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Best Meromorphic Approximation of Markov Functions
on the Unit Circle

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Abstract. Let $E \subset (-1, 1)$ be a compact set, let μ be a positive Borel measure with support $\text{supp } \mu = E$, and let $H_p(G)$, $1 \leq p \leq \infty$, be the Hardy space of analytic functions on the open unit disk G with circumference $\Gamma = \{z: |z| = 1\}$. Let $\Delta_{n,p}$ be the error in best approximation of the Markov function

$$\frac{1}{2\pi i} \int_E \frac{d\mu(x)}{z-x}$$

in the space $L_p(\Gamma)$ by meromorphic functions that can be represented in the form $h = P/Q$, where $P \in H_p(G)$, Q is a polynomial of degree at most n , $Q \not\equiv 0$. We investigate the rate of decrease of $\Delta_{n,p}$, $1 \leq p \leq \infty$, and its connection with n -widths. The convergence of the best meromorphic approximants and the limiting

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distribution of poles of the best approximants are described in the case when $1 < p \leq \infty$ and the measure μ with support $E = [a, b]$ satisfies the Szegő condition

$$\int_a^b \frac{\log(d\mu/dx)}{\sqrt{(x-a)(b-x)}} dx > -\infty.$$

1. Introduction

1.1. Overview

Let G denote the open unit disk with center at zero in the complex plane \mathbb{C} , and let Γ denote its boundary. Our goal is to investigate problems of best meromorphic approximation on Γ to Markov functions of the form

$$f(z) = \frac{1}{2\pi i} \int_E \frac{d\mu(x)}{z-x}, \quad (1.1)$$

where μ is a positive Borel measure with support $\text{supp } \mu = E \subset (-1, 1)$. The approximating meromorphic functions are assumed to belong to the class

$$\mathcal{M}_{n,p} = \left\{ \frac{P}{Q} : P \in H_p(G), Q \text{ a polynomial, } \deg \leq n \right\},$$

where $H_p(G)$ denotes the p -norm Hardy space of analytic functions on G , $1 \leq p \leq \infty$, and $n = 0, 1, 2, \dots$

With $\Delta_{n,p}$ denoting the error in best $L_p(\Gamma)$ approximation from $\mathcal{M}_{n,p}$ to the Markov function f , we show in Sections 1.5 and 1.6 that there is a fundamental connection between $\Delta_{n,p}$ and the Kolmogorov, Gel'fand, and linear n -widths in $L_2(\mu, E)$ of the restriction to E of the unit ball of $H_{2q}(G)$, where $1/p + 1/q = 1$. We remark that n -widths are important in approximation theory since they enable one to obtain best or near-best methods of approximation and interpolation as well as to estimate the errors in these methods (see, e.g., Fisher and Micchelli [18], Fisher [19], Fisher and Stessin [20]). In connection with n -widths we also mention the papers of Levin and Saff [32], and Parfenov [37]–[40] relating to an investigation of Szegő-type asymptotics for minimal Blaschke products.

For the special case when $p = \infty$, there is a well-known connection between the best meromorphic approximation problem and the theory of Hankel operators, which is provided by the so-called AAK theory [2], [3]. In Section 1.5 we further describe the relationship between the Hankel operator with symbol f and the embedding operator of $H_2(G)$ into $L_2(\mu, E)$.

Section 2 presents more formulas involving the best approximation error as well as the orthogonality properties inherent to this approximation problem.

In the case when $1 \leq p \leq \infty$, the measure μ has support $E = [a, b]$ and satisfies the Szegő condition

$$\int_a^b \frac{\log(d\mu/dx)}{\sqrt{(x-a)(b-x)}} dx > -\infty,$$

we give in Theorem 4 of Section 3 the precise rate of decrease (as $n \rightarrow \infty$) of the best approximation errors $\Delta_{n,p}$. Moreover, for $1 < p \leq \infty$, we find the limiting distribution of the poles of the best approximants (cf. (3.11) in Section 3), and in Theorem 6 of Section 3 we describe the region and pointwise rate of convergence of these approximants. The proofs of Theorems 4 and 6 are given, respectively, in Sections 5 and 6. The methods used are based on an investigation of Szegő-type asymptotics for orthogonal polynomials with varying weight functions. In this connection we appeal to results of Totik [48] and Stahl [46].

Problems concerning the best uniform rational approximation of Markov functions were studied by Gonchar [24] and W. Barrett [14]. One of the basic results relating to the convergence of Padé approximants is the classical theorem of Markov (see [35], [5]). Convergence questions for multipoint Padé approximants for Markov functions were considered by Gonchar and Lopez [25], Totik [48], Stahl and Totik [47]. We also mention the papers of D. Braess [15] and J.-E. Andersson [4] relating to the case when the Markov function is approximated by meromorphic functions. The results of the latter paper play a useful role in our present investigation (see Section 1.6). Questions of approximation of Markov functions in the space $L_2(\Gamma)$ by rational functions were investigated by Baratchart, Stahl, and Wielonsky [11], [12], [13].

1.2. Motivation

In this section, we describe some motivation for our results that pertain to computational problems in control.

First of all, analytic approximation in the uniform norm on the unit circle or the imaginary axis, whose link to spectral properties of Hankel operators was initially stressed in [36], has become a cornerstone of robust control design, ever since the connection with norm minimization of transfer functions under output feedback was recognized in [52], see, for instance, [21], [17] for a comprehensive account. Shortly after, the remarkable generalization of the Nehari theory to meromorphic approximation, known as AAK theory [3], became a popular method for model reduction in control design as it provided a suboptimal but easy method for uniform rational approximation on the circle or on the line. Indeed, the required spectral decomposition of the Hankel operator was nicely constructive, at least for rational functions, and the seminal work in [23] made it possible to bound the distance to the optimal error when truncation of the analytic part of a best meromorphic approximant was used to obtain a rational function which is analytic at infinity (this last requirement being essential for control purposes).

Meanwhile, rational approximation on the circle or the line in the (usually weighted) L^2 -norm has been the traditional setting for identification of linear control systems, mean square criteria being here induced by the minimization of the variance of the error in a stochastic context, see, for instance, [27], [33].

The difficulty is to relate these two types of approximation, and the fact that estimates of the Fourier coefficients, when corrupted by noise, yield very poor

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the case when $1 < p < \infty$. The Szegő condition

in the complex plane \mathbb{C} , and let $\mathcal{M}_{n,p}$ be the class of best meromorphic

(1.1)

$\Gamma \subset (-1, 1)$. The class

$\{f_n\}$,

functions on G ,

from $\mathcal{M}_{n,p}$ to the class $\mathcal{M}_{n,p}$ is a fundamental problem. Near n -widths in L^p spaces, since they enable us to study interpolation as well as approximation, and since they also relate to the n -widths we also have [10] relating to an approximation problem.

The connection between the approximation of Hankel operators, in Section 1.5 we further study the approximation of a meromorphic function f and the

approximation error as a function of n .

$E = [a, b]$ and

bounds via [23], is largely responsible for almost separate algorithmic developments between identification and design in the field of control.

Now, meromorphic approximation in L^p of the circle, or of the line, stresses a continuous link between AAK theory, when $p = \infty$, and L^2 best meromorphic approximation which is in fact equivalent to L^2 best rational approximation by the orthogonality between the Hardy space of the disk and its shifted conjugate. Moreover, this link is continuous with respect to p in a natural sense [10], [43]. Therefore, the effort made in the present paper to compare these approximants can be regarded as an attempt to unify the identification and the design point of view. Of course, the only results available so far concern Markov functions, and in the scalar (i.e., nonmatrix) case only. However, Markov functions have an interesting system-theoretic status as transfer functions of so-called relaxation systems [51], [16], and the authors believe that the present study lays ground for more general classes of functions.

It is of course natural to ask whether computing error rates, as is done in the present paper, are helpful to solve constructively approximation problems. One obvious answer is that it allows one to decide if a given procedure is far from being optimal. There is yet another answer which is less obvious, namely, that the regularity of the decay of the error is deeply linked with uniqueness of a critical point in rational and meromorphic approximation, and such uniqueness issues are themselves the key to convergence properties of a numerical search, see [9], [12], [10] and also [6] for a general discussion.

In another connection, let us finally mention that meromorphic approximation was recently proposed in [7], [34] as a tool to approach some inverse problems for the two-dimensional Laplacian such as "crack detection." Roughly speaking, computing a meromorphic approximant to the complexified solution of a Dirichlet-Neuman problem on the outer boundary of a plane domain raises the question as to whether the poles of the approximant give information on the singular set of the solution, namely, the unknown boundary (i.e., the crack). The answer is "yes" under appropriate regularity assumptions on the crack, and determining the asymptotic distribution of these poles, as is done here in the case of Markov functions, becomes crucial in this respect.

1.3. Notation

Let E be a compact set, let $E \subset (-1, 1)$, and let μ be a finite positive Borel measure with the support $\text{supp } \mu = E$. We assume that E contains infinitely many points. Let $1 \leq p \leq \infty$ and $1/p + 1/q = 1$, and let n be a positive integer.

Here and in what follows we adopt to the following notation:

$L_p(\Gamma)$, $1 \leq p < \infty$, is the Lebesgue space of functions φ measurable on Γ such that

$$\|\varphi\|_p = \left(\int_{\Gamma} |\varphi(\xi)|^p |d\xi| \right)^{1/p} < \infty. \quad (1.2)$$

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L^p of the circle, or of the line, stresses a given $p = \infty$, and L^2 best meromorphic approximation to L^2 best rational approximation by the disk and its shifted conjugate. Respect to p in a natural sense [10], [43]. In order to compare these approximants can be done with respect to the design point of view. In the case of concern Markov functions, and in the case of Markov functions have an interesting connection of so-called relaxation systems [51], the present study lays ground for more general

computing error rates, as is done in the context of approximation problems. One can decide if a given procedure is far from the case which is less obvious, namely, that the problem is closely linked with uniqueness of a critical approximation, and such uniqueness issues are the subject of a numerical search, see [9], [12],

attention that meromorphic approximation is a good tool to approach some inverse problems such as "crack detection." Roughly speaking, the complexified solution of a Dirichlet problem of a plane domain raises the question as to how much information on the singular set of the domain (i.e., the crack). The answer is "yes" under the condition of crack, and determining the asymptotic behavior in the case of Markov functions, becomes

1), and let μ be a finite positive Borel measure. We assume that E contains infinitely many points, and let n be a positive integer. We use the following notation: $L_p(\mu, E)$ is the space of functions φ measurable on E

$$\left(\int_E |\varphi|^p d\mu \right)^{1/p} < \infty. \tag{1.2}$$

We caution the reader that the norm in (1.2) is slightly different from the more common definition since it does not include the factor $1/2\pi$ before the integral.

$L_\infty(\Gamma)$ is the space of essentially bounded functions, with the norm

$$\|\varphi\|_\infty = \text{ess sup}_\Gamma |\varphi(\xi)|.$$

$H_p(G)$, $1 \leq p \leq \infty$, is the Hardy space of analytic functions on G . We regard $H_p(G)$ as a subspace of $L_p(\Gamma)$, with norm (1.2) applied to the boundary values of functions in $H_p(G)$.

$L_p(\mu, E)$, $1 \leq p < \infty$, is the Lebesgue space of functions on E , with the norm

$$\|\varphi\|_{p,\mu} = \left(\int_E |\varphi(x)|^p d\mu(x) \right)^{1/p} \quad \varphi \in L_p(\mu, E).$$

4. N -Widths

Let X be a Banach space and let A be a convex, compact, centrally symmetric subset of X .

The Kolmogorov n -width of A in X is defined as follows:

$$d_n(A, X) := \inf_{X_n} \sup_{\varphi \in A} \inf_{g \in X_n} \|\varphi - g\|,$$

where X_n is an arbitrary n -dimensional subspace of X .

The Gel'fand n -width of A in X is given by

$$d^n(A, X) := \inf_{X_{-n}} \sup_{x \in X_{-n} \cap A} \|x\|,$$

where X_{-n} runs over all subspaces of X of codimension n .

The linear n -width of A in X is defined by

$$\delta_n(A, X) := \inf_K \sup_{\varphi \in A} \|\varphi - K\varphi\|$$

where $K: X \rightarrow X$ varies over all linear operators of rank n (see [41], [42] for more details about n -widths).

The restriction to E of the closed unit ball of $H_{2q}(G)$ forms a compact, convex, centrally symmetric subset A_{2q} of $L_2(\mu, E)$. It is possible to determine the size of the set A_{2q} as measured by its n -width. It follows from paper [20] of Fisher and Stessin that

$$\begin{aligned} d_n(A_{2q}, L_2(\mu, E)) &= d^n(A_{2q}, L_2(\mu, E)) = \delta_n(A_{2q}, L_2(\mu, E)) \\ &= \inf_{\{z_1, \dots, z_n\} \subset G} \sup\{\|\varphi\|_{2,\mu} : \varphi \in A_{2q} \text{ and } \varphi(z_k) = 0, k = 1, \dots, n\}. \end{aligned} \tag{1.3}$$

In the next two sections we shall see how these n -widths are related to the rate of best meromorphic approximation to Markov functions.

1.5. Connection with Hankel Operators

The deviation of f in $L_p(\Gamma)$, $1 \leq p \leq \infty$, from the class $\mathcal{M}_{n,p}(G)$ will be denoted by $\Delta_{n,p}$:

$$\Delta_{n,p} \quad \Delta_{n,p}(f; G) = \inf_{h \in \mathcal{M}_{n,p}(G)} \|f - h\|_p, \quad (1.4)$$

where $\|\cdot\|_p$ is the norm in $L_p(\Gamma)$.

For the case when $p = \infty$, there is a connection between the extremal problem (1.4) and results from the theory of Hankel operators. This connection was first explored in papers by Adamyan, Arov, and Kreĭn [2], [3] (see also [44]) and is known as the AAK method. To describe it, we first write the space $L_2(\Gamma)$ as the direct sum

$$L_2(\Gamma) = H_2(G) \oplus H_2^\perp(G),$$

where $H_2^\perp(G)$ is the orthogonal complement of $H_2(G)$ in $L_2(\Gamma)$, and $H_2(G)$ is regarded as a subspace of $L_2(\Gamma)$. For f continuous on Γ , the Hankel operator $A_f: H_2(G) \rightarrow H_2^\perp(G)$ is defined by

$$A_f(g) = P_-(fg), \quad (1.5)$$

where P_- is the orthogonal projection of $L_2(\Gamma)$ onto $H_2^\perp(G)$. An essential role is played by the singular numbers of this compact operator.

In general, if X and Y are Hilbert spaces and $A: X \rightarrow Y$ is a compact linear operator, then the n th singular number $s_n = s_n(A)$ is defined by

$$s_n(A) = \inf_K \|A - K\|, \quad (1.6)$$

where the infimum is taken over all linear operators $K: X \rightarrow Y$ having rank at most n . Here $\|\cdot\|$ denotes the norm of the corresponding linear operator. Equivalently, s_n is given by

$$s_n(A) = \inf_{X_{-n}} \|A|_{X_{-n}}\|, \quad (1.7)$$

where X_{-n} runs over all subspaces of X of codimension n (see, e.g., [50, Theorem 7.7]). Moreover, the sequence $\{s_n(A)\}$, $n = 0, 1, 2, \dots$, coincides with the sequence of eigenvalues of the operator $(A^*A)^{1/2}$, where $A^*: Y \rightarrow X$ is the adjoint of A .

With the above notation, the AAK theorem asserts that for $f \in C(\Gamma)$, we have

$$\Delta_{n,\infty}(f; G) = s_n(A_f), \quad n = 0, 1, 2, \quad (1.8)$$

For Markov functions f of the form (1.1), the singular numbers $s_n(A_f)$ turn out to be related to the singular numbers of the embedding operator $J: H_2(G) \rightarrow L_2(\mu, E)$ defined by the restriction

$$J(\varphi) = \varphi|_E, \quad (1.9)$$

where $E = \text{supp } \mu$. This relationship is described in the following result.

Theorem 1. For the Markov function (1.1) and Borel measure with $E = \text{supp } \mu$ there holds

where A_f and J denote, respectively, the Hankel operator and the embedding operator defined in (1.5) and (1.9). Consequently, for $n = 0, 1, \dots$

$$\begin{aligned} \Delta_{n,\infty}(f; G) &= s_n(A_f) \\ &= d_n(A_f|_E) \end{aligned}$$

where $A_f|_E$ is the restriction to $L_2(E)$ of the operator A_f .

Proof. We establish (1.10) by determining the singular numbers of the two operators appearing in (1.10). For the adjoint operator $A_f^*: H_2^\perp(G) \rightarrow H_2(G)$

$$A_f^*(h)(z) = P_+(\bar{f}h)(z) =$$

where $h \in H_2^\perp(G)$ and P_+ denotes the orthogonal projection onto $H_2(G)$. Furthermore, for $g \in H_2(G)$, we have

$$A_f(g)(\xi) = P_-(fg)(\xi)$$

which, on substituting the integral representation (1.1) for f , yields

$$\begin{aligned} A_f(g)(\xi) &= \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)g(\zeta)}{\zeta - \xi} d\zeta \\ &= \frac{1}{2\pi i} \int_\Gamma \frac{g(\zeta)}{\zeta - \xi} d\zeta \\ &= \frac{1}{2\pi i} \int_\Gamma \frac{g(\zeta)}{\zeta - \xi} d\zeta \end{aligned}$$

where in the last step we used the fact that f is a Markov function. This implies that $A_f(g)$ has an analytic continuation to the interior of E . We deduce that for $g \in H_2(G)$ and $z \in E$

$$\begin{aligned} (A_f^*A_f)(g)(z) &= \frac{1}{2\pi i} \int_\Gamma \frac{\overline{f(\zeta)}g(\zeta)}{\zeta - z} d\zeta \\ &= \frac{1}{2\pi i} \int_\Gamma \frac{g(\zeta)}{\zeta - z} d\zeta \\ &= \frac{1}{4\pi^2} \int_E \int_E \frac{g(\zeta)}{\zeta - z} d\mu(\zeta) d\mu(z) \end{aligned}$$

the class $\mathcal{M}_{n,\rho}(G)$ will be denoted

$$\|f - h\|_p, \quad (1.4)$$

connection between the extremal problem operators. This connection was first given in [2], [3] (see also [44]) and is first write the space $L_2(\Gamma)$ as the

$H_2^+(G)$, of $H_2(G)$ in $L_2(\Gamma)$, and $H_2(G)$ is continuous on Γ , the Hankel operator

onto $H_2^+(G)$. An essential role is played by the operator

$A: X \rightarrow Y$ is a compact linear operator. A is defined by

$$\|A\|, \quad (1.6)$$

rank of $K: X \rightarrow Y$ having rank at most n . Equivalently,

$$\|A\|, \quad (1.7)$$

dimension n (see, e.g., [50, Theorem 1.1]), coincides with the dimension of K where $A^*: Y \rightarrow X$ is the adjoint operator.

states that for $f \in C(\Gamma)$, we have

$$s_n(A_f) = 0, 1, 2, \dots \quad (1.8)$$

singular numbers $s_n(A_f)$ turn out to be the singular values of the embedding operator $J: H_2(G) \rightarrow L_2(\Gamma)$.

$$\|A_f\|, \quad (1.9)$$

is given in the following result.

Theorem 1. For the Markov function f of (1.1), where μ is a finite positive Borel measure with $E = \text{supp } \mu \subset (-1, 1)$ containing infinitely many points, there holds

$$A_f^* A_f = (J^* J)^2, \quad (1.10)$$

where A_f and J denote, respectively, the Hankel operator and the embedding operator defined in (1.5) and (1.9).

Consequently, for $n = 0, 1, \dots$,

$$\begin{aligned} \Delta_{n,\infty}(f; G) &= s_n(A_f) = s_n(J)^2 = d^n(A_2, L_2(\mu, E))^2 \\ &= d_n(A_2, L_2(\mu, E))^2 = \delta_n(A_2, L_2(\mu, E))^2, \end{aligned} \quad (1.11)$$

where A_2 is the restriction to $L_2(\mu, E)$ of the unit ball in $H_2(G)$.

Proof. We establish (1.10) by deriving and then comparing integral formulas for the two operators appearing in (1.10).

For the adjoint operator $A_f^*: H_2^+(G) \rightarrow H_2(G)$ it is easily verified that

$$A_f^*(h)(z) = P_+(\bar{f}h)(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\overline{f(\xi)}h(\xi)}{\xi - z} d\xi, \quad |z| < 1, \quad (1.12)$$

where $h \in H_2^+(G)$ and P_+ denotes the orthogonal projection of $L_2(\Gamma)$ onto $H_2(G)$. Furthermore, for $g \in H_2(G)$, we have (cf. (1.5)):

$$A_f(g)(\xi) = P_-(fg)(\xi) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(t)g(t)}{\xi - t} dt, \quad |\xi| > 1,$$

which, on substituting the integral representation (1.1) for f , can be written as

$$\begin{aligned} A_f(g)(\xi) &= \frac{1}{2\pi i} \int_{\Gamma} \frac{g(t)}{\xi - t} \left(\frac{1}{2\pi i} \int_E \frac{d\mu(x)}{t - x} \right) dt \\ &= \frac{1}{2\pi i} \int_E \left(\frac{1}{2\pi i} \int_{\Gamma} \frac{g(t) dt}{(\xi - t)(t - x)} \right) d\mu(x) \\ &= \frac{1}{2\pi i} \int_E \frac{g(x)}{\xi - x} d\mu(x), \end{aligned} \quad (1.13)$$

where in the last step we used the Cauchy integral formula. Notice that (1.13) implies that $A_f(g)$ has an analytic continuation to $\mathbb{C} \setminus E$. Using (1.12) and (1.13) we deduce that for $g \in H_2(G)$ and $|z| < 1$:

$$\begin{aligned} (A_f^* A_f)(g)(z) &= \frac{1}{2\pi i} \int_{\Gamma} \frac{\overline{f(\xi)}}{\xi - z} \left(\frac{1}{2\pi i} \int_E \frac{g(x)}{\xi - x} d\mu(x) \right) d\xi \\ &= \frac{1}{2\pi i} \int_{\Gamma} \left(\frac{-1}{2\pi i} \int_E \frac{\xi d\mu(y)}{1 - y\xi} \right) \left(\frac{1}{2\pi i} \int_E \frac{g(x)}{\xi - x} d\mu(x) \right) \frac{d\xi}{\xi - z} \\ &= \frac{1}{4\pi^2} \int_E \int_E \left(\frac{1}{2\pi i} \int_{\Gamma} \frac{\xi g(x) d\xi}{(1 - y\xi)(\xi - x)(\xi - z)} \right) d\mu(x) d\mu(y) \end{aligned}$$

The integral over Γ can be computed using the Cauchy residue theorem from which we obtain

$$(A_f^* A_f)(g)(z) = \frac{1}{4\pi^2} \int_E \int_E \frac{g(x)}{(1-yx)(1-yz)} d\mu(x) d\mu(y), \quad (1.14)$$

for $|z| < 1$.

On the other hand, it is easily verified that for the adjoint $J^*: L_2(\mu, E) \rightarrow H_2(G)$ of the embedding operator J of (1.9), we have for $\psi \in L_2(\mu, E)$:

$$J^*(\psi)(z) = \frac{1}{2\pi} \int_E \frac{\psi(y)}{1-yz} d\mu(y), \quad |z| < 1.$$

Hence for $g \in H_2(G)$:

$$(J^* J)(g)(z) = \frac{1}{2\pi} \int_E \frac{g(y)}{1-yz} d\mu(y), \quad (1.15)$$

from which it follows that $(J^* J)^2$ is likewise given by the right-hand side of (1.14). Thus (1.10) is proved. (See [8] for further generalizations.)

Concerning the equalities (1.11), we first mention that it is a general fact (cf. [19, Theorem 6.1, p. 251]), that if L is a compact linear operator mapping a Hilbert space X into a Hilbert space Y , and A is the unit ball in X , then the Kolmogorov n -width of the image $L(A)$ in Y is the same as the $(n+1)$ st eigenvalue of LL^* . From (1.15) it is easily seen that the operators $J^* J$ and $J J^*$ have the same eigenvalues. The equalities (1.11) therefore follow from these observations along with (1.3), (1.8), and (1.10). \square

It is worth noting that, in the case of Markov functions that do not reduce to rational functions, all singular numbers involved are simple, that is,

$$\Delta_{n,\infty}(f; G) > \Delta_{n+1,\infty}(f; G). \quad (1.16)$$

This property follows from the simplicity of the eigenvalues for the $J^* J$; cf. [19, Theorem 6.2, p. 252], where it is also mentioned that if $\Delta_n = Q_{n,\infty} \in H_2(G)$ generates the eigenspace for the $(n+1)$ st eigenvalue of $J^* J$, then Q_n has exactly n zeros in G . Thus the singular vector Q_n , of the unit norm for A_f associated to $s_n(A_f)$, is uniquely determined up to a constant of modulus one by the equation

$$s_n(A_f) Q_n(z) = \frac{1}{2\pi} \int_E \frac{Q_n(x)}{1-zx} d\mu(x). \quad (1.17)$$

From this last equation we can easily further deduce that Q_n has a certain orthogonality property, and that its n zeros lie in the smallest closed interval containing the support E of μ . These facts, as well as (1.11), are special cases of the general L_p results presented in the next sections (see, e.g., Corollary 2 and Lemma 3).

We remark that L_p , $2 \leq p \leq \infty$, generalizations of the AAK theorem were obtained by Le Merdy [30], [31], and by Baratchart and Seyfert [10].

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 the λ_n^* have the same eigenvalues.
 observations along with (1.3),
 □

actions that do not reduce to
 simple, that is,

$$G). \quad (1.16)$$

$$d\mu(x). \quad (1.17)$$

that Q_n has a certain orthog-
 onal closed interval containing
 the special cases of the general
 Corollary 2 and Lemma 3).
 results of the AAK theorem were
 and Seyfert [10].

Finally, we mention that when the measure μ has support $E = [a, b] \subset (-1, 1)$
 and satisfies the Szegő condition on E , then the special case $p = \infty$ of Theorem 4
 in Section 3 describes the asymptotic behavior (as $n \rightarrow \infty$) of the singular values
 $s_n(A_f)$.

1.6. A Corollary of a Result of Andersson

For any nonnegative integer n , denote by B_n the collection of Blaschke products
 of the form

$$B(z) = \prod_{i=1}^n \frac{z - \xi_i}{1 - \bar{\xi}_i z}, \quad \xi_i \in G.$$

For the general case when $1 \leq p \leq \infty$, the starting point for our investigation
 of the best meromorphic approximation problem (1.4) for Markov functions f is
 the following result of Andersson [4]:

Theorem A. Let $1 \leq p \leq \infty, 1/p + 1/q = 1$, and

$$f(z) = \frac{1}{2\pi i} \int_E \frac{d\mu(x)}{z - x},$$

where μ is a finite positive Borel measure with support $\text{supp } \mu = E \subset (-1, 1)$
 containing infinitely many points. Then there exists a best approximant $h_n = h_{n,p}$
 in $\mathcal{M}_{n,p}(G)$ to the function f in the space $L_p(\Gamma)$, $1 \leq p \leq \infty$:

$$\Delta_{n,p} = \|f - h_n\|_p,$$

such that:

- (i) all poles $x_{1,n}, \dots, x_{n,n}$ of h_n in G are simple and belong to the smallest
 interval $K(E)$ containing the support E of μ ;
- (ii)

$$\Delta_{n,p} = \inf_{B \in B_n} \sup_{g \in A_q} \left| \int_E g(x) |B(x)|^2 d\mu(x) \right|;$$

- (iii) there exists an extremal function g_n in $H_q(G)$ with $\|g_n\|_q = 1$ so that

$$\Delta_{n,p} = \int_E g_n(x) B_n^2(x) d\mu(x),$$

where

$$B_n(x) = \prod_{k=1}^n \frac{x - x_{k,n}}{1 - \bar{x}_{k,n}x}; \quad (1.18)$$

- (iv) the function g_n is positive on $(-1, 1)$ and without zeros in G .

As a consequence of Theorem A and the results of [20] we have

Corollary 2. *Let $1 \leq p \leq \infty$. Then the following assertions hold:*

(i)

$$\begin{aligned} \sqrt{\Delta_{n,p}} &= d^n(A_{2q}, L_2(\mu, E)) = d_n(A_{2q}, L_2(\mu, E)) = \delta_n(A_{2q}, L_2(\mu, E)) \\ &= \inf_{B \in \mathcal{B}_n} \left(\sup_{\varphi \in A_{2q}} \int_E |(B\varphi)(x)|^2 d\mu(x) \right)^{1/2}, \end{aligned} \tag{1.19}$$

where A_{2q} is the restriction to E of the closed unit ball in $H_{2q}(G)$;

(ii) there exists $Q_n = Q_{n,p} \in H_{2q}(G)$, $\|Q_n\|_{2q} = 1$, satisfying on Γ the equation

$$(Q_n^2(f - h_n))(\xi) d\xi = \Delta_{n,p} |Q_n(\xi)|^{2q} |d\xi| \quad \text{if } 1 < p \leq \infty, \tag{1.20}$$

and

$$(Q_n^2(f - h_n))(\xi) d\xi = |(f - h_n)(\xi)| |d\xi| \quad \text{if } p = 1;$$

(iii) furthermore,

$$Q_n = \varphi_n B_n,$$

where $\varphi_n \in H_{2q}(G)$ is positive on $(-1, 1)$ and $\varphi_n \neq 0$ in G , and the extremal for the right-hand side of (1.19) is attained for $\varphi = \varphi_n$, $B = B_n$. In particular,

$$\Delta_{n,p} = \int_E Q_n^2(x) d\mu(x); \tag{1.21}$$

(iv) there is a constant $C > 0$ not depending on n such that

$$|Q_n(\xi)| \geq C, \quad \xi \in \Gamma, \tag{1.22}$$

and

$$|\varphi_n(\xi)| \geq C, \quad \xi \in \bar{G}. \tag{1.23}$$

Proof. The desired equalities (1.19) are an immediate consequence of the relation (1.3) and the following assertion:

Let σ be a positive Borel measure with support $\text{supp } \sigma = E \subset (-1, 1)$. Then

$$\begin{aligned} \sup_{\varphi \in A_{2q}} \int_E |\varphi(x)|^2 d\sigma(x) &= \sup_{u, v \in A_{2q}} \int_E |(uv)(x)| d\sigma(x) \\ &= \sup_{g \in A_q} \int_E |g(x)| d\sigma(x) \\ &= \sup_{g \in A_q} \left| \int_E g(x) d\sigma(x) \right|. \end{aligned} \tag{1.24}$$

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Taking $\sigma = |B_n|^2 d\mu$ now yields (1.19). To prove (1.24) it suffices to provide some remarks.

First, any function $g \in A_q$ can be represented in the form $g = uv$, where $u, v \in A_{2q}$.

Second, there is the unique solution $\bar{g} \in A_q$ of the extremal problem

$$\sup_{g \in A_q} \int |g(x)| d\sigma(x) \tag{1.25}$$

which is positive at the origin (see [20]). Since $\overline{\bar{g}(z)}$ is also a solution of (1.25), we can conclude that the extremal function \bar{g} is real on $(-1, 1)$. Taking into account now that $\bar{g} \neq 0$ in G (see [20]), we obtain that \bar{g} is positive on $(-1, 1)$ and

$$\sup_{g \in A_q} \int_E |g(x)| d\sigma(x) = \int_E \bar{g}(x) d\sigma(x).$$

Third, we can represent \bar{g} in the form

$$\bar{g} = \chi^2,$$

where $\chi = \sqrt{\bar{g}} \in A_{2q}$.

Let $\varphi_n = \sqrt{g_n}$ (cf. Theorem A(iii)) and let $Q_n = \varphi_n B_n$, where we take that branch of the square root that is positive on the positive part of the real line. It is easy to see that φ_n is positive on $(-1, 1)$, $\varphi_n \neq 0$ in G , $\|\varphi_n\|_{2q} = \|Q_n\|_{2q} = 1$, and

$$\Delta_{n,p} = \int_E Q_n^2(x) d\mu(x). \tag{1.26}$$

We now show that in the case when $1 < p \leq \infty$, Q_n and h_n satisfy on Γ the equation (1.20). The case $p = 1$ can be investigated analogously. We also remark that it follows immediately from (1.19) that for $p = 1$ the function $\varphi_n \equiv 1$.

By the Cauchy formula we get

$$\begin{aligned} \Delta_{n,p} &= \int_E Q_n^2(x) d\mu(x) \\ &= \int_{\Gamma} (Q_n^2(f - h_n))(\xi) d\xi \leq \|Q_n\|_{2q}^2 \|f - h_n\|_p = \Delta_{n,p}. \end{aligned}$$

From this we obtain that almost everywhere on Γ :

$$(Q_n^2(f - h_n))(\xi) d\xi = \Delta_{n,p} |Q_n(\xi)|^{2q} |d\xi|.$$

Further, since f is holomorphic on Γ , the function $Q_n^2(f - h_n)$ can be continued analytically across Γ , and the relation (1.20) holds for all $\xi \in \Gamma$ (see, e.g., [22], [23] for more details).

It is a well-known fact (see, e.g., [18], [19]) that the extremal function of (1.19) satisfies the inequality (1.22). From (1.22), on the basis of the principle of the minimum of analytic functions, we get (1.23). \square

2. Some Formulas

Here we assume that $1 < p \leq \infty$. In this section we consider some formulas concerning the function Q_n and the best meromorphic approximant h_n .

According to the following lemma the function φ_n^q can be extended analytically to $\bar{C} \setminus E^{-1}$, where E^{-1} is the reflection of E in the unit circle Γ . By this lemma we also obtain an analytic continuation of the function $Q_n(f - h_n)/\varphi_n^{q-1}$.

Here and throughout we select the branch of the power of φ_n that is positive in $(-1, 1)$.

Lemma 3. *Let $1 < p \leq \infty$. The function $Q_n(f - h_n)/\varphi_n^{q-1}$ can be extended analytically to $\bar{C} \setminus E$ and has n zeros at the points $1/x_{k,n}$, $k = 1, \dots, n$. For an arbitrary polynomial T_n of degree at most n :*

$$\frac{T_n Q_n(f - h_n)}{w_n \varphi_n^{q-1}}(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{(T_n Q_n)(z) / \varphi_n^{q-1}(x) d\mu(x)}{w_n(x)(z - x)}, \quad z \in \bar{C} \setminus E, \tag{2.1}$$

where

$$w_n(z) = \prod_{k=1}^n (1 - x_{k,n} z). \tag{2.2}$$

In particular,

$$B_n^2(z)(\varphi_n^{2-q}(f - h_n))(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{B_n^2(x) \varphi_n^{2-q}(x) d\mu(x)}{z - x}, \quad z \in \bar{C} \setminus E. \tag{2.3}$$

The function φ_n^q can be extended analytically to $\bar{C} \setminus E^{-1}$, and satisfies the equation

$$\frac{1}{2\pi} \int_E \frac{B_n^2(x) \varphi_n^{2-q}(x) d\mu(x)}{1 - \xi x} = \Delta_{n,p} \varphi_n^q(\xi), \quad \xi \in \bar{C} \setminus E^{-1}. \tag{2.4}$$

The following orthogonality relations are valid:

$$\int_E x^v w_n^*(x) \frac{\varphi_n^{2-q}(x)}{w_n^2(x)} d\mu(x) = 0 \quad \text{for } v = 0, 1, \dots, n-1, \tag{2.5}$$

where $w_n^*(z) = \prod_{k=1}^n (z - x_{k,n})$.

Proof. It follows from the relations (1.2f) and (1.18) that almost everywhere on Γ :

$$\begin{aligned} Q_n(f - h_n)/\varphi_n^{q-1}(\xi) &= \overline{\Delta_{n,p} Q_n(\xi) \varphi_n^{q-1}(\xi)} \frac{1}{i\xi} \\ &= \overline{\Delta_{n,p} \frac{1}{B_n(\xi)} \varphi_n^q(\xi)} \frac{1}{i\xi} = \Delta_{n,p} \frac{1}{B_n(\xi)} \varphi_n^q \left(\frac{1}{\xi} \right) \frac{1}{i\xi} \\ &= \Delta_{n,p} \prod_{k=1}^n \frac{1 - \xi x_{k,n}}{\xi - x_{k,n}} \varphi_n^q \left(\frac{1}{\xi} \right) \frac{1}{i\xi}, \end{aligned} \tag{2.6}$$

where we used the fact that we can conclude that Q_n has zeros at the points $1/x_{k,n}$. From this we also see that (2.6) holds for all ξ . The function

is holomorphic on \bar{C}

where the right-hand side is holomorphic in the region G which lies outside γ . It is also holomorphic in the region G .

$$\frac{T_n Q_n(f - h_n)}{w_n \varphi_n^{q-1}}(z)$$

So, we have

$$\frac{T_n Q_n(f - h_n)}{w_n \varphi_n^{q-1}}(z)$$

Letting $T_n(z) = u$

$$(B_n^2 \varphi_n^{2-q})(z)(f - h_n)$$

In this section we consider some formulas for meromorphic approximant h_n . The function φ_n^q can be extended analytically in the unit circle Γ . By this lemma we know that the function $Q_n(f - h_n)/\varphi_n^{q-1}$ is holomorphic in the region $\bar{C} \setminus E$ and has n zeros at the points $1/x_{k,n}$, $k = 1, \dots, n$, belonging to E^{-1} and one zero at infinity. From this we also obtain that $\varphi_n^q(\xi)$ can be extended analytically to $\bar{C} \setminus E^{-1}$ and (2.6) holds for all $\xi \in \Gamma$.

The function

$$\frac{Q_n(f - h_n)}{w_n \varphi_n^{q-1}}$$

is holomorphic on $\bar{C} \setminus E$, and in a neighborhood of infinity

$$\frac{Q_n(f - h_n)}{w_n \varphi_n^{q-1}}(z) = \frac{A_n}{z^{n+1}} + \dots \quad (2.7)$$

where the right-hand side is a series in increasing powers of $1/z$.

Let T_n be an arbitrary polynomial of degree at most n . Fix $z \in \bar{C} \setminus K(E)$, where $K(E)$ is the smallest interval containing the support E . Let γ be an arbitrary contour lying in G and surrounding the interval $K(E)$ and such that the point z lies outside γ . It will be assumed that γ is positively oriented with respect to a region containing infinity. Using now the fact that $T_n Q_n(f - h_n)/w_n \varphi_n^{q-1}$ is holomorphic in the region $\bar{C} \setminus E$, by the Cauchy formula, we obtain

$$\begin{aligned} \frac{T_n Q_n(f - h_n)}{w_n \varphi_n^{q-1}}(z) &= \frac{1}{2\pi i} \int_{\gamma} \frac{(T_n Q_n(f - h_n))(\xi) d\xi}{w_n(\xi) \varphi_n^{q-1}(\xi)(\xi - z)} \\ &= \frac{1}{2\pi i} \int_{\gamma} \frac{(T_n Q_n f)(\xi) d\xi}{w_n(\xi) \varphi_n^{q-1}(\xi)(\xi - z)} \\ &= \frac{1}{2\pi i} \int_E \frac{1}{2\pi i} \int_{\gamma} \frac{(T_n Q_n)(\xi) d\xi}{w_n(\xi) \varphi_n^{q-1}(\xi)(\xi - x)(\xi - z)} d\mu(x) \\ &\quad + \frac{1}{2\pi i} \int_E \frac{(T_n Q_n)(x) d\mu(x)}{w_n(x) \varphi_n^{q-1}(x)(z - x)} \end{aligned}$$

So, we have

$$\frac{T_n Q_n(f - h_n)}{w_n \varphi_n^{q-1}}(z) = \frac{1}{2\pi i} \int_E \frac{(T_n Q_n)(x) d\mu(x)}{w_n(x) \varphi_n^{q-1}(x)(z - x)} \quad z \in \bar{C} \setminus E.$$

Letting $T_n(z) = w_n^*(z) = \prod_{k=1}^n (z - x_{k,n})$, we get

$$\begin{aligned} (B_n^2 \varphi_n^{2-q})(z)(f - h_n)(z) &= (B_n Q_n(f - h_n))(z)/\varphi_n^{q-1}(z) \\ &= \frac{1}{2\pi i} \int_E \frac{(B_n Q_n)/\varphi_n^{q-1}(x) d\mu(x)}{z - x} \\ &= \frac{1}{2\pi i} \int_E \frac{B_n^2(x) \varphi_n^{2-q}(x) d\mu(x)}{z - x} \quad z \in \bar{C} \setminus E. \end{aligned}$$

$Q_n(f - h_n)/\varphi_n^{q-1}$ can be extended to points $1/x_{k,n}$, $k = 1, \dots, n$. For an

$$\frac{Q_n(f - h_n)(x) d\mu(x)}{(x)(z - x)} \quad z \in \bar{C} \setminus E, \quad (2.1)$$

$$-x_{k,n} \quad (2.2)$$

$$\frac{\varphi_n^{2-q}(x) d\mu(x)}{z - x} \quad z \in \bar{C} \setminus E. \quad (2.3)$$

satisfies the equation

$$\Delta_{n,p} \varphi_n^q(\xi), \quad \xi \in \bar{C} \setminus E^{-1} \quad (2.4)$$

is valid:

$$\text{for } v = 0, 1, \dots, n-1. \quad (2.5)$$

and (1.18) that almost everywhere on

$$\begin{aligned} \frac{1}{\varphi_n^{q-1}(\xi) i\xi} &= \Delta_{n,p} \frac{1}{B_n(\xi) \varphi_n^q(\xi)} \left(\frac{1}{\xi} \right) \frac{1}{i\xi} \\ &= \frac{-\xi x_{k,n}}{-x_{k,n}} \varphi_n^q \left(\frac{1}{\xi} \right) \frac{1}{i\xi}, \quad (2.6) \end{aligned}$$

From this, the fact that φ_n is real on E , and with the help of (2.6), we get

$$\frac{1}{2\pi} \int_E \frac{B_n^2(x) \varphi_n^{2-q}(x) d\mu(x)}{1 - \xi x} = \Delta_{n,p} \varphi_n^q(\xi), \quad \xi \in \Gamma.$$

Further, since in a neighborhood of infinity

$$\frac{Q_n(f - h_n)}{w_n \varphi_n^{q-1}}(z) = \frac{A_n}{z^{n+1}} + \dots$$

for sufficiently large $r > 0$, we have

$$\int_{|\xi|=r} \xi^\nu \frac{Q_n(f - h_n)(\xi)}{w_n(\xi) \varphi_n^{q-1}(\xi)} d\xi = 0, \quad \nu = 0, 1, \dots, n - 1$$

and

$$\int_E \frac{x^\nu Q_n(x) d\mu(x)}{w_n(x) \varphi_n^{q-1}(x)} = \int_E x^\nu w_n^*(x) \frac{\varphi_n^{2-q}(x)}{w_n^2(x)} d\mu(x) = 0, \quad \nu = 0, 1, \dots, n - 1 \tag{2.8}$$

The last relations imply that the polynomial $w_n^*(z) = \prod_{k=1}^n (z - x_{k,n})$ is the n th orthogonal polynomial with respect to the measure $\varphi_n^{2-q} d\mu/w_n^2$. \square

3. Statements of Convergence Theorems

In this section and in what follows we assume that the support $\text{supp } \mu$ of μ is a closed interval $E = [a, b]$ and the measure μ satisfies the Szegő condition

$$\int_E \frac{\log(d\mu/dx)}{\sqrt{(x-a)(b-x)}} dx > -\infty, \tag{3.1}$$

where $d\mu/dx$ denotes the Radon–Nikodym derivative of μ with respect to Lebesgue measure. It will be assumed without loss of generality that $E = [a, b] \subset (0, 1)$. Let $E^{-1} = [1/b, 1/a]$.

Let $\Phi: \bar{C} \setminus (E \cup E^{-1}) \rightarrow \{r < |z| < 1/r\}$, $\Phi(1) = 1$, be the conformal mapping of the region $\bar{C} \setminus (E \cup E^{-1})$ onto the annulus with the inner radius r and the outer radius $1/r$. We mention that $|\Phi| = r$ on E and $r = e^{-1/C(E, \Gamma)}$, where $C(E, \Gamma)$ is the capacity of the condenser (E, Γ) . We also point out the formula

$$r = \exp\left(-\frac{\pi K}{2K'}\right)$$

where

$$K = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-\tau^2 x^2)}}, \quad \tau^2 = \frac{(1-a^2)(1-b^2)}{\dots}$$

and K' is the corresponding elliptic integral for τ' $\sqrt{1-\tau^2}$ (see, e.g., [1], [45]).

Let ω be the unit equilibrium potential

where $g(z, \xi) = \log|1 - \xi z|$ on the disk G with singularity at the point ξ .

There is a constant m such that ω has the equilibrium property:

The following formula holds

where $C(E, \Gamma)$ is the capacity of the condenser (E, Γ) . It is remarked that the measure $d\omega$

$$d\omega(x) = \frac{dx}{\sqrt{(x-a)(b-x)}}$$

where the constant k is chosen so that $\int_E d\omega = 1$.

(cf. [45, Section II.5]).

Let $d\mu = \psi d\omega + d\mu_s$, where ψ denotes the Radon–Nikodym derivative of μ with respect to ω , and μ_s is a singular measure. From (3.1), we have

Let $\mathcal{G}_\omega(\psi)$ be the geometric mean of ψ with respect to ω .

\mathcal{G}_ω

It follows from (3.3) that

In the asymptotic formula (3.1) by the Szegő function $D_\psi(z)$. This function was first introduced by Szegő.

(1) $D_\psi(z)$ is analytic in the region $\bar{C} \setminus (E \cup E^{-1})$.

of (2.6), we get

$$\xi \in \Gamma.$$

Let ω be the unit equilibrium measure for E , corresponding to the Green's potential

$$V_g^\omega(z) = \int_E g(z, \xi) d\omega(\xi),$$

where $g(z, \xi) = \log|(1 - \bar{\xi}z)/(z - \xi)|$ is the Green's function of the open unit disk G with singularity at the point $\xi \in G$.

There is a constant m such that the Green's potential V_g^ω has the following equilibrium property:

$$V_g^\omega = m \quad \text{on } E.$$

The following formula holds:

$$m = \frac{1}{C(E, \Gamma)},$$

where $C(E, \Gamma)$ is the capacity of the condenser (E, Γ) (see [29], [45], [49]). We remark that the measure $d\omega$ can be represented in the form

$$d\omega(x) = \frac{k}{\sqrt{(x-a)(b-x)(1-ax)(1-bx)}} dx \quad \text{for } x \in [a, b], \quad (3.2)$$

where the constant k is chosen so that $d\omega$ has unit mass, that is,

$$k = \frac{(1-ab)}{2K'}$$

(cf. [45, Section II.5]).

Let $d\mu = \psi d\omega + d\mu_s$ be the Riesz decomposition of μ , where $\psi = d\mu/d\omega$ denotes the Radon-Nikodym derivative of μ with respect to the equilibrium measure ω , and μ_s is a singular measure. Since the measure μ satisfies the Szegő condition (3.1), we have

$$\int_E \log \psi d\omega > -\infty. \quad (3.3)$$

Let $\mathcal{G}_\omega(\psi)$ be the geometric mean of the function ψ with respect to ω :

$$\mathcal{G}_\omega(\psi) = \exp\left(\int_E \log \psi d\omega\right).$$

It follows from (3.3) that

$$\mathcal{G}_\omega(\psi) > 0. \quad (3.4)$$

In the asymptotic formulas that we shall derive, an important role will be played by the Szegő function $\mathcal{D}_\psi(z)$ of ψ for the doubly connected domain $\bar{C} \setminus (E \cup E^{-1})$. This function was first introduced in [32] and has the following properties:

- (1) $\mathcal{D}_\psi(z)$ is analytic and nonvanishing in $\bar{C} \setminus (E \cup E^{-1})$;

$\prod_{k=1}^n (z - x_{k,n})$ is the n th $d\mu/w_n^2$. \square

support $\text{supp } \mu$ of μ is a Szegő condition

μ with respect to Lebesgue that $E = [a, b] \subset (0, 1)$.

the conformal mapping inner radius r and the outer $C(E, \Gamma)$, where $C(E, \Gamma)$ is the formula

$$\frac{-a^2(1-b^2)}{(1-ab)^2},$$

$-\tau^2$ (see, e.g., [1], [45]).

- (2) the increment $\Delta_\Gamma \arg D_\psi$ of the argument of D_ψ along Γ is equal to 0;
- (3) $|D_\psi(z)|^2 = \psi(z)$ almost everywhere on E ; more precisely, the nontangential limits of $|D_\psi(\xi)|^2$ as ξ approaches $z \in E$ equal $\psi(z)$ almost everywhere;
- (4) $|D_\psi(z)|^2 = \mathcal{G}_\omega(\psi)$ on Γ .

Since E is an interval $[a, b]$ it is possible to give the following integral representation for $D_\psi(z)$ (cf. [11]):

$$D_\psi(z) = \sqrt{\mathcal{G}_\omega(\psi)} \exp \left(\sqrt{(z-a)(z-b)(1-az)(1-bz)} \right. \\ \times \frac{1}{2\pi} \int_a^b \frac{\log(\psi(x)/\mathcal{G}_\omega(\psi))}{\sqrt{(x-a)(b-x)(1-ax)(1-bx)}} \\ \left. \times \frac{1-2xz+x^2}{(z-x)(1-xz)} dx \right).$$

Denote by φ^* the function maximizing the functional $\mathcal{G}_\omega(|\varphi|^2)$ in the class $\{\varphi: \varphi \in H_{2q}(G), \|\varphi\|_{2q} = 1\}$:

$$\mathcal{G}_\omega(|\varphi^*|^2) = \sup_{\varphi \in H_{2q}(G), \|\varphi\|_{2q}=1} \mathcal{G}_\omega(|\varphi|^2). \tag{3.5}$$

It is easy to prove that $\varphi^*(z) \equiv 1$ for $p = 1$. In the case when $1 < p \leq \infty$, we have (see Lemma 7):

$$\varphi^*(z) = \frac{k_1}{\sqrt[2q]{(1-az)(1-bz)}}, \quad z \in G,$$

where $k_1 = \sqrt[2q]{k/2}$. Here and in what follows we take that branch of the $2q$ th root that is positive on the positive part of the real line. We remark that then φ^* is positive on E .

We formulate a theorem characterizing the rate of convergence to zero of the best meromorphic approximation errors $\Delta_{n,p}$ and the limiting behavior of the functions B_n , φ_n , and φ_n^2 .

Theorem 4. *We have, for $1 \leq p \leq \infty$:*

$$\frac{\Delta_{n,p}}{2r^{2n} \mathcal{G}_\omega(\psi) \mathcal{G}_\omega((\varphi^*)^2)} \rightarrow 1 \quad \text{as } n \rightarrow \infty. \tag{3.6}$$

For $1 < p \leq \infty$, the sequence of measures $B_n^2 \varphi_n^2 / \Delta_{n,p} d\mu$ converges weak* to $d\omega$:

$$\frac{B_n^2 \varphi_n^2}{\Delta_{n,p}} d\mu \xrightarrow{*} d\omega \quad \text{as } n \rightarrow \infty. \tag{3.7}$$

Furthermore, in this case,

$$\frac{(B_n D_\psi D_{(\varphi^*)^2})(z)}{\Phi^n(z) \sqrt{\mathcal{G}_\omega(\psi) \mathcal{G}_\omega(\varphi^*)}} \rightarrow 1 \tag{3.8}$$

Best Meromorphic Approx

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interest (cf. [32], [38])

Corollary 5. *Let μ
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The next assertion

Theorem 6.

$$\frac{2(D_\psi^2 D_{\varphi_n^2})(z) r^{2n}}{i \Phi^{2n}(z)}$$

and for $z \in \bar{C} \setminus (E \cup I$

(φ
2)

where both limits as n

long Γ is equal to 0; precisely, the nontangential $\psi(z)$ almost every-

uniformly on compact subsets of $\tilde{C} \setminus (E \cup E^{-1})$ as $n \rightarrow \infty$, and

$$\varphi_n(z) \rightarrow \varphi^*(z) \tag{3.9}$$

and

$$\varphi_n^q(z) \rightarrow (\varphi^*)^q(z) \tag{3.10}$$

uniformly on compact subsets of G and $\tilde{C} \setminus E^{-1}$, respectively, as $n \rightarrow \infty$.

Concerning the asymptotic distribution of the poles of the best approximants h_n , that is, the points $\{x_{k,n}\}_{k=1}^n$, we have the following immediate consequence of (3.8):

$$\nu^*(h_n) := \frac{1}{n} \sum_{k=1}^n \delta_{x_{k,n}} \xrightarrow{*} d\omega \quad \text{as } n \rightarrow \infty, \tag{3.11}$$

where δ_ξ denotes the unit measure whose support is the point ξ . Moreover, (3.11) holds under much weaker assumptions. For example, if $d\mu/dx > 0$ almost everywhere on $E = [a, b]$, then it can be shown from Corollary 2, and the results of Gonchar and Rakhmanov [26], that (3.11) remains true.

Finally, we mention the following corollary of (3.7) which is of independent interest (cf. [32], [38]).

Corollary 5. *Let μ be a finite positive Borel measure with $\text{supp } \mu = [a, b]$, where $[a, b] \subset (-1, 1)$ and assume μ satisfies the Szegő condition (3.1). Then there exists a sequence of Blaschke products \tilde{B}_n with all its zeros on $[a, b]$ such that*

$$\frac{\tilde{B}_n^2 d\mu}{2r^{2n} \mathcal{G}_\omega \left(\frac{d\mu}{d\omega} \right)} \xrightarrow{*} d\omega \quad \text{as } n \rightarrow \infty.$$

The next assertion describes the convergence of h_n to f .

Theorem 6. *Let $1 < p \leq \infty$. We have for $z \in G \setminus E$:*

$$\frac{(f - h_n)(z)}{\frac{2(\mathcal{D}_\psi^2 \mathcal{D}_{\varphi^{2-1}}^2)(z)r^{2n}}{i\Phi^{2n}(z)} \sqrt{\frac{(1-az)(1-bz)}{(z-a)(z-b)} \left(\frac{(1-az)(1-bz)}{k/2} \right)^{(1-q)/q}}} \rightarrow 1, \tag{3.12}$$

and for $z \in \tilde{C} \setminus (E \cup E^{-1})$:

$$\frac{(\varphi_n^{2-q}(f - h_n))(z)}{\frac{2(\mathcal{D}_\psi^2 \mathcal{D}_{\varphi^{2-1}}^2)(z)r^{2n}}{\Phi^{2n}(z)}} \rightarrow \frac{\sqrt{k/2}}{i\sqrt{(z-a)(z-b)}}, \tag{3.13}$$

where both limits as $n \rightarrow \infty$ are locally uniform.

4. The Extremal Function for (3.5)

In this section we assume that $1 < p \leq \infty$. Let us consider the following extremal problem:

$$\begin{aligned} \delta &= \sup_{\varphi \in H_{2q}(G), \|\varphi\|_{2q}=1} \mathcal{G}_\omega(|\varphi|^2) \\ &= \sup_{\varphi \in H_{2q}(G), \|\varphi\|_{2q}=1} \left(\exp \left(2 \int_E \log|\varphi(x)| d\omega(x) \right) \right). \end{aligned} \tag{4.1}$$

The characteristics of the solution to this extremal problem will be used in the proof of Theorem 4.

Lemma 7. *Let $1 < p \leq \infty$. There is a uniquely determined function φ^* (up to unimodular scalar multiples) which attains the value δ . The function $(\varphi^*)^q$ can be extended analytically to $\bar{C} \setminus E^{-1}$ and can be found from the formula*

$$(\varphi^*)^q(z) = \frac{k_1^q}{\sqrt{(1-az)(1-bz)}}, \tag{4.2}$$

where $k_1 = \sqrt[2]{k/2}$. Moreover, φ^* satisfies, on Γ , the equation

$$(d\omega)^* = |\varphi^*(\xi)|^{2q} |d\xi|, \tag{4.3}$$

where $(d\omega)^*$ is the balayage of the measure $d\omega$ on Γ .

Proof. By the Cauchy formula

$$\varphi(x) = \frac{1}{2\pi i} \int_\Gamma \frac{\varphi(\xi) d\xi}{\xi - x}, \quad x \in E.$$

Since $\|\varphi\|_{2q} = 1$, it follows from the last formula that

$$|\varphi(x)| \leq C, \quad C > 0, \quad x \in E,$$

where C denotes a positive quantity not depending on n . From this we can conclude that $\delta < \infty$.

A normal families argument shows that at least one solution $\varphi^* \in H_{2q}(G)$, $\|\varphi^*\|_{2q} = 1$, to (4.1) exists:

$$\delta = \exp \left(2 \int_E \log|\varphi^*(x)| d\omega(x) \right)$$

Moreover, φ^* cannot vanish in G , since otherwise dividing by a suitable Blaschke product would increase the value of the integral in (4.1). Let us write δ in another form

$$\delta = \sup_{\varphi \in H_{2q}(G), \|\varphi\|_{2q} \neq 0} \left(\exp \left(2 \int_E \log(|\varphi(x)|/\|\varphi\|_{2q}) d\omega(x) \right) \right)$$

=
 $\varphi \in H_{2q}(G)$
 =
 $\varphi \in H_{2q}(G)$

To see that φ^* is continuous on the circle, denote by ϵ

The function φ_ϵ lies

$$\begin{aligned} \|\varphi_\epsilon\|_{2q}^2 &= \left(\int_\Gamma |\varphi| \right) \\ &= 1 + 2\epsilon \end{aligned}$$

It is not hard to see

$$\mathcal{G}_\omega(|\varphi_\epsilon|^2) = \epsilon$$

By the formula (4.1) for sufficiently small ϵ :

Letting $\epsilon \rightarrow 0$ for δ

Next, assuming that

which implies the

Since u is an arbitrary harmonic function on G and continuous on \bar{G} , we can write

$$(d\omega)^* = |\varphi^*(\xi)|^{2q} |d\xi|,$$

where $(d\omega)^*$ is the balayage of the measure $d\omega$ on Γ (see, e.g., [45]).

We now prove that $(\varphi^*)^q$ is given by formula (4.2).

The balayage of $d\omega$ on Γ is given (see [45, Section II.4]) by

$$(d\omega)^*(\xi) = \left(\frac{1}{2\pi} \int_E \frac{1-x^2}{|\xi-x|^2} d\omega(x) \right) |d\xi|. \tag{4.8}$$

Then, using (4.3), we can write

$$\frac{1}{2\pi} \int_E \frac{1-x^2}{|\xi-x|^2} d\omega(x) = |\varphi^*(\xi)|^{2q}, \quad \xi \in \Gamma.$$

By the relation

$$\frac{1-x^2}{|\xi-x|^2} = \frac{\xi}{\xi-x} - \frac{\xi}{\xi-1/x} \quad \text{for } |\xi|=1, \quad x \in E,$$

and the formula (3.2) for the equilibrium measure, we get

$$\begin{aligned} \frac{1}{2\pi} \int_E \left(\frac{\xi}{\xi-x} - \frac{\xi}{\xi-1/x} \right) \frac{k}{\sqrt{(x-a)(b-x)(1-ax)(1-bx)}} dx \\ = |\varphi^*(\xi)|^{2q}, \quad \xi \in \Gamma. \end{aligned}$$

It follows from this that

$$\begin{aligned} \frac{k\xi}{2\pi} \int_E \frac{1}{\xi-x} \frac{1}{\sqrt{(x-a)(b-x)(1-ax)(1-bx)}} dx \\ - \frac{k\xi}{2\pi} \int_E \frac{1}{\xi-1/x} \frac{1}{\sqrt{(x-a)(b-x)(1-ax)(1-bx)}} dx \\ = |\varphi^*(\xi)|^{2q}, \quad \xi \in \Gamma. \end{aligned} \tag{4.9}$$

With help of the substitution $x \rightarrow 1/x$ we can rewrite the last equation in the form

$$\begin{aligned} \frac{k\xi}{2\pi} \int_E \frac{1}{\xi-x} \frac{1}{\sqrt{(x-a)(b-x)(1-ax)(1-bx)}} dx \\ - \frac{k\xi}{2\pi} \int_E \frac{1}{\xi-1/x} \frac{1}{\sqrt{(x-a)(b-x)(1-ax)(1-bx)}} dx \\ = |\varphi^*(\xi)|^{2q}, \quad \xi \in \Gamma. \end{aligned}$$

From this, on the basis of the Cauchy formula, we get

$$\frac{k\xi i}{2} \frac{1}{\sqrt{(\xi-a)(b-\xi)(1-a\xi)(1-b\xi)}} = |\varphi^*(\xi)|^{2q}, \quad \xi \in \Gamma,$$

and

$$\frac{k}{2\sqrt{|1-a\xi|}}$$

Therefore, since φ^* does not

$$\varphi^*(\xi) =$$

where $k_1 = \sqrt[2]{k/2}$. Moreover

Remark. In connection with to obtain in terms of geomet We have

$$\Delta_{n,p} \geq r^{2n} \sup_{\varphi \in H_{2q}(G), \|\varphi\|}$$

In fact, according to Corolla

$$\Delta_{n,p} = \inf_{B \in \mathcal{B}_n}$$

By the inequality between th

$$\Delta_{n,p} \geq \inf_{B \in \mathcal{B}_n} \mathcal{G}$$

Since

$$\inf_{B \in \mathcal{B}_n} \mathcal{G}_\omega(|B|^2) =$$

from (4.11) we get (4.10). In twice as large as we have or

5. Proof of Theorem 4

5.1. Auxiliary Results

Let $\Psi: \bar{C} \setminus E \rightarrow \{z: |z| > 1\}$ the exterior of the unit disk, that

We mention that $|\Psi| = 1$ on

continuous on \bar{G} , we can write

(see, e.g., [45]).

II.4] by
 $c) |d\xi|$. (4.8)

$\xi \in \Gamma$.

$| = 1, x \in E,$

get

$$\frac{k}{\sqrt{(1-ax)(1-bx)}} dx$$

dx

$$\frac{dx}{-ax(1-bx)} \quad (4.9)$$

write the last equation in the

$$\frac{dx}{-ax(1-bx)}$$

$$(\xi)^{2q}, \quad \xi \in \Gamma,$$

and

$$\frac{k}{2\sqrt{|1-a\xi|^2|1-b\xi|^2}} = |\varphi^*(\xi)|^{2q}, \quad \xi \in \Gamma.$$

Therefore, since φ^* does not vanish in G , we have

$$\varphi^*(\xi) = \frac{k_1}{\sqrt[2q]{(1-az)(1-bz)}}, \quad z \in G,$$

where $k_1 = \sqrt[2q]{k/2}$. Moreover, $(\varphi^*)^q$ can be extended analytically to $\bar{C} \setminus E^{-1}$. \square

Remark. In connection with the extremal problem (3.5) we note that it is easy to obtain in terms of geometric means some rough estimate of $\Delta_{n,p}$ from below. We have

$$\Delta_{n,p} \geq r^{2n} \sup_{\varphi \in H_{2q}(G), \|\varphi\|_{2q}=1} \mathcal{G}_\omega(|\varphi|^2) \mathcal{G}_\omega(\psi) = r^{2n} \mathcal{G}_\omega(\varphi^{*2}) \mathcal{G}_\omega(\psi). \quad (4.10)$$

In fact, according to Corollary 2:

$$\Delta_{n,p} = \inf_{B \in \mathcal{B}_n} \sup_{\varphi \in H_{2q}(G), \|\varphi\|_{2q} \leq 1} \int_E |(B\varphi)(x)|^2 d\mu(x).$$

By the inequality between the arithmetic mean and the geometric mean

$$\Delta_{n,p} \geq \inf_{B \in \mathcal{B}_n} \mathcal{G}_\omega(|B|^2) \sup_{\varphi \in H_{2q}(G), \|\varphi\|_{2q} \leq 1} \mathcal{G}_\omega(|\varphi|^2) \mathcal{G}_\omega(\psi). \quad (4.11)$$

Since

$$\inf_{B \in \mathcal{B}_n} \mathcal{G}_\omega(|B|^2) = \inf_{B \in \mathcal{B}_n} \exp\left(2 \int_E \log|B(x)| d\omega(x)\right) = r^{2n},$$

from (4.11) we get (4.10). In Section 5 it will be proved that $\Delta_{n,p}$ is asymptotically twice as large as we have on the right-hand side of (4.10).

5. Proof of Theorem 4

5.1. Auxiliary Results

Let $\Psi: \bar{C} \setminus E \rightarrow \{z: |z| > 1\}$ be the conformal mapping of the region $\bar{C} \setminus E$ onto the exterior of the unit disk, with infinity corresponding to infinity in such manner that

$$\lim_{z \rightarrow \infty} \frac{\Psi(z)}{z} > 0.$$

We mention that $|\Psi| = 1$ on E .

Denote by ω_1 the unit equilibrium measure for E , corresponding to the logarithmic potential

$$V^{\omega_1}(z) = \int_E \log \frac{1}{|\xi - z|} d\omega_1(\xi).$$

There is a constant (the Robin constant) γ such that the logarithmic potential V^{ω_1} has the equilibrium property

$$V^{\omega_1} = \gamma \quad \text{on } E.$$

The following formula is valid:

$$C(E) = e^{-\gamma},$$

where $C(E) = (b - a)/4$ is the logarithmic capacity of the closed interval E (see [29], [45], [49]). The measure $d\omega_1$ can be represented in the form

$$d\omega_1(x) = \frac{1}{\pi \sqrt{(x - a)(b - x)}} dx \quad \text{for } x \in [a, b]. \quad (5.1)$$

It is not hard to see that on E :

$$d\omega(x) = \frac{\pi k}{\sqrt{(1 - ax)(1 - bx)}} d\omega_1(x) = \pi \sqrt{2k(\varphi^*)^q(x)} d\omega_1(x). \quad (5.2)$$

Let $d\mu = \psi_1 d\omega_1 + d\mu_{s,1}$ be the Riesz decomposition of μ , where $\psi_1 = d\mu/d\omega_1$ is the Radon-Nikodym derivative of μ with respect to the equilibrium measure ω_1 , and $\mu_{s,1}$ is a singular measure.

Denote by $\mathcal{G}_{\omega_1}(\psi_1)$ the geometric mean of the function ψ_1 with respect to ω_1 :

$$\mathcal{G}_{\omega_1}(\psi_1) = \exp \left(\int_E \log \psi_1 d\omega_1 \right)$$

It follows from the Szegő condition (3.1) that

$$\mathcal{G}_{\omega_1}(\psi_1) > 0. \quad (5.3)$$

Let $D_{\psi_1}(z)$ be the Szegő function of ψ_1 for the domain $\bar{C} \setminus E$. This function satisfies the following properties:

- (1) $D_{\psi_1}(z)$ is analytic and nonvanishing in $\bar{C} \setminus E$;
- (2) $|D_{\psi_1}(z)|^2 = \psi_1(z)$ almost everywhere on E and $D_{\psi_1}^2(\infty) = \mathcal{G}_{\omega_1}(\psi_1)$.

The following formula for $D_{\psi_1}(z)$ holds (see, e.g., [48]):

$$D_{\psi_1}(z) = \exp \left(\sqrt{(z - a)(z - b)} \frac{1}{2\pi} \int_a^b \frac{\log(\psi_1(x))}{\sqrt{(x - a)(b - x)}} \frac{1}{(z - x)} dx \right)$$

5.2. The Rate of I

In this subsection v meromorphic approuse the method base polynomials with v Let

$$\delta_n = \int_E$$

Since the polyn thogonal polynomi from below (see (1. of the results of To

$$\overline{2C}$$

and

uniformly on comp remark that (5.5) f from (5.5) togethe totics follow from [46] are given onl our purpose one r the φ_n form a nor can have a zero c when the measur convergent and be tions of Theor identifying formu get (5.5) and (5.6 case of absolutely mentioned above weak* converges

First we show

5.2. The Rate of Decrease of $\Delta_{n,p}$ for $1 < p \leq \infty$

In this subsection we assume that $1 < p \leq \infty$. To prove that the error in best meromorphic approximation $\Delta_{n,p}$ satisfies the asymptotic formula (3.6) we will use the method based on an investigation of Szegő-type asymptotics for orthogonal polynomials with varying weight functions (see [48], [46]).

Let

$$\delta_n = \int_E \frac{w_n^{*2}(x)}{w_n^2(x)} \varphi_n^{2-q}(x) d\mu(x) = \int_E B_n^2(x) \varphi_n^{2-q}(x) d\mu(x).$$

Since the polynomial $w_n^*(z) = \prod_{k=1}^n (z - x_{k,n})$ satisfying (2.8) is the n th orthogonal polynomial with respect to the measure $\varphi_n^{2-q} d\mu/w_n^2$, and φ_n is bounded from below (see (1.23)) and above on E by constants not depending on n , by virtue of the results of Totik [48, Sections 14 and 16] and Stahl [46]:

$$\frac{\delta_n \mathcal{G}_{\omega_1}(w_n^2)}{2C(E)^{2n} \mathcal{G}_{\omega_1}(\varphi_n^{2-q}) \mathcal{G}_{\omega_1}(\psi_1)} \rightarrow \quad \text{as } n \rightarrow \infty \quad (5.5)$$

and

$$\frac{w_n^{*2}(z) D_{\varphi_n^{2-q}}^2(z) D_{\psi_1}^2(z) \mathcal{G}_{\omega_1}(w_n^2)}{C(E)^{2n} \Psi^{2n}(z) \mathcal{G}_{\omega_1}(\varphi_n^{2-q}) \mathcal{G}_{\omega_1}(\psi_1) D_{w_n^2}^2(z)} \rightarrow 1 \quad (5.6)$$

uniformly on compact subsets of $\bar{C} \setminus E$ as $n \rightarrow \infty$. For absolutely continuous μ , we remark that (5.5) follows essentially from Theorem 13.1 of [48] and (5.6) follows from (5.5) together with Theorem 14.3 of [48]. For arbitrary μ , the desired asymptotics follow from [46] after suitable modification. More precisely, the results of [46] are given only for the case when the orthogonality measure is $d\mu/|w_n^2|$. For our purpose one must replace $d\mu$ by $\varphi_n^{2-q} d\mu$, which varies with n . However, since the φ_n form a normal family in G and, by (1.23), no limit function of this family can have a zero on $[a, b]$, it's enough to extend Theorem 1 of [46] to the case when the measure is multiplied by a weight, varying with n , that is uniformly convergent and bounded away from zero. This extension, which necessitates extensions of Theorem 3 and Proposition 2.48 of [46], is straightforward and, on identifying formula (2.15) of [46] with the formula of Theorem 14.3 of [48], we get (5.5) and (5.6) in the general case. Moreover (cf. [48, Theorem 14.1] for the case of absolutely continuous μ , and [46, Proposition 2.21] suitably extended as mentioned above for the general case), the sequence of measures $(B_n^2 \varphi_n^{2-q} / \delta_n) d\mu$ weak* converges to $d\omega_1$:

$$\frac{B_n^2 \varphi_n^{2-q}}{\delta_n} d\mu \xrightarrow{*} d\omega_1 \quad \text{as } n \rightarrow \infty. \quad (5.7)$$

First we show that

$$\varphi_n^q(z) \rightarrow (\varphi^*)^q(z)$$

$$\int dx \quad (5.4)$$

uniformly on compact subsets of $\bar{C} \setminus E^{-1}$ as $n \rightarrow \infty$. To do this we use the formulas (2.4) and (5.7), and then for $\xi \in \bar{C} \setminus E^{-1}$ we obtain

We now est

$$\begin{aligned} \frac{\Delta_{n,p}}{\delta_n} \varphi_n^q(\xi) &= \frac{1}{2\pi} \int_E \frac{(B_n^2(x) \varphi_n^{2-q}(x) / \delta_n) d\mu(x)}{1 - \xi x} \\ &\rightarrow \frac{1}{2\pi} \int_E \frac{d\omega_1(x)}{1 - \xi x} = \frac{1}{2\pi \sqrt{(1 - a\xi)(1 - b\xi)}} \quad \text{as } n \rightarrow \infty. \end{aligned}$$

where $(d\omega)^*$ is

$$I(\xi, x)$$

From this, since $\|\varphi_n\|_{2q} = 1$ and $\|1/\sqrt{(1 - ax)(1 - bx)}\|_{2q} = 1/k_1 = \sqrt[2q]{2/k}$, we get

$$\frac{\Delta_{n,p}}{\delta_n} \rightarrow \frac{1}{\pi \sqrt{2k}} \quad \text{as } n \rightarrow \infty,$$

and

$$\varphi_n^q(z) \rightarrow (\varphi^*)^q = \frac{\sqrt{k/2}}{\sqrt{(1 - az)(1 - bz)}} \quad (5.9)$$

for $x \in E$, it fo

uniformly on compact subsets of $\bar{C} \setminus E^{-1}$ as $n \rightarrow \infty$. We note also that (5.9) implies

$$\int_{\Gamma} \log |$$

$$\varphi_n(z) \rightarrow \varphi^*(z)$$

uniformly on compact subsets of G as $n \rightarrow \infty$.

Set

$$m_n(z) = \frac{2B_n^2(z)w_n^2(z)D_{\varphi_n^{2-q}}^2(z)D_{\psi_1}^2(z)}{\delta_n \Psi^{2n}(z)D_{w_n^2}^2(z)} \quad (5.10)$$

Similarly,

Using (5.5) and (5.6), we can write

$$m_n(z) \rightarrow 1 \quad (5.11)$$

From this, by

uniformly on compact subsets of $\bar{C} \setminus E$ as $n \rightarrow \infty$.

Denote by $g_E(z, a)$, $a \in \bar{C} \setminus E$, the Green's function of $\bar{C} \setminus E$ with a pole at a . Since for $z \in \bar{C} \setminus E$:

for $|a| > 1$, ar

$$\left| \frac{w_n^2(z)}{\Psi^{2n}(z)D_{w_n^2}^2(z)} \right| = \exp \left(-2 \sum_{k=1}^n g_E(z, 1/x_{k,n}) \right),$$

$$\min_{\xi \in \Gamma} k_n(\xi)$$

we obtain that

$$\frac{2}{\delta_n} \left| B_n^2(z)D_{\varphi_n^{2-q}}^2(z)D_{\psi_1}^2(z) \right| \exp \left(-2 \sum_{k=1}^n g_E(z, 1/x_{k,n}) \right) \rightarrow 1$$

Now, taking the help of (5.

uniformly on compact subsets of $\bar{C} \setminus E$ as $n \rightarrow \infty$. In particular,

$$\begin{aligned} k_n(z) &:= \log 2 - \log \delta_n + \log |D_{\varphi_n^{2-q}}^2(z)| \\ &\quad + \log |D_{\psi_1}^2(z)| - 2 \sum_{k=1}^n g_E(z, 1/x_{k,n}) \rightarrow 0 \end{aligned}$$

and, by (5.8):

uniformly on Γ as $n \rightarrow \infty$.

to this we use the formulas

$$\overline{\overline{bx}}\|_{2q} = 1/k_1 = \sqrt[2]{2/k},$$

∞,

$$\overline{\overline{bz}} \tag{5.9}$$

We note also that (5.9)

$$\overline{\overline{z}} \tag{5.10}$$

$$\tag{5.11}$$

of $\bar{C} \setminus E$ with a pole at a .

$$1/x_{k,n}$$

$$1/x_{k,n} \rightarrow 1$$

articular,

$$1/x_{k,n} \rightarrow 0$$

We now estimate the function $k_n(z)$ for $z \in \Gamma$. First, it is not hard to see that

$$\min_{\xi \in \Gamma} k_n(\xi) \leq \int_{\Gamma} k_n(\xi) (d\omega)^*(\xi) \leq \max_{\xi \in \Gamma} k_n(\xi), \tag{5.12}$$

where $(d\omega)^*$ is the balayage of the measure $d\omega$ on Γ . Second, note that since

$$\begin{aligned} I(\xi, x) &= \frac{1}{2\pi} \int_{\Gamma} \frac{\sqrt{(\xi-a)(\xi-b)}}{(\xi-x)(1-a\xi)(1-b\xi)} |d\xi| \\ &= \frac{1}{2\pi i} \int_{\Gamma} \frac{1}{(\xi-x)\sqrt{(1-\xi a)(1-\xi b)}} d\xi \\ &= \frac{1}{\sqrt{(1-ax)(1-bx)}} \end{aligned}$$

for $x \in E$, it follows (see also (4.2), (4.3), and (5.4)) that

$$\begin{aligned} \int_{\Gamma} \log |D_{\psi_1}^2(\xi)| (d\omega)^*(\xi) &= \int_{\Gamma} \log |D_{\psi_1}^2(\xi)| |\varphi^*(\xi)|^{2q} |d\xi| \\ &= k \int_a^b \frac{\log(\psi_1(x))}{\sqrt{(x-a)(b-x)}} \operatorname{Re}(I(\xi, x)) dx \\ &= \log \mathcal{G}_{\omega}(\psi_1). \end{aligned} \tag{5.13}$$

Similarly,

$$\int_{\Gamma} \log |D_{\varphi_n}^{2-q}(\xi)| (d\omega)^*(\xi) = \log \mathcal{G}_{\omega}(\varphi_n^{2-q}). \tag{5.14}$$

From this, by the relation

$$\begin{aligned} \min_{\xi \in \Gamma} k_n(\xi) &\leq \log 2 - \log \delta_n + \log \mathcal{G}_{\omega}(\varphi_n^{2-q}) + \log \mathcal{G}_{\omega}(\psi_1) - 2n \log \frac{1}{r} \\ &\leq \max_{\xi \in \Gamma} k_n(\xi). \end{aligned}$$

Now, taking into account that $k_n(z) \rightarrow 0$ uniformly on Γ as $n \rightarrow \infty$, and with the help of (5.9), we obtain that

$$\frac{\pi \sqrt{2k} \Delta_{n,p}}{2 \mathcal{G}_{\omega}((\varphi^*)^{2-q}) \mathcal{G}_{\omega}(\psi_1) r^{2n}} \rightarrow 1,$$

as $n \rightarrow \infty$. Consequently, on the basis of the relation

$$\pi\sqrt{2k}\mathcal{G}_\omega((\varphi^*)^q)\mathcal{G}_\omega(\psi) = \mathcal{G}_\omega(\psi_1)$$

(see (5.2)), we get

$$\frac{\Delta_{n,p}}{2\mathcal{G}_\omega((\varphi^*)^2)\mathcal{G}_\omega(\psi)r^{2n}} \rightarrow 1 \tag{5.15}$$

as $n \rightarrow \infty$.

5.3. The Case $p = 1$

We prove that

$$\lim_{n \rightarrow \infty} \frac{\Delta_{n,1}}{2r^{2n}\mathcal{G}_\omega(\psi)} = 1. \tag{5.16}$$

Since, for any function $\varphi \in H_{2q}(G)$:

$$\max_{z \in E} |\varphi(z)| \leq C^{1/2q} \|\varphi\|_{2q}$$

where

$$C = \frac{1}{2\pi} \max_{z \in E} \left(\frac{1 + |z|}{1 - |z|} \right)$$

(see [22, Chapter II]), it follows from (1.19) that for $1 < p \leq \infty$:

$$(1/2\pi)^{1/q} \Delta_{n,1} \leq \Delta_{n,p} \leq C^{1/q} \Delta_{n,1}$$

and

$$\begin{aligned} \frac{(1/2\pi)^{1/q} \Delta_{n,1}}{\mathcal{G}_\omega((\varphi^*)^2)2r^{2n}\mathcal{G}_\omega(\psi)} &\leq \frac{\Delta_{n,p}}{2r^{2n}\mathcal{G}_\omega(\psi)\mathcal{G}_\omega(\varphi^{*2})} \\ &\leq \frac{C^{1/q} \Delta_{n,1}}{\mathcal{G}_\omega((\varphi^*)^2)2r^{2n}\mathcal{G}_\omega(\psi)} \end{aligned}$$

Letting $n \rightarrow \infty$ and then $p \rightarrow 1$, with the help of the relation

$$\lim_{p \rightarrow 1} \mathcal{G}_\omega((\varphi^*)^2) = \lim_{q \rightarrow \infty} \exp \left(\frac{1}{q} \int_E \log \left| \frac{k/2}{(1-ax)(1-bx)} \right| d\omega(x) \right) = 1,$$

we obtain (5.16).

5.4. The Limiting Di

Here and in what foll investigate the limiting asymptotics for Blasch

First we establish

It is easy to see that

$$\frac{B_n^2 \varphi_j}{\Delta_n}$$

From this, the fact th

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We now prove (3 formula

Since

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and

5.4. *The Limiting Distribution of Poles of h_n*

Here and in what follows we assume that $1 < p \leq \infty$. In this subsection we investigate the limiting distribution of poles of h_n . Namely, we obtain Szegő-type asymptotics for Blaschke products B_n associated with the poles of h_n .

First we establish

$$(5.15) \quad \frac{B_n^2 \varphi_n^2 d\mu}{\Delta_{n,p}} \xrightarrow{*} d\omega \quad \text{as } n \rightarrow \infty.$$

It is easy to see that

$$\frac{B_n^2 \varphi_n^2 d\mu}{\Delta_{n,p}} = \frac{\delta_n}{\Delta_{n,p} \pi \sqrt{2k}} \pi \sqrt{2k} \varphi_n^q \frac{B_n^2 \varphi_n^{2-q}}{\delta_n} d\mu.$$

From this, the fact that

$$(5.16) \quad \frac{\delta_n}{\Delta_{n,p} \pi \sqrt{2k}} \pi \sqrt{2k} \varphi_n^q \rightarrow \pi \sqrt{2k} (\varphi^*)^q$$

uniformly on E as $n \rightarrow \infty$ (see (5.8) and (5.9)), and the fact that

$$\frac{B_n^2 \varphi_n^{2-q}}{\delta_n} d\mu \xrightarrow{*} d\omega_1 \quad \text{as } n \rightarrow \infty,$$

and with the help of (5.2), we get

$$\frac{B_n^2 \varphi_n^2 d\mu}{\Delta_{n,p}} \xrightarrow{*} d\omega \quad \text{as } n \rightarrow \infty.$$

We now prove (3.8). The functions u_n , $n = 1, 2, \dots$, are introduced by the formula

$$u_n = \frac{B_n^2 \mathcal{D}_\psi^2 \mathcal{D}_{(\varphi^*)^2}}{\Phi^{2n} \mathcal{G}_\omega(\psi) \mathcal{G}_\omega((\varphi^*)^2)}.$$

Since

$$|\mathcal{D}|^2 = \mathcal{G}_\omega(\psi) \quad \text{and} \quad |\mathcal{D}|^2 = \mathcal{G}_\omega((\varphi^*)^2) \quad \text{on } \Gamma$$

and

$$|\mathcal{D}|^2 = \psi \quad \text{and} \quad |\mathcal{D}|^2 = (\varphi^*)^2$$

almost everywhere on E , we obtain that for each nonnegative integer n :

$$|u_n| = 1 \quad \text{on } \Gamma$$

and

$$|u_n| = B_n^2 (\varphi^*)^2 \psi / r^{2n} \mathcal{G}_\omega(\psi) \mathcal{G}_\omega((\varphi^*)^2) \quad (5.17)$$

(ψ)

$< p \leq \infty$:

$\Delta_{n,1}$

$$\frac{\int_{\Gamma} \frac{d\omega(x)}{-bx} \Big|_{d\omega(x)} = 1,$$

relation

almost everywhere on E . We observe that the function u_n is analytic in $\bar{C} \setminus (E \cup E^{-1})$. We prove that $u_n \rightarrow 1$ uniformly on compact subsets of $\bar{C} \setminus (E \cup E^{-1})$ as $n \rightarrow \infty$. Since $|u_n| = 1$ on Γ , it is sufficient to show that $u_n \rightarrow 1$ uniformly on compact subsets of $G \setminus E$ as $n \rightarrow \infty$.

Set

$$v_n(z) = \frac{B_n^2 u_n^2(z) D_{(\varphi^*)^2}^2(z) D_\psi^2(z)}{\Psi^{2n}(z) D_{\varphi_n^2}^2(z) r^{2n} \mathcal{G}_\omega((\varphi^*)^2) \mathcal{G}_\omega(\psi)}$$

We remark that the function $v_n(z)$ is analytic in $\bar{C} \setminus E$ and

$$|v_n| = B_n^2 (\varphi^*)^2 \psi / r^{2n} \mathcal{G}_\omega((\varphi^*)^2) \mathcal{G}_\omega(\psi)$$

almost everywhere on E . Using the relation

$$D_{\psi_1}^2(z) = \pi \sqrt{2k} D_{(\varphi^*)^2}^2(z) D_\psi^2(z)$$

(see (5.2)) and the formula (5.10) for $m_n(z)$, we can rewrite $v_n(z)$ in the form

$$v_n(z) = m_n(z) \cdot \frac{\delta_n}{\pi \sqrt{2k} 2r^{2n} \mathcal{G}_\omega((\varphi^*)^2) \mathcal{G}_\omega(\psi)} \cdot \frac{D_{(\varphi^*)^{2-q}}^2(z)}{D_{\varphi_n^{2-q}}^2(z)}$$

Since

$$\frac{\delta_n}{\pi \sqrt{2k} 2r^{2n} \mathcal{G}_\omega((\varphi^*)^2) \mathcal{G}_\omega(\psi)} \rightarrow 1 \quad \text{as } n \rightarrow \infty$$

(see (5.8) and (5.15)) and

$$\varphi_n \rightarrow \varphi^*$$

uniformly on E as $n \rightarrow \infty$, from the last equality together with (5.11), we obtain that

$$v_n(z) \rightarrow 1 \tag{5.18}$$

uniformly on compact subsets of $\bar{C} \setminus E$ as $n \rightarrow \infty$.

Let us consider the function u_n/v_n . This function is analytic in $\bar{C} \setminus (E \cup E^{-1})$, $u_n/v_n \neq 0$ in $\bar{C} \setminus (E \cup E^{-1})$, $|u_n/v_n| = 1$ almost everywhere on E , and $|u_n/v_n| \rightarrow 1$ uniformly on Γ as $n \rightarrow \infty$. From this, the fact that $\{u_n/v_n\}_{n=1}^\infty$ and $\{v_n/u_n\}_{n=1}^\infty$ form normal families in G , and the fact that $(u_n/v_n)(x) > 0$ for $x \in (b, 1)$, we get

$$u_n/v_n \rightarrow 1$$

uniformly on compact subsets of $G \setminus E$ as $n \rightarrow \infty$. Therefore, by (5.18):

$$u_n(z) \rightarrow 1 \tag{5.19}$$

uniformly on compact subsets of $G \setminus E$ as $n \rightarrow \infty$. It remains to remark that the relation (3.8) follows directly from (5.19). \square

Proof of Theor

The arguments are b
 h_n , where $h_n =$

$$(f - h_n)(z) = \frac{1}{(z)}$$

According to Th

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we get

$$\frac{(f - h_n)}{(D_\psi^2 D_{(\varphi^*)^2}^2)} \Phi^{2n}(z)$$

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 We have

$$d\omega(x) = - \sqrt{}$$

The constants k and

on $\bar{C} \setminus (E \cup E^{-1})$ as uniformly on

6. Proof of Theorem 6

The arguments are based on the following representation (see (2.3)) of the function $f - h_n$, where $h_n = P_n/Q_n$:

$$(f - h_n)(z) = \frac{1}{(B_n^2 \varphi_n^{2-q})(z)} \cdot \frac{1}{2\pi i} \int_E \frac{B_n^2(x) \varphi_n^{2-q}(x) d\mu(x)}{z-x}, \quad z \in G \setminus E.$$

According to Theorem 4:

$$\frac{(B_n \mathcal{D}_\psi \mathcal{D}_{(\varphi^*)^2})(z)}{\Phi^n(z) \sqrt{\mathcal{G}_\omega(\psi) \mathcal{G}_\omega(\varphi^*)}} \rightarrow 1 \tag{6.1}$$

uniformly on compact subsets of $\bar{C} \setminus (E \cup E^{-1})$ as $n \rightarrow \infty$. Using the fact that

$$\varphi_n \rightarrow \varphi^*$$

uniformly on compact subsets of G as $n \rightarrow \infty$, and weak* convergence of the measures

$$\frac{B_n^2(\varphi^*)^2 d\mu}{2r^{2n} \mathcal{G}_\omega(\psi) \mathcal{G}_\omega((\varphi^*)^2)} \xrightarrow{*} d\omega \quad \text{as } n \rightarrow \infty, \tag{6.2}$$

we get

$$\frac{(f - h_n)(z)}{\Phi^{2n}(z)} \rightarrow \frac{1}{\pi i (\varphi^*)^{2-q}(z)} \int_E \frac{(1/(\varphi^*)^q)(x) d\omega(x)}{z-x} \tag{6.3}$$

), we obtain

(5.18)

uniformly on compact subsets of $G \setminus E$ as $n \rightarrow \infty$.

We have

$$\varphi^*(z) = \frac{k_1}{\sqrt{(1-az)(1-bz)}}, \quad z \in G,$$

and

$$d\omega(x) = \frac{k}{\sqrt{(x-a)(b-x)(1-ax)(1-bx)}} dx, \quad x \in [a, b].$$

(8):

(5.19)

The constants k and k_1 are connected by the relation

$$k_1^{2q} = \frac{k}{2}.$$

mark that the

□

ie form

$(E \cup E^{-1})$,
 $u_n/v_n \rightarrow 1$
 $i \{v_n/u_n\}_{n=1}^\infty$
 $b, 1)$, we get

We now compute the expression on the right-hand side of the relation (6.3). We obtain

$$\begin{aligned} & \frac{1}{\pi i (\varphi^*)^{2-q}(z)} \int_E \frac{(1/(\varphi^*)^q)(x) d\omega(x)}{z-x} \\ &= \frac{((1-az)(1-bz))^{(2-q)/2q}}{\pi i k_1^{2-q}} \\ & \quad \times \int_E \frac{k \sqrt{(1-ax)(1-bx)} dx}{\sqrt{(x-a)(b-x)(1-ax)(1-bx)} k_1^q (z-x)} \\ &= \frac{2((1-az)(1-bz))^{(2-q)/2q}}{\pi i (k/2)^{(1-q)/q}} \int_E \frac{dx}{\sqrt{(x-a)(b-x)(z-x)}}. \end{aligned}$$

By the Cauchy formula, the last integral is equal to

$$\frac{\pi}{\sqrt{(z-a)(z-b)}}.$$

So, we get

$$\frac{(f - \dots)}{(D_{\psi}^2 D_{(\varphi^*)^2}^2(z)) r^{2n}} \rightarrow \frac{2}{i} \sqrt{\frac{(1-az)(1-bz)}{(z-a)(z-b)}} \left(\frac{(1-az)(1-bz)}{k/2} \right)^{(1-q)/q} \quad (6.4)$$

$$\Phi^{2n}(z)$$

uniformly on compact subsets of $G \setminus E$ as $n \rightarrow \infty$. The relation (3.12) follows immediately from (6.4).

As above, using (2.3), (6.1), and (6.2), we get (3.13). \square

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relation (6.3). We

$$\frac{-x}{-x(z-x)} \quad (6.4)$$

relation (3.12) follows

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On the Power for Weighted

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Abstract. We study the power of standard and order natural approximation for two classes of functions. We obtain bounds for starlikeness and tractability for

1. Introduction

The weighted approximation of functions $f: D \rightarrow \mathbb{C}$ uses a finite number of

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