THE CONVERGENCE OF PARTICULAR PADÉ APPROXIMANTS TO $e^{-z}$ ON UNBOUNDED REGIONS

In this section, we shall be concerned with the convergence, in the uniform norm, of particular Padé approximants to $e^{-z}$ on unbounded sets in the complex plane which are symmetric with respect to the positive ray $0 \leq x < \infty$. To begin, let $s_n(z) = \sum_{k=0}^{n} \frac{z^k}{k!}$ denote the familiar $n$-th partial sum of $e^z$. Then, it is clear from (2.2)–(2.3) that the $(0, n)$-th Padé approximant $R_{0,n}(z)$ of $e^{-z}$ is given by

$$R_{0,n}(z) = \frac{1}{s_n(z)}. \quad (4.1)$$

Thus, the poles of the Padé approximant $R_{0,n}$ are the zeros of $s_n$. It is further known that the parabolic region $T$ in the complex plane, defined by

$$T = \{ z = x + iy : x \geq 0 \quad \text{and} \quad |y| \leq dx^{1/2} \}, \quad (4.2)$$

where

$$d < 0.863\,369\,712, \quad (4.3)$$

contains no zeros of any $s_n$, i.e., $1/s_n$ is analytic in $T$ for all $n$ sufficiently large. That such a parabolic region with this property could exist was first indicated by the numerical results of Iverson [4], and the existence of this region was later established$^1$ by Newman and Rivlin [9].

The special case $\nu = 0$ of (2.8) of Theorem 2.3, coupled with (4.1), implies that

$$\left\| e^{-z} - \frac{1}{s_n(z)} \right\|_{L_{\infty}[0, \infty)} \leq \frac{1}{2^n}, \quad (4.4)$$

for all $n \geq 0$, and moreover, from Theorem 1 of Meinardus and Varga [6], we have that

$$\lim_{n \to \infty} \left( \left\| e^{-z} - \frac{1}{s_n(x)} \right\|_{L_{\infty}[0, \infty)} \right)^{1/n} = \frac{1}{2}. \quad (4.5)$$

It is natural to ask if the sequence $\{1/s_n\}_{n=1}^\infty$ converges geometrically to $e^{-z}$ on some larger set in the complex plane, especially when we know that $1/s_n$ is analytic in the parabolic region $T$ of (4.2), for all $n$. That this is so is

$^1$ Strictly speaking, the above-mentioned property of $T$, as stated in [9], does not follow completely from results of [9], but depends additionally on a subsequent note by Newman and Rivlin [10].
established in the following result. For added notation, if $S$ is any set in the complex plane and $f$ is defined on $S$, we write

$$\|f\|_{L_\infty(S)} = \sup\{|f(z)| : z \in S\}.$$  

**Theorem 4.1.** Let $g$ be a positive continuous function on $[0, +\infty)$ which satisfies

$$\lim_{x \to +\infty} \frac{g(x)}{(x)^{1/2}} = d^*,$$  

and let $G = \{z = x + iy : x \geq 0 \text{ and } |y| \leq g(x)\}$. If (cf. (4.2))

$$d^* < d\left(\frac{(2)^{1/2} - 1}{(2)^{1/2} + 1}\right), \quad \text{e.g.,} \quad d^* < 0.184 \ 130 \ 824,$$  

then the sequence $\{1/s_{n}\}_{n=1}^{\infty}$ converges geometrically to $e^{-z}$ on $G$. In particular, if $d^*$ of (4.6) is positive, then

$$\lim_{n \to \infty} \frac{\left(\|e^{-z} - \frac{1}{s_{n}}\|_{L_\infty(G)}\right)^{1/n}}{\left(\frac{d + d^*}{d - d^*}\right)^{2}} < 1,$$  

while if $d^* = 0$, then

$$\lim_{n \to \infty} \frac{\left(\|e^{-z} - \frac{1}{s_{n}}\|_{L_\infty(G)}\right)^{1/n}}{2} = \frac{1}{2}.$$  

**Proof.** By way of construction, it is possible from (4.7) to choose positive numbers $d_0$ and $d_1$ such that $d^* < d_0 < d_1 < d$ and such that $\frac{1}{2}((d_1 + d_0)/(d_1 - d_0))^2 < 1$. With these positive numbers, the sets $T_i$ are defined:

$$T_i = \{z = x + iy : x \geq 0 \quad \text{and} \quad |y| \leq d_i x^{1/2}\}, \quad i = 0, 1,$$

and hence $T_0 \subset T_1$. Next, it is clear from (4.6) that there is a finite $\sigma > 0$ such that the subset $G_\sigma = \{z = x + iy : z \in G \text{ and } x \geq \sigma\}$ of $G$ satisfies

$$G_\sigma \subset T_0.$$  

Next, since the zeros of the $s_n$'s have no finite limit point, i.e., if $\{z^{(n)}_j\}_{j=1}^{n}$ denote the zeros of $s_n$, then $\lim_{n \to \infty} \{|z^{(n)}_j|\} = +\infty$, then for all $n$ sufficiently large, say $n \geq n_0$, each $s_n$ is free of zeros in the sets $T_0, T_1, \text{ and } G$.

Continuing our construction, for each $t \geq 0$ and each $\beta > 0$, let $m(t, \beta)$ be the interval $[t - \beta t^{1/2}, t + \beta t^{1/2}]$ of the real axis. For $t \geq \beta^2$, $m(t, \beta)$ lies entirely on the nonnegative axis. Next, for each $\mu > 1$, let $m_\mu(t, \beta)$ denote the level curve of $m(t, \beta)$ in the complex plane, i.e., $m_\mu(t, \beta)$ is an ellipse given by

$$m_\mu(t, \beta) = \left\{z = x + iy : \frac{(x - t)^2}{a^2} + \frac{y^2}{b^2} = 1\right\}, \quad (4.10)$$
where
\[
a = a(t, \beta, \mu) = \frac{\beta t^{1/2}}{2} (\mu + \mu^{-1}), \quad \text{and} \quad b = b(t, \beta, \mu) = \frac{\beta t^{1/2}}{2} (\mu - \mu^{-1}).
\]  
(4.11)

For each \( t \geq \beta^2 \), we seek the largest value of \( \mu \geq 1 \) such that \( m_\mu(t, \beta) \subseteq T_1 \). This value of \( \mu \), which we call \( A_1 = A_1(t, \beta, d_1) \) is obtained when \( m_\mu(t, \beta) \) is tangent to the parabola \( y^2 = d_1 t x \) which defines \( T_1 \). In particular, as is readily shown, for \( \beta^2 \leq t \leq \beta^2 M \) where \( M = 1 + d_1^2/2\beta^2 \), \( A_1 \) is obtained by making \( m_\mu(t, \beta) \) tangent to \( T_1 \) at the origin, and \( A_1 \) is given in this case by
\[
A_1 = \frac{t^{1/2}}{\beta} + \left( \frac{t}{\beta^2} - 1 \right)^{1/2}, \quad \beta^2 \leq t \leq \beta^2 M.
\]  
(4.12)

For \( t > \beta^2 M \), \( m_\mu(t, \beta) \) will have exactly two points of intersection with \( T_1 \), i.e.,
\[
(x - t)^2/a^2 + d_1^2 x/b^2 = 1
\]
will have exactly one (nonnegative) root for \( x \), precisely when the discriminant of the above quadratic in \( x \), equals zero:
\[
\{2t - a^2 d_1^2/b^2\}^2 - 4(t^2 - a^2) = 0,
\]
or equivalently, solving for \( b^2 \),
\[
b^2 = \frac{d_1^2}{2} \left\{ t + (t^2 - a^2)^{1/2} \right\}.
\]

Thus, with (4.11), the largest value of \( \mu \geq 1 \), i.e., \( A_1 \), for which \( m_\mu(t, \beta) \subseteq T_1 \) satisfies
\[
\left( A_1 - \frac{1}{A_1} \right)^2 = 2 \left( \frac{d_1}{\beta} \right)^2 \left\{ 1 + \left( 1 - \frac{1}{4u^2} \left( A_1 + \frac{1}{A_1} \right)^2 \right)^{1/2} \right\}, \quad u^2 \equiv \frac{t}{\beta^2} > M,
\]  
(4.12')

which gives rise to a polynomial equation in \( A_1 \) of degree 6. It is apparent from (4.12) and (4.12') that \( A_1 = A_1(t, \beta, d_1) \) is in reality a function of \( u \equiv t^{1/2}/\beta \) and \( d_1/\beta \), and we also write \( A_1 = A_1(u, d_1/\beta) \). It is also clear from (4.12') that \( A_1 \) is a continuous strictly increasing function of \( u \). Next, to obtain an upper bound for \( A_1 \), one sees geometrically that forcing the ellipse \( m_\mu(t, \beta) \) to intersect the curve \( y = d_1 t^{1/2} \) in the particular point \((t, b)\) must give an upper bound for \( A_1 \). Thus, \( b = d_1 t^{1/2} = \beta t^{1/2}(\hat{a} - 1/\hat{a})/2 \) implies \( A_1 < \hat{a} \), or equivalently
\[
A_1 \left( u, \frac{d_1}{\beta} \right) < \left( \frac{d_1}{\beta} \right) + \left( 1 + \left( \frac{d_1}{\beta} \right)^2 \right)^{1/2} \quad \text{for all} \quad u \geq 1.
\]
Hence, \( A_1(u, d_1/\beta) \) is bounded, for fixed \( d_1/\beta \), as \( u \to +\infty \). Using this fact, it follows from (4.12') that the above upper bound is asymptotically sharp:

\[
\lim_{u \to +\infty} A_1 \left( u, \frac{d_1}{\beta} \right) = \alpha_1 \left( \frac{d_1}{\beta} \right) = \left( \frac{d_1}{\beta} \right) + \left(1 + \left( \frac{d_1}{\beta} \right)^2 \right)^{1/2}. \tag{4.13}
\]

Similarly, if \( A_0(t, \beta, d_0) = A_0(u, d_0/\beta) \) denotes the largest value of \( \mu \geq 1 \) such that \( m_\mu(t, \beta) \) is contained in \( T_0 \) for all \( t \geq \beta^2 \), the argument above directly gives

\[
\lim_{u \to +\infty} A_0 \left( u, \frac{d_0}{\beta} \right) = \alpha_0 \left( \frac{d_0}{\beta} \right) = \left( \frac{d_0}{\beta} \right) + \left(1 + \left( \frac{d_0}{\beta} \right)^2 \right)^{1/2}. \tag{4.13'}
\]

Next, it is straightforward to deduce from (4.13) and (4.13') that

\[
\lim_{\beta \to +\infty} \left[ \frac{\alpha_1(d_1/\beta)}{\alpha_1(d_1/\beta) - \alpha_0(d_0/\beta)} - 1 \right] = \frac{d_1 + d_0}{d_1 - d_0} < (2)^{1/2}, \tag{4.14}
\]

the last inequality following from our choice of \( d_0 \) and \( d_1 \).

Now, with the inequality of (4.4), we have

\[
\left| \frac{1}{s_{n+1}(x)} - \frac{1}{s_n(x)} \right| \leq \left| \frac{1}{s_{n+1}(x)} - e^{-x} \right| + \left| e^{-x} - \frac{1}{s_n(x)} \right| \\
\leq \frac{1}{2^{n+1}} + \frac{1}{2^n} = \frac{3}{2^{n+1}},
\]

for any \( x \geq 0 \) and any \( n \geq 0 \). In particular, for any \( t \geq \beta^2 \) (so that \( m(t, \beta) \) lies entirely on the nonnegative axis),

\[
\left| \frac{1}{s_{n+1}(x)} - \frac{1}{s_n(x)} \right| \leq \frac{3}{2^{n+1}}, \quad x \in m(t, \beta), \ t \geq \beta^2, \ n \geq 0.
\]

In addition, we know that the rational function \((1/s_{n+1} - 1/s_n) \in \pi_{n+1,2n+1}\) has, for any \( n \geq n_0 \) all its poles outside of \( T_1 \). Then, applying Walsh's Lemma (cf. [16; Eq. (41), p. 250]) to this rational function on the set \( m(t, \beta) \) yields

\[
\left| \frac{1}{s_{n+1}(z)} - \frac{1}{s_n(z)} \right| \leq \frac{3}{2^{n+1}} \left( \frac{A_1(t, \beta, d_1) A_0(t, \beta, d_0) - 1}{A_1(t, \beta, d_1) - A_0(t, \beta, d_0)} \right)^{2n+1},
\]

for all \( z \in \overline{m}_n(t, \beta), \ t \geq \beta^2, \ n \geq n_0 \), where \( \overline{m}_n(t, \beta) \) denotes all points \( z \) on or inside \( m_n(t, \beta) \) i.e.,

\[
\overline{m}_n(t, \beta) = \left\{ z = x + iy : \frac{(x - t)^2}{a^2} + \frac{y^2}{b^2} \leq 1 \right\}.
\]
Hence, given any $\epsilon > 0$ sufficiently small, so that
\[(d_1 + d_0)/(d_1 - d_0) + \epsilon < (2)^{1/2},\]
it follows from (4.13)-(4.14) that there is a $\tilde{\beta}$ and a $\tilde{u}$ sufficiently large so that
\[
\left| \frac{1}{s_{n+1}(z)} - \frac{1}{s_n(z)} \right| \leq \frac{3}{2^{n+1}} \left\{ \frac{d_1 + d_0}{d_1 - d_0} + \frac{\epsilon}{2^{n+1}} \right\},
\]
for all $n \geq n_0 + 1$, for all $z \in \overline{m}_{A_0}(t, \tilde{\beta})$, and for all $t \geq \tilde{\beta}^2 \tilde{u}^2$. Thus, since
\[
\left| \frac{1}{s_{n+r}(z)} - \frac{1}{s_n(z)} \right| \leq \sum_{j=0}^{r-1} \left| \frac{1}{s_{n+j+1}(z)} - \frac{1}{s_{n+j}(z)} \right|
\]
for any $r \geq 1$, then applying the inequality of (4.15) in the above sum and summing the resultant geometric series gives
\[
\left| \frac{1}{s_{n+r}(z)} - \frac{1}{s_n(z)} \right| \leq \frac{3\gamma^{2n+1}}{2^{n+1}} \left\{ \frac{2}{2 - \gamma^2} \right\}, \quad \gamma = \left[ \frac{d_1 + d_0}{d_1 - d_0} + \epsilon \right].
\]
Consequently, letting $r \to \infty$,
\[
\left| e^z - \frac{1}{s_n(z)} \right| \leq \frac{3\gamma^{2n+1}}{2^{n+1}} \left\{ \frac{2}{2 - \gamma^2} \right\}, \quad z \in \overline{m}_{A_0}(t, \beta), \ t \geq \beta^2 u^2, \ n \geq n_0.
\]
Now, by construction, the closed ellipses $\overline{m}_{A_0}(t, \beta)$ trace out the set $T_0$, i.e., for every $\beta > 0$,
\[
\bigcup_{t \geq \beta^2 u^2} \{ \overline{m}_{A_0}(t, \beta) \} = T_0.
\]
Hence, the set $\bigcup_{t \geq \beta^2 u^2} \{ \overline{m}_{A_0}(t, \beta) \}$ can be expressed as $T_0 - C$, where $C = C(\epsilon)$ is some compact set in the complex plane. Thus, (4.16) can be equivalently expressed as
\[
\left\| e^z - \frac{1}{s_n} \right\|_{L^\infty(T_0 - C)} \leq \frac{3\gamma^{2n+1}}{2^{n+1}} \left\{ \frac{2}{2 - \gamma^2} \right\}, \quad n \geq n_0.
\]
Recalling that the set $G$ of Theorem 4.1 is a subset of $T_0 - C$ with the exception of some compact set $C'$, this implies that
\[
\limsup_{n \to \infty} \left\{ \left\| e^z - \frac{1}{s_n} \right\|_{L^\infty(G - C')} \right\}^{1/n} \leq \frac{1}{2} \left[ \frac{d_1 + d_0}{d_1 - d_0} + \epsilon \right]^2.
\]
On the other hand, for any compact set $C$,
\[
\lim_{n \to \infty} \left\{ \left\| e^z - \frac{1}{s_n} \right\|_{L^\infty(C)} \right\}^{1/n} = 0.
\]
To see this, define $0 < \delta \equiv \inf\{|e^{z} : z \in C\}$, and $\rho \equiv \sup\{|z : z \in C\}$. Because of the uniform convergence of $s_n$ to $e^z$ on $C$, then $\delta/2 \leq |s_n(z)|$ for all $z \in C$, all $n \geq n_1$. Thus, for $n \geq \max\{\rho - 2, n_1\}$,

$$
\left| e^{-z} - \frac{1}{s_n(z)} \right| = \frac{|s_n(z) - e^z|}{|e^{z} \cdot s_n(z)|} \leq \frac{2}{\delta^2} |s_n(z) - e| = \frac{2}{\delta^2} \sum_{k=n+1}^{\infty} z^k/k!
$$

$$
\leq \frac{2}{\delta^2} \sum_{k=n+1}^{\infty} \rho^k/k! \leq \frac{2(n + 2) \rho^{n+1}}{\delta^2(n + 1)!} (n + 2 - \rho),
$$

for all $z \in C$. Thus, using Stirling’s formula, (4.18) follows. Hence, combining (4.17) and (4.18), we deduce that

$$
\limsup_{n \to \infty} \left\{ \left\| e^{-z} - \frac{1}{s_n} \right\|_{L_\infty(G)} \right\}^{1/n} \leq \frac{1}{2} \left[ \frac{d_1 + d_0}{d_1 - d_0} + \epsilon \right]^2.
$$

(4.19)

Thus, letting both $\epsilon \to 0$ in (4.19) yields

$$
\limsup_{n \to \infty} \left\{ \left\| e^{-z} - \frac{1}{s_n} \right\|_{L_\infty(G)} \right\}^{1/n} \leq \frac{1}{2} \left( \frac{d_1 + d_0}{d_1 - d_0} \right)^2.
$$

Finally letting $d_1 \to d$ and $d_0 \to d^*$ in the above expression then establishes

$$
\limsup_{n \to \infty} \left\{ \left\| e^{-z} - \frac{1}{s_n} \right\|_{L_\infty(G)} \right\}^{1/n} \leq \frac{1}{2} \left( \frac{d + d^*}{d - d^*} \right)^2 < 1,
$$

the desired result of (4.8). Of course, if $d^* = 0$, then

$$
\limsup_{n \to \infty} \left\{ \left\| e^{-z} - \frac{1}{s_n} \right\|_{L_\infty(G)} \right\}^{1/n} \leq \frac{1}{2}.
$$

But as $[0, +\infty)$ is a subset of $G$, it follows from (4.5) and the above inequality, that

$$
\frac{1}{2} \leq \limsup_{n \to \infty} \left\{ \left\| e^{-z} - \frac{1}{s_n} \right\|_{L_\infty(G)} \right\}^{1/n} \leq \limsup_{n \to \infty} \left\{ \left\| e^{-z} - \frac{1}{s_n} \right\|_{L_\infty(G)} \right\}^{1/n} \leq \frac{1}{2},
$$

whence $\liminf_{n \to \infty} \left\{ \left\| e^{-z} - 1/s_n \right\|_{L_\infty(G)} \right\}^{1/n} = \frac{1}{2}$, the desired result of (4.9).

Q.E.D.

As a special case of Theorem 4.1, we have

**Corollary 4.2.** For any semi-infinite strip

$$
I_\tau \equiv \{z = x + iy : x \geq 0, |y| \leq \tau\},
$$
where \( 0 \leq \tau < \infty \),
\[
\lim_{n \to \infty} \left( \left\| e^{-z} - \frac{1}{S_n \cdot t_n(t_\tau)} \right\|^{1/n} \right) = \frac{1}{2}.
\]

It is again natural to ask if the geometric convergence of (4.8)–(4.9) of Theorem 4.1 holds for similar unbounded domains in the complex plane, for other Padé approximations of \( e^{-z} \). Such a result, which would extend Theorem 3.2 to larger sets in the complex plane, of course depends on a precise knowledge of the location of the poles of other Padé approximations of \( e^{-z} \), which seems not to be known in the general case. On the other hand, the uniform convergence of Padé approximants to \( e^{-z} \) on \([0, \pm \infty)\) of Theorem 3.1 can be similarly extended to larger sets in the complex plane for particular Padé approximations, as we now show.

**Theorem 4.3.** Given any \( \delta \) with \( 0 < \delta \leq \pi/2 \), the sequences
\[
\{ R_{n-1,n}(z) \}_{n=1}^\infty \quad \text{and} \quad \{ R_{n-2,n}(z) \}_{n=2}^\infty
\]
converge uniformly to \( e^{-z} \) on the sector \( S_\delta \equiv \{ z = re^{i\theta} : |\theta| \leq \pi/2 - \delta \} \).

**Proof.** It was originally shown by Birkhoff and Varga [1] that all the Padé approximants \( R_{n,n}(z) \) of \( e^{-z} \) are analytic in the right-half plane \( \Re z > 0 \), and are bounded in modulus there by unity. More recently, Ehle [3] has extended both of these results to \( \{ R_{n-1,n}(z) \}_{n=1}^\infty \) and \( \{ R_{n-2,n}(z) \}_{n=2}^\infty \). Dealing for definiteness with \( \{ R_{n-1,n}(z) \}_{n=1}^\infty \), we thus have that each
\[
f_n(z) = e^{-z} - R_{n-1,n}(z)
\]
is analytic in the open first quadrant \( S \equiv \{ z = x + iy : x > 0 \ \text{and} \ y > 0 \} \), and that sup\(|f_n(z)| : z \in S\) \( \leq 2 \), for all \( n \geq 1 \). Since the boundary of \( S \) consists of the rays \( \gamma_1 \equiv \{ z = x + iy : x \geq 0, y = 0 \} \) and
\[
\gamma_2 \equiv \{ z = x + iy : x = 0, y \geq 0 \},
\]
the harmonic measure \( w(z) \) of \( \gamma_1 \) with respect to \( S \), defined as a function which is harmonic and bounded in \( S \) and for which \( w(z) = 1 \) for all \( z \in \text{int} \ \gamma_1 \) and \( w(z) = 0 \) for all \( z \in \text{int} \ \gamma_2 \), is obviously given by
\[
w(z) = 1 - \frac{2}{\pi} \arg z. \quad (4.20)
\]

Then, by the Nevanlinna Two-Constants Theorem (cf. [7, p. 41]), if
\[
M_i = \sup\{|f_n(z)| : z \in \text{int} \ \gamma_i\}, \quad i = 1, 2,
\]
then
\[
|f_n(z)| \leq M_1^{w(z)} \cdot M_2^{1-w(z)}, \quad \text{for all} \ z \in S. \quad (4.21)
\]
Strictly speaking the Two-Constants Theorem is stated for bounded domains. Therefore, the validity of (4.21) follows by considering an appropriate conformal mapping of \( S \).

Now since \( M_1 = \eta_{n-1,n} \) (cf. (2.5)), and \( M_2 \leq 2 \), it follows from (4.20) and (4.21) that
\[
|f_n(z)| \leq \eta_{n-1,n}^{1-(2/n)} \arg z \cdot 2^{(2/n)} \arg z, \quad \text{for all} \quad z \in S.
\]

Thus, as \( \arg(z) < \pi/2 \) in \( S \),
\[
|f_n(z)| \leq 2\eta_{n-1,n}^{1-(2/n)} \arg(z), \quad \text{all} \quad z \in S.
\]

Now, from Proposition 2.4, there exists an \( n_0 > 0 \) such that \( \eta_{n-1,n} < 1 \) for all \( n \geq n_0 \). Thus, restricting \( z \) to be in the sector \( S_\delta^+ = \{ z = re^{i\theta} : 0 \leq \theta \leq \pi/2 - \delta \} \) where \( 0 < \delta \leq \pi/2 \), then
\[
|f_n(z)| \leq 2 \cdot \eta_{n-1,n}^{2/n}, \quad \text{all} \quad z \in S_\delta^+,
\]

and, as the same result evidently holds for the reflected sector \( S_\delta^- = \{ z = re^{i\theta} : -(\pi/2 - \delta) \leq \theta \leq 0 \} \), we have
\[
\| e^{-z} - R_{n-1,n} \|_{L_\infty(S_\delta)} \leq 2\eta_{n-1,n}^{2/n}.
\]

Thus, since \( \eta_{n-1,n} \to 0 \) as \( n \to \infty \) from Proposition 2.4, then \( \{ R_{n-1,n}(z) \}_{n=1}^{\infty} \) converges uniformly to \( e^{-z} \) on \( S_\delta \), the same conclusion being true also for \( \{ R_{n-2,n}(z) \}_{n=2}^{\infty} \).

Q.E.D.

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