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Zero Location for Nonstandard Orthogonal Polynomials

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A method to locate the zeros of orthogonal polynomials with respect to non-standard inner products is discussed and applied to Sobolev orthogonal polynomials and polynomials satisfying higher-order recurrence relations.

Key Words: zeros; higher-order recurrence relation; matrix polynomials; Sobolev inner product.

1. INTRODUCTION

In this note (Section 2) we develop a method for the location of the set of zeros of a sequence of orthogonal polynomials with respect to an inner product of the form

$$\langle p, q \rangle = \int (T_0(p), ..., T_N(p))\, W(z)(T_0(q), ..., T_N(q))^*\, d\mu(z), \tag{1.1}$$

where:

(i) $T_0, ..., T_N$ are linear operators on the space $\mathbb{P}$ of algebraic polynomials with complex coefficients, and

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(ii) $W(z)$ is a positive definite matrix of integrable functions with respect to the positive measure μ supported on a subset of the complex plane.

The method is applied to locate zeros of Sobolev orthogonal polynomials and polynomials satisfying higher-order recurrence relations (assuming that the matrix of functions W is diagonal). In both cases the bounds for the zeros are sharp.

In Section 3, we consider orthogonal polynomials with respect to a Sobolev inner product of the form

$$\langle p, q \rangle = \sum_{k=0}^{N} \int p^{(k)}(z)\, \overline{q^{(k)}(z)}\, w_k(z)\, d\mu(z), \qquad N \geqslant 1, \tag{1.2}$$

where μ is a finite positive Borel measure with support $\Delta \subset \mathbb{C}$ (Δ containing infinitely many points) and $w_k \in L^1(\mu)$, $w_k \geqslant 0$, $k = 0, ..., N$.

These nonstandard inner products have attracted much attention in the last few years. Regarding the zero location, some results have been obtained for the case when μ has real support and $w_k(z)\, d\mu(z)$ reduces to a Dirac's delta, $k = 1, ..., n$, (discrete Sobolev inner product); see for instance ([AMRR1], [AMRR2]). For a continuous Sobolev inner product, W. Gautschi and A. B. J. Kuijlaars in [GK] have found zero asymptotic properties for $N = 1$ under the hypothesis that $w_0\, d\mu$ and $w_1\, d\mu$ are regular measures (in the sense of [ST]). More recently, G. Lopez and H. Pijeira in [LP] have obtained some estimates on the zeros of Sobolev orthogonal polynomials with respect to an inner product of the form given in (1.2) assuming that μ is supported on a compact set of the real line and that $w_k / w_{k-1} \in L^\infty(\mu)$, $k = 1, ..., N$. They used the boundedness of the multiplication operator defined in a certain Banach space associated with the Sobolev inner product. G. Lopez has pointed out to the authors a new and very short proof for the boundedness of the zeros assuming the boundedness of the multiplication operator which, for the sake of completeness, we include here (see also [LPP]): *let* $(p_n)_n$ *be a sequence of orthogonal polynomials with respect to an inner product* $\langle \cdot, \cdot \rangle$ *for which the multiplication operator is bounded, that is, there exists* $c > 0$ *such that* $\langle zq(z), zq(z) \rangle \leqslant c \langle q(z), q(z) \rangle$ *for any polynomial* q*; then* $|a| \leqslant \sqrt{c}$ *for any zero* a *of* p_n, $n \in \mathbb{N}$. Indeed, we can write $p_n(z) = (z - a)\, q(z)$ for some polynomial q of degree $n - 1$; then $\langle p_n(z), q(z) \rangle = 0$ and since $zq(z) = p_n(z) + aq(z)$ we get that

$$\langle zq(z), zq(z) \rangle = \langle p_n(z), p_n(z) \rangle + \langle aq(z), aq(z) \rangle \geqslant |a|^2 \langle q(z), q(z) \rangle.$$

Using the boundedness of the multiplication operator we have that

$$|a|^2 \langle q(z), q(z)\rangle \leqslant \langle zq(z), zq(z)\rangle \leqslant c\langle q(z), q(z)\rangle,$$

and since $\langle q(z), q(z)\rangle \neq 0$, we finally deduce that $|a| \leqslant \sqrt{c}$.

Let us remark that, for Sobolev inner product, the boundness of the multiplication operator forces the measures which define the inner product to have compact support.

Using our method, we can extend the latter results (improving also their bounds) for measures μ with support in any set of the complex plane (compact or not):

THEOREM 1.1. *Consider a Sobolev inner product of the form given in* (1.2), *where μ is a finite Borel measure with support Δ in the complex plane $\mathbb{C}$ (Δ containing infinitely many points) and $w_k \in L^1(\mu)$, $w_k \geqslant 0$, $k = 0, ..., N$. Assume $w_k/w_{k-1} \in L^\infty(\mu)$, $k = 1, ..., N$, and write $C_k = \|w_k/w_{k-1}\|_\infty$, $k = 1, ..., N$. If z_0 is a zero of an orthogonal polynomial with respect to $\langle , \rangle$, then*

$$d(z_0, \mathrm{Co}(\Delta)) \leqslant (1/2) \sqrt{\sum_{k=1}^{N} k^2 C_k}, \tag{1.3}$$

where Co(Δ) *is the convex hull of the support of μ.*

An example is presented in Section 3 showing that the estimate (1.3) is sharp.

We complete Section 3 obtaining other different bounds (see Theorem 3.2) and extending them for general Sobolev inner products of size 2×2 (see Theorem 3.3).

In Section 4 we consider sequences of polynomials $(p_n)_n$ satisfying a $(2N+3)$-term recurrence relation of the form

$$t^{N+1} p_n(t) = c_{n,0}\, p_n(t) + \sum_{k=1}^{N+1} [c_{n,k}\, p_{n-k}(t) + c_{n+k,k}\, p_{n+k}(t)], \tag{1.4}$$

with initial conditions $p_k = 0$, for $k < 0$, and $p_k(t)$ a polynomial of degree k, for $k = 0, ..., N+1$. One of the authors proved in [D2] that the sequence $(p_n)_n$ is then orthonormal with respect to an inner product of the form (1.1), where

$$T_m(p) = \sum_n \frac{p^{(n(N+1)+m)}(0)}{(n(N+1)+m)!}\, t^n, \tag{1.5}$$

and μ a positive measure supported on the real line. Discrete Sobolev orthonormal polynomials are a particular case of polynomials satisfying a higher order recurrence relation (see [DV], Sect. 3).

Polynomials satisfying higher-order recurrence relations such as (1.4) are closely related to orthogonal matrix polynomials (see [D1], [D2], [DV]).

We next assume that the matrix of functions W is diagonal and μ has compact support. Using our method we get the following bound:

THEOREM 1.2. *Consider an inner product of the form*

$$\langle p, q \rangle = \sum_{k=0}^{N} \int T_k(p(z))\, \overline{T_k(q(z))}\, w_k(z)\, d\mu(z), \tag{1.6}$$

where the operators T_m, $m = 0, ..., N$, *are given by* (1.5), μ *is a finite positive Borel measure with compact support* $\Delta \subset \mathbb{C}$ (Δ *containing infinitely many points*) *and* $w_k \in L^1(\mu)$, $w_k \geqslant 0$. *Assume that* $w_k/w_{k-1} \in L^\infty(\mu)$, $k = 1, ..., N$, *and* $|z|^2\, w_0(z)/w_N(z) \in L^\infty(\mu)$. *If* z_0 *is a zero of an orthogonal polynomial with respect to* $\langle , \rangle$, *then*

$$|z_0| \leqslant \max\{\sqrt{C_0}, \sqrt{C_1}, ..., \sqrt{C_N}\}, \tag{1.7}$$

where $C_k = \|w_k/w_{k-1}\|_\infty$, $k = 1, ..., N$, *and* $C_0 = \| |z|^2\, w_0(z)/w_N(z)\|_\infty$.

We provide an example that shows (1.7) is sharp.

Finally, we extend Theorem 1.2 for the nondiagonal case of size 2×2 (see Theorem 4.1).

2. DESCRIPTION OF THE METHOD

We consider inner products $\langle , \rangle$, defined on the linear space of polynomials with complex coefficients, of the form given by (1.1).

We will assume that for each z in the support of μ and fixed $w \in \mathbb{C}$ there exists a $(N+1) \times (N+1)$ matrix $\Gamma(z, w)$ such that for all $p \in \mathbb{P}$

$$\begin{aligned}(T_0((z-w)\, p(z)), ..., T_N((z-w)\, p(z)))\\ = (T_0(p(z)), ..., T_N(p(z)))\, \Gamma(z, w).\end{aligned} \tag{2.1}$$

Let us remark that for an inner product these matrices Γ do not exist in general (e.g. consider $T_0(p) = \int_0^1 p(t)\, dt$ and $T_1(p) = p'(0)$; then for $p(t) = 1 - 3t^2$ one has $T_0(p) = T_1(p) = 0$, but $T_0((z-w)\, p) = -1/4$, $T_1((z-w)\, p) = 1$). However for the most interesting cases of inner products of the form (1.2) and (1.6) such matrices Γ always exist (see (3.1) for inner products of the form (1.2) and (4.1) for those of the form (1.6)).

Let z_0 be a zero of an orthogonal polynomial p for the inner product $\langle , \rangle$. Write $p(z)=(z-z_0)\,q(z)$. Then from the orthogonality of p and (2.1) we get for any $a \in \mathbb{C}$ that

$$\begin{aligned}
0 &= a\langle (z-z_0)\,q(z), q(z)\rangle + \bar{a}\langle q(z), (z-z_0)\,q(z)\rangle \\
&= a \int [(T_0((z-z_0)\,q(z)), ..., T_N((z-z_0)\,q(z)))\, W(z)(T_0(q(z)), ..., T_N(q(z)))^* \\
&\qquad + \bar{a}(T_0(q(z)), ..., T_N(q(z)))\, W(z)(T_0((z-z_0)\,q(z)), ..., T_N((z-z_0)\,q(z)))^*]\, d\mu \\
&= \int (T_0(q(z)), ..., T_N(q(z)))(a\Gamma(z,z_0)\, W(z) + \bar{a}W(z)\,\Gamma(z,z_0)^*) \\
&\qquad\qquad (T_0(q(z)), ..., T_N(q(z)))^*\, d\mu.
\end{aligned}$$

We can then conclude that for some z in the support of μ the hermitian matrix

$$(a\Gamma(z,z_0)\, W(z) + \bar{a}W(z)\,\Gamma(z,z_0)^*) \tag{2.2}$$

cannot be positive definite or negative definite. From this fact one can obtain some bounds on z_0 under additional hypotheses on the inner product.

In the next sections we illustrate the method with some examples.

3. SOBOLEV INNER PRODUCTS

In this section we prove the estimate (1.3) stated in Theorem 1.1 (we thus assume that the matrix of functions W is diagonal).

Indeed, for a Sobolev inner product we have that $T_m = D^{(m)}$ and then

$$((z-w)\,p(z))^{(m)} = (z-w)\,p^{(m)}(z) + mp^{(m-1)}(z),$$

from which we deduce:

$$\Gamma(z,w) = \begin{pmatrix} z-w & 1 & 0 & \cdots & 0 & 0 \\ 0 & z-w & 2 & \cdots & 0 & 0 \\ 0 & 0 & z-w & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & z-w & N \\ 0 & 0 & 0 & \cdots & 0 & z-w \end{pmatrix}. \tag{3.1}$$

Let z_0 be a zero of an orthogonal polynomial with respect to the Sobolev inner product given by (1.2). If $z_0 \notin \mathrm{Co}(\Delta)$ there exists $z_1 \in \mathrm{Co}(\Delta)$ such that $d(z_0, \mathrm{Co}(\Delta)) = d(z_0, z_1) > 0$.

We first assume that $\Im z_0 = \Im z_1$ and that $\Re z_1 \geqslant \Re z_0$. This implies that $\Re z - \Re z_0 > \Re z_1 - \Re z_0 > 0$ for $z \in \mathrm{Co}(\Delta)$ and that

$$d(z_0, \mathrm{Co}(\Delta)) = \inf\{\Re z - \Re z_0 : z \in \mathrm{Co}(\Delta)\}. \tag{3.2}$$

Taking $a = 1$ in (2.2) and according to our method we have that the hermitian matrix

$$\Gamma(z, z_0)\, W(z) + W(z)\, \Gamma(z, z_0)^*$$

cannot be positive definite for some $z \in \Delta$.

Formula (3.1) now gives that:

(3.3)

$$\Gamma(z, z_0)\, W(z) + W(z)\, \Gamma(z, z_0)^* = \begin{pmatrix} 2\Re(z-z_0)\, w_0(z) & w_1(z) & 0 & 0 & \cdots & 0 \\ w_1(z) & 2\Re(z-z_0)\, w_1(z) & 2w_2(z) & \cdots & \cdots & 0 \\ 0 & 2w_2(z) & 2\Re(z-z_0)\, w_2(z) & 3w_3(z) & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & Nw_N(z) & 2\Re(z-z_0)\, w_N(z) \end{pmatrix}.$$

To prove (1.3) we need the following bound on the principal determinant of this matrix.

Lemma 3.1. *Write $M_n = M_n(z, z_0)$ for the nth principal determinant of the matrix $\Gamma(z, z_0)\, W(z) + W(z)\, \Gamma(z, z_0)^*$. If $M_n > 0$ for $1 \leqslant n \leqslant m$, where $0 \leqslant m \leqslant N$, then*

$$M_{m+1} \geqslant (2(\Re z - \Re z_0))^{m-1} \times \left(4(\Re z - \Re z_0)^2 \prod_{k=0}^{m} w_k(z) - \sum_{k=1}^{m} k^2 w_k^2(z) \prod_{i=0;\, i \neq k, k-1}^{m} w_i(z) \right).$$

Proof. Let $\alpha = \Re z - \Re z_0$. Expanding M_{n+1} by the last row we get the following recurrence formula:

$$M_{n+1} = 2\alpha w_n M_n - n^2 w_n^2 M_{n-1}. \tag{3.4}$$

Since $M_n > 0$ for $1 \leqslant n \leqslant m$, it follows that

$$M_{n+1} \leqslant 2\alpha w_n M_n. \tag{3.5}$$

This gives for $n=m$:

$$M_{m+1}=2\alpha w_m M_m-m^2w_m^2M_{m-1}$$
$$\geqslant 2\alpha w_m M_m-2\alpha m^2 w_m^2 w_{m-2}M_{m-2}.$$

Using the recurrence relation (3.4) for $n=m-1$ we get:

$$M_{m+1}\geqslant(2\alpha)^2\,w_m w_{m-1}M_{m-1}$$
$$-(2\alpha)(m^2w_m^2w_{m-2}+(m-1)^2\,w_m w_{m-1}^2)\,M_{m-2}.$$

Again from (3.5) for $n=m-3$ we get

$$M_{m+1}\geqslant(2\alpha)^2\,w_m w_{m-1}M_{m-1}$$
$$-(2\alpha)^2\,(m^2w_m^2w_{m-2}w_{m-3}+(m-1)^2\,w_m w_{m-1}^2w_{m-3})\,M_{m-3}.$$

Applying successively (3.4) and (3.5) we find that

$$M_{m+1}\geqslant(2\alpha)^{m-2}\,w_m\cdots w_3M_3-(2\alpha)^{m-2}\left(\sum_{k=3}^{m}k^2w_k^2\prod_{i=1;\,i\neq k,\,k-1}^{m}w_i\right)M_1.$$

Lemma 3.1 follows now by taking into account that

$$M_3=(2\alpha)^3\,w_2w_1w_0-(2\alpha)(2^2w_2^2w_0+w_1^2w_2),$$

and $M_1=2\alpha w_0$. ∎

Since the hermitian matrix $\Gamma(z,z_0)\,W(z)+W(z)\,\Gamma(z,z_0)^*$ cannot be positive definite, we have that for some $m+1$, $1\leqslant m\leqslant N$, and for some z in Δ, $M_{m+1}(z,z_0)\leqslant 0$. We take the smallest m with this property.

From Lemma 3.1 and taking into account that $\Re z-\Re z_0>0$ for $z\in\Delta$, we deduce that

$$4(\Re z-\Re z_0)^2\prod_{k=0}^{m}w_k(z)-\sum_{k=1}^{m}k^2w_k^2(z)\prod_{i=0;\,i\neq k,\,k-1}^{m}w_i(z)\leqslant 0.$$

This inequality finally gives for some $z\in\Delta$ and some m, $1\leqslant m\leqslant N$, that

$$\Re z-\Re z_0\leqslant\frac{1}{2}\sqrt{\sum_{k=1}^{m}k^2\,\frac{w_k(z)}{w_{k-1}(z)}}.$$

To get (1.3), it is enough to take into account (3.2) and that the right-hand side in the previous inequality is an increasing function of m.

If $\Im z_0 \neq \Im z_1$, or $\Re z_1 < \Re z_0$, we proceed as follows. We write r for the line joining z_0 and z_1 and s for the line perpendicular to r at z_1. We now make a rotation τ such that $\tau(r)$ and $\tau(s)$ are parallel to the real and imaginary axis respectively and $\Re(\tau(z_1)) > \Re(\tau(z_0))$, that is, we take $\theta \in [0, 2\pi)$ such that $\Im(e^{i\theta} z) = \text{constant}$ for $z \in r$, $\Re(e^{i\theta} z) = \text{constant}$ for $z \in s$ and $\Re(e^{i\theta} z_1) > \Re(e^{i\theta} z_0)$. Then we have that

$$\Re(e^{i\theta}(z - z_0)) \geqslant \Re(e^{i\theta}(z_1 - z_0)) > 0$$

for $z \in \text{Co}(\Delta)$ and

$$d(z_0, \text{Co}(\Delta)) = \inf\{\Re(e^{i\theta}(z - z_0)) \colon z \in \text{Co}(\Delta)\}.$$

We now take $a = e^{i\theta}$ in (2.2) and proceed as before. ∎

We now give an example proving that (1.3) is sharp. Indeed, let us consider the Sobolev inner product

$$\langle p, q \rangle = \int_0^a p(t)\, q(t)\, dt + M p(1)\, q(1) + N p'(1)\, q'(1),$$

where $0 < a < 1$, $M, N > 0$ (see [AMRR1] for an study of this type of inner product). With the notation of Theorem 1.1 we have $N = 1$, $\mu = \chi_{[0, a]}\, dt + \delta_1$, $\Delta = [0, a] \cup \{1\}$, $\text{Co}(\Delta) = [0, 1]$, $w_0(t) = \chi_{[0, a]}(t) + M \chi_{\{1\}}(t)$, $w_1(t) = N \chi_{\{1\}}(t)$. Hence, Theorem 1.1 gives the bound $d(z_0, \text{Co}(\Delta)) \leqslant (1/2)\sqrt{N/M}$.

We now compute an orthogonal polynomial of degree two with respect to $\langle , \rangle$. To do that we need the following data:

$$\langle 1, 1 \rangle = a + M, \qquad \langle t, 1 \rangle = a^2/2 + M, \qquad \langle t^2, 1 \rangle = a^3/3 + M$$

$$\langle t, t \rangle = a^3/3 + M + N, \qquad \langle t^2, t \rangle = a^4/4 + M + 2N.$$

We then have that

$$p_2(t) = \begin{vmatrix} a + M & a^2/2 + M & a^3/3 + M \\ a^2/2 + M & a^3/3 + M + N & a^4/4 + M + 2N \\ 1 & t & t^2 \end{vmatrix}$$

is the orthogonal polynomial of degree two with respect to $\langle , \rangle$ (up to a nonzero multiplicative constant). Hence, the polynomial

$$q_2(t) = \begin{vmatrix} 1 + M/a & a/2 + M/a & a^2/3 + M/a \\ a/2 + M/a & a^2/3 + M/a + N/a & a^3/4 + M/a + 2N/a \\ 1 & t & t^2 \end{vmatrix}$$

is also orthogonal with respect to $\langle , \rangle$.

We take $M = N = \varepsilon a$, $\varepsilon > 0$. The bound given by Theorem 1.1 is now

$$d(z_0, \mathrm{Co}(\Delta)) \leqslant 1/2. \tag{3.6}$$

Taking the limit as a tends to 0, we get that the largest zero of q_2 tends to the largest zero of the polynomial

$$\begin{vmatrix} 1+\varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 2\varepsilon & 3\varepsilon \\ 1 & t & t^2 \end{vmatrix},$$

or what is equivalent, to the largest zero of the polynomial

$$\begin{vmatrix} 1+\varepsilon & \varepsilon & \varepsilon \\ 1 & 2 & 3 \\ 1 & t & t^2 \end{vmatrix}.$$

Taking again the limit as ε tends to 0, we get that the largest zero of q_2 tends to the largest zero of the polynomial

$$\begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 3 \\ 1 & t & t^2 \end{vmatrix} = t(2t-3),$$

that is, the largest zero of q_2 tends to $3/2$, and since $\mathrm{Co}(\Delta) = [0, 1]$, we deduce that (3.6) is sharp.

We can proceed in a slightly different way than in the proof of Theorem 1.1 to obtain another bound on the zeros:

THEOREM 3.2. *Let* $0 < \alpha_i, \beta_i < 2$ *satisfy* $\alpha_i + \beta_i = 2$, $i = 1, ..., N-1$, *and* $\alpha_0 = 2$, $\beta_N = 2$. *Then, with the hypotheses of Theorem* 1.1 *we also have*

$$d(z_0, \mathrm{Co}(\Delta)) \leqslant \max\left\{\frac{k\sqrt{C_k}}{\sqrt{\alpha_{k-1}\beta_k}}, k = 1, ..., N\right\}. \tag{3.7}$$

Proof. We again assume (3.2) and that $\Re z - \Re z_0 > \Re z_1 - \Re z_0 > 0$ for $z \in \mathrm{Co}(\Delta)$.

To prove the estimate (3.7) we split up the matrix $\Gamma(z, z_0)\, W(z) + W(z)\, \Gamma(z, z_0)^*$ (see (3.3)) as follows:

$$\Gamma(z, z_0)\, W(z) + W(z)\, \Gamma(z, z_0)^* = X_1 + X_2 + \cdots + X_N,$$

where

$$X_1 = \begin{pmatrix} 2\Re(z-z_0)\, w_0 & w_1 & 0 & \dots & 0 \\ w_1 & \beta_1 \Re(z-z_0)\, w_1 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix},$$

$$X_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & \alpha_1 \Re(z-z_0)\, w_1 & 2w_2 & 0 & \dots & 0 \\ 0 & 2w_2 & \beta_2 \Re(z-z_0)\, w_2 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix},$$

$$\vdots$$

$$X_N = \begin{pmatrix} 0 & \dots & 0 & 0 & 0 \\ 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & 0 \\ 0 & \dots & 0 & \alpha_{N-1} \Re(z-z_0)\, w_{N-1} & N w_N \\ 0 & \dots & 0 & N w_N & 2\Re(z-z_0)\, w_N \end{pmatrix}.$$

Since for some $z \in \Delta$ the hermitian matrix $\Gamma(z, z_0)\, W(z) + W(z)\, \Gamma(z, z_0)^*$ can not be positive definite, at least one of the matrices X_m, $m = 1, \dots, N$, should not be positive semidefinite for some $z \in \Delta$. Now (3.7) can be easily deduced taking into account that $\Re z - \Re z_0 > 0$ in Δ. ∎

Our method can also be applied to locate the zeros of orthogonal polynomials with respect to general Sobolev inner product of size 2×2.

THEOREM 3.3. *Consider a Sobolev inner product of the form*

$$\langle p, q \rangle = \sum_{k=0}^{1} \int p^{(k)}(z)\, \overline{q^{(k)}(z)}\, w_k(z)\, d\mu(z)$$
$$+ \int (p'(t)\, \overline{q(t)} + p(t)\, \overline{q'(t)})\, v(z)\, d\mu(z),$$

where μ is a finite positive Borel measure with support $\Delta \subset \mathbb{C}$ (Δ containing infinitely many points), $w_0, w_1, v \in L^1(\mu)$, $w_0, w_1 \geqslant 0$ and $w_0 w_1 - v^2 \geqslant 0$. Also assume that $w_1^2/(w_0 w_1 - v^2) \in L^\infty(\mu)$. If z_0 is a zero of an orthogonal polynomial with respect to $\langle , \rangle$, then

$$d(z_0, \mathrm{Co}(\Delta)) \leqslant \frac{1}{2} \left\| \frac{w_1^2}{w_0 w_1 - v^2} \right\|_\infty^{1/2}. \tag{3.8}$$

Proof. Indeed, proceeding as before, we find that

$$\begin{aligned}\Gamma(z, z_0)\, W(z) + W(z)\, \Gamma(z, z_0)^* \\ = \begin{pmatrix} 2\Re(z - z_0)\, w_0 + 2v & 2\Re(z - z_0)\, v + w_1 \\ 2\Re(z - z_0)\, v + w_1 & 2\Re(z - z_0)\, w_1 \end{pmatrix}.\end{aligned}$$

(3.8) can be deduced taking into account that that matrix can not be positive definite for some $z \in \Delta$. ∎

4. POLYNOMIALS SATISFYING HIGHER-ORDER RECURRENCE RELATIONS

In this section we prove Theorem 1.2.

Indeed, taking into account the definition of the operators T_m (see (1.5)) we get for $m = 1, ..., N$, that

$$T_m((z - w)\, p(z)) = -w T_m(p(z)) + T_{m-1}(p(z)),$$

and for $m = 0$ that

$$T_0((z - w)\, p(z)) = -w T_0(p(z)) + z T_N(p(z)).$$

From this we find:

$$\Gamma(z, w) = \begin{pmatrix} -w & 1 & 0 & \cdots & 0 & 0 \\ 0 & -w & 1 & \cdots & 0 & 0 \\ 0 & 0 & -w & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -w & 1 \\ z & 0 & 0 & \cdots & 0 & -w \end{pmatrix}.$$

Proceeding as in the previous section we obtain:

$$\Gamma(z, z_0)\, W(z) + W(z)\, \Gamma(z, z_0)^*$$

$$= \begin{pmatrix} -2\Re(z_0)\, w_0 & w_1 & 0 & \dots & 0 & \bar{z} w_0 \\ w_1 & -2\Re(z_0)\, w_1 & w_2 & \dots & 0 & 0 \\ 0 & w_2 & -2\Re(z_0)\, w_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ z w_0 & 0 & 0 & \dots & w_N & -2\Re(z_0)\, w_N \end{pmatrix}.$$

We now split up the hermitian matrix $\Gamma(z, z_0)\, W(z) + W(z)\, \Gamma(z, z_0)^*$ as follows:

$$\Gamma(z, z_0)\, W(z) + W(z)\, \Gamma(z, z_0)^* = X + X_1 + X_2 + \cdots + X_N,$$

where

$$X = \begin{pmatrix} -\Re(z_0)\, w_0 & 0 & \dots & 0 & \bar{z} w_0 \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ z w_0 & 0 & \dots & 0 & -\Re(z_0)\, w_N \end{pmatrix},$$

$$X_1 = \begin{pmatrix} -\Re(z_0)\, w_0 & w_1 & 0 & \dots & 0 \\ w_1 & -\Re(z_0)\, w_1 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix},$$

$$X_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & -\Re(z_0)\, w_1 & w_2 & 0 & \dots & 0 \\ 0 & w_2 & -\Re(z_0)\, w_2 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix},$$

$$\vdots$$

$$X_N = \begin{pmatrix} 0 & \dots & 0 & 0 & 0 \\ 0 & \dots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 \\ 0 & \dots & 0 & -\Re(z_0)\, w_{N-1} & w_N \\ 0 & \dots & 0 & w_N & -(\Re z_0)\, w_N \end{pmatrix}.$$

And so, if $\Re z_0 < 0$ at least one of the matrices X_m, $m = 1, \dots, N$, or X cannot be positive semidefinite (respectively negative semidefinite if $\Re z_0 > 0$), from which (1.7) can be deduced. ▌

We now prove that (1.7) is sharp. To do that we consider a positive measure v whose support is the interval $[-1, 1]$ and write $(r_n)_n$ for its sequence of orthonormal polynomials. Hence the set of zeros of $(r_n)_n$ is dense in $[-1, 1]$. We define the sequence of polynomials $(p_n)_n$ as follows:

$$p_{2n}(t) = r_n(t^2), \qquad p_{2n+1}(t) = t r_n(t^2).$$

It is clear that $(p_n)_n$ are orthonormal with respect to the inner product (1.6) when $N = 1$, $\mu = v$, $\Delta = [-1, 1]$ and $w_0 = w_1 = 1$. If z_0 is a zero of some p_n, Theorem 1.2 gives that

$$|z_0| \leqslant 1. \tag{4.1}$$

Since the set of zeros of $(r_n)_n$ is dense in $[-1, 1]$ we can take a sequence of zeros $(x_k)_k$ converging to 1 (or -1). Each square root of x_k is then a zero of p_n for certain n, and then (4.1) is sharp.

Finally, we study the general case of size 2×2.

THEOREM 4.1. *Consider an inner product of the form*

$$\langle p, q\rangle = \int [T_0(p(z))\, \overline{T_0(q(z))}\, w_0(z) + T_1(p(z))\, \overline{T_1(q(z))}\, w_1(z) + (T_0(p(z))\, \overline{T_1(q(z))} + T_1(p(z))\, \overline{T_0(q(z))})\, v(z)]\, d\mu(z), \tag{4.2}$$

where the operators T_0 and T_1 are given by (1.5) *for $N = 1$, μ is a finite positive Borel measure with compact support $\Delta \subset \mathbb{C}$, ($\Delta$ containing infinitely many points) and $w_0, w_1, v \in L^1(\mu)$, $w_0, w_1 \geqslant 0$ and $w_0 w_1 - v^2 \geqslant 0$. We assume that,*

$$|v|/w_0, \qquad \frac{|zw_0(z) + w_1(z)|^2 - 4\Re(z)\, v^2(z)}{w_0(z)\, w_1(z) - v^2(z)}, \frac{|zw_0(z) - w_1(z)|^2}{w_0(z)\, w_1(z) - v^2(z)} \in L^\infty(\mu).$$

If z_0 is a zero of an orthogonal polynomial with respect to $\langle , \rangle$, then

$$|z_0| \leqslant \sqrt{C_1^2 + C_2^2}, \tag{4.3}$$

where

$$C_1 = \max\left\{ \| |v|/w_0 \|_\infty,\ (1/2) \left\| \frac{|zw_0(z)+w_1(z)|^2 - 4\Re(z)\, v^2(z)}{w_0(z)\, w_1(z) - v^2(z)} \right\|_\infty^{1/2} \right\}$$

and

$$C_2 = (1/2) \left\| \frac{|zw_0(z)-w_1(z)|^2}{w_0(z)\, w_1(z)-v^2(z)} \right\|_\infty^{1/2}.$$

Proof. We assume $\Re z_0 > 0$. Taking $a = -1$ in (2.2) ($a = 1$ if $\Re z_0 \leqslant 0$), and applying our method we deduce that for some $z \in \Delta$ the hermitian matrix

$$\begin{pmatrix} 2\Re z_0 w_0 - 2v & 2\Re z_0 v - (\bar{z} w_0 + w_1) \\ 2\Re z_0 v - (z w_0 + w_1) & 2\Re z_0 w_1 - 2\Re z v \end{pmatrix}$$

can not be positive definite.

An easy computation gives that $\Re z_0 \leqslant |v|/w_0$, or

$$\Re z_0 \leqslant \tfrac{1}{2}\, |[|zw_0(z)+w_1(z)|^2 - 4\Re(z)\, v^2(z)]/[w_0(z)\, w_1(z) - v^2(z)]|^{1/2},$$

for some $z \in \Delta$.

Taking now $a = i$ in (2.2) we deduce that for some $z \in \Delta$ the hermitian matrix

$$\begin{pmatrix} 2\Im z_0 w_0 & 2\Im z_0 v - i\bar{z} w_0 + i w_1 \\ 2\Im z_0 v + i z w_0 - i w_1 & 2\Im z_0 w_1 - 2\Im z v \end{pmatrix}$$

can not be positive definite, from where we find that

$$\Im z_0 \leqslant \tfrac{1}{2}\, |[|zw_0(z) - w_1(z)|^2]/[w_0(z)\, w_1(z) - v^2(z)]|^{1/2},$$

for some $z \in \Delta$. ∎

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