# The Distribution of Zeros and Poles of Asymptotically Extremal Rational Functions for Zolotarev's Problem

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*Communicated by Doron S. Lubinsky*

We investigate the possible limit distributions of zeros and poles associated with ray sequences of rational functions that are asymptotically optimal for weighted Zolotarev problems. For disjoint compacta $E_1$, $E_2$ in the complex plane, the Zolotarev problem entails minimizing the ratio of the sup over $E_1$ of the modulus of a weighted rational to its inf over $E_2$. Potential theoretic tools are utilized in the analysis.     © 2001 Academic Press

*Key Words:* Zolotarev problem; rational functions; asymptotic distributions; zeros and poles; potential theory.

## 1. INTRODUCTION

Let $E_1$, $E_2$ be closed sets in the complex plane **C** that are a positive distance apart. Given a pair $(m, n)$ of non-negative integers, denote by **R**$_{mn}$ the class of all rational functions in the complex variable $z$ whose numerator and denominator degrees are $m$ and $n$, respectively. Let $w$ be an

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admissible weight (see Definition 2.1) on $E_1 \cup E_2$. Given $r \in \mathbf{R}_{mn}$, consider the quantity

$$Z_{mn}(r; w) := \frac{\sup_{z \in E_1}\{|r(z)| \, w(z)^{m+n}\}}{\inf_{z \in E_2}\{|r(z)| \, w(z)^{m+n}\}} \tag{1.1}$$

and set

$$Z_{mn}(w) := \inf\{Z_{mn}(r; w): r \in \mathbf{R_{mn}}\}. \tag{1.2}$$

We call $Z_{mn}(w)$ the *weighted Zolotarev constant* of type $(m, n)$ for the condenser $(E_1, E_2)$. Next, fix $0 < \tau < 1$ and consider a "ray sequence" of pairs $(m, n)$, namely a sequence $N_\tau$ for which

$$m + n \to \infty, \qquad \frac{m}{n} \to \frac{\tau}{1 - \tau}. \tag{1.3}$$

Our first task will be to show that for any such $N_\tau$,

$$\lim_{(m, n) \in N_\tau} \{Z_{mn}(w)\}^{1/(m+n)} = \exp(-F_{w, \tau}), \tag{1.4}$$

where $F_{w, \tau}$ is a quantity that arises in the solution to a certain energy problem discussed in Section 2. This generalizes a previous result obtained by the authors [LeSa].

Next, to each

$$r_{mn}(z) = \frac{\prod_{i=1}^m (z - \alpha_{im})}{\prod_{i=1}^n (z - \beta_{in})} \tag{1.5}$$

we associate the normalized distribution

$$v(r_{mn}) := \frac{1}{m+n}\left\{\sum_{i=1}^m \delta_{\alpha_{im}} - \sum_{i=1}^n \delta_{\beta_{in}}\right\}, \tag{1.6}$$

where, generically, $\delta_z$ stands for the point distribution with total mass 1 at $z$. Given a ray sequence $N_\tau$ (cf. (1.3)), we call the sequence $\{r_{mn}\}$, $r_{mn} \in \mathbf{R_{mn}}$, *asymptotically extremal* if

$$\lim_{(m, n) \in N_r} \{Z_{mn}(r_{mn}; w)\}^{1/(m+n)} = \exp(-F_{w, \tau}). \tag{1.7}$$

We shall see, in the course of the proof of (1.4), that there exists such $\{r_{mn}\}$ for which $v(r_{mn})$ converges to the equilibrium distribution $\mu^*$ for the above mentioned energy problem. The convergence is understood in the weak-star sense on the Riemann sphere $\bar{\mathbf{C}}$, that is we write $\mu_n \to \mu$ if $\int f \, d\mu_n \to \int f \, d\mu$,

for every $f$ that is continuous on $\bar{\mathbf{C}}$. Simple examples show that there may be other asymptotically extremal sequences $\{r_{mn}\}$ for which $v(r_{mn}) \to \mu \neq \mu^*$.

The main objective of this paper is to describe all possible weak-star limits of sequences $\{v(r_{mn})\}$ associated with asymptotically extremal sequences $\{r_{mn}\}$. We will concentrate on the case of *bounded* $E_1$, $E_2$. The case of unbounded sets requires additional assumptions on the weight. These are briefly discussed at the end of the paper.

## 2. TWO EXTREMAL PROBLEMS OF POTENTIAL THEORY

We take the weight in the form $w = \exp(-Q)$, where $Q$ is a function from $E_1 \cup E_2$ to the extended real line $[-\infty, \infty]$.

DEFINITION 2.1. Let $E_1$, $E_2$ be disjoint compacta in $\mathbf{C}$, both of positive logarithmic capacity. A weight $w = \exp(-Q)$ is called *admissible* if the following conditions hold:

(i)  $Q$ is a lower (upper) semicontinuous function on $E_1$ (on $E_2$)

(ii)  $Q < \infty$ $(Q > -\infty)$ on a subset of $E_1$ (of $E_2$) that has positive logarithmic capacity.

We remark that a lower (upper) semicontinuous function does not attain, by definition, the value $-\infty$ $(+\infty)$.

Given $0 < \tau < 1$, let $M_\tau$ denote the set of all signed measures $\mu = \mu_1 - \mu_2$ that have a compact support in $\mathbf{C}$ and satisfy $\|\mu_1\| = \tau$, $\|\mu_2\| = 1 - \tau$. If, additionally, $S(\mu_i) \subset E_i$, $i = 1, 2$, we write $\mu \in M_\tau(E_1, E_2)$. Here and throughout, $S$ stands for the support of indicated measure.

With the usual notation

$$U^\mu(z) := \int \log \frac{1}{|z - t|} \, d\mu(t),$$

$$I(\mu) := \int U^\mu \, d\mu = \iint \log \frac{1}{|z - t|} \, d\mu(z) \, d\mu(t),$$

consider the following extremal problems:

$$V = V_{w, \tau} := \inf \left\{ I(\mu) + 2 \int Q \, d\mu : \mu \in M_\tau(E_1, E_2) \right\} \tag{2.1}$$

$$F = F_{w, \tau} := \sup \left\{ \underset{E_1}{\text{``inf''}} (U^\mu + Q) - \underset{E_2}{\text{``sup''}} (U^\mu + Q) : \mu \in M_\tau \right\}, \tag{2.2}$$

where "inf" and "sup" mean, respectively, inf and sup neglecting sets of zero capacity.

The following result is well-known (cf. [SaTo, p. 383]):

THEOREM 2.2. *Let $E_1$, $E_2$, and $w$ be as above. Then for any $0 < \tau < 1$,*

(i) $V_{w,\tau}$ *is finite and there exists a unique*

$$\mu^* = \mu^*_{w,\tau} = \mu^*_1 - \mu^*_2 \in M_\tau(E_1, E_2)$$

*for which $V_{w,\tau} = I(\mu^*) + 2 \int Q \, d\mu^*$.*

(ii) $\mu^*$ *has finite logarithmic energy and both $U^{\mu^*}$ and $Q$ are bounded on $S(\mu^*)$. Consequently, $U^{\mu^*}$ is bounded on compact subsets of* **C**.

(iii) *There exist constants $F_1$, $F_2$ (depending on $w$, $\tau$) such that*

$$U^{\mu^*} + Q \leqslant F_1 \qquad \text{on } S(\mu^*_1), \qquad U^{\mu^*} + Q \geqslant F_1 \qquad \text{q.e. on } E_1,$$

$$U^{\mu^*} + Q \geqslant -F_2 \qquad \text{on } S(\mu^*_2), \qquad U^{\mu^*} + Q \leqslant -F_2 \quad \text{q.e. on } E_2$$

*(here and throughout q.e. means neglecting sets of zero capacity). Consequently,*

$$U^{\mu^*} + Q = \begin{cases} F_1 & \text{q.e. on } S(\mu^*_1) \\ -F_2 & \text{q.e. on } S(\mu^*_2). \end{cases} \tag{2.3}$$

(iv) *For any $\mu \in M_\tau$,*

$$\underset{E_1}{\text{"inf"}}(U^\mu + Q) - \underset{E_2}{\text{"sup"}}(U^\mu + Q) \leqslant F_1 + F_2. \tag{2.4}$$

We see from Theorem 2.2(iii), (iv) that the value $F_{w,\tau}$ in problem (2.2) is equal to $F_1 + F_2$ and it is attained for $\mu = \mu^*$. However, an extremal measure for this problem may be not unique (see Examples 2.4, 2.5 below); hence we introduce

DEFINITION 2.3. Given $\mu \in M_\tau$, we say that $\mu \in M^*_\tau = M^*_\tau(w)$ if

$$\underset{E_1}{\text{"inf"}}(U^\mu + Q) - \underset{E_2}{\text{"sup"}}(U^\mu + Q) = F_1 + F_2.$$

We adopt the simplified notation

$$F_1(\mu) := \underset{E_1}{\text{"inf"}}(U^\mu + Q), \qquad -F_2(\mu) := \underset{E_2}{\text{"sup"}}(U^\mu + Q), \tag{2.5}$$

so that the above definition takes the form

$$M_\tau^* = M_\tau^*(w) = \{\mu \in M_\tau : F_1(\mu) + F_2(\mu) = F_1 + F_2\} \qquad (2.6)$$

(where $F_i = F_i(\mu^*)$, $i = 1, 2$).

EXAMPLE 2.4.   Let $E_1$ be the circle $|z| = 1$ and let $E_2$ be the union of the circles $|z| = R_1$, $|z| = R_2$, $1 < R_1 < R_2$. Assuming $\tau < 1/2$, take any $\alpha$ that satisfies

$$0 \leqslant \alpha \leqslant \frac{\tau}{1-\tau} \qquad (<1) \qquad (2.7)$$

and consider $\mu = \mu_1 - \mu_2 \in M_\tau$ that is defined by

$$\mu_1 = \tau \frac{1}{2\pi} d\theta \qquad \text{on} \quad |z| = 1,$$
$$\mu_2 = (1-\tau) \frac{1}{2\pi} \begin{cases} \alpha \, d\theta & \text{on} \quad |z| = R_1 \\ (1-\alpha) \, d\theta & \text{on} \quad |z| = R_2 \end{cases} \qquad (2.8)$$

($d\theta$ stands for the angular measure on the indicated circle). Simple calculation gives (we take $Q \equiv 0$ in this example)

$$\inf_{E_1} U^\mu - \sup_{E_2} U^\mu = \tau \log R_1. \qquad (2.9)$$

On the other hand, the extremal measure $\mu^* = \mu_1^* - \mu_2^*$, being unique, must have the form (2.8), for some $\alpha \in [0, 1]$. Utilizing Theorem 2.2(iii), we find that this $\alpha$ is equal to $\tau/(1-\tau)$. Therefore (2.7) to (2.9) show that any $\mu$ that is given by (2.7), (2.8) belongs to $M_\tau^*$. Note that $S(\mu) = S(\mu^*)$ if $\alpha \neq 0$, while $S(\mu_1) = S(\mu_1^*)$, $S(\mu_2) \subset S(\mu_2^*)$ if $\alpha = 0$.

Our next example shows that there may be $\mu \in M_\tau^*$ for which $S(\mu_2)$ intersects $E_1$ or even intersects Int $E_1$ (the interior of $E_1$).

EXAMPLE 2.5.   Let $E_1$ be the union of the circle $|z| = r < 1$, and the set $R \leqslant |z| \leqslant 2R$, $R > 1$. Let $E_2$ be the circle $|z| = 1$. Again, we take $Q \equiv 0$. It can be shown (cf. [LeSa, Example 5.2], where $|z| = R$ rather than $R \leqslant |z| \leqslant 2R$ was considered) that for

$$\tau < \tau_0 := \frac{\log R}{\log R + \log(R/r)} \qquad \left( < \frac{1}{2} \right)$$

we have $F_1 + F_2 = \tau \log(1/r)$, while $\mu^*$ is given by

$$\mu_1^* = \tau \frac{1}{2\pi} d\theta \qquad \text{on} \quad |z| = r,$$

$$\mu_2^* = (1 - \tau) \frac{1}{2\pi} d\theta \qquad \text{on} \quad |z| = 1.$$

Define $\mu \in M_\tau$ by

$$\mu_1 = \tau \frac{1}{2\pi} d\theta \qquad \text{on} \quad |z| = r,$$

$$\mu_2 = (1 - \tau) \frac{1}{2\pi} \begin{cases} \alpha \, d\theta & \text{on} \quad |z| = 1 \\ (1 - \alpha) \, d\theta & \text{on} \quad |z| = R_1, \quad R_1 \geqslant R \end{cases} \qquad (2.10)$$

with $\alpha$ satisfying

$$1 > \alpha \geqslant \frac{\tau}{1 - \tau} \frac{1 - \tau_0}{\tau_0}. \qquad (2.11)$$

Calculation shows that the corresponding potential satisfies

$$U^\mu(z) = \begin{cases} \tau \log \dfrac{1}{r} - (1 - \tau)(1 - \alpha) \log \dfrac{1}{R_1}, & |z| = r \\[2mm] \tau \log \dfrac{1}{|z|} - (1 - \tau) \alpha \log \dfrac{1}{|z|} - (1 - \tau)(1 - \alpha) \log \dfrac{1}{R_1}, & |z| \geqslant R \\[2mm] -(1 - \tau)(1 - \alpha) \log \dfrac{1}{R_1}, & |z| = 1. \end{cases}$$

The condition (2.11) ensures that the value of $U^\mu$ on $|z| = r$ does not exceed any of its values on $|z| \geqslant R$. Therefore,

$$F_1(\mu) + F_2(\mu) = \tau \log \frac{1}{r},$$

so that $\mu \in M_\tau^*$.

Note that $S(\mu_2)$ has points on the boundary of $E_1$ (if $R_1 = R$, say) or in the interior of $E_1$ (if $R < R_1 < 2R$). In the latter case no asymptotically extremal sequence $\{r_{mn}\}$ exists, for which $\nu(r_{mn}) \to \mu$ (otherwise, $r_{mn}$ would have poles on $E_1$, which would imply $Z_{mn} = \infty$). Note also, that as $R_1 \to \infty$, the corresponding part of $\mu_2$ converges to a point mass at $\infty$. Since the measures in $M_\tau^*$ are supposed to have a compact support, we see that $M_\tau^*$ may not be closed in the weak-star topology.

Finally, we could introduce here a weight by setting $Q = 0$ for $|z| = r$ and for $|z| = 1$, and by choosing $Q$ to be an arbitrary *positive* lower semicontinuous function for $R \leqslant |z| \leqslant 2R$. Since the value of the original $U^{\mu^*}$ on $|z| = r$ did not exceed any of its values on $R \leqslant |z| \leqslant 2R$, we easily deduce that, for the new problem, $\mu^*$, $F_1$, $F_2$ remain the same. Therefore $\mu$ that is given by (2.10), (2.11) is again in $M_\tau^*$, but now $Q$ may be unbounded on $S(\mu_2) \cap \text{Int } E_1$. Compare this with the assertion (ii) of Theorem 2.2, according to which $Q$ must be bounded on $S(\mu^*)$.

## 3. ASYMPTOTICS OF $Z_{mn}(w)$

THEOREM 3.1.  *Let $w$ be an admissible weight on $E_1 \cup E_2$, and let $N_\tau$ be a ray sequence (cf. (1.3)). Then*

$$\lim_{(m, n) \in N_\tau} \{Z_{mn}(w)\}^{1/(m + n)} = \exp(-F_{w, \tau}). \tag{3.1}$$

*Moreover, let $\mu = \mu_1 - \mu_2 \in M_\tau^*$ be such that*

$$S(\mu_1) \cap E_2 = \varnothing, \qquad S(\mu_2) \cap E_1 = \varnothing. \tag{3.2}$$

(*In particular, this holds for $\mu = \mu^*$.*) *Then an asymptotically extremal sequence $\{r_{mn}\}$, $(m, n) \in N_\tau$, exists for which*

$$v_{mn} := v(r_{mn}) \to \mu, \qquad (m, n) \in N_\tau. \tag{3.3}$$

*Proof.*   This follows the same lines as the proof in [LeSa, Sect. 6], so we will be brief.

For any $r \in \mathbf{R_{mn}}$ we have (recall (1.1), (1.5), (1.7), (2.4))

$$-\frac{1}{m + n} \log Z_{mn}(r; w) = \inf_{E_1}(U^{v_{mn}} + Q) - \sup_{E_2}(U^{v_{mn}} + Q)$$

$$\leqslant \text{``}\inf_{E_1}\text{''}(U^{v_{mn}} + Q) - \text{``}\sup_{E_2}\text{''}(U^{v_{mn}} + Q)$$

$$\leqslant F_{w, \tau_{mn}},$$

where

$$\tau_{mn} := \frac{m}{m + n} \to \tau, \qquad (m, n) \in N_\tau,$$

by (1.3).

Since $F_{w, \tau}$ is a concave (therefore continuous) function of $\tau$ on $(0, 1)$ (the proof is the same as in [LeSa, p. 242]), we conclude that for any sequence $N_\tau$ there holds

$$\liminf_{(m, n) \in N_\tau} \{Z_{mn}(w)\}^{1/(m+n)} \geqslant \exp(-F_{w, \tau}). \tag{3.4}$$

Next, since $\mu \in M_\tau^*$, we have $F_1(\mu) + F_2(\mu) = F_{w, \tau}$, and the exceptional sets

$$E_1(\varepsilon) := \{z \in E_1: U^\mu(z) + Q(z) \leqslant F_1(\mu) - \varepsilon\}$$
$$E_2(\varepsilon) := \{z \in E_2: U^\mu(z) + Q(z) \geqslant -F_2(\mu) + \varepsilon\}$$

have zero capacity, for any $\varepsilon > 0$.

Let $\sigma_\varepsilon \in M_\tau(E_1, E_2)$ be such that $U^{\sigma_\varepsilon} = +\infty$ on $E_1(\varepsilon)$, $U^{\sigma_\varepsilon} = -\infty$ on $E_2(\varepsilon)$. Then, given any $0 < \alpha < 1$, one can construct a sequence $\{v_{mn}\} \in M_{\tau_{mn}}$,

$$v_{mn} = \frac{1}{m+n} \left\{ \sum_{i=1}^m \delta_{\alpha_{im}} - \sum_{i=1}^n \delta_{\beta_{in}} \right\} =: v_{mn, 1} - v_{mn, 2},$$

where $\alpha_{im} \in E_1(\varepsilon) \cup S(\mu_1)$, $\beta_{in} \in E_2(\varepsilon) \cup S(\mu_2)$, and such that

$$v_{mn} \to (1-\alpha) \mu + \alpha\sigma_\varepsilon, \qquad (m, n) \in N_\tau.$$

By (3.2), we obtain that $U^{v_{mn, 1}} \to U^{(1-\alpha)\mu_1 + \alpha\sigma_{\varepsilon, 1}}$ on $E_2$, and $U^{v_{mn, 2}} \to U^{(1-\alpha)\mu_2 + \alpha\sigma_{\varepsilon, 2}}$ on $E_1$. Using these relations, the principle of descent and semicontinuity property of $Q$, one obtains as in [LeSa] that

$$\limsup_{(m, n) \in N_\tau} (Z_{mn}(r_{mn}, w))^{1/(m+n)} \leqslant \exp(-F_{w, \tau}(\alpha, \varepsilon)),$$

where

$$F_{w, \tau}(\alpha, \varepsilon) := (1-\alpha) F_{w, \tau} - 2(1-\alpha) \varepsilon - \alpha c(\varepsilon)$$

and $r_{mn} \in \mathbf{R_{mn}}$ is constructed by (1.5).

Now, on first letting $\alpha \to 0$ and then $\varepsilon \to 0$ and utilizing the standard diagonal procedure, one can construct some sequence $N_\tau$ of the form (1.3) such that $v_{mn} \to \mu$, $(m, n) \in N_\tau$, while

$$\limsup_{(m, n) \in N_\tau} \{Z_{mn}(r_{mn}; w)\}^{1/(m+n)} \leqslant \exp(-F_{w, \tau}).$$

Together with (3.4) this proves (3.1) as well as the fact that $r_{mn}$ is a desired sequence. The passage from *some* $N_\tau$ to *any* $N_\tau$ is simple (cf. [LeSa, pp. 255–256]). ∎

In many cases one can relax (3.2) to the condition

$$S(\mu_1) \cap \text{Int } E_2 = \varnothing, \qquad S(\mu_2) \cap \text{Int } E_1 = \varnothing. \qquad (3.5)$$

The idea is as follows. Given a set $D$ and $\varepsilon > 0$, let

$$D^\varepsilon := \{z: \text{dist}(z, D) < \varepsilon\} \qquad (3.6)$$

denote the open $\varepsilon$-neighborhood of $D$.

Take $\varepsilon > 0$ small enough so that $E_1^\varepsilon \cap E_2^\varepsilon = \varnothing$. Then, replace that part of $\mu_1$ (of $\mu_2$) that sits on $E_2^\varepsilon$ (on $E_1^\varepsilon$) by its balayage on the boundary $\partial E_2^\varepsilon$ ($\partial E_1^\varepsilon$). Let $\mu_1^\varepsilon$, $\mu_2^\varepsilon$ be the resulting measures. By the known properties of balayage, we have that $\mu^\varepsilon := \mu_1^\varepsilon - \mu_2^\varepsilon \in M_\tau$, while

$$F_1(\mu^\varepsilon) + F_2(\mu^\varepsilon) \geqslant F_1(\mu) + F_2(\mu).$$

Since $\mu \in M_\tau^*$, we obtain (see Theorem 2.2(iv)) that $\mu^\varepsilon \in M_\tau^*$ as well, but now $\mu^\varepsilon$ satisfies (3.2). If we knew that $\mu^\varepsilon \to \mu$ as $\varepsilon \to 0$, we could apply Theorem 3.1 to $\mu^\varepsilon$ and then, on letting $\varepsilon \to 0$ and using he diagonal procedure, we could construct the desired asymptotically extremal sequence $\{r_{mn}\}$.

Obviously, (3.5) is necessary for $\mu^\epsilon \to \mu$. Following are several conditions each of which ensures that $\mu^\epsilon \to \mu$.

THEOREM 3.2. *Let $\mu^\varepsilon$ be as above. Assume that* (3.5) *holds and, in addition, one of the following conditions is satisfied*:

(i) *We have*

$$\mu_1(\partial(\text{Int } E_2)^I) = \mu_2(\partial(\text{Int } E_1)^I) = 0, \qquad (3.7)$$

*where $\partial(\text{Int } E_i)^I \subset \partial E_i$ denotes the set of irregular points of all components of* Int $E_i$, $i = 1, 2$.

(ii) *$U^{\mu_1}$ is bounded on every component of* Int $E_2$, *while $U^{\mu_2}$ is bounded on every component of* Int $E_1$.

(iii) *$Q$ is bounded on the set $\partial E_1 \cup \partial E_2$.*

*Then $\mu^\varepsilon \to \mu$ as $\varepsilon \to 0$ and, consequently, the second assertion of Theorem* 3.1 *holds true.*

COROLLARY 3.3. *Assume*

$$\text{Int } E_1 = \text{Int } E_2 = \varnothing. \qquad (3.8)$$

*Then the second assertion of Theorem* 3.1 *holds for any $\mu \in M_\tau^*$.*

COROLLARY 3.4. *If every component of* Int $E_i$, $i = 1, 2$, *is regular, then the second assertion of Theorem* 3.1 *holds for any* $\mu \in M_\tau^*$ *that satisfies* (3.5).

COROLLARY 3.5. *If $Q$ is bounded on $E_1 \cup E_2$, then the condition* (3.5) *is necessary and sufficient for the second assertion of Theorem* 3.1 *to be true.*

These corollaries are obvious. Concerning the necessity of (3.5) in Corollary 3.5, we refer to the end of Section 2.

*Proof of Theorem* 3.2. We will show that our assumptions imply $\mu_1^\varepsilon \to \mu_1$ as $\varepsilon \to 0$. The proof of $\mu_2^\varepsilon \to \mu_2$ is similar. Let $\varepsilon_n \downarrow 0$. Then $\{U^{\mu_1^{\varepsilon_n}}\}$ is an increasing sequence majorized by $U^{\mu_1}$. Therefore there is a measure $\tilde{\mu}_1$ (obviously, $S(\tilde{\mu}_1) \cap \text{Int } E_2 = \varnothing$) such that $\mu_1^{\varepsilon_n} \to \tilde{\mu}_1$ and $U^{\mu_1^{\varepsilon_n}} \to U^{\tilde{\mu}_1}$ in **C**.

Since $U^{\mu_1^{\varepsilon_n}} = U^{\mu_1}$ outside $E_2^{\varepsilon_n}$, we see that $U^{\tilde{\mu}_1} = U^{\mu_1}$ in $\mathbf{C} \backslash E_2$. This implies that $U^{\tilde{\mu}_1} = U^{\mu_1}$ q.e. on $\partial E_2$. (Indeed, this is true, by definition for any point of $\partial E_2$ that is regular for $\mathbf{C} \backslash E_2$, and the set of irregular points has zero capacity, by Kellogg's Lemma [La, p. 232]).

It remains to show that $U^{\tilde{\mu}_1} = U^{\mu_1}$ in Int $E_2$, since then we obtain $U^{\tilde{\mu}_1} = U^{\mu_1}$ q.e. in **C**, which yields $\tilde{\mu}_1 = \mu_1$.

Let $G$ be any (connected) component of Int $E_2$. Assuming (i), we see that the set of irregular points of $G$ has $\mu_1$-measure zero. Let $z \in G$, $\delta_z$ be the point mass at $z$, and $\hat{\delta}_z$ be its balayage on $\partial G$. Since $U^{\delta_z} = U^{\hat{\delta}_z}$ outside $G$ and at regular points of $\partial G$, we have

$$U^{\mu_1}(z) = \int U^{\delta_z} \, d\mu_1 = \int U^{\hat{\delta}_z} \, d\mu_1 = \int U^{\mu_1} \, d\hat{\delta}_z,$$

the last equality following by Fubini's theorem. Next, as $\hat{\delta}_z$ is C-absolutely continuous (cf. [SaTo, p. 115]) and $U^{\mu_1} = U^{\tilde{\mu}_1}$ q.e. on $\partial G$, we obtain

$$\int U^{\mu_1} \, d\hat{\delta}_z = \int U^{\tilde{\mu}_1} \, d\hat{\delta}_z = \int U^{\hat{\delta}_z} \, d\tilde{\mu}_1.$$

Finally, since $U^{\hat{\delta}_z} \leqslant U^{\delta_z}$ everywhere, we have

$$\int U^{\hat{\delta}_z} \, d\tilde{\mu}_1 \leqslant \int U^{\delta_z} \, d\tilde{\mu}_1 = U^{\tilde{\mu}_1}(z).$$

It then follows that

$$U^{\mu_1}(z) \leqslant U^{\tilde{\mu}_1}(z), \qquad z \in G.$$

Since we also have $U^{\mu_1} \geqslant U^{\tilde{\mu}_1}$, we conclude that these potentials coincide in $G$.

Assume now that (ii) holds. Since $U^{\mu_1}$ is bounded from above on $G$ and $U^{\tilde{\mu}_1}$ is bounded from below, we see that $U^{\mu_1} - U^{\tilde{\mu}_1}$ is a nonnegative harmonic function in $G$ that is bounded in $G$ and equals 0 q.e. on $\partial G$. Since $G$ is connected, every point of $\partial G$ is a fine limit point of $G$. The continuity of potentials in the fine topology then yields, for q.e. $z \in \partial G$,

$$\lim_{\substack{\zeta \to z \\ \zeta \in G}} (U^{\mu_1} - U^{\tilde{\mu}_1})(\zeta) = U^{\mu_1}(z) - U^{\tilde{\mu}_1}(z) = 0.$$

The maximum principle then implies that $U^{\mu_1} = U^{\tilde{\mu}_1}$ in $G$.

Finally, we show that (iii) $\Rightarrow$ (ii). It suffices to prove that (iii) implies finiteness of $U^{\mu_1}$ at every point of $\partial E_2$, since then the above fine topology argument yields boundedness of $U^{\mu_1}$ in any component $G$ of Int $E_2$. Assume $U^{\mu_1}(z) = \infty$, $z \in \partial E_2$. Then $z \in \partial E_2 \cap S(\mu_1)$. We shall see in the next section that $S(\mu_1) \cap S(\mu_2) = \varnothing$ for any $\mu \in M_\tau^*$. Thus, $U^{\mu_2} \leqslant C$ in some neighborhood of $z$. Also, $Q \geqslant -C_1$ on $\partial E_2$. Since

$$U^{\mu_1} - U^{\mu_2} + Q \leqslant -F_2(\mu) \qquad \text{q.e. on } E_2,$$

we conclude that there exists a disk $D_z$, centered at $z$, such that $\text{cap}(D_z \cap \partial E_2) = 0$ (otherwise, the lower semicontinuity of $U^{\mu_1}$ would imply $U^{\mu_1}(z) \leqslant -F_2(\mu) + C + C_1$). But this is impossible, since $D_z$ contains points of $G \subset \text{Int } E_2$ as well as points of $\mathbf{C} \backslash E_2$. Therefore $D_z$ must contain a continuum belonging to $\partial E_2$, which implies that $\text{cap}(D_z \cap \partial E_2) > 0$. ∎

Unfortunately, we were not able to find a satisfactory necessary and sufficient condition for the second assertion of Theorem 3.1 to be true.

## 4. SOME PROPERTIES OF $M_\tau^*$

Consider the condenser $(S(\mu_1^*), S(\mu_2^*))$, where $\mu^* = \mu_1^* - \mu_2^*$ is the extremal measure for Problem (2.1). Let $U^\sigma$ be the corresponding condenser potential, that is $\sigma = \sigma_1 - \sigma_2$, $\|\sigma_i\| = 1$, $\sigma_i \geqslant 0$, $S(\sigma_i) \subseteq S(\mu_i^*)$, $i = 1, 2$, while for some constants $a_1 \geqslant 0$, $a_2 \geqslant 0$,

$$U^\sigma = \begin{cases} a_1 & \text{q.e. on } S(\mu_1^*) \\ -a_2 & \text{q.e. on } S(\mu_2^*). \end{cases} \tag{4.1}$$

Therefore

$$-a_2 \leqslant U^\sigma(z) \leqslant a_1, \qquad z \in \bar{\mathbf{C}}. \tag{4.2}$$

Now, by (2.3) of Theorem 2.2, we have

$$\int U^{\mu^*} d\sigma = \int (U^{\mu^*} + Q) \, d\sigma - \int Q \, d\sigma = F_1 + F_2 - \int Q \, d\sigma,$$

while

$$\int U^\sigma \, d\mu^* = \tau a_1 + (1 - \tau) \, a_2$$

by (4.1). The Fubini theorem then yields

$$\tau a_1 + (1 - \tau) \, a_2 = F_1 + F_2 - \int Q \, d\sigma. \tag{4.3}$$

Next, by the definition of $M_\tau^*$, we have for any $\mu \in M_\tau^*$,

$$\int U^\mu \, d\sigma \geq F_1(\mu) + F_2(\mu) - \int Q \, d\sigma = F_1 + F_2 - \int Q \, d\sigma,$$

while

$$\int U^\sigma \, d\mu \leq \tau a_1 + (1 - \tau) \, a_2$$

by (4.2). The Fubini theorem and (4.3) then yield

$$\int U^\sigma \, d\mu = \tau a_1 + (1 - \tau) \, a_2, \tag{4.4}$$

$$\int (U^\mu + Q - F_1(\mu)) \, d\sigma_1 - \int (U^\mu + Q + F_2(\mu)) \, d\sigma_2 = 0. \tag{4.5}$$

After these preliminaries we turn to the study of $M_\tau^*$.

LEMMA 4.1. *Let* $\mu = \mu_1 - \mu_2 \in M_\tau^*$. *Then*

(i) $S(\mu_1) \subseteq S(\sigma_1) \cup \{z : U^\sigma(z) = a_1\}$, $S(\mu_2) \subseteq S(\sigma_2) \cup \{z : U^\sigma(z) = -a_2\}$. *In particular,* $S(\mu_1) \cap S(\mu_2) = \varnothing$.

(ii) $U^\mu + Q \leq F_1(\mu)$ *everywhere on* $S(\sigma_1)$, $U^\mu + Q \geq -F_2(\mu)$ *everywhere on* $S(\sigma_2)$. *In particular,* $U^\mu$ *is bounded on* $S(\sigma)$.

(iii)

$$U^\mu + Q = \begin{cases} F_1(\mu) & \text{q.e. on } S(\sigma_1) \\ -F_2(\mu) & \text{q.e. on } S(\sigma_2). \end{cases}$$

*Proof.* Assume $z \notin S(\sigma_1) \cup \{z : U^{\sigma}(z) = a_1\}$. Then $U^{\sigma}(z) < a_1$ and $z \notin S(\sigma_1)$. Therefore $U^{\sigma}$ is upper semicontinuous in some disk $D_z$ centered at $z$. This implies that, for some $\varepsilon > 0$, $U^{\sigma} \leqslant a_1 - \varepsilon$ in $D_z$. By (4.4), (4.2) we conclude that $\mu_1(D_z) = 0$, so that $z \notin S(\mu_1)$. The proof of the second inclusion in (i) is similar.

Next, since $S(\mu_1) \cap S(\mu_2) = \varnothing$, the function $U^{\mu} + Q$ is lower (upper) semicontinuous on $S(\sigma_1)$ (on $S(\sigma_2)$). Since the first integrand in (4.5) is non-negative q.e. on $S(\sigma_1)$ and the second is non-positive q.e. on $S(\sigma_2)$, we obtain

$$U^{\mu} + Q = F_1(\mu), \qquad \sigma_1\text{-a.e. on } S(\sigma_1),$$
$$U^{\mu} + Q = -F_2(\mu), \qquad \sigma_2\text{-a.e. on } S(\sigma_2).$$

These equalities and the semicontinuity of $U^{\mu}$ and $Q$, prove (ii). Part (iii) then follows by the definitions of $F_1(\mu)$, $F_2(\mu)$. ∎

COROLLARY 4.2. *Assume that* $\operatorname{Int} E_1 = \operatorname{Int} E_2 = \varnothing$, *and the complement of* $E_1 \cup E_2$ *is connected. Then* $M_\tau^* = \{\mu^*\}$; *that is, the solution of problem* (2.2) *is unique.*

*Proof.* Let $\mu = \mu_1 - \mu_2 \in M_\tau^*$. Our assumption implies (via Lemma 4.1(i)):

$$S(\mu_i) \subseteq S(\mu_i^*) = S(\sigma_i), \qquad i = 1, 2. \tag{4.6}$$

Therefore, the potential $U^{\mu - \mu^*}$ is harmonic in $\overline{\mathbf{C}} \backslash S(\mu^*)$ and equals 0 at $\infty$.

Since $\mu, \mu^* \in M_\tau^*$, we have

$$F_1(\mu) - F_1(\mu^*) = -F_2(\mu) + F_2(\mu^*).$$

Then, by Lemma 4.1(iii), we obtain that $U^{\mu - \mu^*} = \operatorname{const}$ q.e. on its support. Next, by Lemma 4.1(ii), $U^{\mu}$ is bounded from above (from below) on $S(\mu_1^*)$ (on $S(\mu_2^*)$). Therefore (see (4.6)), $U^{\mu}$ is bounded on compact subsets of $\mathbf{C}$. So is $U^{\mu^*}$. Hence $U^{\mu - \mu^*}$ is a bounded harmonic function in $\overline{\mathbf{C}} \backslash S(\mu^*)$, which is constant q.e. on $S(\mu^*)$ and 0 at $\infty$. Therefore it is identically zero, which gives $\mu = \mu^*$. ∎

LEMMA 4.3. *All the measures from* $M_\tau^*$ *have the same balayage on* $S(\sigma)$.

*Remark.* It is not true that they have the same balayage on $S(\mu^*)$. See Example 2.4, in which

$$S(\mu_1^*) = \{z : |z| = 1\}, \qquad S(\mu_2) = \{z : |z| = R_1\} \cup \{z : |z| = R_2\}$$

and there is $\mu \in M_\tau^*$ such that $S(\mu_1) = \{z : |z| = 1\}$, $S(\mu_2) = \{z : |z| = R_2\}$.

*Proof of Lemma* 4.3. We prove that for any $\mu \in M_\tau^*$, $\hat{\mu} = \hat{\mu}^*$, where the hat symbol stands for the balayage on $S(\sigma)$. It is known, that for some constants $c_{1i}$, $c_{2i}$

$$U^{\hat{\mu}_i} = U^{\mu_i} + c_{1i} \qquad \text{q.e. on } S(\sigma_i)$$

$$U^{\hat{\mu}_i^*} = U^{\mu_i^*} + c_{2i} \qquad \text{q.e. on } S(\sigma_i),$$

while $\leqslant$ sign holds everywhere in $\mathbf{C}$. Reasoning as in the proof of Corollary 4.2, we see that $U^{\hat{\mu} - \hat{\mu}^*}$ is a bounded harmonic function in $\bar{\mathbf{C}} \backslash S(\sigma)$, equal to 0 at $\infty$, and constant q.e. on $S(\sigma)$. Hence $\hat{\mu} = \hat{\mu}^*$. ∎

We have seen (in Example 2.5) that, for $\mu_1 - \mu_2 \in M_\tau^*$, the set $S(\mu_2)$ may intersect $E_1$, but then "$\inf_{E_1}$"$(U^\mu + Q)$ is attained on $E_1 \backslash S(\mu_2)$. This is a general feature of $\mu \in M_\tau^*$.

LEMMA 4.4. *Let* $\mu \in M_\tau^*$, *and assume* $S(\mu_2) \cap E_1 \neq \varnothing$. *Then*

(i) $E_1$ *is not connected*;

(ii) $E_1$ *can be decomposed into the union of two disjoint compacta* $E_1'$, $E_1''$, *such that*

$$\text{cap } E_1' > 0, \qquad S(\mu_2) \cap E_1' = \varnothing, \qquad E_1'' \neq \varnothing$$

*and*

$$\text{"}\inf_{E_1'}\text{"}(U^\mu + Q) = F_1(\mu). \tag{4.7}$$

*Similar assertions hold for* $E_2$, *given that* $S(\mu_1) \cap E_2 \neq \varnothing$.

*Proof.* Fix $\varepsilon > 0$ small enough, so that $E_1^\varepsilon \cap E_2 = \varnothing$ (recall the notation (3.6)). $E_1^\varepsilon$ is a finite union of disjoint domains. Let $G_1, ..., G_k$ be those domains for which

$$S(\mu_2) \cap G_i = \varnothing, \qquad i = 1, ..., k,$$

and let $G_{k+1}, ..., G_l$ be the remaining domains, so that

$$S(\mu_2) \cap G_i \neq \varnothing, \qquad i = k+1, ..., l.$$

Let

$$E_1' = E_1 \cap \left( \bigcup_{i=1}^{k} G_i \right), \qquad E_1'' = E_1 \cap \left( \bigcup_{i=k+1}^{l} G_i \right).$$

By assumption, $E_1'' \neq \varnothing$. For $k+1 \leqslant i \leqslant l$, let $K_i := E_1^{\varepsilon/2} \cap G_i$. Then $K_i$ is a compact subset of $G_i$, and, by assumption, $\mu_2(K_i) > 0$. Let us sweep out that part of $\mu_2$ onto $\partial G_i$, and let $\tilde{\mu} = \mu_1 - \tilde{\mu}_2$ be the resulting measure.

Inside each $G_i$ we have

$$U^{\tilde{\mu}} = U^\mu + \int_{K_i} g_{G_i}(\cdot, t) \, d\mu_2(t), \tag{4.8}$$

where $g_G(\cdot, t)$ stands for the Green function of a domain $G$ with a pole at $t$. We shall prove in a moment that for any compact $K \subset G$,

$$\inf_{z, \, t \in K} g_G(z, t) \geqslant c > 0, \qquad c = c(K). \tag{4.9}$$

Assuming this, we deduce from (4.8) that

$$\underset{E_1''}{\text{"inf"}} (U^{\tilde{\mu}} + Q) > F_1(\mu).$$

Also, $U^{\tilde{\mu}} = U^\mu$ outside $\bigcup_{i=k+1}^l G_i$. In particular this holds on $E_2$ and on $E_1'$. Therefore, if we had

$$\operatorname{cap} E_1' = 0 \qquad \text{or} \qquad \underset{E_i'}{\text{"inf"}} (U^\mu + Q) > F_1(\mu),$$

we would obtain that

$$F_1(\tilde{\mu}) + F_2(\tilde{\mu}) > F_1(\mu) + F_2(\mu),$$

contradicting (2.4) of Theorem 2.2 (recall that $\mu \in M_\tau^*$). It thus remains to prove (4.9).

Write $g(z, t) = \log |z - t|^{-1} + u(z, t)$, where $t \in K$ and $u(z, t) = \log |z - t|$ for q.e. $z$ on $\partial G$. Note that for any $t \in K$, $u(z, t)$ is a *bounded* harmonic function in $G$ (cf. [HaKe, p. 250]).

Since for $t, \, t' \in K$, $z \in \partial G \setminus I$, $\operatorname{cap} I = 0$ we have

$$u(z, t) - u(z, t') = \log \left| \frac{z - t}{z - t'} \right| = O(|t - t'|)$$

uniformly for $t, \, t' \in K$, $z \in \partial G \setminus I$, the maximum principle yields

$$|u(z, t) - u(z, t')| \leqslant c \, |t - t'|, \qquad z \in G, \quad t, \, t' \in K.$$

Applying the symmetry of $u(z, t)$ we see that $u$ is jointly continuous on $K \times K$. This immediately gives (4.9). ∎

We also mention a simple property that follows directly from Lemma 4.1(i):

LEMMA 4.5. *Let* $\mu \in M_\tau^*$ *and let* $G$ *be any* (*connected*) *component of* $\mathbf{C} \backslash S(\sigma)$. *Then* $\mu_2(G) = 0$ *if* $\partial G \subset S(\mu_1)$, *and* $\mu_1(G) = 0$ *if* $\partial G \subset S(\mu_2)$. *If* $\partial G$ *contains points of both* $S(\sigma_1)$ *and* $S(\sigma_2)$, *then* $\mu_1(G) = \mu_2(G) = 0$.

Finally, we present a result that strengthens part (iv) of Theorem 2.2.

THEOREM 4.6. *For any* $\mu \in M_\tau$ *there holds*

$$\text{``}\inf_{S(\sigma_1)}\text{''}(U^\mu + Q) - \text{``}\sup_{S(\sigma_2)}\text{''}(U^\mu + Q) \leqslant F_1 + F_2. \tag{4.10}$$

*If equality holds, and* $S(\mu) \subset S(\sigma)$, *then* $\mu = \hat{\mu}^*$ (*the balayage of* $\mu^*$ *on* $S(\sigma)$).

*Proof.* We have

$$\tau a_1 + (1 - \tau)\, a_2 \geqslant \int U^\sigma \, d\mu = \int U^\mu \, d\sigma$$

$$\geqslant \text{``}\inf_{S(\sigma_1)}\text{''}(U^\mu + Q) - \text{``}\sup_{S(\sigma_2)}\text{''}(U^\mu + Q) - \int Q \, d\sigma.$$

The result now follows, in view of (4.3).

If equality holds and $S(\mu) \subset S(\sigma)$, we consider $U^{\mu - \hat{\mu}^*}$ and, reasoning as in the proof of Corollary 4.2, obtain that $U^{\mu - \hat{\mu}^*} = 0$. ∎

## 5. CHARACTERIZATION OF WEAK-STAR LIMIT POINTS OF $v(R_{mn})$

THEOREM 5.1. *Let* $w$ *be an admissible weight on* $E_1 \cup E_2$, *and let* $\{r_{mn}\}$ *be an asymptotically extremal sequence. Assume* $v_{mn} := v(r_{mn}) \to \mu$, $(m, n) \in N_\tau$. *Then we have*:

   (i)   $\mu \in M_\tau^*$, *if* $S(\mu)$ *is compact*;
   (ii)  $\mu \in \bar{M}_\tau^*$, *if* $S(\mu)$ *is not compact*.

*More precisely, let* $R > 0$ *be large enough so that the disk* $D_R = \{z : |z| < R\}$ *contains* $E_1$, $E_2$. *Replace that part of* $\mu$ *that sits outside* $D_R$ *by its balayage onto* $\partial D_R$, *and let* $\mu^R$ *denote the resulting measure. Then* $\mu_R \in M_\tau^*$, *and* $\mu^R \to \mu$ *as* $R \to \infty$.

*Proof.*   It is given that

$$v_{mn} =: v_{mn, 1} - v_{mn, 2} \to \mu_1 - \mu_2 = \mu, \tag{5.1}$$

while

$$F_1(v_{mn}) + F_2(v_{mn}) \to F_1 + F_2, \qquad (m, n) \in N_\tau. \tag{5.2}$$

(i)   If $S(\mu)$ is compact, take $R$ large enough so that the disk $D_R$ contains $S(\mu)$ as well as $E_1$, $E_2$. Then $v_{mn}$ has a mass $o(1)$ on $|z| > R$. Replace this part of $\mu$ by its balayage on $|z| = R$. This will add a constant to $U^{v_{mn}}$ in $D_R$, so that (5.2) will hold for a new $\mu$ as well. Also, the limit of $v_{mn}$ remains the same. Thus, we may assume that $S(v_{mn}) \subset D_R$, for all $(m, n) \in N_\tau$.

Let $\varepsilon > 0$ be small enough. If we know that

$$v_{mn, 1}(E_2^\varepsilon) = o(1), \qquad v_{mn, 2}(E_1^\varepsilon) = o(1), \qquad (m, n) \in N_\tau, \tag{5.3}$$

we proceed as follows. Take the balayage of $v_{mn, 1}$ (of $v_{mn, 2}$) from $E_2^\varepsilon(E_1^\varepsilon)$ onto $\partial E_2^\varepsilon(\partial E_1^\varepsilon)$. This will not change the limit measure, while the quantity $F_1(v_{mn}) + F_2(v_{mn})$ can only increase. Therefore (by (5.2) and Theorem 2.2 (iv)), the relation (5.2) will hold for a new $v_{mn}$. But now we have, for $z \in E_1$,

$$U^{v_{mn, 1}}(z) - U^{v_{mn, 2}}(z) + Q(z) \geqslant \frac{m}{m+n} \log \frac{1}{R} - \frac{n}{m+n} \log \frac{1}{\varepsilon} + \min_{E_1} Q$$

which implies that the sequence $\{F_1(v_{mn})\}$ is bounded from below. Similarly, one can show that $\{F_2(v_{mn})\}$ is bounded from below. Then (5.2) implies that these sequences are bounded. Passing to a subsequence, we may assume that for some constant $A$,

$$F_1(v_{mn}) \to F_1 + A, \qquad F_2(v_{mn}) \to F_2 - A, \qquad (m, n) \in N_\tau.$$

Since

$$U^{v_{mn, 2}}(z) \leqslant U^{v_{mn, 1}}(z) + Q(z) - F_1(v_{mn}) \qquad \text{q.e. on } E_1,$$

the lower envelope theorem yields

$$U^{\mu_2}(z) \leqslant U^{\mu_1}(z) + Q(z) - F_1 - A, \qquad \text{q.e. on } E_1.$$

Thus, $F_1(\mu) \geqslant F_1 + A$. Similarly, $F_2(\mu) \geqslant F_2 - A$, and we conclude (by Theorem 2.2(iv)) that $\mu \in M_\tau^*$. If (5.3) does not hold, we apply the same

reasoning as in the proof of Lemma 4.4 and show that there exist $E_1' \subset E_1$, $E_2' \subset E_2$ such that $F_1(v_{mn})$, $F_2(v_{mn})$ are attained on $E_1'$, $E_2'$ respectively, while

$$v_{mn, 1}(E_2') = o(1), \qquad v_{mn, 2}(E_1') = o(1), \qquad (m, n) \in N_\tau.$$

Reasoning as above, we conclude that the sequences $\{F_i(v_{mn})\}$, $i = 1, 2$, are bounded, and the rest of the proof remains the same.

(ii) If $S(\mu)$ is not compact, take $R$ large enough and replace that part of $v_{mn}$ that sits on $|z| > R$ by its balayage on $|z| = R$. Let $v_{mn}^R$ be the resulting measure. Then

$$F_1(v_{mn}^R) + F_2(v_{mn}^R) = F_1(v_{mn}) + F_2(v_{mn}) \to F_1 + F_2,$$

and $v_{mn}^R \to \mu^R$, $(m, n) \in N_\tau$. By what we have proved, $\mu^R \in M_\tau^*$. The relation $\mu^R \to \mu$ as $R \to \infty$ is obvious. ∎

COROLLARY 5.2. *Assume* Int $E_1 = $ Int $E_2 = \varnothing$ *and* $\mathbf{C} \backslash (E_1 \cup E_2)$ *is connected. Then for any asymptotically extremal sequence* $\{r_{mn}\}$ *we have* $v(r_{mn}) \to \mu^*$.

*Proof.* Apply Corollary 4.2. ∎

Other corollaries concerning the limit points of $v(r_{mn})$ can be drawn, using the results of Section 3. However, we do not have a complete description of all limit points of $v(r_{mn})$ in the general case. Yet, we do have the following result, which generalizes the result due to Mhaskar and Saff [MhSa] for the polynomial case.

THEOREM 5.3. *Let* $\{r_{mn}\}$ *be asymptotically extremal. Modify* $v_{mn}$ *in the following way. Let* $G$ *be any connected component of* $\mathbf{C} \backslash S(\sigma)$. *If* $\partial G \subset S(\sigma_1)$ *($\partial G \subset S(\sigma_2)$) replace* $v_{mn, 1}$ *($v_{mn, 2}$) restricted to* $G$ *by its balayage onto* $\partial G$. *Then the resulting distribution,* $\tilde{v}_{mn}$, *converges weak-star to* $\hat{\mu}^*$, *the balayage of* $\mu^*$ *onto* $S(\sigma)$.

*Proof.* Clearly, we have

$$\text{``}\inf_{S(\sigma_1)}\text{''}(U_{mn}^{\tilde{v}} + Q) - \text{``}\sup_{S(\sigma_2)}\text{''}(U_{mn}^{\tilde{v}} + Q) \geqslant F_1(v_{mn}) + F_2(v_{mn}).$$

Note that if $G$ is as above, then $|\tilde{v}_{mn}(K)| = o(1)$ for any compact $K \subset G$. Indeed, if $\partial G \subset S(\sigma_i)$, $i = 1$ or $i = 2$, this is true by construction, and otherwise we have $-a_2 < U^\sigma < a_1$ in $K$, and we may appeal to the beginning of Section 4. Thus, if $v$ is any weak-star limit of $\{\tilde{v}_{mn}\}$, then $S(v) \subset S(\sigma)$.

Reasoning as in the proof of Theorem 5.1, we find that for this $v$, equality holds in (4.10), and the result follows from Theorem 4.6.   ∎

## 6. THE CASE OF UNBOUNDED SETS

Assume first that only one set, say $E_1$, is unbounded, and $Q$ satisfies (i), (ii) of Definition 2.1.

Pick $c \notin E_1 \cup E_2$ and apply the Möbius transformation

$$z = c + (\zeta - c)^{-1} =: f(\zeta).$$

This will transform (cf. [LeSa]) problems (2.1), (2.2) into similar ones, with compacta $\tilde{E}_1$, $\tilde{E}_2$ and with $Q$ replaced by

$$\tilde{Q}(\zeta) := Q(f(\zeta)) + (1 - 2\tau) \log \frac{1}{|\zeta - c|}, \qquad c \in \tilde{E}_1.$$

By the assumptions (i), (ii) on $Q$, $\tilde{Q}$ satisfies these as well, except perhaps at $\zeta = c$. To ensure that $\tilde{Q}$ is lower semicontinuous at $c$, one can impose the following additional condition on $Q$:

   (iii)   $Q(z) + (1 - 2\tau) \log |z|$ is lower semicontinuous at $\infty$.

If both sets are unbounded (but a positive distance apart) one can impose a stronger condition (cf. [SaTo]) on $Q$, namely

   (iii)′   $Q(z) - \log |z| \to \infty$ as $z \in E_1 \cup E_2 \to \infty$.

This forces $\mu^*$ to have a compact support (see [SaTo] for details).

## 7. APPLICATION TO MINIMAL BLASCHKE PRODUCTS

Let $\Omega$ be an arbitrary domain in $\mathbf{C}$ and let $E \subset \Omega$ be compact with cap $E > 0$. Given $w = \exp(-Q)$ (with $Q$ satisfying (i), (ii) of Definition 2.1 on $E_1 := E$), consider the quantity

$$\delta_n(E) := \inf_{B_n} \|B_n w^n\|_E, \tag{7.1}$$

where

$$B_n(z) := \exp \left\{ -\sum_1^n g(z; \alpha_k) - i \sum_1^n \tilde{g}(z; \alpha_k) \right\}, \qquad \alpha_k \in \Omega$$

($\tilde{g}$ stands for the (multiple-valued) conjugate function of the generalized Green function $g$ of $\Omega$).

Let $v_n := n^{-1} \sum_{k=1}^{n} \delta_{\alpha_k}$, and let $\hat{v}_n$ be the balayage of $v_n$ onto $\partial\Omega$. It is known that

$$\frac{1}{n} \sum_{1}^{n} g(z; \alpha_k) = U_n^v(z) - U_n^{\hat{v}}(z) + c_n, \tag{7.2}$$

where $c_n = 0$ if $\Omega$ is bounded and $c_n = n^{-1} \sum_{1}^{n} g(\infty; \alpha_k)$ otherwise.

Thus, if we define $Q$ to be zero on $\partial\Omega$ and set $\mu_n := v_n - \tilde{v}_n$, we can rewrite (7.1) as

$$\frac{1}{n} \log \frac{1}{\delta_n} = \sup_{\mu_n} \left\{ \inf_E (U^\mu + Q) - \sup_{\partial\Omega} (U^\mu + Q) \right\}. \tag{7.3}$$

Consider the energy problem

$$V := \inf \left\{ I(\mu) + 2 \int Q \, d\mu \colon \mu = \mu_1 - \mu_2, \, S(\mu_1) \subset E, \, S(\mu_2) \subset \partial\Omega \right\},$$

$$\|\mu_1\| = \|\mu_2\| = 1.$$

This problem coincides with (2.1) for $\tau = 1/2$, except that $\|\mu_i\| = 1$ instead of $1/2$.

Let $\mu^* = \mu_1^* - \mu_2^*$ be the equilibrium distribution for this problem. Then [SaTo] $\mu_2^*$ coincides with $\hat{\mu}_1^*$, the balayage of $\mu_1^*$ onto $\partial\Omega$. With this observation in mind, the following results follow from Theorems 3.1 and 5.3:

(i) $\lim_{n \to \infty} n^{-1} \log(\delta_n^{-1}) = V$;

(ii) Let $\{B_n\}$ be asymptotically extremal. Note that in our case $S(\sigma_2) = \partial\Omega$, $S(\sigma_1) = $ boundary of polynomial convex hull (relative to $\Omega$) of $E$. Modifying $v_n$ as usual, we obtain $\hat{v}_n \to \hat{\mu}^*$ ($\wedge$ means balayage on $S(\sigma)$).

These assertions should be compared with those of Fisher and Saff [FiSa].

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