On Hankel Operators Associated with Markov Functions

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Abstract

In this paper some questions related to Hankel operators associated with Markov functions are considered. Let G be a bounded multiply connected domain with a boundary Γ consisting of closed analytic Jordan curves. We assume that G is symmetric with respect to the real axis. Let μ be a positive Borel measure with the support $\operatorname{supp}\mu = E \subset \mathbf{R}, E \subset G$. We investigate a connection between the Hankel operator A_f constructed from the Markov function

$$f(z) = \frac{1}{2\pi i}\int_E \frac{d\mu(x)}{z-x}$$

and the embedding operator $J : E_2(G) \to L_2(\mu, E)$, where $E_2(G)$ is the Smirnov class of functions analytic on G. Moreover, in the case when G is the open unit disk we state results characterizing the rate of decrease of the sequence of singular numbers of the Hankel operator A_f constructed from the Markov function with the measure μ satisfying the Szegő condition: $\operatorname{supp}\mu = [a,b] \subset (-1,1)$ and

$$\int_a^b \frac{\log(d\mu/dx)}{\sqrt{(x-a)(b-x)}}\,dx > -\infty.$$

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1 Hankel and Embedding Operators.

1.1 Notation and Statement of Theorem 1

Let $G \subseteq \mathbf{C}$ be a bounded domain symmetric with respect to the real axis whose boundary Γ consists of finitely many closed analytic Jordan curves. We assume that Γ is positively oriented with respect to G. Let $L_p(\Gamma), 1 \leq p < \infty$, be the Lebesgue space of functions φ on Γ such that

$$||\varphi||_p = \left(\int_\Gamma |\varphi(\xi)|^p |d\xi| \right)^{1/p} < \infty.$$

Denote by $L_\infty(\Gamma)$ the space of essentially bounded functions φ,

$$||\varphi||_\infty = \operatorname*{ess\,sup}_\Gamma |\varphi(\xi)| < \infty.$$

Let $E_p(G), 1 \leq p \leq \infty$, be the Smirnov class of analytic functions on G. Here and in what follows we consider $E_p(G)$ as a subspace of the space $L_p(\Gamma)$. The condition

$$\int_\Gamma \frac{\varphi(\xi)\, d\xi}{\xi - z} = 0 \qquad \text{for all} \quad z \in \overline{\mathbf{C}} \setminus \overline{G}$$

is necessary and sufficient for a function $\varphi \in L_1(\Gamma)$ to be the boundary value of a function in the Smirnov class $E_1(G)$ (see [16] and [19] for more details about the classes $E_p(G)$).

We represent $L_2(\Gamma)$ as the direct sum $L_2(\Gamma) = E_2(G) \oplus E_2^\perp(G)$, where $E_2^\perp(G)$ is the orthogonal complement of $E_2(G)$ in $L_2(\Gamma)$. Denote by $\mathbf{P}_-$ the orthogonal projection from $L_2(\Gamma)$ onto $E_2^\perp(G)$. Let f be continuous on Γ. The Hankel operator A_f with symbol f is the map from $E_2(G)$ to $E_2^\perp(G)$ which takes $\varphi \in E_2(G)$ to $A_f\varphi = \mathbf{P}_-(\varphi f)$. Note that A_f is a compact operator.

Let μ be a positive Borel measure with the support $\operatorname{supp}\mu = E \subset \mathbf{R}, E \subset G$. We assume that the support of μ contains infinitely many points. Denote by $L_2(\mu, E)$ the Hilbert space with the inner product

$$(\varphi, \psi)_{2,\mu} = \int_E (\varphi\overline{\psi})(x) d\mu(x), \quad \varphi, \psi \in L_2(\mu, E),$$

and the norm $||\varphi||_{2,\mu}$.

For the Markov function

$$f(z) = \frac{1}{2\pi i} \int_E \frac{d\mu(x)}{z - x} \tag{1.1}$$

there is a connection between the singular numbers $s_n(A_f)$ of the Hankel operator A_f and the singular numbers of the *embedding operator* $J : E_2(G) \to L_2(\mu, E)$,

$$J\varphi = \varphi|_E, \qquad \varphi \in E_2(G).$$

The compact operator J is given by restricting an element $\varphi \in E_2(G)$ to E.

Let X and Y be the Hilbert spaces, and let A be a compact linear operator from X to Y. For any nonnegative integer n denote by $s_n(A)$ the *n-th singular number* of the operator A:

$$s_n(A) = \inf_K ||A - K||,$$

where the infimum is taken over the collection of all linear operators $K : X \to Y$ of rank at most n, and $|| \cdot ||$ is the norm of the corresponding linear operator. It can be seen that the sequence $\{s_n(A)\}$, $n = 0, 1, 2, \dots,$ coincides with the sequence of eigenvalues (counting multiplicity) of the operator $(A^*A)^{1/2}$, where $A^* : Y \to X$ is the adjoint of A, and

$$s_n(A) = \inf_{X_{-n}} ||A|_{X^n}||,$$

where X_{-n} runs over all possible subspaces of codimension n of X (see [12] for more details about singular numbers).

We now formulate the main result of this section establishing a connection between the operators $A_f^* A_f$ and $(J^*J)^2$.

Theorem 1 *Let G be a domain symmetric with respect to real axis and let f be the Markov function of* (1.1), *where μ is a positive Borel measure with the support* $\operatorname{supp} \mu = E \subset \mathbf{R}$, $E \subset G$, *containing infinitely many points. Then*

$$A_f^* A_f = (J^*J)^2. \tag{1.2}$$

Consequently,

$$s_n(J)^2 = s_n(A_f), \qquad n = 0, 1, 2, \dots. \tag{1.3}$$

Remark. In the case when G is the open unit disk $\{z : |z| < 1\}$ and $E \subset (-1, 1)$ it is not hard to prove (see [4]) that for $|z| < 1$

$$(A_f^* A_f)(\varphi)(z) = (J^*J)^2(\varphi)(z) = \frac{1}{4\pi^2} \int_E \int_E \frac{\varphi(x)}{(1 - yx)(1 - yz)} d\mu(x) d\mu(y).$$

1.2 Some Properties of A_f

In this subsection we investigate some properties of Hankel operators. We first state a result giving necessary and sufficient conditions for a function a belonging to the space $L_2(\Gamma)$ to be an element of the subspace $E_2^{\perp}(G)$ (see [17]).

Suppose that $a \in L_2(\Gamma)$. Then $a \in E_2^{\perp}(G)$ if and only if there exists a function $b \in E_2(G)$ such that

$$\overline{a}(\xi)\,|d\xi| = b(\xi)\,d\xi \quad \textit{a.e. on} \quad \Gamma. \tag{1.4}$$

Let $\varphi \in E_2(G)$. Since $A_f\varphi = \mathbf{P}_-(\varphi f)$,

$$A_f\varphi = \varphi f - \psi = u, \tag{1.5}$$

where $u \in E_2^{\perp}(G)$ and $\psi \in E_2(G)$. On the basis of the fact that $A_f^* u = \mathbf{P}_+(u\overline{f})$, where $\mathbf{P}_+$ is the orthogonal projection of $L_2(\Gamma)$ onto $E_2(G)$, we obtain that

$$A_f^* u = u\overline{f} - v = \omega, \tag{1.6}$$

where $\omega \in E_2(G)$ and $v \in E_2^{\perp}(G)$. Since $u, v \in E_2^{\perp}(G)$, we can assert (see (1.4)) that there exist $\alpha, \beta \in E_2(G)$ such that

$$\overline{u}(\xi)|d\xi| = \alpha(\xi)d\xi \quad \text{and} \quad \overline{v}(\xi)|d\xi| = \beta(\xi)d\xi$$

a.e. on Γ. Therefore, by (1.5) and (1.6), we get

$$(\varphi f - \psi)(\xi)d\xi = \overline{\alpha(\xi)}\;|d\xi|, \text{ and } (\alpha f - \beta)(\xi)d\xi = \overline{\omega(\xi)}\;|d\xi|$$

a.e. on Γ, and so for the Markov function (1.1) we have

$$\left(\varphi(\xi)\frac{1}{2\pi i}\int_E \frac{d\mu(x)}{\xi - x} - \psi(\xi)\right)\; d\xi = \overline{\alpha(\xi)}\;|d\xi| \quad \text{a.e. on} \quad \Gamma, \tag{1.7}$$

$$\left(\alpha(\xi)\frac{1}{2\pi i}\int_E \frac{d\mu(x)}{\xi - x} - \beta(\xi)\right)\; d\xi = \overline{\omega(\xi)}\;|d\xi| \quad \text{a.e. on} \quad \Gamma. \tag{1.8}$$

Remark 1 It follows immediately from (1.4) and (1.5) that α and ψ are uniquely determined by equation (1.7). Namely, if for $\tilde{\psi}, \tilde{\alpha} \in E_2(G)$ we have

$$\left(\varphi(\xi)\frac{1}{2\pi i}\int_E \frac{d\mu(x)}{\xi - x} - \tilde{\psi}(\xi)\right) d\xi = \overline{\tilde{\alpha}(\xi)}|d\xi|, \quad \text{a.e. on} \quad \Gamma,$$

then $\tilde{\psi} = \psi$ and $\tilde{\alpha} = \alpha$. Analogously, β and ω are uniquely determined by (1.8).

1.3 The Embedding Operator J

Let $J : E_2(G) \to L_2(\mu, E)$ be the embedding operator $J\varphi = \varphi|_E, \varphi \in E_2(G)$. By the Cauchy formula, for any function $\varphi \in E_2(G)$ we have

$$(J\varphi)(x) = \frac{1}{2\pi i}\int_\Gamma \frac{\varphi(\xi)d\xi}{\xi - x}, \quad x \in E.$$

The adjoint operator $J^* : L_2(\mu, E) \to E_2(G)$ of J can be represented in the form

$$(J^*g)(\xi) = \mathbf{P}_+(\Phi), \qquad g \in L_2(\mu, E), \tag{1.9}$$

where

$$\Phi(\xi) = \left(\frac{1}{2\pi i}\int_E \frac{g(x)d\mu(x)}{x - \overline{\xi}}\right)\frac{|d\xi|}{d\xi}.$$

Let $\varphi \in E_2(G)$. Equality (1.9) implies that

$$(J^*J)\varphi(\xi) = (\Phi - \Psi)(\xi) = \alpha_1(\xi), \quad \text{a.e. on} \quad \Gamma,$$

where $\alpha_1 \in E_2(G)$ and $\Psi \in E_2^\perp(G)$. By (1.4), there exists a function $\psi_1 \in E_2(G)$ such that $\overline{\Psi}(\xi)|d\xi| = \psi_1(\xi)d\xi$ a.e. on Γ. Therefore, we get

$$\left(\frac{1}{2\pi i}\int_E \frac{\overline{\varphi(x)}d\mu(x)}{\xi - x} - \psi_1(\xi)\right)d\xi = \overline{\alpha_1(\xi)}|d\xi| \quad \text{a.e. on} \quad \Gamma. \tag{1.10}$$

Analogously, using the formula $(J^*J)\alpha_1(\xi) = \omega_1(\xi)$, where $\omega_1 \in E_2(G)$, we obtain that there exists $\beta_1 \in E_2(G)$ such that

$$\left(\frac{1}{2\pi i}\int_E \frac{\overline{\alpha_1(x)}d\mu(x)}{\xi - x} - \beta_1(\xi)\right)d\xi = \overline{\omega_1(\xi)}|d\xi| \quad \text{a.e. on} \quad \Gamma. \tag{1.11}$$

1.4 Proof of Theorem 1

We shall prove that $(J^*J)^2\varphi = (A_f^*A)\varphi$ for every $\varphi \in E_2(G)$. Let us consider equation (1.10). Replacing ξ by $\overline{\xi}$ and then taking the conjugate, we get

$$\left(-\frac{1}{2\pi i}\int_E \frac{\varphi(x)d\mu(x)}{\xi - x} - \psi_1^*(\xi)\right)(-d\xi) = \overline{\alpha_1^*(\xi)}|d\xi| \quad \text{a.e. on} \quad \Gamma,$$

where $\psi^*(\xi) = \overline{\psi_1(\overline{\xi})} \in E_2(G)$ and $\alpha_1^*(\xi) = \overline{\alpha_1(\overline{\xi})} \in E_2(G)$. Therefore,

$$\left(\frac{1}{2\pi i}\int_E \frac{\varphi(x)d\mu(x)}{\xi - x} + \psi_1^*(\xi)\right)d\xi = \overline{\alpha_1^*(\xi)}|d\xi| \quad \text{a.e. on} \quad \Gamma. \tag{1.12}$$

We now note that for $\varphi \in E_2(G)$ the function

$$\int_E \frac{\varphi(x)d\mu(x)}{\xi - x} - \varphi(\xi)\int_E \frac{d\mu(x)}{\xi - x}$$

belongs to $E_2(G)$. Then, from (1.7) and (1.12) we can conclude with the help of Remark 1 that $\alpha_1^* = \alpha$. Using (1.11) and the fact $\overline{\alpha_1(x)} = \alpha_1^*(x)$ for $x \in E$, we obtain

$$\left(\frac{1}{2\pi i}\int_E \frac{\alpha(x)d\mu(x)}{\xi - x} - \beta_1(\xi)\right) d\xi = \overline{\omega_1(\xi)}|d\xi| \quad \text{a.e. on} \quad \Gamma.$$

From this, on account of (1.8) and Remark 1, we get $\omega_1 = \omega$. Consequently, for any $\varphi \in E_2(G)$ we have $(J^*J)^2\varphi = (A_f^*A_f)\varphi$, which implies (1.2) and (1.3).

□

1.5 Connection between the Best Meromorphic Approximation Problem, the Theory of Hankel Operators, and n-widths

There is a connection between the singular numbers $s_n(A_f)$ of the Hankel operator A_f with symbol $f \in C(\Gamma)$ and the error

$$\Delta_{n,\infty} = \Delta_{n,\infty}(f;G) = \inf_{h\in\mathcal{M}_{n,\infty}(G)} ||f - h||_\infty$$

in best meromorphic approximation of f in $L_\infty(\Gamma)$ in the class

$$\mathcal{M}_{n,\infty}(G) = \{h = \alpha/\beta : \alpha \in E_\infty(G),\ \beta \text{ is a polynomial, } \deg\beta \le n, \beta \not\equiv 0\}.$$

This connection was first investigated by Adamyan, Arov, and Kreĭn (see [2], [3]) and is known as the AAK method. The Adamyan-Arov-Kreĭn theorem relates to the situation when the Hankel operator is constructed from a function given on the boundary Γ of the unit disk G.

Theorem (Adamyan, Arov, Kreĭn) *Let f be continuous on the unit circle $\Gamma = \{z : |z| = 1\}$. Then*

$$\Delta_{n,\infty}(f;G) = s_n(A_f), \quad n = 0, 1, 2, \ldots,$$

and a unique best approximant $h_n \in \mathcal{M}_{n,\infty}(G)$ satisfying

$$\Delta_{n,\infty}(f;G) = ||f - h_n||_\infty$$

is given by $h_n = \mathbf{P}_+(Q_n f)/Q_n$, *where* Q_n *is the first element of some* $(n+1)$*-st Schmidt pair of* $A_f, ||Q_n||_2 = 1$.

As shown in [17] we have the following connection between the quantities $\Delta_n(f;G)$ and the singular numbers $s_n(A_f)$ in the case when G is a N-connected domain.

Let G *be a* N*-connected domain with a boundary* Γ *consisting of closed analytic Jordan curves and let* f *be continuous on* Γ*. Then, for all integers* $n \geq N-1$,

$$\Delta_{n+N-1}(f;G) \leq s_n(A_f) \leq \Delta_n(f;G). \tag{1.13}$$

Let G be a bounded domain whose boundary Γ consists of N closed analytic Jordan curves. In this case the Smirnov class $E_p(G), 1 \leq p \leq \infty$, coincides with the Hardy space $H_p(G)$ of analytic functions on G (see [11], [13], [20] for more details about the Hardy spaces). Let μ be a positive Borel measure with support $\operatorname{supp}\mu = E \subset G$. There is a link between the singular numbers $s_n(J)$ of the embedding operator J and n-widths of the unit ball of the Hardy space $H_2(G)$ in $L_2(\mu, E)$ as we now describe.

Let X be a Banach space and $\mathbf{A}$ be a convex, compact, centrally symmetric subset of X. The *Kolmogorov* n*-width* of $\mathbf{A}$ in X is given by

$$d_n(\mathbf{A}, X) := \inf_{X_n} \sup_{\varphi \in \mathbf{A}} \inf_{g \in X_n} ||\varphi - g||,$$

where the infimum is taken over all n dimensional subspaces X_n of X. The *Gel'fand* n*-width* of $\mathbf{A}$ in X is defined as follows:

$$d^n(\mathbf{A}, X) := \inf_{X_{-n}} \sup_{x \in X_{-n} \cap \mathbf{A}} ||x||,$$

where X_{-n} is an arbitrary subspace of X of codimension n. The *linear* n*-width* of $\mathbf{A}$ in X is given by

$$\delta_n(\mathbf{A}, X) := \inf_{K} \sup_{\varphi \in \mathbf{A}} ||\varphi - K\varphi||,$$

where $K : X \to X$ varies over all linear operators of rank n (see [15] for more details about n-widths).

Let A_2 be the restriction to E of the closed unit ball of $H_2(G)$. We have (see, for example, [10], [15])

$$s_n(J) = d_n(A_2, L_2(\mu, E)) = d^n(A_2, L_2(\mu, E)) \leq \delta_n(A_2, L_2(\mu, E)).$$

It is proved by S.D. Fisher [10] that there is a constant $C > 0$ depending on G and E but not n or μ such that

$$\frac{1}{C}\gamma_n \le d_n(A_2, L_2(\mu, E)) \le \delta_n(A_2, L_2(\mu, E)) \le C\gamma_n,$$

where

$$\gamma_n = \inf_{\{z_1,\dots,z_n\}\subset G} \sup\{||\varphi||_{2,\mu} : \varphi \in A_2 \text{ and } \varphi(z_k) = 0, k = 1, \dots, n\}.$$

Moreover, if E is small enough (see [10, Theorem 4]), then

$$\begin{aligned} s_n(J) &= d_n(A_2, L_2(\mu, E)) = d^n(A_2, L_2(\mu, E)) \\ &= \delta_n(A_2, L_2(\mu, E)) = \gamma_n. \end{aligned}$$

In the case when G is the open unit disk, S.D. Fisher and C.A. Micchelli (see [8], [9], and [7]) proved that always

$$\begin{aligned} s_n(J) &= d_n(\mathbf{A}_2, L_2(\mu, E)) = d^n(\mathbf{A}_2, L_2(\mu, E)) \\ &= \delta_n(\mathbf{A}_2, L_2(\mu, E)) = \gamma_n. \end{aligned}$$

2 Best Meromorphic Approximation of Markov Functions on the Unit Circle

2.1 Some Formulas

Here and in what follows we consider the case when G is the open unit disk with the center at 0. Let μ be a positive Borel measure whose support $\operatorname{supp}\mu = E \subset (-1, 1)$ contains infinitely many points and let f be the Markov function associated with μ. In this case the singular values $s_n(A_f)$ of A_f are strictly decreasing (i.e. the eigenspaces are one dimensional). It follows from Theorem 1 and results of S. D. Fisher and C.A. Micchelli (see [8], [9], and [7]), where it is proved that the eigenvalues for $J^*J : H_2(G) \to H_2(G)$,

$$(J^*J)g(z) = \frac{1}{2\pi}\int_E \frac{g(x)}{1 - xz} d\mu(x), \quad g \in H_2(G), \quad |z| < 1,$$

are simple, the corresponding eigenspaces are one dimensional and the $(n+1)$-st eigenfunction has exactly n zeros in G. Thus (cf. [5]) an eigenfunction

$Q_n = Q_{n,\infty}, ||Q_n||_2 = 1$, for A_f corresponding to the singular value $s_n(A_f)$ satisfies the equation

$$s_n(A_f)Q_n(z) = \frac{1}{2\pi}\int_E \frac{Q_n(x)}{1-zx}d\mu(x), \quad |z|<1. \tag{2.1}$$

We remark that Q_n is unique up to multiplication by a unimodular scalar. As shown in [5], equation (2.1) implies that the polynomial $w_n^*(z) = \prod_{k=1}^n (z - x_{k,n})$ constructed from the zeros of Q_n is an orthogonal polynomial with a varying weight function (see Subsection 2.2). It is possible to show further that all n zeros $x_{1,n}, \ldots, x_{n,n}$ of Q_n are simple, lie on the smallest closed interval containing the support of μ and that Q_n can be analytically continued from G to $\overline{\mathbf{C}} \setminus E^{-1}$, where $E^{-1} = \{1/x : x \in E\}$ (see [4] for more details about properties of Q_n).

2.2 Orthogonality

We can represent Q_n in the form $Q_n = B_n\varphi_n$, where φ_n is an outer function and B_n is a Blaschke product constructed from the zeros Q_n:

$$B_n(z) = \prod_{k=1}^n \frac{z - x_{k,n}}{1 - x_{k,n}z} = \frac{w_n^*(z)}{w_n(z)},$$

$$w_n^*(z) = \prod_{k=1}^n (z - x_{k,n}), \quad w_n(z) = \prod_{k=1}^n (1 - x_{k,n}z).$$

By (2.1),

$$\int_E \frac{Q_n(x)}{1 - x_{k,n}x}d\mu(x) = 0 \quad \text{for} \quad k = 1, \ldots, n.$$

Thus the following orthogonality relations are valid:

$$\int_E T(x)\frac{Q_n(x)}{w_n(x)}d\mu(x) = 0$$

and

$$\int_E T(x)w_n^*(x)\frac{\varphi_n(x)}{w_n^2(x)}d\mu(x) = 0 \tag{2.2}$$

for any polynomial T, $\deg T \le n-1$.

3 Potential Theory

To describe the asymptotic behavior of the singular values and corresponding singular functions for a Hankel operator whose symbol is a Markov function, we need to recall some fundamental quantities from potential theory.

Let $g(z,\xi)$ be the Green function for the unit disk G with pole at $\xi \in G$:

$$g(z,\xi) = \log\left|\frac{1-\overline{\xi}z}{z-\xi}\right|.$$

Let $E \subset G$ be a compact set. Denote by $\mathcal{P}(E)$ the set of all positive unit Borel measures σ with support $\operatorname{supp}\sigma \subset E$. The Green potential of a measure $\sigma \in \mathcal{P}(E)$ is denoted by

$$V_g^\sigma(z) = \int_E g(z,\xi)d\sigma(\xi).$$

We consider the energy of a measure $\sigma \in \mathcal{P}(E)$ with respect to the Green potential

$$I_g^\sigma = \int_E V_g^\sigma(z)d\sigma(z) = \int_E\int_E g(z,\xi)d\sigma(\xi)d\sigma(z).$$

It is a well known fact (see, for example, [18]) that if logarithmic capacity $\operatorname{cap}(E) > 0$, then there exists a unique measure $\omega \in \mathcal{P}(E)$ such that

$$I_g^\omega = \inf\{I_g^\sigma : \sigma \in \mathcal{P}(E)\}.$$

Moreover,

$$V_g^\omega(z) = I_g^\omega \quad \text{q.e. on} \quad E,$$

where q.e. (quasi-everywhere) means neglecting sets of zero logarithmic capacity. The measure ω is called the *Green equilibrium measure.* The *condenser capacity* is $C(E,\Gamma) := 1/I_g^\omega$. Let $r = e^{-1/C(E,\Gamma)} = e^{-I_g^\omega}$

Let us consider the case when $E = [a,b] \subset (-1,1)$. Then

$$I_g^\omega = \frac{\pi K}{2K'} \quad \text{or} \quad r = \exp\left(-\frac{\pi K}{2K'}\right),$$

where

$$K = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-\tau^2x^2)}}, \quad \tau^2 = \frac{(1-a^2)(1-b^2)}{(1-ab)^2},$$

and K' is the corresponding elliptic integral for $\tau' = \sqrt{1-\tau^2}$ (see [1], [18]). Moreover, the Green equilibrium measure is

$$d\omega(x) = \frac{k dx}{\sqrt{(x-a)(b-x)(1-ax)(1-bx)}}, \quad x \in [a,b],$$

where

$$k = \frac{(1-ab)}{2K'}.$$

4 Statements of Convergence Theorems

4.1 Szegő Function $D_\psi(z)$

We assume that the support of μ is $E = [a,b] \subset (0,1)$ and the measure μ satisfies the Szegő condition

$$\int_a^b \frac{\log(d\mu/dx)}{\sqrt{(x-a)(b-x)}} dx > -\infty.$$

Let $E^{-1} = [1/b, 1/a]$. Let $\Phi : \overline{\mathbf{C}} \setminus (E \cup E^{-1}) \to \{r < |z| < 1/r\}, \Phi(1) = 1$, be the conformal mapping of the region $\overline{\mathbf{C}} \setminus (E \cup E^{-1})$ onto the annulus with the interior radius r and the exterior radius $1/r$. Then $|\Phi| = r$ on E and $r = e^{-1/C(E,\Gamma)}$, where $C(E,\Gamma)$ is the capacity of the condenser (E,Γ).

Let $d\omega$ be the Green equilibrium measure for $E = [a,b]$. We can write the Riesz decomposition of μ

$$d\mu = \psi d\omega + d\mu_s,$$

where $\psi = d\mu/d\omega$ denotes the Radon-Nikodym derivative of μ with respect to ω, and μ_s is a singular measure. We also need the *geometric mean* $\mathcal{G}_\omega(\psi)$ of ψ with respect to ω:

$$\mathcal{G}_\omega(\psi) = \exp\left(\int_E \log\psi \, d\omega\right)$$

and the Szegő function D_ψ for a doubly-connected region $\overline{\mathbf{C}} \setminus (E \cup E^{-1})$ (see [14]). The Szegő function $\mathcal{D}_\psi(z)$ for $\overline{\mathbf{C}} \setminus (E \cup E^{-1})$ has the following properties:

1) $\mathcal{D}_\psi(z)$ is analytic and non-vanishing in $\overline{\mathbf{C}} \setminus (E \cup E^{-1})$;

2) the increment $\Delta_\Gamma \arg \mathcal{D}_\psi$ of the argument of $\mathcal{D}_\psi$ along Γ is equal to 0;

3) $|\mathcal{D}_\psi(z)|^2 = \psi(z)$ a.e. on E; more precisely, the nontangential limits of $|\mathcal{D}_\psi(\xi)|^2$ as ξ approaches $z \in E$ equal $\psi(z)$ a.e.;

4) $|\mathcal{D}_\psi(z)|^2 = \mathcal{G}_\omega(\psi)$ on Γ.

We have (cf. [6])

$$\mathcal{D}_\psi(z) = \sqrt{\mathcal{G}_\omega(\psi)} \exp \Bigg(\sqrt{(z-a)(z-b)(1-az)(1-bz)} \times \frac{1}{2\pi} \int_a^b \frac{\log(\psi(x)/\mathcal{G}_\omega(\psi))(1-2xz+x^2)dx}{\sqrt{(x-a)(b-x)(1-ax)(1-bx)}(z-x)(1-xz)} \Bigg).$$

4.2 Main Results

In this subsection we formulate theorems characterizing the limiting distribution of poles and degree of convergence of best meromorphic approximants $h_n = h_{n,p}, 1 \le p \le \infty$, to the Markov function of (1.1) in the space $L_p(\Gamma)$. The methods used are based on an investigation of Szegő type asymptotics for the orthogonal polynomials (cf. (2.2)) associated with the poles of h_n.

Let $1 \le p \le \infty$ and let $\mathcal{M}_{n,p}(G)$ be the following class of meromorphic functions on the unit disk G:

$$\mathcal{M}_{n,p}(G) = \{h = \alpha/\beta : \alpha \in H_p(G),\ \beta \text{ is a polynomial, } \deg \beta \le n, \beta \not\equiv 0\}.$$

The deviation of $f \in L_p(\Gamma)$ from the class $\mathcal{M}_{n,p}(G)$ will be denoted by $\Delta_{n,p}$:

$$\Delta_{n,p} = \Delta_{n,p}(f;G) = \inf_{h \in \mathcal{M}_{n,p}(G)} ||f-h||_p. \tag{4.1}$$

Let $1/p + 1/q = 1$. There exists a best approximant $h_n = h_{n,p}$ in $\mathcal{M}_{n,p}(G)$ to f in the space $L_p(\Gamma)$:

$$\Delta_{n,p} = ||f-h_n||_p$$

such that all n poles of h_n in G are simple and belong to E. The function h_n can be represented in the form $h_n = P_n/Q_n$, where $P_n = P_{n,p} \in H_p(G)$ and $Q_n = Q_{n,p} \in H_{2q}(G), ||Q_n||_{2q} = 1$. We have $Q_n = B_n\varphi_n$, where B_n is a Blaschke product degree n constructed from the zeros of Q_n, and $\varphi_n \in H_{2q}(G)$, $||\varphi_n||_{2q} = 1$, φ_n is positive on $(-1,1)$. It is possible to show that

φ_n^q and $B_n\varphi_n^{2-q}(f-h_n)$ can be extended analytically to $\overline{\mathbf{C}}\setminus E^{-1}$ and $\overline{\mathbf{C}}\setminus E$, respectively (see [4]).

Let $\varphi^* \in H_{2q}(G), \varphi^*(0) > 0$, satisfy

$$\mathcal{G}_\omega(|\varphi^*|^2) = \sup_{\varphi\in H_{2q}(G),||\varphi||_{2q}=1} \mathcal{G}_\omega(|\varphi|^2).$$

It is easy to prove that $\varphi^*(z) \equiv 1$ for $p = 1$. In the case when $1 < p \leq \infty$ we have

$$\varphi^*(z) = \frac{k_1}{\sqrt[2q]{(1-az)(1-bz)}}, \quad z \in G,$$

where $k_1 = \sqrt[2q]{k/2}$.

We now state a theorem characterizing the degree of convergence of $\Delta_{n,p}$ to zero as $n \to \infty$ and the limiting behavior of B_n and φ_n (see [4]). In the following theorems we assume that f is a Markov function generated by a measure μ satisfying the Szegő condition in Subsection 4.1.

Theorem 2 *We have for* $1 \leq p \leq \infty$,

(i)

$$\frac{\Delta_{n,p}}{2r^{2n}\mathcal{G}_\omega(\psi)\mathcal{G}_\omega((\varphi^*)^2)} \to 1 \quad as \quad n \to \infty. \tag{4.2}$$

In particular, for $p = \infty$, (4.2) *describes the asymptotic behavior of the singular values* $s_n(A_f) = \Delta_{n,\infty}$.

(ii) *For* $1 < p \leq \infty$, *the sequence* $B_n^2\varphi_n^2 d\mu/\Delta_{n,p}$ *converges weak* to* $d\omega$:

$$\frac{B_n^2\varphi_n^2}{\Delta_{n,p}} d\mu \xrightarrow{*} d\omega \quad as \quad n \to \infty.$$

Moreover,

$$\frac{(B_n\mathcal{D}_\psi\mathcal{D}_{(\varphi^*)^2})(z)}{\Phi^n(z)\sqrt{\mathcal{G}_\omega(\psi)}\mathcal{G}_\omega(\varphi^*)} \to 1$$

uniformly on compact subsets of $\overline{\mathbf{C}} \setminus (E \cup E^{-1})$ *as* $n \to \infty$, *and*

$$\varphi_n(z) \to \varphi^*(z)$$

and

$$\varphi_n^q(z) \to (\varphi^*)^q(z)$$

uniformly on compact subsets of G *and* $\overline{\mathbf{C}} \setminus E^{-1}$, *respectively, as* $n \to \infty$.

The next theorem describes the convergence of h_n to f (see [4]).

Theorem 3 *Let $1 < p \leq \infty$. We have for $z \in G \setminus E$,*

$$\frac{(f-h_n)(z)}{\frac{2(\mathcal{D}_\psi^2 \mathcal{D}_{\varphi^{*2}}^2)(z) r^{2n}}{i\Phi^{2n}(z)} \sqrt{\frac{(1-az)(1-bz)}{(z-a)(z-b)}} \left(\frac{(1-az)(1-bz)}{k/2}\right)^{(1-q)/q}} \to 1,$$

and for $z \in \overline{\mathbf{C}} \setminus (E \cup E^{-1})$

$$\frac{(\varphi_n^{2-q}(f-h_n))(z)}{\frac{2(\mathcal{D}_\psi^2 \mathcal{D}_{\varphi^{*2}}^2)(z) r^{2n}}{\Phi^{2n}(z)}} \to \frac{\sqrt{k/2}}{i\sqrt{(z-a)(z-b)}},$$

where both limits as $n \to \infty$ are locally uniform.

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